The Cosmic Microwave Background

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Contents

Chapter 1

introduction

The aim of these lectures is to give a complete overview of the physics of the relic radiation in cosmology, the Cosmic Microwave Background, in the framework of modern cosmology. At the end of the course, a student should be able to understand most of the modern scientific papers on this matter, ranging from the theoretical details of the CMB anisotropies, down to the physical content and implications of the current experimental data.
In this chapter, we give an overview of the course, specifying the most important literature to look at, as well as reviewing the plan of the lectures. In the end, we also fix the notation we adopt, and introduce very basic elements of general relativity.

1.1 things to know for attending the course

From the point of view of cosmology, these lectures are almost self consistent, altough who's attending might benefit from having familiarity with the physics of the Friedmann Robertson Walker (FRW) background cosmology, as well as any knowledge of cosmological perturbations. Moreover, the physics of the CMB is well known, as the energy scales at which the CMB decouples from the other components, corresponds to a temperature of about 3000 K.
On the other hand, some knowledge of general relativity is necessary in order to apprach these lectures properly. Indeed, the whole cosmology is rooted deeply in general relativity, and with the level of precision of cosmological measurements, getting to the percent or better, it is no longer possible to take Newtonian shortcuts. This is relevant in particular for the whole scheme of cosmological perturbations, which is a relevant part of the present work.

The main text where studying while attending the lectures should be represented by the present notes. The students are on the other hand welcome to consult the books and papers from where these notes have been taken. They are:

- textbook by Scott Dodelson, Modern Cosmology, Elsevier Science, 2003,
- review paper by Hideo Kodama and Misao Sasaki, Progresses of Theoretical Physics Supplement 78, 1, 1984,
- paper by Wayne Hu and Martin White 1997, *Phys. Rev. D* 56, 596.

The

- textbook by Andrew R. Liddle and David H. Lyth, Cosmological Inflation and Large Scale Structure, Cambridge Press 2000,

may be also relevant.

1.2 plan of the lectures

The first part of the course is devoted to a summary of the FRW cosmological background, setting the notation of the course and yielding also a continuity with possible previous courses that the students may have attended. The lectures then cover the following topics:

- cosmological perturbation theory,
- black-body radiation in cosmology,
- Boltzmann equation, gravitational and Thomson scattering,
- harmonic expansion and anisotropies,
- large scale anisotropies and acoustic oscillations,
- CMB observables,
- status of CMB observations.

1.3 notation and micro-elements of general relativity

Spacetime is described by three spatial dimensions plus the time coordinate. Greek indeces run from 0 to 3, while latin indeces are used for spatial directions, from 1 to 3. We use x to indicate a generic spacetime point, $\vec{x}$ and $\hat{x}$ for its spatial component and versor, respectively. The fundamental constants we use are the Boltzmann and Gravitational ones, indicated with k_B and G, respectively. We work with unitary light speed velocity, $c = 1$, never using the Planck constant $\hbar$.

In general relativity, the infinitesimal distance from two spacetime points is defined as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \tag{1.1}$$

where $g_{\mu\nu}(x)$ is the metric tensor. By an appropriate change on reference frame, it is always possible to reduce the metric tensor to the Minkowski one, meaning that the system changes to the one which in free fall locally in x. The signature of the metric tensor we adopt is the following:

$$(-, +, +, +) \ . \tag{1.2}$$

The inverse of the metric tensor is represented with the indeces up:

$$g_{\mu\rho} g^{\rho\nu} = \delta_\mu^\nu \ , \tag{1.3}$$

where δ_μ^ν is the Kronecker delta. We shall use the Kronecker delta in arbitrary index configuration:

$$\delta_\mu^\nu = \delta_{\mu\nu} = \delta^{\mu\nu} = 1 \text{ if } \mu = \nu, \ 0 \text{ otherwise.} \tag{1.4}$$

The Christoffel symbols are defined as usual as

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\right) \ , \ \Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\alpha'}\left(\frac{\partial g_{\alpha'\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha'\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha'}}\right) . \tag{1.5}$$

The Riemann, Ricci and Einstein tensors are given by

$$R^{\alpha}_{\beta\mu\nu} = \frac{\Gamma^{\alpha}_{\beta\nu}}{\partial x^{\mu}} - \frac{\Gamma^{\alpha}_{\beta\mu}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\lambda}_{\beta\nu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\mu} \ , \tag{1.6}$$

$$R_{\mu\nu} = R^{\alpha}_{\alpha\mu\nu} \ , \tag{1.7}$$

and

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \ , \tag{1.8}$$

where $R = R^{\mu}_{\mu}$ is the Ricci scalar and the repeated indeces are summed.
To simplify the notation, let us introduce the following conventions for derivation in general relativity:

- $_{;\mu} \equiv \nabla_{\mu}$ means covariant derivative with respect to x^{μ},
- $_{|a} \equiv {}^{s}\nabla_{a}$ means covariant derivative with respect to the spatial metric, i.e. the 3×3 array obtained removing the time column and row from the metric tensor in (1.1),
- $_{,\mu}$ means ordinary derivative with respect to x^{μ} .

A uni-dimensional tensor, or vector, v_{μ} can be obtained via covariant derivation of a scalar quantity s as

$$v_{\mu} = s_{;\mu} = s_{,\mu} \ , \tag{1.9}$$

where the last equality holds for scalars only. A tensor can be obtained via covariant derivation of a vector as

$$t_{\mu\nu} = v_{\mu;\nu} = v_{\mu,\nu} - v_{\alpha}\Gamma^{\alpha}_{\mu\nu} \ . \tag{1.10}$$

Further covariant derivative raises the rank of tensors:

$$\begin{aligned} u_{\mu\nu\rho} &= t_{\mu\nu;\rho} = t_{\mu\nu,\rho} - t_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - t_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu} \\ u^{\nu}_{\mu\rho} &= t^{\nu}_{\mu;\rho} = t^{\nu}_{\mu,\rho} - t^{\nu}_{\alpha}\Gamma^{\alpha}_{\mu\rho} + t^{\alpha}_{\mu}\Gamma^{\nu}_{\rho\alpha} \ . \end{aligned} \tag{1.11}$$

Chapter 2

Homogeneous and isotropic cosmology

The FRW metric is built upon the hypothesis that space is homogeneous and isotropic at all times. The first condition means that at a given time, the physical properties, e.g. particle number density, are the same in each point. The second condition means that any physical quantity does not depend on the direction of an observer locate in any spacetime point x.
These assumptions simplify dramatically the structure of the metric tensor $g_{\mu\nu}$. A spherical symmetry around each spacetime location is necessary, and no off-diagonal terms are left; homegeneity and isotropy leave essentially only two degrees of freedom to the system. The first one is a global scale factor, fixing at each time the value of physical lengths. The second one is related to the spacetime curvature, as an homogeneous metric can be globally more or less curved. The form of the fundamental length element is therefore

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{1}{1 - Kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) , \qquad (2.1)$$

where $a(t)$ and K represent the global scale factor and the curvature, respectively; r, θ and ϕ are the usual spherical coordinates for radius, polar and azimut angle, respectively.
The physical meaning of the scale factor can be read straightforwardly from the metric, and the only point to discuss concerns its dimension. One may assign physical dimensions to the scale factor a or to the radial coordinate r; in this lectures, we choose the second option. Concerning the curvature, some more discussion is needed. The first point is about dimensions again; if r is dimensionless, K is also dimensionless. If r is a length, then K is the inverse of the square of a length. Moreover, if $K = 0$, then the spatial part of the length (2.1) is Minkowskian, and in this case the FRW metric is flat. If $K > 0$, there is an horizon in the metric, given by $r_H = \pm 1/\sqrt{K}$; this means that an infinite physical distance corresponds to those coordinates, regardless of the value of the scale factor a, and the FRW metric is closed; note that this does not conflict with the assumption of homogeneity, as this property is the same as seen from all spacetime locations. If $K < 0$, the opposite happens, as there is no horizon, and the distance between two space points vanishes at infinity; in this case the FRW metric is open. Finally, note that one may always change the overall normalization of a or r in (2.1), and therefore, as a pure convention, we can restrict our attention to three relevant cases for K:

$$\begin{aligned} K &= -1 \quad \text{open FRW} , \\ K &= 0 \quad \text{flat FRW} , \\ K &= +1 \quad \text{closed FRW} . \end{aligned} \qquad (2.2)$$

2.1 comoving coordinates

Of course one might apply any change of coordinate to the FRW metric, making that appearing differently. On the other hand the form (2.1) is the form that is common for cosmological purposes. The reason is that the expansion, represented by the scale factor a, has been factored out of the spatial dependence. This leads us to the concept of comoving coordinate, i.e. at rest with respect to the cosmic expansion, or in other words characterized by the spacetime points for which

$$r = \text{constant} \ , \ \theta = \text{constant} \ , \ \phi = \text{constant} \ , \tag{2.3}$$

where r, θ and ϕ are coordinates in the frame where the metric assumes the form 2.1. To visualize, one may think that galaxies are the tracers of the cosmic expansion, or in other words, their motion is approximately described by 2.3. In the original Hubble view of the cosmic expansion, this corresponds to assign the whole motion of galaxies to the cosmic expansion, giving them a fixed comoving coordinate.

2.1.1 conformal time

Although time does not enter in the discussion about comoving coordinates above, there is a very common time variable which may replace the ordinary time in (2.1). By performing the coordinate change

$$d\tau = \frac{dt}{a(t)} \ , \tag{2.4}$$

the FRW metric may be easily written as

$$g_{\mu\nu} \equiv a^2 \cdot \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & (1-Kr^2)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \equiv a^2 \gamma_{\mu\nu} \tag{2.5}$$

so that the cosmic expansion is completely factored out of the comoving part of the metric, which we define as $\gamma_{\mu\nu}$; τ is the conformal time, and is our time variable in the following, unless otherwise specified. We will indicate the conformal time derivative with $\dot{}$, while those with respect to the ordinary time are indicated with the subscript $_t$. It is also useful to define two different quantities describing the velocity of the expansion, i.e.

$$H = \frac{a_t}{a} \quad , \quad \mathcal{H} = \frac{\dot{a}}{a} \ , \tag{2.6}$$

named ordinary and conformal Hubble expansion rates, respectively; as it is easy to see, the two are related by $H = \mathcal{H}/a$.

2.2 stress energy tensor

The stress energy tensor specifies the contents of spacetime, in terms of physical entities, i.e. particles and their properties. We limit ourselves here to describe a perfect relativistic fluid, homogeneous and isotropic. These assumptions restrict quite alot the complexity of a general expression for a stress energy tensor. The quantities that characterized it are just the energy density, ρ, and the pressure p. The quantities in the stress energy tensor which has direct physical meaning are those with covariand and controvariant indeces. In this form, T_μ^ν is most easy as the $(0,0)$ components represent the energy density, while p is isotropically assigned to all directions as

$$T_\mu^\nu \equiv \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} , \tag{2.7}$$

where the minus to the energy density is due to the choice of our signature (1.2). The stress energy tensor may also be written as

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} , \tag{2.8}$$

where u^μ represents the quadri-velocity of a fluid element, with an affine parameter which for convenience may be taken as the conformal time itself:

$$u^\mu = \frac{dx^\mu}{d\tau} . \tag{2.9}$$

In analogy with the normalization of the quadri-impulse of a particle with mass m, $p_\mu p^\mu = -m^2$, and since the energy is absorbed by the term $\rho + p$ in (2.9), the quadri-velocities are normalized as $u_\mu u^\mu = -1$. In comoving coordinates, where the $u^a = 0$, this condition implies

$$u^\mu \equiv \left(\frac{1}{a}, 0, 0, 0 \right) . \tag{2.10}$$

2.3 expansion and conservation

The Einstein and conservation equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \ T_{\mu\nu}^{;\nu} = 0 \tag{2.11}$$

reduce to two differential equations only, where the independent variable is the time τ, expressing the dynamics of the expansion, plus the conservation of energy, respectively. The first one is the Friedmann equation

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \rho - \frac{K}{a^2} , \tag{2.12}$$

which is equivalent to the equation ruling the acceleration of the expansion:

$$\dot{\mathcal{H}} - \mathcal{H}^2 = -4\pi G (\rho + p) . \tag{2.13}$$

The conservation equation becomes

$$\dot\rho + 3\mathcal{H}(\rho + p) = 0 . \tag{2.14}$$

As it is evident, it is impossible to solve this system if some relation between pressue and energy density is given, $p(\rho)$. For interesting cases, as those we shall see in the next Section, pressure is proportional to the energy density:

$$p = w\rho \ , \tag{2.15}$$

where w is the equation of state of the fluid.

