The early universe

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Chapter 1

Introduction

The aim of these lectures is to give an overview of the physics of the early universe, in the context of the inflationary cosmology. At the end of the course, a student should be able to understand most of the modern scientific papers on this matter, ranging from the theoretical details of quantum fields in cosmology, down to the physical content and implications of the current experimental data. In this chapter, we give an overview of the course, specifying the most important literature to look at, as well as reviewing the plan of the lectures. In the end, we also fix the notation we adopt, and introduce very basic elements of general relativity and field theory.

1.1 Things to know for attending the course

These lectures aim at being self consistent, although some knowledge of general relativity and field theory is necessary in order to apprach them properly. Moreover, who's attending might benefit from having familiarity with the physics of the Friedmann Robertson Walker (FRW) background cosmology, as well as any knowledge of cosmological perturbations.
The main text where studying while attending the lectures should be represented by the present notes. The students are on the other hand welcome to consult the books and papers from where these notes have been taken. They are:

- textbook by Andrew R. Liddle and David H. Lyth, Cosmological Inflation and Large Scale Structure, Cambridge Press 2000,
- review paper by Hideo Kodama and Misao Sasaki, Progresses of Theoretical Physics Supplement 78, 1, 1984.

1.2 Plan of the lectures

The first part of the course contains a summary of the FRW cosmological background, focusing on the expansion dynamics and in particular on the cosmic acceleration. This initial part sets the notation of the course and grants also a continuity with possible previous courses that the students may have attended. The lectures then cover the following topics:

- pre-inflationary cosmology,
- kinematic of inflation,
- scalar fields in cosmology,
- inflationary models,
- the origin of cosmological perturbations,
- testing inflation with cosmic microwave background anisotropies,
- inflation and dark energy.

1.3 General relativity and quantum field theory

Spacetime is described by three spatial dimensions plus the time coordinate. Greek indeces run from 0 to 3, while latin indeces are used for spatial directions, from 1 to 3. We use x to indicate a generic spacetime point, $\vec{x}$ and $\hat{x}$ for its spatial component and versor, respectively. The fundamental constants we use are the Boltzmann and gravitational ones, indicated with k_B and G, respectively. Unless specified otherwise, we work with unitary light speed velocity, $c = 1$, and unitary Planck constant, $\hbar = 1$.
Fields are function of the spacetime point x, and under coordinate transformation they may behave as scalars, vectors or tensors if they have zero, one or more than one Lorentz indeces, respectively. A tensor with two indeces, indicated with $g_{\mu\nu}(x)$ and called metric tensor, sets the infinitesimal distance from two spacetime points, defined as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \tag{1.1}$$

where repeated indeces are summed. By an appropriate change on reference frame, it is always possible to reduce the metric tensor to the Minkowski one, meaning that the system changes to the one which in free fall locally in x. The signature of the metric tensor we adopt is the following:

$$(-, +, +, +) \ . \tag{1.2}$$

The inverse of the metric tensor is represented with the indeces up:

$$g_{\mu\rho} g^{\rho\nu} = \delta_\mu^\nu \ , \tag{1.3}$$

where δ_μ^ν is the Kronecker delta. We shall use the Kronecker delta in arbitrary index configuration:

$$\delta_\mu^\nu = \delta_{\mu\nu} = \delta^{\mu\nu} = 1 \text{ if } \mu = \nu, \ 0 \text{ otherwise.} \tag{1.4}$$

The Christoffel symbols are defined as usual as

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2} \left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \right) \ ,$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\alpha'} \left(\frac{\partial g_{\alpha'\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha'\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha'}} \right) \ . \tag{1.5}$$

The Riemann, Ricci and Einstein tensors are given by

$$R^{\alpha}_{\beta\mu\nu} = \frac{\partial\Gamma^{\alpha}_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial\Gamma^{\alpha}_{\beta\mu}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\lambda}_{\beta\nu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\mu} \ , \tag{1.6}$$

$$R_{\mu\nu} = R^{\alpha}_{\alpha\mu\nu} \ , \tag{1.7}$$

and

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \ , \tag{1.8}$$

where $R = R^{\mu}_{\mu}$ is the Ricci scalar.
To simplify the notation, let us introduce the following conventions for derivation in general relativity:

- $_{;\mu} \equiv \nabla_{\mu}$ means covariant derivative with respect to x^{μ},
- $_{|a} \equiv {}^{s}\nabla_{a}$ means covariant derivative with respect to the spatial metric, i.e. the 3×3 array obtained removing the time column and row from the metric tensor in (1.1),
- $_{,\mu}$ means ordinary derivative with respect to x^{μ} .

A vector, v_{μ} can be obtained via covariant derivation of a scalar quantity s as

$$v_{\mu} = s_{;\mu} = s_{,\mu} \ , \tag{1.9}$$

where the last equality holds for scalars only. A tensor can be obtained via covariant derivation of a vector as

$$t_{\mu\nu} = v_{\mu;\nu} = v_{\mu,\nu} - v_{\alpha}\Gamma^{\alpha}_{\mu\nu} \ . \tag{1.10}$$

Covariant derivative raise further the rank of tensors as

$$\begin{aligned} u_{\mu\nu\rho} &= t_{\mu\nu;\rho} = t_{\mu\nu,\rho} - t_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - t_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu} \\ u^{\nu}_{\mu\rho} &= t^{\nu}_{\mu;\rho} = t^{\nu}_{\mu,\rho} - t^{\nu}_{\alpha}\Gamma^{\alpha}_{\mu\rho} + t^{\alpha}_{\mu}\Gamma^{\nu}_{\rho\alpha} \ , \end{aligned} \tag{1.11}$$

and the process of course is not limited in the number of Lorentz indeces.

Field quantization is performed conveniently in the Fourier space. For a real scalar field $\psi(x)$ with mass m it reads

$$\psi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[a_{\vec{p}}u(\vec{p})e^{i\vec{p}\cdot\vec{x}} + a^{+}_{\vec{p}}u^{*}(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right] \ , \tag{1.12}$$

where $a_{\vec{p}}$ and its Hermitian conjugate $a^{+}_{\vec{p}}$ are the annihilation and creation operators of a quantum with momentum $\vec{p}$, with energy $E_{\vec{p}} = \sqrt{m^2 + \vec{p}^2}$. The vacuum state in the Fock space is indicated as usual as $|0>$. The $u\vec{p}e^{i\vec{p}\cdot\vec{x}}$ function and its complex conjugate $u^{*}(\vec{p})e^{-i\vec{p}\cdot\vec{x}}$ are eigenfunctions of the Hamiltonian of the system; for a massless and non-interacting field, $u(\vec{p})e^{i\vec{p}\cdot\vec{x}} = e^{i(-|\vec{p}|t+\vec{p}\cdot\vec{x})}$ The commutation relations are

$$[\psi(x), \psi^{+}(x')] = i\hbar\delta^3(\vec{x} - \vec{x}') \ , \tag{1.13}$$

related as usual to the space coordinate only.

Chapter 2

Cosmology with repulsive gravity

In this chapter we review the main aspects of the FRW cosmology, focusing on the species which may induce a repulsive gravity effect, appearing as an accelerated cosmic expansion.

The FRW metric is built upon the hypothesis that space is homogeneous and isotropic at all times. The first condition means that at a given time, the physical properties, e.g. expansion rate, particle density etc., are the same in each point. The second condition means that any physical observable does not depend on the direction of an observer located in any spacetime point x.
These assumptions simplify dramatically the structure of the metric tensor $g_{\mu\nu}$. A spherical symmetry around each spacetime location is necessary, so that no off-diagonal terms are left; homegeneity and isotropy leave essentially only two degrees of freedom to the system. The first one is a global scale factor, fixing at each time the value of physical lengths. The second one is related to the spacetime curvature, as an homogeneous metric can be globally more or less curved. The form of the fundamental length element is therefore

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{1}{1 - Kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) , \qquad (2.1)$$

where $a(t)$ and K represent the global scale factor and the curvature, respectively; r, θ and ϕ are the usual spherical coordinates for radius, polar and azimut angle, respectively.
The physical meaning of the scale factor can be read straightforwardly from the metric, and the only point to discuss concerns its dimension. One may assign physical dimensions to the scale factor a or to the radial coordinate r; in this lectures, we choose the second option. Concerning the curvature, some more discussion is needed. The first point is about dimensions again; if r is dimensionless, K is also dimensionless. If r is a length, then K is the inverse of the square of a length. Moreover, if $K = 0$, then the spatial part of the length (2.1) is Minkowskian, and in this case the FRW metric is flat. If $K > 0$, there is an horizon in the metric, given by $r_H = \pm 1/\sqrt{K}$; this means that an infinite physical distance corresponds to those coordinates, regardless of the value of the scale factor a, and the FRW metric is closed; note that this does not conflict