2.4 relativistic and non-relativistic matter, dark energy

So far we did not consider the case in which the stress energy tensor is made by more than one component, although in a realistic case, several components are present at the same time. In this case, the stress energy tensor we treated so far corresponds to the total one, sum over those corresponding to each of the components, labeled by c as follows:

$$T_{\mu\nu} = \sum_c {}_cT_{\mu\nu} \ . \tag{2.16}$$

The single stress energy tensors may not be conserved as a result of mutual interactions between the different component. Therefore, the conservation equation for each component may be written as

$${}_cT^{;\nu}_{\mu\nu} = {}_cQ_\mu \ , \tag{2.17}$$

where ${}_sQ_\mu$ expresses the non-conservation. Since the total stress energy tensor must be conserved, the interactions between the different components must be such that

$$\sum_c {}_cQ_\mu = 0 \ . \tag{2.18}$$

The simplest example of cosmological component is represented by the non-relativistic (nr) component. The usual example is that of particles at thermal equilibrium, and characterized by a temperature giving rise to a thermal agitation which is negligible with respect to the mass m, so that the impulse of each of those particles is

$$p^2 = p_\mu p^\mu \simeq m^2 \ . \tag{2.19}$$

Whatever the interaction is, in this regime collisions are negligible. No collisions means no pressure, therefore for this species, the equation of state is simply zero. Such component in cosmology is commonly known as Cold Dark Matter (CDM). As it is easy to verify, the time dependence of this component, assuming that it is decoupled from the others, may be expressed as a function of the scale factor as

$$\rho_{nr} \propto a^{-3} \ . \tag{2.20}$$

The next example is opposite in many respects. Relativistic (r) particles at thermal equilibrium are characterized by an energy which is dominated by thermal agitation rather than mass. By applying the laws of statistical quantum mechanics for relativistic particles at thermal equilibrium, one finds that pressure and energy density are related by the following relation:

$$p_r = \frac{1}{3}\rho_r \ . \tag{2.21}$$

As it is easy to verify, this implies

$$\rho_r \propto a^{-4} \ , \tag{2.22}$$

which has a direct intuitive meaning. Indeed, taking photons as an example, each of those carries an energy $h\nu$ where h is the Planck constant, thus redshifting as a result of the stretching of the wavelenght. This is responsible for the extra-power in (2.22) with respect to (2.20), which contains only the contribution from the dilution as a result of the expansion of the volume.
A third case, most interesting and dense of theoretical implications, is the case in which the energy density is conserved, i.e.

$$p_{de} = -\rho_{de} \ , \tag{2.23}$$

where the subscript means dark energy, which is the class of cosmological species which produces an equation of state close or equal to -1. A constant vacuum energy density appeared for the first time in the form of a cosmological constant Λ, introduced by Einstein himself in the Einstein equations as a pure geometrical term:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{2.24}$$

Indeed, bringing it to the right hand side, and passing to the mixed form for indeces, one gets

$$G_\mu^\nu = 8\pi G \left(T_\mu^\nu - \frac{\Lambda}{8\pi G}\delta_\mu^\nu \right) \ , \tag{2.25}$$

and looking at the form of the stress energy tensor (1.10) it is straightforward to verify that the one related to the cosmological constant is characterized by $p_\Lambda = -\rho_\Lambda = -\Lambda/8\pi G$.

Chapter 3

Cosmological perturbation theory

The whole concept is to classify and understand small perturbations around the FRW metric, in a general relativistic framework. The perturbed FRW metric tensor may be defined as

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} = a^2(\gamma_{\mu\nu} + h_{\mu\nu}) \ , \tag{3.1}$$

where the bar means background plus perturbations, i.e. the complete system. We limit our analysis to the linear regime: since $\gamma_{\mu\nu}$ contains coefficients of order 1, e.g. in the flat FRW it coincides with the Minkoski metric having ones on the main diagonal, linearity may be defined as

$$\gamma_{\mu\nu} \text{ containing terms of order 1 or larger, } h_{\mu\nu} \ll 1 \ . \tag{3.2}$$

At the same time, the stress energy tensor is perturbed as

$$\bar{T}_\mu^\nu = T_\mu^\nu + \delta T_\mu^\nu \ , \tag{3.3}$$

and in this case linearity may be expressed in terms of the non-zero quantities in the background tensor, $T_{\mu\nu}$:

$$\delta T_\mu^\nu \ll \rho, p \ . \tag{3.4}$$

One may question what is the motivation for dealing with linear perturbations only in cosmology. Indeed, structures we see around us are markedly non-linear in the density distribution, like galaxies of cluster of galaxies. The underlying assumption is that most of the cosmological evolution occurs in linear regime, while non-linearity is confined on very small scales, reaching the typical size of a galaxy or a cluster of galaxies only in cosmologically recent epochs. The great support for this assumption comes from the CMB itself. As it is well known nowadays (see e.g. Bennett et al., 2003, and references therein) the CMB temperature anisotropies are at the 10^5 level relatively to the average temperature over the whole sky, down to an angular resolution of a few arcminutes. This remarkable isotropy suffests that the density fluctuations, and cosmological perturbations in general, where in a linear regime at the epoch in which the CMB radiation decoupled from the rest of the system. Then, the linear approximation should be valid to describe the bulk of physical cosmology on a large interval of time and physical scales, before and after the CMB origin, breaking down only recently and on scales smaller than those of galaxy cluster.

3.1 background and perturbation dynamics

Linearity allows to neglect the terms of second or higher orders, i.e. involving products of fluctuations. A direct consequence concerns the relativistic algebra of cosmological perturbations. Of course, in a perturbed FRW enviroment, tensor indeces are raised and lowered using the complete metric, $\bar{g}_{\mu\nu}$. On the other hand, linearity allows to neglect the second or higher order terms. This means that for cosmological perturbations, represented by $\delta g_{\mu\nu}$ and $\delta T_{\mu\nu}$, indeces are raised and lowered using the background metric $g_{\mu\nu}$. Another, most important consequence of the linear scheme is that background and perturbation dynamics do not mix: the prescription is no longer valid whenever $h_{\mu\nu}$ approaches unity or δT_μ^ν gets close to the background energy density or pressure. Therefore, any equation in cosmology splits in two, concerning the background and perturbation dynamics, respectively:

$$\bar{G}_{\mu\nu} = 8\pi G\bar{T}_{\mu\nu} \equiv \left\{ \begin{array}{c} G_{\mu\nu} = 8\pi G T_{\mu\nu} \\ \delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \end{array} \right. , \tag{3.5}$$

$$\bar{T}_{\mu\nu}^{;\nu} = 0 \equiv \left\{ \begin{array}{c} T_{\mu\nu}^{;\nu} = 0 \\ \delta T_{\mu\nu}^{;\nu} = 0 \end{array} \right. . \tag{3.6}$$

This does not mean that the background and perturbation dynamics are fully decoupled. The scale factor evolution, as well as the dynamics of energy density and pressure do affect the perturbation equations, appearing explicitely in those, as we shall see in the following.

3.2 decomposition of cosmological perturbations in general relativity

We now carry out a general discussion concerning the different components of the cosmological perturbations, $h_{\mu\nu}$ and $\delta T_{\mu\nu}$. This is done on the basis of their physical properties in a general relativistic framework, yielding a general classification, which is one of the basis of modern cosmology. The fluctuations depend on the spacetime point $x \equiv (\tau, \vec{x})$. The decomposition is made considering the spatial part only, while the perturbed Einstein and conservation equations (3.5, 3.6) give the time dependence of each species.
Let us consider in general the space dependence of functions in general relativity, but restricting ourselves to the spatial metric at a fixed time, γ_{ij}. Under spatial rotations, a function of the space may behave as a scalar, vector or tensor:

$$s(\vec{x}) \ , \ v^i(\vec{x}) \ , \ t^{ij}(\vec{x}) \ . \tag{3.7}$$

While the physical nature of scalars is unique, the classifiction gets non-trivial already for vectors. Indeed, each vector function may be divided in full generality in a part which may be derived by space covariant derivation of a scalar, and the rest. The first component is called scalar-type component of vectors, while the second one is called vector-type component of vectors. In a similar way, tensors may be divided in three components. The first one, called scalar-type component of tensors, comes from double space covariant derivation of a scalar function. The second one, called vector-type component of tensors, comes from space covariant derivation of a vectory-type component of a vector. The remaining part is called tensor-type component of tensors. Let us see all that in formulas now. For vectors, one has

$$v^i = s^{|i} + v_*^i \ , \tag{3.8}$$

indicating scalar-type and vector-type components, respectively. Still maintining full generality, s may be chosen is such in order to be responsivle for the divergence of the whole vector function:

$$\Box s \equiv s^{|i}_{|i} = v^{i}_{|i} = v \ . \tag{3.9}$$

This implies that the vector-type component is divergenceless:

$$v^{i}_{*|i} = 0 \ . \tag{3.10}$$

As we will see in a moment, this has a direct and intuitive physical meaning. For tensors, the decomposition may be written as

$$t^{ij} = s^{|ij} + v^{i|j}_{*} + t^{ij}_{*} \ . \tag{3.11}$$

In analogy with what has been done for vectors, in general the scalar-type component is responsible for the trace of the tensor, obeying the equation

$$\Box s = t^{i}_{i} \equiv t \ , \tag{3.12}$$

while the vector-type is divergenless and obeys the same condition as in (3.10). This implies that the tensor-type component is traceless:

$$t^{i}_{*i} = 0 \ . \tag{3.13}$$

Moreover, the tensor-type components may be also shown such that

$$t^{|j}_{*ij} = 0 \ . \tag{3.14}$$

The physical interpretation of this decomposition is rather simple. Consider for example the vector field of velocities caused by gravitational infall toward the center of some overdensity. That is an example of scalar-type component of a vector field, as the divergence of this field is non-null. Physically, the nature of such motion is clearly of scalar origin, because it is completely caused by the overdensity in the center. Consider now a vortex. In this case, no static density perturbation can cause such motion, which is divergenless by construction. This is an example of a vector field which is made completely by its vector-type component. To finish with the examples, the tensor-type component of tensors correspond to the tensor fluctuation modes in general relativity, i.e. the gravitational waves, as we see below.

3.3 Fourier expansion of cosmological perturbations

The perturbed Einstein and conservation equations will have in general the form

$$A \cdot \delta_1 + B \cdot \delta_2 + C \cdot \delta_3 + ... = 0 \ , \tag{3.15}$$

where $A, B, C, ...$ are function of time only, while the δ_i represent perturbations and their time derivatives; no products between perturbations are allowed in a linear theory. As a consequence, working in the Fourier space becomes feasible, as the absence of products between perturbed quantities makes each Fourier mode evolving independently on the others. Working in the Fourier space is particularly useful in the present context, since in cosmology observables are often probed by studying their angular distribution, which correspond to cosmological scales, or Fourier wavenumbers, at a given redshift. The CMB is a striking example of this. Indeed, as we will see, the last scattering is a rather sudden event in cosmology, and the thickness of the last scattering surface is just about 10 Mpc, compared with almost 10000 Mpc of distance between us and that epoch in space. The angular distribution of CMB anisotropies directly corresponds to its Fourier expansion, expecially on small angular scales. In this Section, therefore, we lay down the formalism necessary to study cosmological perturbations in the real space, and that will be our framework in the following.

We expand perturbations into a complete set of functions which are the eigenfunctions of the Laplace operator ${}^s\Box f = \gamma^{ij} f_{|ij}$. For scalar functions, those eigenfunctions $Y_{\vec{k}}(\vec{x})$ are the solutions of the differential equation

$$({}^s\Box + k^2)Y = 0 \ . \tag{3.16}$$

where $k = |\vec{k}|^2$ and we drop the explicit arguments of $Y_{\vec{k}}(\vec{x})$ in the following. Note that in flat cosmology ${}^s\Box$ is just the double derivative, and the solutions of the equation above are just plane waves:

$$Y \propto e^{i\vec{k}\cdot\vec{x}} \text{ in flat FRW} \ . \tag{3.17}$$

The scalar-type components of vectors is expanded in Fourier modes exploiting the simple derivation of the functions above, expressed as

$$Y_i = -\frac{1}{k} Y_{|i} \ , \tag{3.18}$$

where the factor and sign in front is purely conventional. Similarily, the scalar-type component of tensors may be expanded in the following Fourier modes:

$$Y_{ij} = \frac{1}{k^2} Y_{|ij} + \frac{1}{3}\gamma_{ij} Y \ . \tag{3.19}$$

Note that the expression above is traceless; indeed it will be convenient to express the trace of tensors separately.
The vector-type component of vectors may be expanded into the vector solutions of the Laplace operator, i.e. obeying

$$({}^s\Box + k^2)Y_i^{(1)} = 0 \ , \tag{3.20}$$

with the divergenceless constraint

$$\gamma^{ij} Y_{i|j}^{(1)} = 0 \ . \tag{3.21}$$

The vector-type component of tensors may be expanded into

$$Y_{ij}^{(1)} = -\frac{1}{2k}\left(Y_{i|j} + Y_{j|i}\right) \ , \tag{3.22}$$

where the sum is made in order to keep symmetricity.
Finally, the tensor-type component of tensors may be expanded into the tensor solutions of the Laplace operator

$$({}^{s}\Box + k^2)Y_{ij}^{(2)} = 0 \ , \tag{3.23}$$

with the transverse traceless constraint

$$Y_{i|j}^{(2)|j} = 0 = Y_{i}^{(2)i} \ . \tag{3.24}$$

From now on, we adopt the Fourier space as our default framework, working out the analysis for one generic mode; unless otherwise specified, we also drop the label $\vec{k}$.