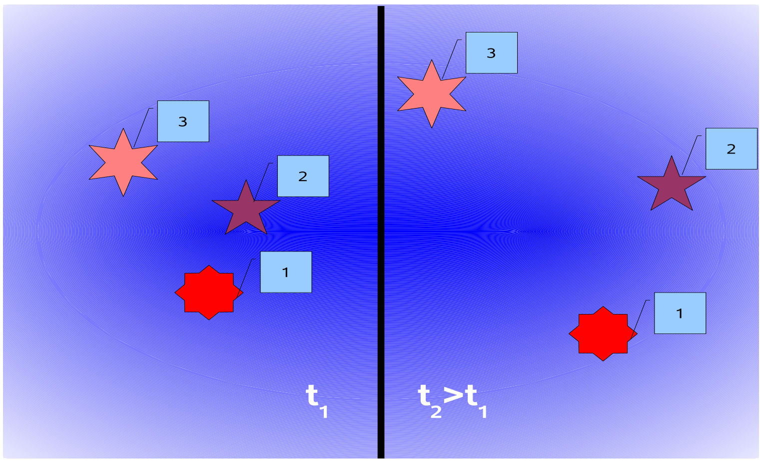

Figure 2.1: A representation of cosmological expansion in comoving coordinates; distances between objects increase with time, but their coordinates, represented by labels here, stay constant.

with the assumption of homogeneity, as this property is the same as seen in all spacetime locations. If $K < 0$, the opposite happens, as there is no horizon, and the distance between two space points vanishes at infinity; in this case the FRW metric is open. Finally, note that one may always change the overall normalization of a or r in (2.1), and therefore, as a pure convention, we can restrict our attention to three relevant cases for K:

$$\begin{aligned} K &= -1 \quad \text{open FRW} \ , \\ K &= 0 \quad \text{flat FRW} \ , \\ K &= +1 \quad \text{closed FRW} \ . \end{aligned} \tag{2.2}$$

2.1 Comoving coordinates

Of course one might apply any change of coordinate to the FRW metric. On the other hand the form (2.1) is the one that is common for cosmological purposes. The reason is that the expansion, represented by the scale factor a, has been factored out of the spatial dependence. This leads us to the concept of comoving coordinate, i.e. at rest with respect to the cosmic expansion, or in other words made by the spacetime points for which

$$r = \text{constant} \ , \ \theta = \text{constant} \ , \ \phi = \text{constant} \ , \tag{2.3}$$

where r, θ and ϕ are coordinates in the frame where the metric assumes the form (2.1). To visualize, one may think that galaxies are the tracers of the cosmic expansion, or in other words, their motion is approximately described by (2.3). In the original Hubble view of the cosmic expansion, this corresponds to assign the whole motion of galaxies to the cosmic expansion, giving them a fixed comoving coordinate, as it is represented in figure 2.1.

2.1.1 Conformal time

Although time does not enter in the discussion about comoving coordinates above, there is a very common time variable which may replace the ordinary time in (2.1). By performing the coordinate change

$$d\tau = \frac{dt}{a(t)} \ , \tag{2.4}$$

the FRW metric may be easily written as

$$g_{\mu\nu} \equiv a^2 \cdot \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & (1-Kr^2)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \equiv a^2\gamma_{\mu\nu} \tag{2.5}$$

so that the cosmic expansion is completely factored out of the comoving part of the metric, which we define as $\gamma_{\mu\nu}$; τ is the conformal time, and is our time variable in the following, unless otherwise specified. We will indicate the conformal time derivative with $\dot{}$, while those with respect to the ordinary time are indicated with the subscript $_t$. It is also useful to define two different quantities describing the velocity of the expansion, i.e.

$$H = \frac{a_t}{a} \ \ , \ \ \mathcal{H} = \frac{\dot{a}}{a} \ , \tag{2.6}$$

named ordinary and conformal Hubble expansion rates, respectively; as it is easy to see, the two are related by $H = \mathcal{H}/a$.

2.2 Stress energy tensor

The stress energy tensor specifies the content of spacetime, in terms of physical entities, i.e. particles, fields and their properties. We limit ourselves here to describe a perfect relativistic fluid, homogeneous and isotropic. These assumptions again restrict dramatically the complexity of the general expression for the stress energy tensor. The quantities that characterize it are just the energy density, ρ, and the pressure p. The quantities in the stress energy tensor which has direct physical meaning are those with covariant and controvariant indeces. In this form, T^ν_μ is most easy as the $(0,0)$ components represent the energy density, while p is isotropically assigned to all directions as

$$T^\nu_\mu \equiv \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \ , \tag{2.7}$$

where the minus to the energy density is due to the choice of our signature (1.2). The stress energy tensor may also be written as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \ , \tag{2.8}$$

where u^μ represents the quadri-velocity of a fluid element, with an affine parameter which for convenience may be taken as the conformal time itself:

$$u^\mu = \frac{dx^\mu}{d\tau} \ . \tag{2.9}$$

In analogy with the normalization of the quadri-impulse of a particle with mass m, $p_\mu p^\mu = -m^2$, and since the energy is represented by the term $\rho + p$ in (2.9), the quadri-velocities are normalized as $u_\mu u^\mu = -1$. In comoving coordinates, where the $u^a = 0$, this condition implies

$$u^\mu \equiv \left(\frac{1}{a}, 0, 0, 0\right) \ . \qquad (2.10)$$

2.3 Dynamics and conservation

The Einstein and conservation equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \ T_{\mu\nu}^{;\nu} = 0 \ , \qquad (2.11)$$

reduce to two differential equations only, where the independent variable is the time τ, expressing the dynamics of the expansion, plus the conservation of energy, respectively. The first one is the Friedmann equation

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \rho - K \ , \qquad (2.12)$$

which is equivalent to the equation ruling the acceleration of the expansion:

$$\dot{\mathcal{H}} - \mathcal{H}^2 = -4\pi G a^2 (\rho + p) \ . \qquad (2.13)$$

The conservation equation becomes

$$\dot{\rho} + 3\mathcal{H}(\rho + p) = 0 \ . \qquad (2.14)$$

As it is evident, it is impossible to solve this system if some relation between pressue and energy density is given, $p(\rho)$. For interesting cases, as those we shall see in the next Section, pressure is proportional to the energy density:

$$p = w\rho \ , \qquad (2.15)$$

where w is the equation of state of the fluid, which may have a time dependence as any other cosmological component in the FRW background.

2.4 Cosmological species

So far we did not consider the case in which the stress energy tensor is made by more than one component, although in a realistic case, several of them are present at the same time. In this case, the stress energy tensor we treated so far corresponds to the total one, a sum over those corresponding to each component, labeled by c as follows:

$$T_{\mu\nu} = \sum_c {}_cT_{\mu\nu} \ . \qquad (2.16)$$

The single stress energy tensors may not be conserved as a result of mutual interactions between the different components. Therefore, the conservation equation for each component may be written as

$${}_cT_{\mu\nu}^{;\nu} = {}_cQ_\mu \ , \qquad (2.17)$$

where ${}_sQ_\mu$ expresses the non-conservation. Since the total stress energy tensor must be conserved, the interactions between the different components must satisfy the constraint

$$\sum_c {}_cQ_\mu = 0 \; . \tag{2.18}$$

2.4.1 Relativistic and non-relativistic matter

The simplest example of cosmological component is represented by the non-relativistic (nr) matter. The usual example is that of particles at thermal equilibrium, and characterized by a temperature giving rise to a thermal agitation which is negligible with respect to their mass m, so that the momentum of each of them is

$$p^2 = p_\mu p^\mu \simeq m^2 \; . \tag{2.19}$$

Whatever the interaction is, in this limit collisions are negligible. No collisions means no pressure, therefore for this species, the equation of state is simply zero. Such component in cosmology is commonly known as Cold Dark Matter (CDM). As it is easy to verify, the time dependence of this component, assuming that it is decoupled from the others, may be expressed as a function of the scale factor as

$$\rho_{nr} \propto a^{-3} \; . \tag{2.20}$$

The next example is opposite in many aspects. Relativistic (r) particles at thermal equilibrium are characterized by an energy which is dominated by thermal agitation rather than mass. By applying the laws of statistical quantum mechanics for relativistic particles at thermal equilibrium, one finds that pressure and energy density are related by the following relation:

$$p_r = \frac{1}{3}\rho_r \; . \tag{2.21}$$

As it is easy to verify, this implies

$$\rho_r \propto a^{-4} \; , \tag{2.22}$$

which has a direct intuitive meaning. Indeed, taking photons as an example, each of those carries an energy $\hbar\omega$ where ω is the frequency of the associated wave, thus redshifting as a result of the stretching of the wavelenght. This is responsible for the extra-power in (2.22) with respect to (2.20), which contains only the contribution from the dilution as a result of the expansion of the volume.

2.4.2 Cosmological constant

A third case, most interesting and dense of theoretical implications, is the one in which the energy density is conserved, i.e.

$$p = -\rho \; . \tag{2.23}$$

A constant vacuum energy density appeared for the first time in the form of a cosmological constant Λ, introduced by Einstein himself in the general relativity equations as a pure geometrical term:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \; . \tag{2.24}$$

Indeed, bringing it to the right hand side, and passing to the mixed form for the indeces, one gets

$$G_\mu^\nu = 8\pi G \left(T_\mu^\nu - \frac{\Lambda}{8\pi G} \delta_\mu^\nu \right) \ , \tag{2.25}$$

and looking at the form of the stress energy tensor (1.10) it is straightforward to verify that the one related to the cosmological constant is characterized by $p_\Lambda = -\rho_\Lambda = -\Lambda/8\pi G$.

2.4.3 Scalar field

The simplest generalization of a constant energy density is represented by a scalar field Ψ. Its stress energy tensor may be obtained by varying the action in general relativity with respect to the metric in order to get the Einstein equations. The Lagrangian density for a scalar field ψ in general relativity is given by

$$\mathcal{L} = \frac{R}{16\pi G} + \psi_{;\mu}\psi^{;\mu} - V(\psi) \ , \tag{2.26}$$

where V represents the potential energy. The variation with respect to the metric leads to the following expression for the scalar field stress energy tensor:

$$T_{\mu\nu} = \psi_{;\mu}\psi_{;nu} + \left(\frac{1}{2}\psi_{;\rho}\psi^{;\rho} - V \right) g_{\mu\nu} \ . \tag{2.27}$$

In a FRW background, the field may depend on the time coordinate only. By looking at the expression above in the FRW limit, one may easily see that the scalar field is equivalent to a fluid which has an energy density

$$\rho = \frac{1}{2a^2}\dot{\psi}^2 + V \ , \tag{2.28}$$

and a pressure given by

$$p = \frac{1}{2a^2}\dot{\psi}^2 - V \ , \tag{2.29}$$

Interestingly, the scalar field reduces to the case of a cosmological constant in the limit $\dot{\psi} = 0$.

2.5 Cosmic acceleration

We close this chapter by pointing out which one among the cosmological species mentioned above is able to induce an accelerated cosmic expansion. The latter condition is simply equivalent to

$$\ddot{a} > 0. \tag{2.30}$$

From (2.13), using (2.12), it is immediately evident that

$$\frac{a_{tt}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right) - \frac{K}{a^2} \ . \tag{2.31}$$

The first thing to note is that an open universe, with $K < 0$, contributes to make $a_{tt} > 0$. The second thing to note is that in order to have acceleration from the content of the stress energy tensor, one must have

$$w < -\frac{1}{3} \, . \tag{2.32}$$

If the energy density is positive, that means that the pressure must be negative as in the case of the cosmological constant, or the scalar field with vanishing kinetic energy.

Chapter 3

Pre-inflationary cosmology

In this chapter we review the main aspects of cosmology as it was before the introduction of the concept of inflation, focusing on the problems of the whole picture and showing how a phase of exponential expansion in the early universe may constitute an elegant solution.

3.1 Thermal history of the universe

The cosmic microwave background represents a very strong indication that the universe was hotter in its early stages. Today the non-relativistic component of the energy density, which we indicate generically as matter, is about 25% of the total one; photons and neutrinos, which are relativistic and indicated generically as radiation, are about 100000 times less abundant. Given that matter scales as the inverse of the cube of the scale factor, while radiation as the inverse of the fourth power, it is easy to calculate that the moment at which matter and radiation were comparable occurs at an epoch corresponding to

$$1 + z_{eq} = \frac{a_0}{a_{eq}} \simeq 10^4 \ , \tag{3.1}$$

where $z = a_0/a_{eq} - 1$ is the redshift coordinate, eq marks the equivalence epoch, and a_0 represents the value of the scale factor at the present, which we assume to be 1 in the following. Given that the CMB temperature scales as the inverse of the scale factor, and that today it is about 2.726 Kelvin, the equivalence epoch corresponds to about $3 \cdot 10^4$ Kelvin. At that temperature, CMB and photons are tightly coupled via Thomson scattering. That means that the mean free path, simply related to the number density of targets and the Thomson cross section, is typically much smaller than the causal scale associated to the cosmic expansion, which corresponds to the inverse of the Hubble length:

$$\lambda_T = 1/(n_e \sigma_T) \ll H^{-1} \ . \tag{3.2}$$

In this regime, photons are tightly coupled with all charged particles, so that a single temperature characterizes the whole system. Going back in time, densities increase for all species, and sooner or later all interactions occur on spacetime scales smaller than the horizon, no matter how small is the cross section.

In other words, the early universe is characterized by a single temperature describing a thermal bath of elementary bosons and fermions tightly coupled by their interactions. Is this picture going to persist at any time, no matter how early it is?
There is a natural limit at which this picture cannot be trusted. The current model of particle physics predicts unification of the different interactions at high energy. An example is represented by the electromagnetic and weak interactions, which are thought to decouple via a spontaneous symmetry breaking at an energy scale comparable to the mass of the particle responsible for that breaking, the Higgs boson, corresponding to a few hundreds of GeV. In units of temperature, 10^2 GeV is equivalent to about 10^{15} K. This scale more or less represents the limits of the physics we can probe directly in laboratories.
Let us review the most important steps in the thermal cosmic history, going backward in time, up to that scale:

- $E = k_B T \simeq 10^{-1}$ eV, photon baryon decoupling,
- $E \simeq 10^0$ eV, matter radiation equality,
- $E \simeq 10^{-1}$ MeV, nucleosynthesis through deuterium formation from neutron proton scattering,
- $E \simeq 10^0$ MeV, neutrino decoupling from neutrino antineutrino annihlilation in electron positron pairs, electron positron annihilation in two photons,
- $E \simeq 10^2$ GeV, symmetry breaking between electromagnetic and weak interaction,
- ...?

At higher energies or at earlier cosmological epochs, the electromagnetic and weak interactions should be described by a single one, named electroweak. Similar models exist for the unification of the strong interaction with the others, which should occur at earlier cosmological epochs. It is conceivable that also the gravitational interaction unifies with the other forces; although there is not a successful model for that, one may guess that such epoch corresponds to the energy scale which may be formed combining the fundamental constant including the gravitational one:

$$E_{Planck} = \sqrt{\frac{\hbar c^5}{G}} \simeq 10^{19} \text{ GeV} . \quad (3.3)$$

In the above expression, we have momentarily re-introduced the use of the speed of light and Planck constants, c, keeping that until the end of this section. If this is indeed the scale at which gravity unifies with other forces, then a large gaps seperates this scale from the highest one which may be probed directly in laboratories on the Earth, which is about 10^2 GeV. The time coordinate which may be associated to the fundamental constant, including G, is given by

$$t_{Planck} = \sqrt{\frac{\hbar G}{c^5}} \simeq 10^{-55} \text{ s} , \ L_{Planck} = c \cdot t_{Planck} = \sqrt{\frac{\hbar G}{c^3}} \simeq 10^{-35} \text{ cm} . \quad (3.4)$$

The relations (3.3) and (3.4) may be combined to obtain the Planck energy density, which is

$$\rho_{Planck} = \frac{E_{Planck}}{L^3_{Planck}} \simeq 10^{123}\rho_0 \ , \tag{3.5}$$

where $\rho_0 = 3(cH_0)^2/8\pi G$ is the present critical energy density, corresponding to about 10^{-8} g/cm^3.
The Planck energy scale represents the limit for our understanding of cosmology, but also for physics as a whole. At this epoch, the physics of spacetime and particles can no longer be thought as decoupled. Since no viable prediction for the physics at this energy has been produced, at the moment this is the border between physical investigations and speculations.