3.4 perturbations in the metric tensor

We have now all the means to define and classify the different species of cosmological perturbations in general relativity. The classification is entirely done in terms of the properties under spatial rotations, defined in section 3.2, and we consider the perturbations to the metric tensor first, $\delta g_{\mu\mu}$; with the addition of the background component, they represent the total metric tensor (3.1), $\bar{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$. As we have seen already, the background non-zero terms are on the diagonal, $\delta g_{00} = -a^2$, $\delta g_{11} = a^2dr^2/(1-Kr^2)$, $\delta g_{22} = a^2r^2$, $\delta g_{33} = a^2r^2(\sin\theta)^2$. We consider $\delta g_{\mu\nu}$ and we define scalar and scalar-type components, vector and vector-type components, tensor and tensor-type components, respectively.

3.4.1 scalar-type components

The δg_{00} quantity is clearly a scalar, since it does not contain any spatial index. In the Fourier space, it may be defined as

$$\delta g_{00} = -2a^2AY \ , \tag{3.25}$$

where A and Y represent the Fourier amplitude and Laplace operator eigenfunction at wavenumber $\vec{k}$, respectively. As we already mentioned, we omit the arguments τ in a, τ and $\vec{k}$ in A, $\vec{k}$ and $\vec{x}$ in Y. In the following, those dependences remain the same for all background terms, perturbations and Fourier eigenfunctions, therefore we omit them unless otherwise specified. Obviously, at any time the real space counterpart of the perturbations in the Fourier space may be got as follows:

$$\delta g_{00}(\tau,\vec{x}) = \int d^3k A(\tau,\vec{k})Y(\vec{k},\vec{x}) \ . \tag{3.26}$$

The quantity δg_{0i} is a vector, and since the background term for that is zero, it coincides with the component of the total metric tensor, $\bar{g}_{0i}$. Just like any vector in our scheme, it contains scalar-type and vector-type components. Its scalar-type component may be defined as

$$\delta g_{0i} = -a^2 B Y_i \ . \tag{3.27}$$

The quantity δg_{ij} contains scalar-type components on the diagonal, also yielding a perturbation in its trace, plus vector-type and tensor-type traceless contributions. It may be written as

$$\delta g_{ij} = a^2(\gamma_{ij} + 2H_L Y \gamma_{ij} + 2H_T Y_{ij}) \ , \tag{3.28}$$

where H_L and H_T represent the amplitude at the Fourier mode $\vec{k}$ of the diagonal and non-diagonal components of the perturbations to the metric tensor. Indeed, in the mixed form the second term in the right hand side appears simply as $2H_L Y \delta_i^j$, and Y_{ij} is traceless as it is evident from (3.19).

3.4.2 vector-type components

In analogy with (3.27), we may define the vector-type contribution to the metric tensor fluctuation as

$$\delta g_{0i}^{(1)} = -a^2 B^{(1)} Y_i^{(1)} \ . \tag{3.29}$$

The vector-type contribution to δg_{ij} does not possess the trace part, reducing to the shear component only, which we may write as

$$\delta g_{ij}^{(1)} = 2a^2 H_T^{(1)} Y_{ij}^{(1)} \ . \tag{3.30}$$

3.4.3 tensor-type components

The tensor-type fluctuations in the metric may be written as

$$\delta g_{ij}^{(2)} = 2a^2 H_T^{(2)} Y_{ij}^{(2)} \ , \tag{3.31}$$

representing the Fourier amplitude of cosmological gravitational waves, corresponding to the transverse traceless perturbation component.

3.5 perturbations in the stress energy tensor

It is convenient to work with the indeces in the mixed form, as they have a direct physical meaning, e.g. represent energy density, pressure, momentum density etc. As for the metric tensor, the total stress energy tensor may be written as in (3.3), where the background non-zero contributions are $T_0^0 = -\rho$, $T_1^1 = T_2^2 = T_3^3 = p$.

3.5.1 scalar-type perturbations

The perturbation component of $\bar{T}^0_0$ concerns the energy density distribution. We may write it as

$$\delta T^0_0 = -\delta Y \ , \tag{3.32}$$

introducing the Fourier amplitude of the spatial fluctuations in the density contrast, $\delta(\tau, \vec{k}) \equiv \delta\rho(\tau, \vec{k})/\rho(\tau)$. Similarily, the perturbed momentum concides with the corresponding component of the total stress energy tensor $\bar{T}^i_0$ and is expressed as

$$\delta T^i_0 = (\rho + p)vY^i \ , \tag{3.33}$$

where v represents the peculiar velocity, i.e. the velocity component with which the perturbed fluid drifts away from the Hubble flow, and $\rho+p$ gives the effective momentum mass. Finally, the scalar-type stress energy perturbations are made by an isotropic or trace part, plus the traceless component, also called shear, given by

$$\delta T^j_i = p(\pi_L \delta^j_i + \pi_T Y^j_i) \ , \tag{3.34}$$

where the first component, π_L may be seen as the Fourier amplitude of the perturbation to the pressure contrast, $\pi_L(\tau, \vec{k}) \equiv \delta p(\tau, \vec{k})/p(\tau)$, in analogy with what has been done for the energy density.

3.5.2 vector-type perturbations

In analogy with the definitions (3.33) and (3.34), but dealing with the vector-type components, we may write

$$\delta T^{i(1)}_0 = (\rho + p)v^{(1)}Y^{i(1)} \ , \tag{3.35}$$

$$\delta T^{j(1)}_i = p\pi_T Y^{j(1)}_i \ , \tag{3.36}$$

which represent the vector-type contributions to the perturbed stress energy tensor.

3.5.3 tensor-type perturbations

Finally, tensor-type perturbations affect the shear component only, and we may write those as

$$\delta T^{j(2)}_i = p\pi_T Y^{j(2)}_i \ , \tag{3.37}$$

which represent the stress energy counterpart of the gravitational waves $H^{(2)}_T$ defined in (3.31).

3.6 gauge transformations

As we have seen so far, the linearization of general relativity yields a quite simple decomposition of the different contributions on the basis of their geometrical properties. Linearity implies that in the Einstein equations, no terms involving the product of two perturbations are considered, and that the equations themselves slipt in two sub-sets, one evolving the cosmological backgrounds, and one dealing with the perturbation evolution. In other words, linearization creates a new set of dynamical variables, the small perturbations around the FRW background, which does not exist in reality, and is done only for simplicity in the calculations, i.e. our convenience. Of course, linear perturbations are different in different frames, and in particular on those differing from the Hubble frame made by r, θ and ϕ in (2.1) by small amounts. Those frames are called gauges. The coordinate transformations between different gauges, involving coordinate shifts of the same order of the perturbations themselves, determine how the perturbed quantities in different gauges are related. We indicate gauge transformations as

$$\tilde{x}^\mu = x^\mu + \delta x^\mu(x) \ , \tag{3.38}$$

where δx^μ, function of the spacetime point x, represents the coordinate shift between the new frame labeled with a tilde, and the original one.

Cosmological perturbations appear different in different gauges. A typical example of this is represented by the cosmological dipole, i.e. the dipole term in the angular expansion of the CMB temperature which is attributed to a Doppler shift due to the local motion, represented by a peculiar velocity v of our group of galaxies; the perturbation to the CMB temperature, and consequently our velocity, are measured to be rather small in units of the light velocity, so that the arguments of linear cosmological perturbation theory may be applied:

$$\left(\frac{\delta T}{T}\right)_{dipole} = v \simeq 10^{-3} \ . \tag{3.39}$$

Indeed, in a frame moving with velocity $-v$ with respect to our local group of galaxies, no dipole would be seen. That is an example of gauge transformation, as the velocity shift is small as in (3.39). This example tells two important aspects. First of all, cosmological perturbations, represented here by the CMB temperature, do depend on the chosen gauge. Second of all, perturbations may be zero in some gauges, and non-zero in others. We now treat more formally these issues.

Gauge transformations are coordinate shifts between two frames, the original one, labeled with f and a new one, labeled by $\tilde{f}$:

$$\tilde{x}^\mu = x^\mu + \delta x^\mu \ . \tag{3.40}$$

The coordinate shifts δx^μ are functions of the spacetime point x and obey the same classification as we did in section 3.2:

$$\delta x^0 \text{ is a scalar function, while} \tag{3.41}$$

$$\delta x^i \text{ is a vector function,} \tag{3.42}$$

composed by scalar-type and a vector-type components. We will express the gauge transformation laws in the Fourier space, therefore it is convenient to define the Fourier amplitudes of δx^μ:

$$\begin{aligned}\delta x^0(\tau,\vec{x}) &= \int d^3k T(\tau,\vec{k})Y(\vec{k},\vec{x}) \\ \delta x^i(\tau,\vec{x}) &= \int d^3k[L(\tau,\vec{k})Y^i(\vec{k},\vec{x}) + L^1(\tau,\vec{k})Y^{i(1)}(\vec{k},\vec{x})] \ .\end{aligned} \quad (3.43)$$

Again we omit the arguments above in the following. To find how perturbed quantities transform under gauge transformations it is enough to use the transformation laws of tensors in general relativity, keeping the linear order both in the perturbations and in the coordinate shifts as well.

3.6.1 metric tensor transformation

The general transformation of $g_{\mu\nu}$ between f and $\tilde{f}$ is

$$\bar{g}_{\mu\nu}(x) = \frac{\partial \tilde{x}^{\mu'}}{\partial x^\mu}\frac{\partial \tilde{x}^{\nu'}}{\partial x^\nu}\tilde{\bar{g}}_{\mu'\nu'}(\tilde{x}) \ , \quad (3.44)$$

where as usual the bar means background plus perturbations. It is easy to express the partial derivatives above as a function of the shifts δx^μ:

$$\frac{\partial \tilde{x}^{\mu'}}{\partial x^\mu} = \delta^{\mu'}_\mu + \delta x^{\mu'}_{,\mu} \ \ , \ \ \frac{\partial \tilde{x}^{\nu'}}{\partial x^\nu} = \delta^{\nu'}_\nu + \delta x^{\nu'}_{,\nu} \ . \quad (3.45)$$

By keeping only the first order terms in the shifts, from (3.44) one obtains

$$\bar{g}_{\mu\nu}(x) = \tilde{\bar{g}}_{\mu\nu}(\tilde{x}) + \tilde{\bar{g}}_{\mu'\nu}(\tilde{x})\delta x^{\mu'}_{,\mu} + \tilde{\bar{g}}_{\mu\nu'}(\tilde{x})\delta x^{\nu'}_{,\nu} \ . \quad (3.46)$$

To solve the relation above and find the transformation laws between perturbations, we need to express all quantities above in the same coordinate point. It is easy to see that to first order the last two terms are the same in x and $\tilde{x}$: indeed, the two coordinates values differ at a linear level, as those terms are already, so that any correction would be second order which we discard. The only operation which is left consists in relating $\bar{g}_{\mu\nu}(x)$ and $\tilde{\bar{g}}_{\mu\nu}(\tilde{x})$. We choose to work out the second one, which is just $\tilde{\bar{g}}_{\mu\nu}(x) + \tilde{\bar{g}}_{\mu\nu}(x)_{,\rho}\delta x^\rho$ to first order in the coordinate shifts. The relation (3.46) finally becomes

$$g_{\mu\nu} + \delta g_{\mu\nu} = g_{\mu\nu} + \delta\tilde{g}_{\mu\nu} + g_{\mu\nu,\rho}\delta x^\rho + g_{\mu'\nu}\delta x^{\mu'}_{,\mu} + g_{\mu\nu'}(\tilde{x})\delta x^{\nu'}_{,\nu} \ , \quad (3.47)$$

where we have made the cosmological perturbations explicit, and kept only the linear terms in them and in the shifts. Note that the background terms in the left and right hand side are identical and cancel out. Equation (3.47) with $\mu = \nu = 0$ gives

$$-a^2 - 2a^2AY = -a^2 - 2a^2\tilde{A}Y - 2a\dot{a}\delta x^0 - 2a^2\delta x^0_{,0} \ , \quad (3.48)$$

which with (3.43) becomes

$$\tilde{A} = A - \dot{T} - \mathcal{H}T \ . \quad (3.49)$$

In the same way, equation (3.47) with $\mu = 0,\ \nu = i$ for the scalar-type components only gives

$$-a^2 B Y_i = -a^2 \tilde{B} Y - a^2 T Y_{,i} + a^2 \dot{L} Y_i \ . \tag{3.50}$$