3.2 Cosmological constant problem

Historically, there are two ways in which a constant vacuum energy appeared in the Einstein equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G(T_{\mu\nu} + V g_{\mu\nu}) \ . \tag{3.6}$$

The first term is the cosmological constant, which we have already seen, introduced by Einstein in 1916. Its conception is entirely geometric. He removed it from the general relativity equations, when he realized that it was not capable of keeping the universe static, as he thought in the beginning. In the FRW cosmology, that is easily seen by noting that the Friedmann equation reduces to $H^2 = constant$, which admits an exponential expansion as a solution. Later on, the case for a constant energy density in the vacuum was raised independently by quantum mechanics, and is represented by V in the equation above; essentially, there is no reason why quantum systems should have the fundamental state at zero energy, and in many cases actually they assign a non-zero energy to it. The simplest example is that of the harmonic oscillator at frequency ω, which has the following energetic levels:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \ . \tag{3.7}$$

The fundamental state at $n = 0$ does possess a non-zero energy. Let us look at the expectations that we may have for Λ and V in (3.6). Basically, there is no expectation at all for Λ. There is no theory for V, but there is the expectation that it is of the order of the Planck energy density, for the following reasons. As we have already stressed, the current hypothesis on the physical processes at high energies do predict the unification of physical interactions. The highest one, involving all interactions including gravity, corresponds to the Planck energy density. Since quantum mechanics generically predicts a vacuum energy density, it is reasonable to expect that it is of the order of the Planck energy density itself.
Whatever are the values of Λ and V, the cosmological constant problem is that they are required to cancel out with a fantastic precision, simply because the vacuum energy density today can at most be comparable with the present critical energy density in order to be compatible with the recent cosmological

history; from (3.5) one gets

$$\frac{|\Lambda/8\pi G - V|}{\rho_{Planck}} \lesssim 10^{-123} \ . \tag{3.8}$$

This problem has been there for almost a century now, and the evidence for cosmic acceleration, saying that the number above is actually different from zero and of the order of 10^{-123}, renewed the interest in it. Since inflation and the models of the early universe deal with an early stage of cosmological expansion which is dominated by some sort of vacuum energy, it is appropriate to mention it here. On the other hand, no model of the early universe, or in general no physical argument has been able to exmplain it so far.

3.3 Flatness problem

In the scenario we have been treating so far, the cosmological expansion is dominated by radiation at arbitrarily small times. This caused several problems connected with the dynamics of the cosmic expansion. The first one is known as the flatness problem, which we describe here.
Indicating with $\rho_c = 3H^2/8\pi G$ the critical density at an arbitrary time, let us consider the quantity

$$\left|\frac{\rho}{\rho_c} - 1\right| = |\Omega - 1| = \frac{1}{a^2H^2} \ . \tag{3.9}$$

where the last equality is taken from the Friedmann equation (2.12), taking into account that by rescaling the coordinates, it is always possible to deal with values of the curvature K equal to 0, ± 1. The present data in cosmology say that the universe is close to flatness, i.e.

$$|\Omega - 1|_0 = O(10^{-2}) \ . \tag{3.10}$$

Let us see how this number goes back in time. From the known scaling of matter and radiation energy densities, it is easy to see that the scaling of H in the matter and radiation dominated eras (MDE, RDE) are

$$H \propto a^{-3/2} \text{ in the MDE } , H \propto a^{-2} \text{ in the RDE } . \tag{3.11}$$

By assuming that the universe is matter dominated after equivalence and radiation dominated before, at an arbitrary epoch during the RDE, one has

$$|\Omega - 1| = \frac{a^2}{a_{eq}^2}|\Omega - 1|_{eq} = \frac{a^2}{a_{eq}^2}\frac{a_{eq}^{3/2}}{a_0^{3/2}}|\Omega - 1|_0 \ . \tag{3.12}$$

The ratios between the scale factors at different epochs may be calculated by using again the energy scale, proportional to the temperature and inversely proportional to the scale factor. The second ratio in (3.12) yields a factor $(10^{-4})^{3/2} = 10^{-6}$. If we compute the first ratio at the Planck scale (3.3), the result is

$$|\Omega - 1|_{Planck} = 10^{-62}|\Omega - 1|_0 \ . \tag{3.13}$$

That means that if $|\Omega - 1|_0$ today is different from zero, then one must have adjusted the initial energy density to be extremely close to the critical one, in order to yield the value observed at present. Other than that, there is no explanation for this problem in the present scenario.

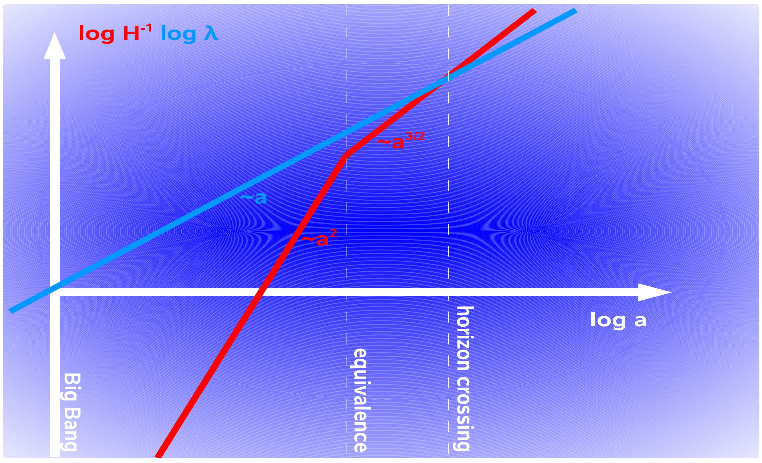

Figure 3.1: A representation of the behavior of horizon and scales as a function of the scale factor, in a logarithmic scale.

3.4 Horizon problem

In figure 3.1 we show the behavior of horizon and physical scales as a function of the scale factor, on a logarithmic scale. The scaling of the Hubble horizon has been taken from (3.11). Physical scales scale of course linearly with a. As it is evident, for any physical scale λ there is only one epoch at which it equals the Hubble horizon, i.e. one moment only in which it is in horizon crossing. After that epoch, a perturbation on that scale is always within the horizon, thus capable to thermalize. Before that epoch, no causal connection exists for structures separated by that scale. The horizon problem comes when one considers in particular the scale λ which is entering the horizon today. On that scale, we see a remarkable isotropy represented by the temperature of the CMB, which is the same with corrections of one part over 10^5 on all scales. From the elementary reasoning in the figure, we also know that that scale was never in causal connection in the past, at least in the present scheme. The horizon problem asks for an explanation of this level of isotropy on super-horizon scales, which in the present scheme where the RDE goes on indefinitely in the past, may be explained only if someone put by hands an extremely high level of homogeneity in the perturbations in the early universe.

3.5 Inflationary kinematics

We condlude this chapter showing how an era of accelerated expansion which occurs before the RDE may solve at least the last two of the problems mentioned above.
Suppose indeed that the cosmological expansion history is described by the representation in figure 3.2, in which the RDE comes after and era in which H is a constant; in an FRW cosmology, that is achieved by having a cosmological constant, regardless of the cosmological curvature. Indeed, if $K = 0$, the

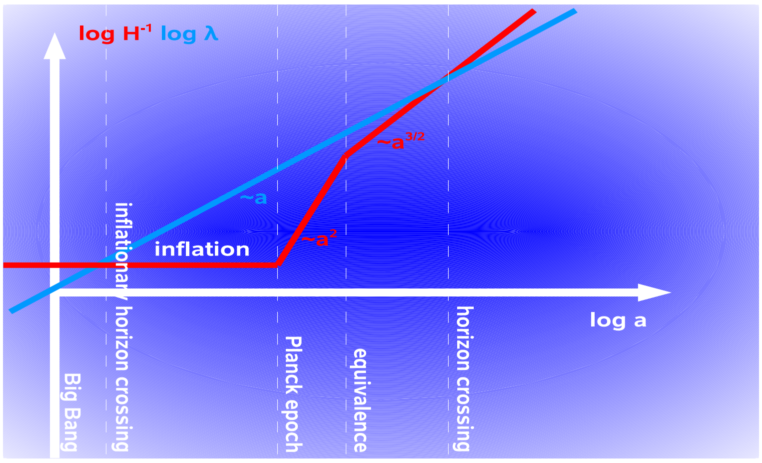

Figure 3.2: A representation of the behavior of horizon and scales as a function of the scale factor, in a logarithmic scale, including an era of exponential expansion in the early universe.