By using (3.18), the result is

$$\tilde{B} = B + \dot{L} + kT \ . \tag{3.51}$$

Finally, equation (3.47) with $\mu = i,\ \nu = j$ for the scalar-type components only gives

$$a^2\gamma_{ij}+2a^2 H_L Y\gamma_{ij}+2a^2 H_T Y_{ij} = a^2\gamma_{ij}+2a^2\tilde{H}_L Y\gamma_{ij}+2a^2\tilde{H}_T Y_{ij}+(a^2\gamma_{ij})_{,\rho}\delta x^\rho+a^2 L(Y_{i,j}+Y_{j,i}) \ . \tag{3.52}$$

The term $(a^2\gamma_{ij})_{,\rho}\delta x^\rho$ splits in two parts: the one with spatial indeces only combines with $a^2 L(Y_{i,j} + Y_{j,i})$ to give covariant derivatives, $a^2 L(Y_{i|j} + Y_{j|i})$. Using (3.18, 3.19) and the fact that $Y_{i|j} = -Y_{|ij}/k = -kY_{ij} + k\gamma_{ij}Y/3$ one gets

$$2a^2 H_L Y\gamma_{ij}+2a^2 H_T Y_{ij} = 2a^2\tilde{H}_L Y\gamma_{ij}+2a^2\tilde{H}_T Y_{ij}+2a^2 L\left(-kY_{ij} + \frac{k}{3}\gamma_{ij}Y\right)+2a\dot{a}TY\gamma_{ij} \ , \tag{3.53}$$

which splits in

$$\tilde{H}_L = H_L - \frac{k}{3}L + \mathcal{H}T \ , \tag{3.54}$$

$$\tilde{H}_T = H_T + kL \ . \tag{3.55}$$

This completes the gauge transformation laws for the scalar-type components of the metric tensor perturbations. In order to get the gauge transformation law for $B^{(1)}$, one can repeat the same passages done for B but starting from the vector-type component of equation (3.47) with $\mu = 0,\ \nu = i$, not considering the term involving T as that is a scalar-type quantity:

$$\tilde{B}^{(1)} = B^1 + \dot{L}^1 \ . \tag{3.56}$$

In the same way, the gauge transformation laws for $H_L^{(1)}$ and $H_T^{(1)}$ can be obtained repeating the same passages done for H_L and H_T but starting from the vector-type component of equation (3.47) with $\mu = i,\ \nu = j$, not considering the term involving T as that is a scalar-type quantity:

$$\tilde{H}_L^{(1)} = H_L^{(1)} - \frac{k}{3}L^{(1)} \ , \tag{3.57}$$

$$\tilde{H}_T^{(1)} = H_T^{(1)} + kL^{(1)} \ . \tag{3.58}$$

Finally, a most important consideration is that since gauge transformations may be scalar-type or vector-type only, any tensor-type cosmological perturbation is gauge invariant.

3.6.2 stress energy tensor transformation

We proceed as for the metric tensor, exploiting the tensor transformation laws (3.44), except that for the stress energy tensor it is convenient to work that out in the mixed form:

$$\bar{T}^{\nu}_{\mu}(x) = \frac{\partial \tilde{x}^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial \tilde{x}^{\nu'}} \tilde{\bar{T}}^{\nu'}_{\ \mu'}(\tilde{x}) \ . \tag{3.59}$$

The first factor in the right hand side is identical to (3.45). It is easy to see that the second one is

$$\frac{\partial x^{\nu}}{\partial \tilde{x}^{\nu'}} = \delta^{\nu}_{\nu'} - \delta x^{\nu}_{,\nu'} \ , \tag{3.60}$$

to the first order in the coordinate shift. Using this in (3.59), one obtains

$$\bar{T}^{\nu}_{\mu}(x) = \tilde{\bar{g}}_{\mu\nu}(\tilde{x}) + \tilde{\bar{T}}^{\nu}_{\ \mu'}(\tilde{x})\delta x^{\mu'}_{,\mu} - \tilde{\bar{T}}^{\nu'}_{\ \mu}(\tilde{x})\delta x^{\nu}_{,\nu'} \ . \tag{3.61}$$

As in the case of the metric tensor, to solve the relation above and find the transformation laws between perturbations, we need to express all quantities in the same coordinate point. Again, to first order the last two terms are the same in x and $\tilde{x}$: indeed, the two coordinates values differ at a linear level, as those terms are already, so that any correction would be second order which we discard. The only operation which is left consists in relating $\bar{T}^{\nu}_{\mu}(x)$ and $\tilde{\bar{T}}^{\nu}_{\ \mu}(\tilde{x})$. We choose to work out the second one, which is just $\tilde{\bar{T}}^{\nu}_{\ \mu}(x) + \tilde{\bar{T}}^{\nu}_{\ \mu}(x)_{,\rho}\delta x^{\rho}$ to first order in the coordinate shifts. The relation (3.61) finally becomes

$$T^{\nu}_{\mu} + \delta T^{\nu}_{\mu} = T^{\nu}_{\mu} + \delta\tilde{T}^{\nu}_{\mu} + T^{\nu}_{\mu\ ,\rho}\delta x^{\rho} + T^{\nu}_{\mu'}\delta x^{\mu'}_{,\mu} - T^{\nu'}_{\mu}\delta x^{\nu}_{,\nu'} \ , \tag{3.62}$$

where we have made the cosmological perturbations explicit, and kept only the linear terms in them and in the shifts. As for the metric tensor, the background terms in the left and right hand side are identical and cancel out. In equation (3.62) with $\mu = \nu = 0$ the two last terms cancel out, too, giving

$$\tilde{\delta} = \delta + \dot{\rho}T/\rho = \delta + 3\mathcal{H}(1+w)T \ . \tag{3.63}$$

In the same way, equation (3.47) with $\mu = 0, \ \nu = i$ for the scalar-type components only gives

$$(\rho + p)vY^i = (\rho + p)\tilde{v}Y^i + \rho(\delta x^i)_{,0} + p(\delta x^i)_{,0} \ , \tag{3.64}$$

which with (3.43) becomes

$$\tilde{v} = v - \dot{L} \ . \tag{3.65}$$

Finally, in equation (3.62) with $\mu = i, \ \nu = j$ the last two terms cancel out again, leaving for the scalar-type components the relation

$$p(\delta^j_i + \pi_L Y\delta^j_i + \pi_T Y^j_i) = p(\delta^j_i + \tilde{\pi}_L Y\delta^j_i + \tilde{\pi}_T Y^j_i) + \dot{p}\delta^j_i TY \ , \tag{3.66}$$

which splits in

$$\tilde{\pi}_L = \pi_L - \dot{p}T/p = \pi_L + 3\mathcal{H}c_s^2\left(1 + \frac{1}{w}\right)T \ , \tag{3.67}$$

$$\tilde{\pi}_T = \pi_T \ , \tag{3.68}$$

where we notice that the anisotropic stress does not depend on the gauge, already at the scalar-type level. This completes the gauge transformation laws for the scalar-type components of the stress energy tensor perturbations. In order to get the gauge transformation law for $v^{(1)}$, one can repeat the same passages done for v but starting from the vector-type component of equation (3.62) with $\mu = 0,\ \nu = i$:

$$\tilde{v}^{(1)} = v^{(1)} - \dot{L}^{(1)} \ . \tag{3.69}$$

Finally the same reasoning done before for equation (3.62) with $\mu = 0, \nu = i$ yields

$$\tilde{\pi}_T^{(1)} = \pi_T^{(1)} \ . \tag{3.70}$$

As for $H_T^{(2)}$, also $\pi_T^{(2)}$ does not depend on the gauge adopted, as gauge transformations do not affect tensor-type components.

3.6.3 gauge dependence, independence, invariance and spuriosity

We may summarize the gauge transformation laws in the following set of equations, concerning metric and stress energy tensors:

$$\begin{aligned}
\tilde{A} &= A - \dot{T} - \mathcal{H}T \ , && (3.71)\\
\tilde{B} &= B + \dot{L} + kT \ , && (3.72)\\
\tilde{B}^{(1)} &= B^{(1)} + \dot{L}^{(1)} \ , && (3.73)\\
\tilde{H}_L &= H_L - \frac{k}{3}L - \mathcal{H}T \ , && (3.74)\\
\tilde{H}_T &= H_T + kL \ , && (3.75)\\
\tilde{H}_T^{(1)} &= H_T^{(1)} + kL^{(1)} \ , && (3.76)\\
\tilde{H}_T^{(2)} &= H_T^{(2)} \ , && (3.77)\\
\tilde{\delta} &= \delta + 3\mathcal{H}(1+w)T \ , && (3.78)\\
\tilde{v} &= v - \dot{L} \ , && (3.79)\\
\tilde{v}^{(1)} &= v^{(1)} - \dot{L}^{(1)} \ , && (3.80)\\
\tilde{\pi}_L &= \pi_L + 3\mathcal{H}c_s^2\left(1 + \frac{1}{w}\right)T \ , && (3.81)\\
\tilde{\pi}_T &= \pi_T \ , && (3.82)\\
\tilde{\pi}_T^{(1)} &= \pi_T^{(1)} \ , && (3.83)\\
\tilde{\pi}_T^{(2)} &= \pi_T^{(2)} \ . && (3.84)
\end{aligned}$$

On the basis of these relations, we distinguish four different and general concepts about gauge transformation in cosmology, which conclude this section and chapter.

The first one is gauge dependence. A perturbation effecting one single element of the metric or stress energy tensor is gauge dependent. Let us consider for example the relation (3.71). Suppose that in the original frame f A is non-zero. In the frame $\tilde{f}$ differing with respect to f because of a time shift T satisfying $\dot{T} + \mathcal{H}T = A$, one has $\tilde{A} = 0$. That means that an observer in $\tilde{f}$ sees clocks running as the unperturved FRW. This same issue clearly holds for other perturbations; an important example concerning the stress energy tensor is represented by the energy density contrast. Although that seems an intuitive and simple concept which does not depend on coordinate issues, a gauge transformation does affect its value: indeed, whatever is δ in f, in the frame $\tilde{f}$ characterized by a time shift T satisfying $T = -\delta/3\mathcal{H}(1+w)$, one has $\tilde{\delta} = 0$. That means that a bump, or a hole in the energy density distribution in a gauge may be absent in another one if the time shift between the two is chosen appropriately. More precisely, for scalar-type components, one has 8 perturbations, 4 for the metric tensor and 4 for the stress energy tensors, and 2 functions describing the gauge shifts, L and T. In general, those may be used to put to zero 2 of the eight scalar-type quantities. For vector-type components, one has 4 perturbations, 2 for the metric tensor and 2 for the stress energy tensor, and 1 function describing the gauge shift, $L^{(1)}$. In general, that may be used to put to zero 1 of the 4 vector-type quantities. On the other hand, tensor-type perturbations are not affected by the coordinate shifts, and have the same value in all gauges.

This brings us to the second general concept, the gauge independence. Simply, that means that since one has not enough degrees of freedom from gauge shifts to nullify all perturbations at once, the concept of perturbation in cosmology is gauge independent; that means that equations and perturbed quantities may differ in different gauges, but if the cosmological system is perturbed in one gauge, then it is perturbed in all the others.