Friedmann equation is

$$H^2 = \frac{\Lambda}{3} \ , \tag{3.14}$$

with the solution

$$a = a_{BB} \exp\left[\sqrt{\frac{\Lambda}{3}}\,(t - t_{BB})\right] \ , \tag{3.15}$$

where BB stays for Big Bang and marks the initial time. If K is bigger or smaller than zero, the Friedmann equation $a_t^2 = \Lambda a^2/3 - K$ is satisfied by

$$a = \sqrt(K) \cosh\left[\sqrt{\frac{\Lambda}{3}}\,(t - t_{BB})\right] \quad \text{and} \quad a = \sqrt(- K) \sinh\left[\sqrt{\frac{\Lambda}{3}}\,(t - t_{BB})\right] \ , \tag{3.16}$$

respectively, so that after a transient time interval corresponding to $\sqrt{3/\Lambda}$ the expansion becomes exponential. We call this epoch inflation in the following. A convenient and very common time coordinate during inflation is represented by the number of $e-$foldings, i.e the number of times the scale factor increases by a factor e, the Nepero number:

$$N = \log_e \left[\frac{a_(t_2)}{a(t_1)}\right] \ . \tag{3.17}$$

It is easy to see that this phase gives an immediate, kinematic solution to the flatness and horizon problem mentioned above, provided that it lasts long enough.
Let us solve the flatness problem first, considering the behavior of the quantity $|\Omega - 1|$ during inflation. We neglect the first transient in (3.16) and take H as a constant. In this regime $|\Omega - 1|$ decreases in time exponentially, making an initially curved cosmology closer and closer to flatness as the expansion proceeds. We do not know what is the cosmological curvature, but let's suppose

that cosmology at the beginning is represented by some value $|\Omega-1|_{BB}$ different from zero. We may as how long the inflation has to last, in order to match the value at the Planck epoch which we extrapolate from the present, in (3.13). It is easy to see that the latter condition implies

$$|\Omega - 1|_{Planck} = |\Omega - 1|_{BB} \left(\frac{a_{BB}}{a_{Planck}}\right)^2 = 10^{-62}|\Omega - 1|_0 \ . \tag{3.18}$$

By giving to the last quantity the present upper limit, $|\Omega-1|_0 \lesssim 10^{-2}$, one gets

$$\frac{a_{Planck}}{a_{BB}} = \sqrt{10^{64}|\Omega - 1|_{BB}} \ . \tag{3.19}$$

If one assumes that initially the cosmological curvature and energy density yielded comparable terms, represented by $|\Omega - 1|_{BB} = O(1)$, a solution to the flatness problem is achieved if inflation lasts for about 74 $e-$foldings at least. Let us now consider the horizon problem. Rejecting the hypothesis of an external intervention setting the initial conditions on super-horizon scales, the whole universe we see today must have been within the Hubble horizon in the past, in order to thermalize and reach the remarkable level of homogeneity we see today through the CMB. As it is evident in figure 3.2, inflation gives you an easy solution to this problem, provided again that it lasts long enough so that the cosmological scale corresponding to the observed universe today was inside the horizon after the Big Bang. The scale of the observed universe today is about twice to the present value of the Hubble horizon, which is about 8200 Mpc. The epoch of the horizon crossing during inflation is obtained by matching that scale with H^{-1} during inflation, which constant and therefore equal to the value it has at the Planck epoch, H^{-1}_{Planck}. This leads to the condition

$$2H_0^{-1} \cdot \frac{a_{IHC}}{a_0} = H^{-1}_{Planck} \ , \tag{3.20}$$

where IHC indicates the inflationary horizon crossing. By using the known scaling in the MDE and RDE up to the Planck epoch, it is easy to see that $H^{-1}_{Planck} = (a_{Planck}/a_{eq})^2(a_{eq}/a_0)^{3/2}H_0^{-1} \simeq 10^{-60}H_0^{-1}$. Moreover, $a_{IHC}/a_0 = (a_{IHC}/a_{Planck}) \cdot (a_{Planck}/a_0) \simeq 10^{-31}(a_{IHC}/a_{Planck})$. Thus the relation (3.20) becomes

$$2\frac{a_{IHC}}{a_{Planck}} = 10^{-29} \ , \tag{3.21}$$

which implies that inflation must have lasted for about 67 $e-$foldings at least in order to have the whole universe we see today in horizon crossing during inflation itself.

The solution to the flatness and horizon problems is due essentially to the exponentially accelerated cosmological expansion, which goes like under the effect of a repulsive gravity, radically different from the behavior induced by the ordinary particles in the RDE and MDE. This tells that the kinematic of inflation is very promising in solving at least two of the three classical problems of the pre-inflationary FRW cosmology. This motivated the construction of some physical models for this process, to be treated in the next chapter.

Chapter 4

Inflationary cosmology

In this chapter we review the simplest inflationary models, focusing on the behavior of the background expansion. We first give the basics of scalar field physics in the context of cosmology, then give the conditions for having a viable inflationary epoch out of a scalar field model, and finally we treat some specific model in some detail.

4.1 Scalar fields in cosmology

A dynamical quantity q of a point-like object moving under the effect of a potential $V(q)$ is described by the action

$$S = \int_{-\infty}^{+\infty} L dt = \int_{-\infty}^{+\infty} \left[\frac{1}{2} q_t^2 - V(q) \right] dt \ , \tag{4.1}$$

where L represents the Lagrangian of the system. The motion equation correspond to the trajectories $q(t)$ which extremize the action above:

$$\frac{d}{dt} \frac{\partial L}{\partial q_t} + V_q = 0 \Leftrightarrow q_{tt} + V_q = 0 \ . \tag{4.2}$$

In field theory, the system above represents the global limit of a local theory, in which the degrees of freedom are different in any spacetime point:

$$q(t) \to \psi(x) \ . \tag{4.3}$$

If ψ is constrained to be homogeneous in space, i.e. in the coordinates $\vec{x}$, field theory reduces to the physical system above. Correspondingly, the action must be generalized to

$$S = \int_{-\infty}^{+\infty} \mathcal{L} \sqrt{-g} d^4 x \ , \tag{4.4}$$

where the integration is over the whole spacetime now, and the term $\sqrt{-g}$ with g indicating the metric tensor determinant is necessary to make S a general relativistic invariant. The Lagrangian becomes a Lagrangian density, where the field dependence on all coordinates is made manifest by the covariant derivatives instead than on time only; the potential $V(\psi)$ still rules the dynamics, and if the argument of the potential is the field only, the field is self-interacting. moreover, the Lagrangian of gravity in general relativity is the scalar quantity $R/16\pi G$. Thus, the dynamics of a self-gravitating and self-interacting scalar fiels is given by

$$\mathcal{L} = \frac{R}{16\pi G} + \frac{1}{2}\psi_\mu\psi^\mu - V(\psi) \ . \tag{4.5}$$

The motion equations again correspond to the trajectories extremizing the action. We have now a double dynamics, induced separately by gravity and the scalar field. The extremization with respect to the metric gives the Einstein field equations

$$\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L} = 0 \Leftrightarrow G_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{4.6}$$

where

$$_\psi T_{\mu\nu} = \psi_{;\mu}\psi_{;\nu} - g_{\mu\nu}\left[\frac{1}{2}\psi_{;\rho}\psi^{;\rho} + V(\psi)\right] \ . \tag{4.7}$$

Similarly, the extremization with respect to the scalar field trajectories give the Klein-Gordon equation

$$\left(\frac{\partial\mathcal{L}}{\partial\psi_\mu}\right)_{;\mu} + \frac{\partial V}{\partial\psi} \Leftrightarrow \psi^{;\mu}_{;mu} + V_\psi = 0 \ , \tag{4.8}$$

where $= \psi^{;\mu}_{;\mu}$. The latter equation coincides with the conservation equation $(_\psi T^\nu_\mu)_{;\nu} = 0$.
The FRW limit of the present picture nicely reduces to a system as simple as the one we described in the beginning of this section. The dependence on $\vec{x}$ disappears and the Klein-Gordon equation becomes

$$\psi_{tt} + 3H\psi_t + V_\psi = 0 \ , \tag{4.9}$$

where with respect to (4.2) we notice effect of gravity, through a cosmological friction equal to the number of spatial dimensions multiplied by the Hubble expansion rate. The latter also corresponds to the continuity equation $\rho_{\psi tt} + 3H(\rho_\psi + p_\psi)$, where

$$\rho_\psi = \frac{1}{2}\psi_t^2 + V(\psi) \tag{4.10}$$

$$p_\psi = \frac{1}{2}\psi_t^2 - V(\psi) \ . \tag{4.11}$$

The Friedmann equation is therefore

$$H^2 = \frac{8\pi G}{3}\rho_\phi + \frac{K}{a^2} = \frac{8\pi G}{3}\left[\frac{1}{2}\psi_t^2 + V(\psi)\right] + \frac{K}{a^2} \ . \tag{4.12}$$

4.2 Inflation from scalar fields

It is clear that in the limit of a static field, $\psi_t \to 0$, the system of equations above reduces to the one of a cosmological constant, where the latter is simply the scalar field potential V. The whole idea of inflation from a scalar field, named the inflaton, is that the latter enters a low dynamic phase in which the potential energy $V > 0$ plays the role of a slowly dynamical cosmological constant; this

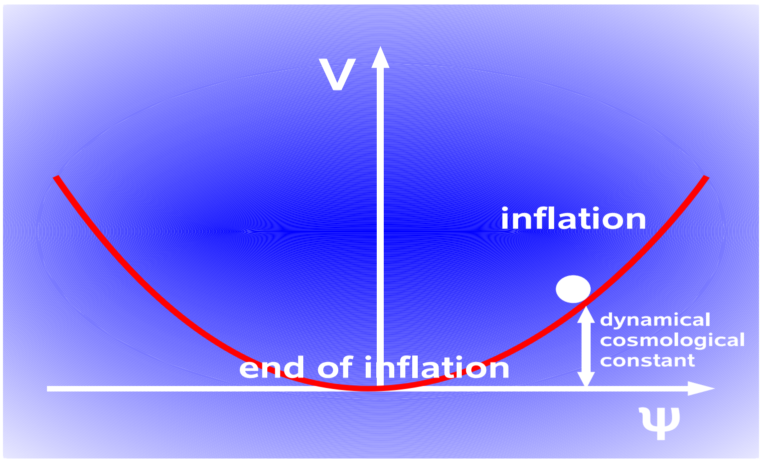

Figure 4.1: The simplest picture of an inflationary model.

phase must last long enough to inflate the space as needed to solve the horizon and flatness problems mentioned in the previous chapter. This mechanism is quite natural in a shallow potential V with a global minimum, as represented in figure 4.1. In this picture, the field value ψ undergoes a slow rolling on the potential, activating the inflationary expansion which ends when it reaches the minimum.

4.2.1 Slow rolling

The concept of slow rolling is actually common to most of the inflationary models proposed so far, and may be casted as a requirement to the shape of the potential in order to be able to activate the inflationary expansion for an amount of time which is enough to solve the horizon and flatness problems. If a scalar field ψ is the only cosmological component obeying the dynamics imposed by the potential energy density V, the Friedmann and Klein Gordon equations are

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\psi_t^2 + V\right) + \frac{K}{a^2} \quad , \quad \psi_{tt} + 3H\psi_t + V_\psi = 0 \tag{4.13}$$

and complete the description of the system. In order to be close to the expansion regime of a cosmological constant, the following three conditions must be satisfied. First, in the Friedmann equation the kinetic energy must be negligible with respect to the potential one. Second, the field acceleration must be small in order to have this condition satisfied for a non-zero time interval. Third, this time interval must be long enough to make the cosmological curvature negligible:

$$H^2 \simeq \frac{8\pi G}{3}V \quad , \quad 3H\psi_t + V_\psi \simeq 0 \ . \tag{4.14}$$

It is possible to express these requirements more formally, defining the slow rolling parameters

$$\epsilon_{sl} = \frac{\psi_t^2}{2V} \ll 1 \quad , \quad \eta_{sl} = \left|\frac{\psi_{tt}}{3H\psi_t}\right| \ll 1 \ . \tag{4.15}$$

These conditions are sufficient to make the slow rolling active in a time interval centered on the time in which they are satisfied. Such time interval must be long enough to solve the flatness and horison problems, i.e. about 60 $e-$foldings. For practical reasons, however, it is convenient to have relations which constrain directly the potential shape. The requirement that the potential is flat enough to allow a dynamics close to the case of a cosmological constant is conveniently expressed in terms of its curvature; specifically, the first and second derivative of potential with respect to the field must be small enough. Two convenient dimensionless conditions are

$$\epsilon = \frac{1}{48\pi G}\left(\frac{V_\phi}{V}\right)^2 \ll 1 \ \ , \ \ \eta = \left|\frac{1}{24\pi G}\frac{V_{\phi\phi}}{V}\right| \ll 1 \ . \tag{4.16}$$

Since these are conditions which affect directly the potential shape, they are often used in any practical application, and named generically slow rolling conditions, while ϵ and η are slow rolling parameters. The general phenomenology is that if they are sufficiently small, for a suitable class of initial conditions inflation starts, makes the initial curvature negligible, and continues long enough to solve horizon and flatness problems. Their choice is not unique however, but convenient because they can be put in direct relation with ϵ_{sl} and η_{sl}. Specifically, the conditions (4.16) are necessary if (4.15) are satisfied, as we now verify:

$$\epsilon_{sl} < 1, \ \eta_{sl} < 1 \Rightarrow \epsilon < 1, \ \eta < 1 \ . \tag{4.17}$$

Indeed, $\eta_{sl} \ll 1$ implies $H^2 = V_\psi^2/9\psi_t^2$ which is also equal to $(8\pi G/3)V$ by virtue of $\epsilon_{sl} \ll 1$. This means $V_\psi^2/24\pi GV = \psi_t^2$, which implies $\epsilon \ll 1$ by using $\epsilon_{sl} \ll 1$ again. Moreover, using $\psi_t = -V_\psi/3H = -V_\psi/\sqrt{24\pi GV}$ and deriving once in time, after dividing both sides by $3H\psi_t = \sqrt{24\pi GV}\psi_t$ and substituting again $\psi_t = -V_\psi/\sqrt{24\pi GV}$ everywhere, one gets $\eta = |\epsilon - \psi_{tt}/3H\psi_t|$; since ϵ and η_{sl} are both much smaller than one, this also implies $\eta \ll 1$. Note finally that the implication in (4.17) is in one sense only. For example, for a potential arbitrarly flat, one could start from a large ψ_t and ψ_{tt} so that the conditions (4.15) are not satisfied.

Let us now get an expression for the number of $e-$foldings N defined in (3.17) during the slow rolling regime. A formal solution for the Friedmann equation is $a = a_{BB}\exp\left(\int_{t_{BB}}^{t} H dt\right)$, so that

$$N = \int_{t_{BB}}^{t} H dt \ . \tag{4.18}$$

If ψ is a monotonic function of t, one may change integration variable, $dt = d\psi/\psi_t$. Therefore, using the slow rolling expressions (4.14) one gets

$$N = -8\pi G\int_{\psi_{BB}}^{\psi} \frac{V}{V_\psi} d\psi \ , \tag{4.19}$$

giving the number of $e-$foldings directly as an integral function of the potential shape and the field motion.

4.2.2 Inflationary models

We now review the simplest models of scalar field inflation. We choose two of them yielding a quasi-exponential and a power law expansion, respectively.

Chaotic inflation

The simplest model of inflation is obtained by assigning a mass to the inflaton field:

$$V = \frac{1}{2} m^2 \psi^2 \ . \quad (4.20)$$

The slow rolling parameters are independent on the mass:

$$\epsilon = \eta = \frac{1}{12\pi G\psi^2} \ . \quad (4.21)$$

That means that in this scenario, a successful inflation depends only on the magnitude of the field, independently on the amplitude of the potential. The solution to the slow rolling equations (4.14)

$$H^2 = \frac{4\pi G}{3} m^2 \psi^2 = 0 \ \ , \ \ 3H\psi_t + m^2\psi = 0 \quad (4.22)$$

is given by

$$\psi = \psi_{BB} - \frac{m}{\sqrt{12\pi G}} (t - t_{BB}) \ \ , \ \ a = a_{BB} \exp\left[2\pi G(\psi_{BB}^2 - \psi^2)\right] \ , \quad (4.23)$$

where the argument of the exponential above also coincides with the number of $e-$foldings, accordingly to (4.18) and (4.19). The last equality may be written also as

$$a = a_{BB} \exp m(t - t_{BB})\sqrt{G/3}(\psi + \psi_{BB}) \ , \quad (4.24)$$

making explicit that the expansion is quasi-exponential, since the linear increase of the first term in parenthesis is counterbalanced by the decrease of the quantity in the second one. From the slow rolling conditions (4.16), in order to have inflation the condition

$$|\psi| \geq \frac{1}{\sqrt{12\pi G}} \quad (4.25)$$

must be satisfied along the trajectory, where the equality marks the exit from the slow rolling regime, and the end of the inflationary expansion itself. The number of $e-$foldings achievable is therefore

$$N_{maximum} = 2\pi G\psi_{BB}^2 - \frac{1}{6} \ . \quad (4.26)$$

In this scenario, a sufficiently long period of inflation is achieved for a large enough initial condition, and no potential parameter. The name of the present chaotic inflationary scenario is due to this remarkable capability of allowing a successful inflation rather independently on the internal details of the theory.