The third general concept is gauge invariance. The gauge invariant nature of cosmological perturbations may be formalized, as it was done for the first time by Bardeen (1980), by combining the cosmological perturbations in a set of variables which do have the same value in all gauges. Such variables are therefore gauge invariant. The two classical examples of scalar-type gauge invariant perturbations in the metric tensor are represented by the Bardeen potentials:

$$\Phi = H_L + \frac{1}{3}H_T + \frac{\mathcal{H}}{k}\left(B - \frac{1}{k}\dot{H}_T\right) \ , \tag{3.85}$$

$$\Psi = A + \frac{\mathcal{H}}{k}\left(B - \frac{1}{k}\dot{H}_T\right) + \frac{1}{k}\left(\dot{B} - \frac{1}{k}\ddot{H}_T\right) \ . \tag{3.86}$$

By exploiting the relations (3.71,3.72,3.74,3.74), it is easy to verify that $\tilde{Phi} = \Phi$, $\tilde{Psi} = \Psi$. Any linear combination of them involving factors made by constants or background quantities is gauge invariant as well. On the other hand, for vector-type perturbations, the gauge invariant metric perturbation is unique:

$$\sigma_g^{(1)} = B^1 - \frac{1}{k}\dot{H}_T^{(1)} \ . \tag{3.87}$$

Note that also the scalar-type version of (3.87), $B - \dot{H}_T/k$ is gauge invariant. Two conventient choices of gauge invariant quantities involving quantities related to the stress energy tensor are

$$\Delta = \delta + \frac{3\mathcal{H}(1+w)}{k}(v - B) \ , \tag{3.88}$$

$$V = v - \frac{1}{k}\dot{H}_T \ , \tag{3.89}$$

which generalize density contrast and peculiar velocity, respectively. As for the metric tensor, such choice is not unique as any combination of the quantities above is gauge invariant as well. It is interesting to build up two gauge invariant quantities involving the stress energy tensor only:

$$\Gamma = \pi_L - \frac{c_s^2}{w}\delta \ , \tag{3.90}$$

$$\pi_T \ . \tag{3.91}$$

Γ has a direct physical meaning in terms of thermal equilibrium of the components in the cosmic fluid. Indeed, the condition $\Gamma = 0$ implies

$$\pi_L \equiv \frac{\delta p}{p} = \frac{c_s^2}{w}\delta \equiv \frac{c_s^2}{w}\frac{\delta\rho}{\rho} \ , \tag{3.92}$$

which is equivalent to

$$\frac{\delta p}{\delta\rho} = \frac{\dot{p}}{\dot{\rho}} \ . \tag{3.93}$$

The condition above is known as adiabaticity. The reason is that it is satisfied when cosmological perturbations keep thermal equilibrium unaltered. Let's suppose to have a radiation thermal bath, characterized by $p = w\rho = \rho/3$. The latter relation comes from statistical mechanics. Γ is zero when also $\delta p = \delta\rho/3$, which means that pressure fluctuations follow adiabatically those of the energy density, keeping thermal equilibrium for radiation, specified by the condition $w = 1/3$, valid everywhere. Note that this is not true in general: $\delta p = (\delta w)\rho + w(\delta\rho)$. That is, if w possesses a dependence on space, then thermal equilibrium is also perturbed, and this is expressed by the fact that the adiabaticity condition (3.93) is violated. To end up on this issue, let us define two relevant vector-type stress energy tensor gauge invariant quantities, which are merely a generalization of (3.89) and (3.91):

$$V^{(1)} = v^{(1)} - \frac{1}{k}\dot{H}_T^{(1)} \ , \tag{3.94}$$

$$\pi_T^{(1)} \ . \tag{3.95}$$

Again the tensor-type stress energy tensor perturbation, $\pi_T^{(2)}$, is gauge invariant as no gauge coordinate shifts is tensor-type.

We now come to the last point, the gauge spuriosity. In extreme synthesis, the issue is that different observers, belonging to different gauges, may see the same numerical value of cosmological perturbations; the coordinate shifts among those observers, not to be confused with cosmological perturbations, are known as spurious gauge modes. The best way to see this point is two consider two most popular example of gauges.
The synchronous gauge is defined by using T, L and $L^{(1)}$ in such a way to confine the perturbations in the metric tensor to the spatial area only:

$$\tilde{A} = \tilde{B} = \tilde{B^{(1)}} = 0 \ . \tag{3.96}$$

As we have already seen, the first condition above means that the time shift between f and $\tilde{f}$ must satisfy the condition

$$\dot{T} + \mathcal{H}T = A \ . \tag{3.97}$$

The solution to this equation is not unique. Actually, there are infinite solutions, made by the sum of a particular one, plus all the solutions of the homogeneous equation, $\dot{T} + \mathcal{H}T = 0$, which is satisfied by $T_g \propto 1/a$. This means that there are an infinite number of frames which are in the synchronous gauge. The gauge mode T_g is spurious in the sense that it is not a cosmological perturbation, but just the time shift between all frames seeing the same numerical value of cosmological perturbations. Spurious gauge modes usually affect several quantities: for example, the energy density fluctuation picks up a spurious gauge mode given by

$$\delta_g = 3\mathcal{H}(1 + w)T_g \ . \tag{3.98}$$

Moreover, the second condition in (3.96), which is written as

$$\dot{L} + kT = -B \ , \tag{3.99}$$

does not determine L uniquely; indeed, all frames differing by a shift given by

$$L_g = -k \int T_g d\tau + \text{constant} \tag{3.100}$$

are in the synchrotron gauge. Note that a constant means a Dirac delta in the real space, meaning that the coordinate shift affects a single point in spacetime. This also produces a gauge spurious mode in the velocity:

$$v_g = -L_g \ . \tag{3.101}$$

Finally, also the third condition in (3.96) leaves a constant spurious vector-type gauge mode, $L_g^{(1)}$ = constant. In principle, one may track the spurious gauge modes in the computations. In practice however, it is much better to performing those in a gauge without spuriosity. We see now an example of gauge where that is absent.
The Newtonian gauge is specified by

$$\tilde{B} = \tilde{H}_T = \tilde{H}_T^{(1)} = 0 \ , \tag{3.102}$$

thus allowing the perturbed metric tensor to be non-null on the diagonal only in the scalar-type component. It is easy to see that the second condition is satisfied performing a coordinate change between f and $\tilde{f}$ such that

$$L = -\frac{1}{k} H_T \ , \tag{3.103}$$

which is uniquely fixed. The second condition also fixed uniquely T:

$$T = -\frac{1}{k}\left(B + \dot{L}\right) \ . \tag{3.104}$$

Similarly, at the vector-type level the coordinate transformation needed to go in the Newtonian gauge is

$$L^{(1)} = -\frac{1}{k} H_T^{(1)} \ , \tag{3.105}$$

which completes the set of coordinate shifts necessary to go in the Newtonian gauge. This is therefore an example where gauge spuriosity is absent. Spuriosity is usually something to avoid when dealing with cosmological perturbations. Spurious gauge modes may activate or accidentally put to zero cosmological perturbation modes in energy density, velocity, or metric fluctuations. By working in a gauge free of spuriosity, one is sure that all the solutions of the cosmological perturbation evolution equations are physical, unaffected by gauge modes.

Chapter 4

Cosmological Boltzmann equation for photons

In this chapter we work out and discuss the Boltzmann equation for photons in cosmology. Unless otherwise specified, we assume a flat FRW metric in the background, and work at first order in cosmological perturbations. We also assume that photons are at thermal equilibrium, obeying a Bose-Einstein distribution statistics; we also assume that cosmological perturbations do not alter the equilibrium.

Let us consider a generic distribution function of photons, $\bar{d}$, as measured by an observer moving with quadri-velocity $\bar{u}^\mu$, normalized as usual, $\bar{u}^\mu \bar{u}^\mu = -1$. If $\bar{p}^\mu$ is the photon quadri-momenta, the energy accessible to the observer (see e.g. Abbott , Wise (1986)) is

$$\bar{E} = -\bar{u}^\mu \bar{g}_{\mu\nu} \bar{p}^\nu \ . \tag{4.1}$$

The distribution $\bar{d}$ gives the photon number per unit volume as a function of spacetime position x^μ, quadri-velocity of the observer $\bar{u}^\mu$, and photon quadri-momentum $\bar{p}^\mu$. One of the four components of $\bar{p}^\mu$ is redundant, as photons are massless: $\bar{g}_{\mu\nu}\bar{p}^\mu\bar{p}^\nu = 0$. The latter relation, together with (4.1) if the observer quadri-velocity is known, give two relations between $\bar{p}^0$ and $\vec{\bar{p}}$. By knowing the photon propagation direction $\hat{\bar{n}}$, which is proportional to $\vec{\bar{p}}$, those two relations fully determine $\vec{\bar{p}}^\mu$. As we shall see later, for our purposes the background value $\hat{n}$ is enough, as the photon propagation direction appears always multiplied by quantities which are at first order; as it may be easily seen from the condition $p_\mu p^\mu = 0$, $\hat{n}$ is simply given by $\vec{p}/p^0$. There are however very important processes affecting the photon propagation direction, which we ignore here, such as the lensing, i.e. the light bending due to forming cosmological structures along the line of sight; that is an example of second order perturbation, as the perturbations in the cosmological structures lens the CMB anisotropies, which are perturbations as well. Being second order, we ignore it in the present treatment, although the CMB lensing is extremely important for cosmology (see e.g. Bartelmann , Schneider, 2001, and references therein). Therefore we keep $\bar{E}$ and $\hat{n}$ as the independent variables for $\bar{d}$:

$$\bar{d}(\tau, \vec{x}, \hat{E}, \hat{n}) \ . \tag{4.2}$$

In absence of interaction, photons simply keep their distribution function unchanged, and the Boltzmann equation assumes its simplest form, $\dot{\bar{d}} = 0$; interactions, or collisions, are usuall described by a term $\bar{C}$ which appears on the right hand side of it. By propagating the total time derivative in all arguments, the Boltzmann equation for photons may be written as

$$\dot{\bar{d}} = \frac{\partial \bar{d}}{\partial \tau} + \frac{\partial \bar{d}}{\partial x^i}\dot{\bar{x}}^i + \frac{\partial \bar{d}}{\partial \bar{E}}\dot{\bar{E}} + \frac{\partial \bar{d}}{\partial n^i}\dot{n}^i = \bar{C} \ . \tag{4.3}$$

This equation may be further simplified by exploiting our assumption on the photon distribution statistics. The unperturbed Bose-Einstein distribution function is given by

$$d = \frac{1}{[\exp{(E/k_B T)} - 1]} \ , \tag{4.4}$$

where E and $T(\tau)$ are the background photon energy and blackbody temperature, depending on time as any background quantity. We may describe in general the linear fluctuations of the Bose-Einstein statistics by introducing the corrections

$$\frac{E}{k_B T} \to \frac{E}{k_B T}(1 - \Theta) \ , \tag{4.5}$$

where formally $\Theta = \delta T/T - \delta E/E$; in our assumption of thermal equilibrium also at a perturbed level, perturbations cannot depend on E; therefore, the distinction between $\delta E/E$ and $\delta T/T$ is redundant and we may keep one of them only, which may be formally seen as the perturbation to E or T or both. By making the arguments of the fluctuation explicit, one gets

$$\frac{\delta T}{T}(\tau, \vec{x}, \hat{n}) - \frac{\delta E}{E}(\tau, \vec{x}, \hat{n}) \equiv \Theta(\tau, \vec{x}, \hat{n}) \ll 1 \ . \tag{4.6}$$

Thus the first order perturbation to the Bose-Einstein distribution assumes the form

$$\delta d = T\frac{\partial d}{\partial T}\Theta = -E\frac{\partial d}{\partial E}\Theta \ . \tag{4.7}$$

As we shall see later, the evolution of $E(\tau)$ is determined by the geodesic equation for photons; thus, equation (4.3) will determine the evolution of the remaining quantities, i.e. $T(\tau)$ for the background part and $\Theta(\tau, \vec{x}, \hat{n})$ for the perturbations. Since from (4.5) the Bose-Einstein statistics depends on position and propagation direction only at first order, we may take the background value of $\dot{\vec{x}}^i$; the latter is $\dot{x}^i = dx^i/d\tau = (dx^i/d\eta) \cdot (d\eta/d\tau) = \bar{p}^i/\bar{p}^0 = n^i$. Moreover, in a flat FRW background the last term in (4.3) contributes only to second order. As above the Bose-Einstein statistics depends on position and propagation direction only at first order; Moreover, in an unperturbed flat FRW background, photons go straight, and $\dot{p}^i \propto \dot{n}^i = 0$, deviating because of metric perturbations again only at first order. Note that if the background were not flat, then $\dot{n}^i \neq 0$ at the background level, and the last term in (4.3) would definitely contribute at first order. Therefore, the Boltzmann equation (4.3) slpits and becomes

$$\frac{\partial d}{\partial E}\dot{E} + \frac{\partial d}{\partial T}\dot{T} = C \ \ , \ \ \frac{\partial \delta d}{\partial \tau} + \frac{\partial \delta d}{\partial x^i}n^i + \frac{\partial d}{\partial E}\dot{\delta E} = \delta C \ . \tag{4.8}$$

The collision terms above describes the transfer of the number densities of photons at different energies and momenta because of the scattering process with other particles; $\dot{E}$ and $\dot{\delta E}$ describe the the change in energy of photons because of all other reasons. In general relativity, a change in the photon energy occurs because of the spacetime geometry. In the following, we work out the photon geodesic equation and collision terms, respectively.