Exponential inflation

Let us consider an exponential shape for the potential:

$$V = W \exp^{\lambda\psi} \ . \quad (4.27)$$

The slow rolling parameters become

$$\epsilon = \frac{\eta}{2} = \frac{1}{48\pi G}\lambda^2 \ , \quad (4.28)$$

while the slow rolling equations are

$$H^2 = \frac{8\pi G}{3} W e^{\lambda\psi} \ , \ 3H\psi_t + \lambda W e^{\lambda\psi} = 0 \ . \tag{4.29}$$

The general solution for ψ is

$$\psi = \frac{-2}{\lambda} \log \left[\frac{\lambda^2}{2} \sqrt{\frac{W}{24\pi G}} (t - t_{BB}) + e^{-\lambda\psi/2} \right] \ , \tag{4.30}$$

from which the general solution for $a(t)$ may be also found in the general case, from the first equation in (4.29):

$$a = a_{BB} \left(\frac{t - t_{BB} + \frac{1}{\lambda^2} \frac{96\pi G}{W} e^{-\lambda\psi_{BB}/2}}{\frac{1}{\lambda^2} \frac{96\pi G}{W} e^{-\lambda\psi_{BB}/2}} \right)^{16\pi G/\lambda^2} \ . \tag{4.31}$$

The number of $e-$foldings in this scenario may be obtained by taking the logarithm of a/a_{BB} from the above expression. Note that, contrary to the case of chaotic inflation, provided that the slow rolling is activated, i.e. the numbers in (4.28) are sufficiently small, there is no upper limit to the number of $e-$foldings. It is convenient to make some simplifying assumptions in order to make the expression above more clear: let us assume $\lambda < 0$, so that the potential makes the field value increasing with time, and work at $\psi \gg \psi_{BB}$, $t \gg t_{BB}$. Then we have the following differential equation for a:

$$\frac{a_t}{a} = \frac{16\pi G}{\lambda^2 t} \ . \tag{4.32}$$

The latter, for $t \gg t_{BB}$ has the power law solution

$$a(t) \propto t^{16\pi G/\lambda^2} \ , \tag{4.33}$$

which by virtue of the slow rolling conditions $\epsilon, \eta \gg 1$ implies an accelerated expansion.

4.2.3 Inflation as an attractor

As we have seen, the inflation has the capability of solving the horizon and flatness problems of pre-inflationary cosmology. However, one may ask how likely is that inflation starts if the universe is inhomogeneous, and how stable is the process after it started.
The first question is answered only qualitatively, through the picture in figure 4.2. Simply, the regions in which the conditions are appropriate for inflation to start undergo inflation, expanding faster than the others, by means of the inflationary mechanism itself. The example in the figure refers to the case of the chaotic inflation, in which the conditions for inflation are given just in terms of the field value. Therefore, after some time the inflating regions occupy a dominant and increasing fraction of volume, while the portion of space which is not undergoing inflation vanishes eventually. In this sense, the onset of inflation is a favourite process in inhomogeneous universe.

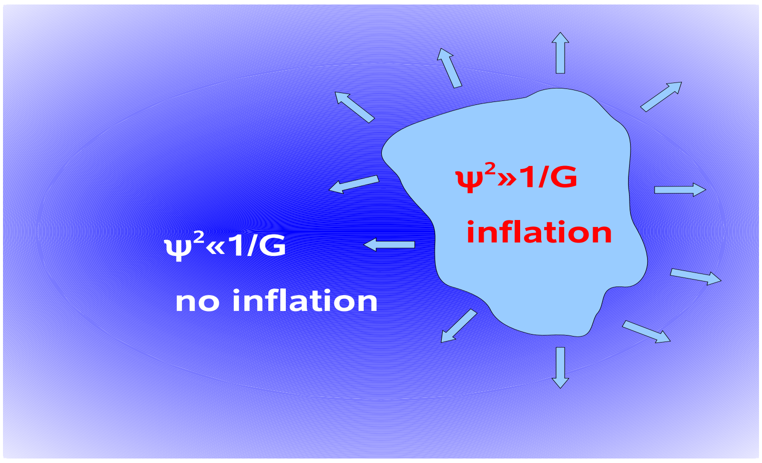

Figure 4.2: In an inhomogeneous universe, the regions where inflation takes place expand faster than the others.

To answer the second question, let us evaluate the stability of the inflationary slow rolling. The time derivative of the Friedmann equation gives

$$2HH_t = \frac{8\pi G}{3}\left(\psi_t\psi_{tt} + V_\psi\psi_t\right) = -8\pi G H\phi_t^2 \ , \tag{4.34}$$

where the last equality follows from the Klein Gordon equation (4.13). If the field evolution is monotonic with time, one may divide both sides by $H\psi_t$ expressing the derivative of H in terms of ψ instead of t:

$$H_\psi = -4\pi G\psi_t \ . \tag{4.35}$$

Thus the Friedmann equation assumes the form

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{32\pi^2 G^2}H_\psi^2 + V\right) \ . \tag{4.36}$$

We may consider the behavior of this equation under perturbations $\delta H(\psi)$ around the inflationary trajectory $H(\psi)$. The variation of (4.36) yields

$$H\delta H = \frac{1}{12\pi G}H_\psi\delta H_\psi \ , \tag{4.37}$$

which is solved by

$$\delta H = \delta H_{BB}\exp\left(12\pi G\int_{\psi_{BB}}^{\psi}\frac{H_{BB}}{H_\psi}\right) \ . \tag{4.38}$$

Now, from (4.15,4.35) and using the Friedmann equation it is easy to verify that the relation

$$\left(\frac{H_{BB}}{H_\psi}\right)^2 = \frac{1}{12\pi G}\left(1 + \frac{1}{\epsilon_{sl}}\right) \gg \frac{1}{12\pi G} \tag{4.39}$$

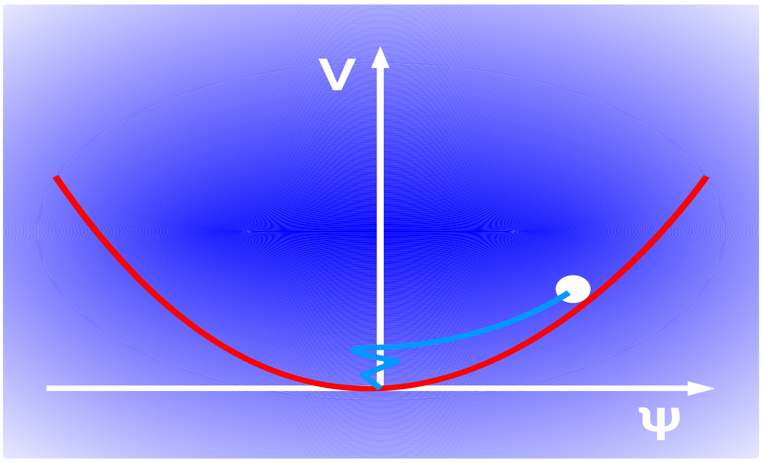

Figure 4.3: After inflation, the inflaton field rapidly oscillates around the minimum of the potential.

holds by virtue of the inflationary expansion. This means that

$$12\pi G \int_{\psi_{BB}}^{\psi} \frac{H_{BB}}{H_\psi} \ll \sqrt{12\pi G} \int_{\psi_{BB}}^{\psi} d\psi = \sqrt{12\pi G}(\psi - \psi_{BB}) \tag{4.40}$$

if $H_\psi > 0$, while if $H_\psi < 0$ a minus is in front of the last quantity. Using again (4.35) one may see that the last quantity in the relation above is always negative, meaning that δH is exponentially damped along the trajectory, and proving the stability of the inflationary trajectory.