4.1 photon geodesic equation

The photon energy dynamics due to the fluctuactions in the metric may be got from the photon geodesic equation

$$\frac{d\bar{p}^\mu}{d\eta} + \bar{\Gamma}^\mu_{\nu\rho}\bar{p}^\nu\bar{p}^\rho = 0 \ , \tag{4.9}$$

where η is an affine parameter parametrizing the trajectory, which by means of linearity splits as usual in background and first order perturbation terms:

$$\frac{dp^\mu}{d\eta} + \Gamma^\mu_{\nu\rho}p^\nu p^\rho = 0 \ , \tag{4.10}$$

$$\frac{d\delta p^\mu}{d\eta} + \delta\Gamma^\mu_{\nu\rho}p^\nu p^\rho + 2\Gamma^\mu_{\nu\rho}\delta p^\nu p^\rho = 0 \ . \tag{4.11}$$

It is easy to see that E possesses a background term scaling inversely proportional to the scale factor. Indeed, by taking (4.10) with $\mu = 0$, one has

$$p^0\frac{dp^0}{d\tau} + \mathcal{H}\left(p^0p^0 + p^ip^i\right) \ , \tag{4.12}$$

where we used $dp^0/d\eta = p^0dp^0/d\tau$ and the Christoffel symbol has been explicited using (1.5). From $p_\mu p^\mu = 0$ one gets $p^ip^i = p^0p^0$, implying $p^0 \propto a^{-2}$. Therefore, using (2.10) it is easy to see that the background component of E is made by u^0 scaling as $1/a$, g_{00} scaling as a^2, and p^0 scaling as $1/a^2$, making the overall scaling as $1/a$. Let us now consider fluctuations. By using the definition (4.1) one has immediately the expression of the fluctuations in E:

$$\delta E = -\delta u^\mu g_{\mu\nu}p^\nu - u^\mu\delta g_{\mu\nu}p^\nu - u^\mu g_{\mu\nu}\delta p^\nu \ . \tag{4.13}$$

The first term comes from peculiar velocities of the observer with respect to a frame at rest with respect to the FRW metric. In order not to introduce any effect coming from the motion of the observer, we choose to describe the fluctuations with respect to that frame, for which $\delta u^i = 0$, while δu^0 may be expressed as a function of the metric fluctuations as follows: the condition $\bar{u}^\mu\bar{u}^\mu = -1$ splits in the background term, $u^\mu u_\mu = -1$, and the perturbed one, which is

$$2\delta u^\mu g_{\mu\nu}u^\nu + u^\mu u^\nu\delta g_{\mu\nu} \ , \tag{4.14}$$

which fixes δu^0 to be $\delta u^0 = \delta g_{00}/2a^3 = h_{00}/2a = AY/a$. The remaining two terms give the relevant contributions in terms of metric fluctuations. By taking the only non-zero terms in the background ones, equation (4.13) simplifies as follows:

$$\delta E = -a(h_{00}p^0 + h_{0\nu}p^\nu + \delta p^0) \ . \tag{4.15}$$

The time derivative of the relation above gives the term in the Boltzmann equation (4.8) we are looking for. By using the definition of the metric perturbations, and integrating the geodesic equations (4.10,4.11), the result is

$$\frac{\dot{(\delta E)}}{E} = -\mathcal{H}\frac{\delta E}{E} + \frac{1}{2}\dot{h}_{ij}n^in^j - \dot{h}_{0i}n^i - \frac{1}{2}h_{00,i}n^i \ , \tag{4.16}$$

where the first term represents the background scaling, affecting perturbations also, and the remaining ones come from perturbations.

4.2 Compton scattering

The CMB decouples at redshift $z \simeq 1100$; the CMB temperature today is $T \simeq 3K$ which corresponds to about 1/4000 eV in energy units. Therefore, using (4.35), at decoupling the CMB temperature is some fraction of eV in energy units. At these energies, according to the present understanding of cosmology, the cosmic budget is composed by photons γ, free electrons e^- in non-relativistic regime as $m_e \simeq 900$ MeV, hydrogen and helium atoms and nuclei, plus many other particles, interacting at most weakly with the former classes, e.g. neutrinos ν, dark matter. The most relevant interaction for photons and these energies and lower is Compton scattering onto non-relativistic electrons, indicated by the following relation

$$e^-(q^\mu) + \gamma(p^\mu) \to e^-(q^{\mu'}) + \gamma(p^{\mu'}) \ , \tag{4.17}$$

where $q^\mu, p^\mu, q^{\mu'}, p^{\mu'}$ are the electron and photon incoming and outcoming quadri-momenta, respectively.
In the Boltzmann equation (4.8), we are interested in the change in the photon distribution $\bar{d}$ from Compton scattering; it will be necessary for a while to consider the electron distribution function, $\bar{d}_e$. In the spacetime point x, in the time interval dt, a number of electron and photon states given by the product between $\bar{d}_e(x, q^{\mu'})$ and $\bar{d}(x, \bar{p}^{\mu'})$, respectively, determine an outgoing photon with quadri-momentum $\bar{p}^\mu$ corresponding to our variables $\bar{E}$ and $\hat{n}$, and therefore an increase in $\bar{d}$. At the same time, a number of states given by the product between $\bar{d}_e(x, \bar{q}^\mu)$ and $\bar{d}(x, \bar{p}^\mu)$ determines an outgoing photon with momentum $\bar{p}^{\mu'}$ and a decrease in $\hat{d}$. Schematically, the term $\bar{C}$ in (4.8) may be written as

$$\bar{C} = \sum_{q^\mu, q^{\mu'}, p^{\mu'}} \mathcal{T}(\bar{d}_e(x, q^{\mu'})\bar{d}(x, p^{\mu'}) - \bar{d}_e(x, q^\mu)\bar{d}(x, p^\mu)) \ , \tag{4.18}$$

where $\mathcal{T}$ is a constant or non perturbative quantity and indicates the strength of the interaction. Let us now make some consideration on the expression above, which simplify the calculations below. First of all, there is no background term in (4.18); as we explain now, the reason is simply that in this problem we may take a perturbative approach not only in the cosmological perturbations, but also in the energy exchange between photons. Indeed, at energies comparable or lower than the eV, the energy transfer between photons is a small quantity with respect to the typical energy of electrons, which is of the order of their mass, $m_e \simeq 900$ MeV. Therefore the photon energy exchange $p^{0'} - p^0 = E' - E$ in (4.18) may be treated perturbatively; in the present analysis, we only keep terms which are at first order in cosmological perturbations or photon energy exchange. Moreover, due to the small spacetime scales involved by the interaction, we make the approximation of special relativity. Thus in this section the metric is Minkowskian unless otherwise specified, and the energy of photons is simply given by $E = |\vec{p}|$, while that of electrons by $E_e = \sqrt{m_e^2 + |\vec{q}|^2}$. Since the background term in (4.18) vanishes, we may write

$$C = 0 \ . \tag{4.19}$$

More simplifications may be found by looking at first order in (4.17); making the background and fluctiation terms in the electron energy distribution explicit, one gets

$$\delta C = \sum_{q^\mu, q^{\mu'}, p^{\mu'}} \mathcal{T}\{[\delta d_e(x, q^{\mu'})]d(x, p^{\mu'}) + d_e(x, q^{\mu'})[\bar{d}(x, p^{\mu'})] -$$

$$-[\delta d_e(x, q^\mu)]d(x, p^\mu)) - d_e(x, q^\mu)[\bar{d}(x, p^\mu)]\} \ . \tag{4.20}$$

Because of the reason above, the difference between the first and third term is first order in the photon energy exchange. That means that the difference between those terms is a product between cosmological perturbations and photon energy exchange, which we neglect in the present treatment. This allows us to use the background electron density distribution only. Applying the laws of electromagnetism, the right form for (4.20) is

$$\delta C = \int \frac{d^3q}{(2\pi)^3 2E_e} \int \frac{d^3q'}{(2\pi)^3 2E_e'} \int \frac{d^3p'}{(2\pi)^3 2E'} \cdot \cdot \frac{16\pi^4 \mathcal{T}}{E} \cdot$$

$$\cdot \delta(E + E_e - E' - E_e')\delta^3(\vec{p} + \vec{q} - \vec{p}' - \vec{q}')[d_e' \bar{d}(x, p^{\mu'}) - d_e \bar{d}(x, p^\mu)] \ , \tag{4.21}$$

Again neglecting terms of order higher than the first in cosmological perturbations and energy transfer, since the electrons are non-relativistic their energy is given by $m_e + |\vec{q}|^2/2m_e$, where the second term is of the order of the thermal energy of the system, the eV, which is much smaller than m_e. Therefore in the denominator of (4.21) may be replaced by m_e. With that we may perform the integration over $\vec{q}'$, which yields

$$\delta C = \frac{\pi}{4m_e^2} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3 2E} \frac{\mathcal{T}}{E} \delta(E + E_e - E' - E_e') \cdot$$

$$\cdot [d_e \bar{d}(x, p^{\mu'}) - d_e \bar{d}(x, p^\mu)] \ , \tag{4.22}$$

where the first electron distribution is now calculated in $|q^0 + p^0 - p^{0'}|$, which is just m_e in our approximation. At energies comparable or lower than the eV, the energy transfer between photons is a small quantity with respect to the typical energy of electrons, which is of the order of their mass, m_e. Therefore, the argument of the Dirac delta above may be treated as follows. In the non-relativistic regime, the energy of electrons if given by $m_e + |\vec{q}|^2/2m_e$; their difference in (4.22) is given by

$$\frac{|\vec{q}|^2}{2m_e} - \frac{|\vec{q} + \vec{p} - \vec{p}'|^2}{2m_e} \simeq \frac{\vec{q}(\vec{p}' - \vec{p})}{m_e} \ , \tag{4.23}$$

so that the Dirac delta becomes

$$\delta(E - E' + \frac{\vec{q}(\vec{p} - \vec{p}')}{m_e}) \ . \tag{4.24}$$

The next simplification concerns the last term in (4.22). Indeed, the difference is first order in $E' - E$ or Θ, even ignoring the spacetime and momentum dependence in the first d_e term. The latter terms would therefore induce a correction of second order $E' - E$ which we ignore. In this way the electron distribution d_e may be factored out of the integral:

$$C = \frac{\pi}{4m_e^2} \int \frac{d_e(q^\mu)}{(2\pi)^3} d^3q \cdot$$

$$\cdot \int \frac{d^3p'}{(2\pi)^3 2E'} \frac{\mathcal{T}}{E} \delta \left[E - E' + \frac{\vec{q}(\vec{p} - \vec{p}')}{m_e} \right] [d(x, p^{\mu'}) - d(x, p^\mu)] \ . \tag{4.25}$$

In order to proceed, we need now a recipe for $\mathcal{T}$. In general the Compton scattering is anisotropic, in the sense that the intensity of the light emitted depends on the angular distribution of the incoming radiation. Moreover, an unpolarized but anisotropic incoming radiation is re-scattered linearly polarized, due to the motion of the electron, following the direction of the incoming electric field. At decoupling, the photons have time to acquire a quadrupole before investing the last scattered electron on our direction, and therefore we do expect a linear polarization in the CMB radiation. The latter is an entire branch of CMB science, and we come back on that later. For now, we are interested in the shift of the temperature of the whole black body spectrum, and therefore we consider only the isotropic term in $\mathcal{T}$, which is

$$\mathcal{T} = 8\pi a \sigma_T m_e^2 \ , \tag{4.26}$$

where $\sigma_T = 8\pi e^4 / 3c^4 m_e^2$ is the Thomson cross section, and the factor a is simply due to the use of the conformal time in the time derivative of the photon distribution function, which is otherwise derived with respect to the ordinary time, $dt = ad\tau$. We are about to conclude this calculation. We expand the photon distributions $d(x, p^\mu)$ and $d(x, p^{\mu'})$ to first order in Θ, and the second one to first order in $E' - E$, too:

$$d(x, p^\mu) \simeq d(E) - E\frac{\partial d}{\partial E}\Theta \ , \tag{4.27}$$

$$d(x, p^{\mu'}) \simeq d(E) + \frac{\partial d}{\partial E}(E' - E) - E\frac{\partial d}{\partial E}\Theta' \ . \tag{4.28}$$

Any other term is second order in the photon momenta difference, or Θ, or their product, and we neglect it. Therefore, the term in parenthesis in (4.25) becomes

$$d(x, p^{\mu'}) - d(x, p^\mu) \simeq \frac{\partial d}{\partial E}(E' - E) - E\frac{\partial d}{\partial E}(\Theta' - \Theta) \ . \tag{4.29}$$

Note that the prime for Θ means that it is calculated on the direction $\hat{n}'$. The fact that the term above does not possess a zero-th order simplifies alot the calculations in the remaining quantities, which may be simply calculated at the zero-th order in $E - E'$ and Θ. Let us go back to the integral (4.25). First we pass to spherical coordinates for $\vec{p}'$: $d^3p = E^{2'} dE' d\Omega'$. In the first integral, done with the first term in (4.29), the Dirac delta makes $E' - E$ calculated in $\vec{q}(\hat{n} - \hat{n}')E/m_e$; indeed, the difference between E and E' is not taken into account in the term dependent on directions, since it is multiplied by the electron momentum, which represents a cosmological perturbation. The second term gives zero when integrated over the directions on the spheres, $d\Omega'$. The integral in d^3q merely represents the definition of the average peculiar velocity of electrons:

$$\int d^3q\vec{q} = n_e\vec{v}_e \ . \tag{4.30}$$

The second integral, done with the second term in (4.29), is already first order in Θ, and may be calculated in $E = E'$. The integration in $d\Omega'$ on the term dependent on Θ simply produces $4\pi\Theta$, as the latter is evaluated along $\hat{n}$. The same integral applied on Θ' produces the interesting term

$$4\pi\Theta_0 = \int \Theta' d\Omega' \ , \tag{4.31}$$

which represents the monopole of the CMB temperature fluctuations, i.e. the displacement that the temperature has locally, in the spacetime point x, from its average value. In this case, the integral in d^3q simply produces the background density of electrons, n_e. Combining all these calculations together, the final result for the collision term from Compton scattering, in our approximations, is

$$\delta C = -an_e\sigma_T E\frac{\partial d}{\partial E}(\Theta_0 - \Theta + \hat{n}\vec{v}_e) \ . \tag{4.32}$$

Using this result and (4.16), we may now write down the Boltzmann equation for photons up to the first order in cosmological perturbations.

4.3 Boltzmann equation

We now put together the gravitational and collision terms in the Boltzmann equation (4.8). As usual, we separate our analyisis in background and perturbations. Concerning the latter, we write down the general expression containing all kinds of fluctuations, and then we further specialize the analysis to the case of scalar-type perturbations only, and Newtonian gauge, restricting the physical degrees of freedom to the metric tensor fluctuations in the diagonal.

$$-2\Psi = h_{00} \quad , \quad 2\Phi = h_{ii} \ . \tag{4.33}$$

The names chosen for the gravitational potentials is conventional and differ from A and H_L because in the Newtonian gauge the two coincide with the gauge invariant Bardeen potentials, see (3.86) and (3.85).

4.3.1 zero-th order

We have already seen the implication of the Boltzmann equation at the background level. The first of the conditions (4.8) implies

$$\frac{\dot{T}}{T} = \frac{\dot{E}}{E} = -\mathcal{H} \ , \tag{4.34}$$

which is equivalent to

$$T \propto \frac{1}{a} \ , \tag{4.35}$$

and tells that the unperturbed bose-einstein distribution keeps its form during the cosmological expansion, with just redshifting temperature.

4.3.2 first order

By putting together (4.16) and (4.32), the first order Boltzmann equation in (4.8) becomes

$$\dot{\Theta} + \frac{\partial \Theta}{\partial x^i} n^i + \frac{1}{2}\dot{h}_{ij} n^i n^j - \dot{h}_{0i} n^i - \frac{1}{2} h_{00,i} n^i = n_e \sigma_T a (\Theta_0 - \Theta + \hat{n} \cdot \vec{v}_e) \ . \tag{4.36}$$

The equation above is general in the sense that collects contributions from all cosmological perturbations, and expressed in the real space. As anticipated above, for an easier understanding of the phenomenology, it is convenient at this point to further simplify this equation assuming scalar-type perturbations only, and also adopting a particular gauge, the Newtonian one, specified for scalar-type components by (4.33). These simplifications affect mostly the expression of the equation (4.36) in the Fourier space, which is our next step.
First, let us define the Fourier transform of the photon temperature fluctuation:

$$\Theta = \int d^3 k \Theta_{\vec{k}}(\tau) Y(\vec{k}, \vec{x}) \ . \tag{4.37}$$

In our hypothesis of flat FRW, Y may be thought as a plane wave, $e^{i\vec{k}\cdot\vec{x}}$; it is easy to handle these in the Fourier space, as spatial derivatives bring down components of $\vec{k}$ multiplied by i. Those also appear contracted with $\hat{n}$ in (4.36). Therefore, indicating with x, y, z the usual Cartesian coordinates, one has

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right) \to i\vec{k} \ \ , \ \ \hat{n} \cdot \nabla = i \hat{n} \cdot \vec{k} \ . \tag{4.38}$$

Therefore it is convenient to define the quantity

$$\cos\theta = \frac{\vec{k} \cdot \hat{n}}{k} = \hat{k} \cdot \hat{n} \ . \tag{4.39}$$

We also define the photon optical depth

$$\tau = \int_0^{\tau_0} n_e(\tau) \sigma_T a(\tau) d\tau = \int_0^{t_0} n_e(t) \sigma_T dt \ . \tag{4.40}$$

which counts the number of Compton scattering targets along the photons line of sight. Finally, by transforming (4.36) in the Fourier space, using the scalar-type components only in the Newtonian gauge, and exploiting the definitions (3.18), (4.39) and (4.40), one gets

$$\dot{\Theta} + ik\cos\theta\Theta + \dot{\Phi} + ik\cos\theta\Psi = \dot{\tau}(\Theta_0 - \Theta - i\cos\theta v_e) \ . \tag{4.41}$$

where the angular dependence is now completely contained in the $\cos\theta$ term, i.e. the cosine of the angle between $\hat{k}$ and $\hat{n}$, mapping directions with respect to the photon propagation direction in the Fourier space.

4.4 relativistic and non-relativistic species other than photons

The photon Boltzmann equation provides the evolution of the photon temperature distribution as a function of the background expansion and the rest of cosmological perturbations. Before concluding this chapter, it is convenient to complete the dynamical system, by writing down the evolution equations for the other components. In cosmology, they are represented by dark matter, neutrinos and the other particles of the standard model of particle physics, i.e. baryons and leptons.

4.4.1 dark matter and neutrinos

Dark matter and neutrinos interact at most weakly with the rest of the system. At the epoch of decoupling, or after, they may be treated as effectively decoupled from the rest of the system. Therefore, the relevant equations for their evolution are given by the conservation of the stress-energy tensor:

$${}_xT^{\nu}_{\mu;\nu} = \delta({}_xT^{\nu}_{\mu;\nu}) = 0 \ . \tag{4.42}$$

While the first one fives as usual $\dot{\rho}_x + 3\mathcal{H}(\rho_x + p_x) = 0$, the second set of equation determines the evolution of energy density and peculiar velocities; their expressions in the Newtonian gauge, and flat FRW, are

$$(\dot{\rho_x\delta_x}) + 3\mathcal{H}\rho_x\delta_x + 3\mathcal{H}p_x\pi_{Lx} + k(\rho_x + p_x)v_x = -3(\rho_x + p_x)\Phi \tag{4.43}$$

$$\frac{d}{d\tau}[(\rho_x+p_x)(v_x)]+4\mathcal{H}(\rho_x+p_x)v_x-ik\rho_x\pi_{Lx}+\frac{2}{3}ikp_x\pi_{Tx} = ik(\rho_x+p_x)\Psi \ . \tag{4.44}$$

Ψ and Φ evolve according to the perturbed Einstein equations, which in the Newtonian gauge are

$$k^2\Psi = 4\pi Ga^2\rho\Delta \ \ , \ \ k^2(\Psi + \Phi) = -a^2 p\Pi \ , \tag{4.45}$$

where $\rho\Delta = \sum_x \rho_x\delta_x + 3(\rho_x + p_x)\mathcal{H}v_x/k$, $p\Pi = \sum_x p_x\pi_{Tx}$. The isotropic and anisotropic stresses are determined by the micro-scopic nature of the particles involved. The dark matter, for our purposes here, may be described as a pressureless component. Neutrinos are like photons, i.e. with equation of state equal to 1/3, and possess an anisotropic stress due to their intrinsic asymmetry.

4.4.2 leptons and baryons

The rest of the particles in the standard model are baryons and leptons. In cosmology it is customary to talk about baryons only; the reason is that electrons and protons are tightly coupled by the Coulomb interaction, which at the energy considered. The fluid is therefore dominated by baryons in terms of mass. Note that, on the other hand, in terms of Compton interaction electrons play the major role, as they are lighter. Since however they stick together to baryons because of the Coulomb interaction, we drop the subscript $_e$ and we substitute it with $_b$. Because of the Compton interaction with photons treated in the previous sections, baryons are not conserved and obey the following equation:

$$\delta({}_bT^{\mu;\nu}_{\nu}) = \delta_b Q^\mu \ . \tag{4.46}$$

The terms of exchange with photons balance out:

$$\delta_e Q^\mu + \delta_\gamma Q^\mu = 0 \ . \tag{4.47}$$

Therefore the one which we are treating here may be obtained from the Compton interaction we treated already. The first aspect to consider is that

$${}_e\delta Q^0 = 0 \ . \tag{4.48}$$

To see this, it is enough to consider that

$$\Theta_0 \propto \delta\rho_\gamma \ , \tag{4.49}$$

as the first term is the integral over all directions of the perturbation to the distribution of photons. Therefore, equation (4.41) becomes an equation for $\delta\rho_\gamma$ when integrated over all directions. It is easy to see that the latter operation puts the right hand side member to zero, which implies (4.48). Therefore, the only non-null term is $\delta_b Q^i$. That is given by the change in the number density of electrons undergoing the Compton interaction with photons, time their spatial momentum. Again ignoring the presence of protons, undergoing a negligible interaction with photons, such quantity is the opposite of the corresponding one for photons, exploiting (4.47). Therefore, all we have to do is taking δC in (4.32) with a minus in front of it, going in the Fourier space, multiply by the photon momentum $\vec{p}$, and integrate over all possible directions, keeping $\hat{k}$ as the polar direction for convenience:

$$\delta_b Q^i = a n_e \sigma_T \int \frac{d^3p}{(2\pi)^3} E^2 \frac{\partial d}{\partial E} \cos\theta \left(\Theta_0 - \Theta - v_b \cos\theta\right) =$$

$$a n_e \sigma_T \int \frac{dE}{2\pi^2} E^4 \frac{\partial d}{\partial E} \int_{-1}^{1} \frac{d\cos\theta}{2} \cos\theta \left(\Theta_0 - \Theta - v_b \cos\theta\right) \ . \tag{4.50}$$

The integral over energies gives $-4\rho_\gamma$. In the integral over the directions, only the second and third term survive. The first one of those highlights the dipole in the CMB temperature anisotropies:

$$\Theta_1 = \int d\Omega(-i\cos\theta)\Theta = i\int_{-1}^{1} d\cos\theta \cos\theta\Theta \ . \tag{4.51}$$

With this, the conservation equation for the velocity term for baryons, in flat FRW, Fourier space, Newtonian gauge and scalar-type perturbations only, becomes

$$\frac{d}{d\tau}(\rho_b v_b) + 4\mathcal{H}\rho_b v_b = k\rho_b\Psi + \dot{\tau}\frac{4\rho_\gamma}{3\rho_b}\left(\Theta_1 - v_b\right) \ , \tag{4.52}$$

where we described the baryons as a pressureless component.

Chapter 5

The cosmic microwave background anisotropies

In this chapter we provide a complete mathematical description of CMB anisotropies, and provide evolution equations for them, which are based on the Boltzmann equation we wrote in the previous chapter. Again we assume a background flat FRW cosmology.

5.1 angular expansion for total intensity

We first provide a basis of expansion for total intensity anisotropies, represented by the temperature fluctuation Θ. In the real space, the arguments of Θ are:

$$\Theta \equiv \Theta(\tau, \vec{x}, \hat{n}) \ . \tag{5.1}$$

We already formalized the Fourier harmonic modes $Y(\vec{k}, \vec{x})$, which are simply plane waves, $e^{i\vec{k}\cdot\vec{x}}$, in our hypotheses. We need to perform the angular expansion. For this, we use the usual spherical harmonics algebra, after fixing the reference axes for $\hat{n}$. For this, it is most convenient to distinguish between the angular expansion which may be performed in a laboratory, and and the one we are making here. Indeed, when expanding in the angular domain the Boltzmann equation (4.41), for reasons which will be clear in the following it is convenient to perform the expansion by keeping each Fourier mode as the polar axis. Therefore we distinguish between the laboratory frame, $lab-$frame, where the anisotropies are observed and expanded for convenience in the angular domain keeping the same frame for all Fourier modes, and the $\hat{k}-$frame, valid for each Fourier mode separately, having the direction of $\vec{k}$ coincident with the polar axis. Therefore, our expansion set for (5.1) is

$$G_l(\vec{x}, \vec{k}, \hat{n}) = (-i)^l Y(\vec{x}, \vec{k}) \sqrt{\frac{4\pi}{2l+1}} Y_l^0(\hat{n}_{\hat{k}}) \tag{5.2}$$

where $Y_l^m(\hat{n}_{\hat{k}})$ are the usual spherical harmonics, where the subscript indicates that $\hat{n}$ is expressed in the $\hat{k}-$frame, and the constants in front are purely conventional. Note that since the Boltzmann equation (4.41) depends on $\cos\theta = chatn \cdot \hat{k}$, the spherical harmonics calculated in $m = 0$ are sufficient. The harmonic coefficients of Θ along the basis (5.2) are given by

$$\Theta_l(\tau, \vec{k}) = \int d^3x \int 2\pi \sin\theta d\theta \Theta(\tau, \vec{x}, \hat{n}) G_l(\vec{k}, \vec{x}, \hat{n}) \ . \tag{5.3}$$

In the following, we explicitate the arguments of the various functions only if necessary.

5.2 angular expansion of the Boltzmann equation

For convenience, let us write here the Boltzmann equation (4.41):

$$\dot{\Theta} + ik\cos\theta\Theta + \dot{\Phi} + ik\cos\theta\Psi = \dot{\tau}(\Theta_0 - \Theta - i\cos\theta v_b) \ . \tag{5.4}$$

Notice that with respect to (4.41) we just substituted the subscript of electrons with the one for baryons, as leptons and baryons are tightly coupled by Coulomb scattering, as we saw in section 4.4.2. The equation is already expressed in the Fourier space, i.e. the first integral in (5.3) has been performed, the Y functions have been simplified on both sides, and Θ is a function of τ, $\vec{k}$ and $\hat{n}$. The gravitational potentials, Φ and Ψ, as well as the baryon peculiar velocity, v_b, do not depend on the photon propagation direction, as they are just metric fluctuations. We need now to exploit the spherical harmonics in (5.2) to perform the angular expansion. Let us do that term by term in (5.4). The first term produces $\dot{\Theta}_l$; similarly, the second term on the right hand side produces $\dot{\tau}\Theta_l$. The third one and the first one in the right hand side are also trivial since they contribute to the monopole only, as they do not depend on $\hat{n}$. For the same reason, the fourth term in the left hand side and the third one in the right hand side represent two pure dipole terms, as they are both a monopole multiplied by $\cos\theta$, which corresponds to Y_1^0 in (5.2), and therefore to G_1. The latter also multiplies the second term in the left hand side, but in this case a more careful analysis is needed: indeed, the product $ik\cos\theta\Theta$ implies products between Y_1^0 and the other sherical harmonics coming from the angular expansion of Θ itself. There exist a relation between spherical harmonics which we exploit here, and allows to express the product of them at a given l as a linear combination of others, computed at l, $l \pm 1$; such relations also involves the spin of the spherical harmonics, as they may correspond to the expansion of scalars, i.e. the ones we are exploiting here, as well as vectors and tensors. Indicating the spherical harmonic spin with s, such relation is

$$\sqrt{\frac{4\pi}{3}} Y_1^0 \cdot {}_sY_l^m = \frac{{}_s\|_l^m}{\sqrt{(2l+1)(2l-1)}}\, {}_sY_l^m - \frac{ms}{l(l+1)}\, {}_sY_l^m + \frac{{}_s\|_{l+1}^m}{\sqrt{(2l+1)(2l+3)}}\, {}_sY_{l+1}^m \ , \tag{5.5}$$

where

$${}_s\|_l^m = \sqrt{\frac{(l^2-m^2)(l^2-s^2)}{l^2}} \ . \tag{5.6}$$

Let us first get an equation from Θ_0. In the right hand side, the monopole cancels out, while v_b contributes to the dipole. For the same reason, only the first gravitational potential in the left hand side contributes at the monopole level. The only non-trivial term is given by the second one on the left hand side, which must be worked out using (5.5). The result is

$$\dot{\Theta}_0 + \frac{k}{3}\Theta_1 + \dot{\Phi} = 0 \ . \tag{5.7}$$

Let us now find an equation for Θ_1. In the right hand side, the last two terms contribute. In the left hand side, everything except $\dot{\Phi}$ gives a contribution, and again one must use (5.5) in order to work things out. The result is

$$\dot{\Theta}_1 - k\Theta_0 + \frac{2}{5}\Theta_2 - k\Psi = \dot{\tau}(\Theta_1 - v_b) \ . \tag{5.8}$$

We may write down the equations for the rest of the system, i.e. $l \geq 3$, this time ignoring the monopole and dipole terms in (5.4), and using (5.5) again. The result is

$$\dot{\Theta}_l - k\left(\frac{{}_0\|_l^m}{2l-1} - \frac{{}_0\|_{l+1}^m}{2l+3}\right) = -\dot{\tau}\Theta_l \ , \tag{5.9}$$

where the numerical coefficients may be obtained from (5.6).

5.3 Solution in the tight coupling approximation

On all cosmologically relevant scales, before decoupling the quantity $\dot{\tau}/k$ is large, which means that the mean Compton scattering free path for photons, $1/\dot{\tau}$, is much smaller than them. For simplicity assuming $1/\dot{\tau} \rightarrow 0$, then from (5.8) one gets $\Theta_1 = v_b$. By calculating the photon peculiar velocity v_γ from their perturbed statistical distribution given by (4.4) and (4.5), i.e. multiplying the number density of states by their momenta, and integrating over all of them, one gets $\Theta_1 = v_\gamma$. Combining that with the previous relation, that means that that photons and baryons move with the same velocity in the limit of vanishing photon mean free path. But then, comparing (4.43) for baryons, (4.52), (5.7) and (5.8), one gets also $\Theta_0 = 3\delta_b$, $\Theta_2 = 0$. From the equations (5.9), also $\Theta_{l\geq 3}$ must be all zero. That means that in the tight coupling regime photons and baryons behave as the same fluid. That has several interesting consequences. First of all, initial conditions must activate either Θ_0 or Θ_1, in order to provide a dynamics. Second, when the free electron number density drops because of recombination, the quantity $\dot{\tau}/k$ decreases rapidly, and power is gradually transmitted to the higher order multipoles, $\Theta_{l\geq 2}$, through the hierarchic equations (5.9). Third, while the tight coupling limit is active, there exist a dynamics in the CMB, although confined to Θ_0 and Θ_1, or equivalently $\delta_\gamma/4$ and v_γ, or equivalently $3\delta_b/4$ and v_b, respectively. Let us solve the system of equations (5.7), (5.8), (5.9) in the tight coupling regime. The zero-th order in $1/\dot{\tau}$ is given by

$$\begin{aligned} \dot{\Theta}_0 &= -\frac{k}{3}\Theta_1 - \dot{\Phi} \ , \\ (m_{\gamma b}\dot{\Theta}_1) &= k(\Theta_0 + m_{\gamma b}\Psi) \ , \end{aligned} \tag{5.10}$$

where $m_{\gamma b} = 1 + 3\rho_b/4\rho_\gamma$. They combine in the following second order equation for Θ:

$$(m_{\gamma b}\dot{\dot{\Theta}}_0) + \frac{k^3}{3}\Theta_0 = -\frac{k^3}{3}m_{\gamma b}\Psi - (m_{\gamma b}\dot{\dot{\Phi}}) \ . \tag{5.11}$$

In absence of metric perturbations, the equation above has exact solutions:

$$\Theta_0 = \frac{1}{m_{\gamma b}^{1/4}}\left[A\cos(ks) + B\sin(ks)\right] \ , \tag{5.12}$$

where A and B are constants, and

$$s = \int_0^\tau \frac{d\tau'}{\sqrt{3m_{\gamma b}}} \ , \tag{5.13}$$

represents the sound horizon for the fluid made by the tightly coupled baryons and photons. Θ_1 may be easily obtained from (5.10). The solution (5.12) may be further simplified by neglecting the presence of baryons at decoupling. Indeed, at decoupling dark matter is about one order of magnitude more abundant than radiation; baryons density is a few percent of the dark matter one and therefore it makes sense to neglect them in order to clearify the phenomenology. In this limit $m_{\gamma b} = 1$, $s = \tau/\sqrt{3}$, therefore

$$\Theta_0 = A\cos\left(\frac{k\tau}{\sqrt{3}}\right) + B\sin\left(\frac{k\tau}{\sqrt{3}}\right) \ . \tag{5.14}$$

The phenomenology does not change much if the metric fluctuations in the right hand side of equation (5.11) are taken into account. In the limit in which the gravitational potentials are constant in time, the solution (5.14) holds for $\Theta+\Psi$, which means that the gravitational potential simply changes the zero level of the oscillations in (5.14).
Let us now consider the evolution of the $\Theta_{l\geq 3}$ coefficients in the tight coupling regime. As we have already seen, at the zero-th order in $1/\dot\tau$ they are simply zero. The power transfers from Θ_0 and Θ_1 gradually to higher multipoles, with increasing powers in $\frac{k}{\dot\tau}$. It is easy to see that an approximate solution for that is given by

$$\Theta_l = -\frac{k}{\dot\tau}\frac{{}_0\|_{l-1}}{2l-1}\Theta_{l-1} \ , \tag{5.15}$$

since in this way, Θ_l is always of order higher than Θ_{l-1}, and the terms involving $\dot\Theta_l$ and Θ_{l+1} may be ignored in (5.9) because they are of order higher than Θ_l.

5.4 The CMB temperature anisotropy angular power spectrum

We are now ready to compute the angular power spectrum of CMB anisotropes in the $lab-$frame. We start from the angular expansion of temperature anisotropy

$$\Theta(\tau,\vec{x},\hat{n}) = \sum_{lm} a_{lm}Y_l^m \ , \ a_{lm} = \int d\Omega Y_{lm}(\hat{n})\Theta(\tau,\vec{x},\hat{n}) \ . \tag{5.16}$$

Note that now the angular expansion is global and not depending on any Fourier wavevector, which does not even appear in (5.16). The coefficients of the angular power spectrum are defined as

$$C_l = \frac{1}{2l+1}\sum_m |a_{lm}|^2 \ , \tag{5.17}$$

and they correspond to the coefficients of the expansion of the correlation function in Legendre polynomials:

$$\int d\Omega d\Omega'[\Theta(\tau,\vec{x},\hat{n})\Theta(\tau,\vec{x},\hat{n}')]_{\hat{n}\hat{n}'=\cos\theta} = \sum_l (2l+1)C_l P_l(\cos\theta)\ . \tag{5.18}$$

The definitions above may be applied to any given sky signal. Usually theory predicts the average of the realizations of possible universes, with a given variance. For the angular power spectrum of the CMB, that means that theory predicts the average of the C_l over all possible realizations, which may be written as

$$C_l =< |a_{lm}|^2 >\ , \tag{5.19}$$

where the average is done over an assumed statistics.
In our case

$$|a_{lm}|^2 = \int d^3k d^3k' d\Omega d\Omega' \sum_{l'l''} Y_l^m(\hat{n}) Y_l^{m*}(\hat{n}')\Theta_{l'}(\tau,\vec{k})G_{l'}(\vec{x},\vec{k}\hat{n}_{\hat{k}})\Theta_{l''}(\tau,\vec{k}')^* G_{l''}(\vec{x},\vec{k}'\hat{n}'_{\hat{k}})^*\ . \tag{5.20}$$

The average in (5.19) operates in the product $\Theta_{l'}(\tau,\vec{k})\Theta_{l'}(\tau,\vec{k}')$. The current pictures of the early universe assume an initial Gaussianity in fluctuations. That means that the correlation between different wavevectors in null, and that the coefficients depend on k only:

$$< \Theta_{l'}(\tau,\vec{k})\Theta_{l''}(\tau,\vec{k}') >= \delta(\vec{k}-\vec{k}')\Theta_{l'}(\tau,k)\Theta_{l''}(\tau,k) >\ . \tag{5.21}$$

This first eliminates one of the two Fourier integrals in (5.20). The angular component of the harmonic functions G_l depends on $\hat{n}\cdot\hat{k}$ only, which is essentially given by the Legendre polynomial $P_l(\hat{n}\cdot\hat{k})$. A useful relation between Legendre polynomials and spherical harmonics allows us to write

$$P_{l'}(\hat{n}\cdot\hat{k}) = \frac{4\pi}{2l+1}\sum_{m'} Y_l^{m'}(\hat{n})Y_l^{m'*}(\hat{k})\ . \tag{5.22}$$

and of course the same expression for $P_{l''}(\hat{n}'\cdot\hat{k})$. This makes it possible to perform the integral in $\hat{n}$ and $\hat{n}'$ above, which selects $l = l' = l''$, $m = m' = m''$. Moreover, the fact that $\Theta_l(\tau,\vec{k})$ does not depend on the direction of the wavevector makes the integrals of $|Y_{lm}(\hat{k})|^2$ over all the directions of $\vec{k}$ trivially equal to 1. Working out the normalization factors, the result is

$$C_l = \frac{2}{\pi}\int k^2 dk |\Theta_l(\tau,k)|^2\ , \tag{5.23}$$

which makes explicit the connection between an observed quantity in the $lab-$frame on the left hand side with the angular expansion coefficients of the temperature anisotropy on the right hand one, evaluated in the $\hat{k}-$frame.

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