4.2.4 Re-heating at the end of inflation

Inflation has to end eventually, with the production of particles, mostly relativistic due to the very high energy at which inflation occur, and the beginning of the radiation dominated era. The end of inflation is called reheating, and in its simplest formulation proceeds as follows. When the field reaches the minimum of the potential, a series of oscillations occur as sketched in figure 4.3. In this phase, when averaged over time interval much longer than the oscillation period, the kinetic and potential energy density equal:

$$< \frac{1}{2}\psi_t >=< V > \ . \tag{4.41}$$

This means that the field behaves as an effective pressurless component, with $< \rho \propto a^{-3} >$, making H decreasing as $1/a^{3/2}$, just as in the matter dominated era. If the inflaton possesses a decay amplitude Γ in particles, the decay acts as an extra-friction with respect to the Hubble expansion rate, so that the Klein Gordon equation may be written as

$$\psi_{tt} + (3H + \Gamma)\psi_t + V_\psi = 0 \ . \tag{4.42}$$

which corresponds to the conservation equation

$$\rho_t + (3H + \Gamma)(\rho + p) = 0 \ . \tag{4.43}$$

We may guess that in order to have inflation, the decay must be negligible with respect to the amplitude of H during inflation itself. But in the oscillatory phase of the field which follows the inflationary expansion, H decreases fast, becoming eventually comparable with the decay amplitude of the field. At this point, the decay takes over, and the energy density stored in the field is transferred in the decay products.

Chapter 5

Inflationary perturbations

In this chapter we make a quick review of how cosmological perturbations are thought to be born during inflation. We need ro review the basics of quantum field theory in curved spacetimes, before applying them to the inflationary process.

5.1 Quantum fields in curved spacetime

Assume a Minkowski background, and consider the evolution of scalar field ψ, self-interacting only through its mass m. Suppose that the field is also not homogeneous in space, $\psi(t, \vec{x})$. The Klein Gordon equation is simply

$$\psi_{tt} - \bigtriangledown^2 \psi + m^2 \psi = 0 \ , \tag{5.1}$$

where $\bigtriangledown^2 = \vec{\bigtriangledown} \cdot \vec{\bigtriangledown}$ is the Laplace operator. It is suitably solved in the Fourier space, decomposing the field as follows:

$$\psi(t, \vec{x}) = \sum_{\vec{k}} \psi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}} \ . \tag{5.2}$$

The choice of plane waves as expansion function is convenient as they are eigengunctions of the at eigenvalue $k^2 = \vec{k} \cdot \vec{k}$ of the Laplace operator in the Fourier space. Thus, still in the Fourier space, the solution to the Klein Gordon equation is analytical:

$$(\psi_{\vec{k}})_{tt} + E^2_{\vec{k}} \psi_{\vec{k}} = 0 \ \ , \ \ E_{\vec{k}} = \sqrt{k^2 + m^2} \ \ , \ \ \psi_{\vec{k}}(t) \propto e^{\pm i E_{\vec{k}} t} \ . \tag{5.3}$$

Since this represents a complete set of solutions to the Klein Gordon equation, the general solution for (5.1) is a linear superposition of them, and may be written as

$$\psi(t, \vec{x}) = \sum_{\vec{k}} \frac{1}{2E_{\vec{k}}} \left[a_{\vec{k}} e^{i(\vec{k}\cdot\vec{x} - E_{\vec{k}} t)} + a^+_{\vec{k}} e^{-i(\vec{k}\cdot\vec{x} - E_{\vec{k}} t)} \right] \ , \tag{5.4}$$

where the $2/\sqrt{2E_{\vec{k}}}$ factor is purely conventional and $a_{\vec{k}}$, $a^+_{\vec{k}}$ represent the amplitude of the field at each mode.
The quantization of the system proceeds as usual elevating the field ψ to the role

of quantum operator on the space of its possible states. , obeying the following commutation relations:

$$\begin{aligned}[\psi(\vec{x},t),\psi(\vec{x}',t)] &= 0\ , \\ [\psi(\vec{x},t),\psi(\vec{x}',t)] &= i\delta(\vec{x}-\vec{x}')\ , \\ [\psi_t(\vec{x},t),\psi_t(\vec{x}',t)] &= 0\ . \end{aligned} \tag{5.5}$$

We must identify what are numbers and what are operators in the expression (5.4). The only possible choice is represented by $a_{\vec{k}}$ and $a^+_{\vec{k}}$; the commutation relations (5.5) impose the corresponding ones on them:

$$\begin{aligned}\left[a_{\vec{k}},a_{\vec{k}'}\right] &= 0\ , \\ \left[a_{\vec{k}},a^+_{\vec{k}'}\right] &= i\delta_{\vec{k}\vec{k}'}\ , \\ \left[a^+_{\vec{k}},a^+_{\vec{k}'}\right] &= 0\ . \end{aligned} \tag{5.6}$$

The space in which these operators act is called the Fock space; it has an empty state, $|0>$, defined by

$$a_{\vec{k}}|0>=0\ \forall\vec{k}\ , \tag{5.7}$$

while a quantum of field at wavenumber is given by

$$|1_{\vec{k}}>=a^+_{\vec{k}}|0>=0\ \forall\vec{k}\ . \tag{5.8}$$

The field status with many quanta at different wavevectors is given by

$$|1_{\vec{k}},1_{\vec{k}},1_{\vec{k}},...>=a^+_{\vec{k}}a^+_{\vec{k}}a^+_{\vec{k}}...|0>\ . \tag{5.9}$$

A status with $n_{\vec{k}}$ equal quanta is indicated as $|n_{\vec{k}}>$ and correspond to have applied n times the operator $a^+_{\vec{k}}$ to $|0>$. All states are normalized so that $<n_{\vec{k}}|n_{\vec{k}}>=1$. In general $a_{\vec{k}}$ and $a^+_{\vec{k}}$ destroy and create one quanta by multiplying the state on which they operate by some numerical factor: $a_{\vec{k}}|n_{\vec{k}}>=\alpha^n_{\vec{k}}|(n-1)_{\vec{k}}>$, $a^+_{\vec{k}}|n_{\vec{k}}>=\beta^n_{\vec{k}}|(n+1)_{\vec{k}}>$. The numbers may are fixed by convention; indeed it is convenient to define the counter operator $N_{\vec{k}}=a^+_{\vec{k}}a_{\vec{k}}$ such that

$$<n_{\vec{k}}|N_{\vec{k}}|n_{\vec{k}}>=n\ . \tag{5.10}$$

It is easy to see that this condition, in addition to the commutation relations (5.6) saying that $1=<n_{\vec{k}}|n_{\vec{k}}>=<n_{\vec{k}}|[a_{\vec{k}},a^+_{\vec{k}}]|n_{\vec{k}}>$ imply

$$\begin{aligned} a^+_{\vec{k}}|n_{\vec{k}}> &= \sqrt{n+1}|(n+1)_{\vec{k}}>\ , \\ a_{\vec{k}}|n_{\vec{k}}> &= \sqrt{n}|(n-1)_{\vec{k}}>\ . \end{aligned} \tag{5.11}$$

It is also useful to compute the vacuum expectation value of the field ψ on the vacuum, exploiting the Fourier treatment we just did. By using the commutation relations (5.6), it is easy to see that

$$<0||\psi|^2|0>=\sum_{\vec{k}}\frac{1}{2E_{\vec{k}}}\ , \tag{5.12}$$

where $|\psi|^2 = \psi\psi^+$. Since the $a_{\vec{k}}$, $a_{\vec{k}}^+$ operators are associated with plane waves in (5.4), a quantum is associated with a plane wave. This is a consequence of having assumed a Minkowski background, i.e. having performed the whole analysis in an inertial frame as special relativity means it; in this framework, inertial frames are protected by the Poincaré symmetry, and are of course invariant in all inertial frames, as it is clear from (5.4). In other words, a quantum is well defined for all observers in inertial frames, regardless of their relative motion.
This no longer holds in general relativity though. No preferred frame exists, and special relativity becomes a local symmetry: an observer can rotate the axes of his reference frame to make the metric tensor in a Minkowskian form, but that happens only in the spacetime point in which such operation is done, and has not anymore a global validity. This has profound implications in quantum mechanics. Indeed, any observer in general relativity performs quantization regardless of relative motion with respect to other frames, and that makes the concept of quantum a relative one, as we now describe briefly.
Suppose to have two observers, a and b, making their own quantization with respect to their two frames in arbitrary motions with respect to each other. Each one solves the Klein Gordon equation written with respect to the local metric tensor $g_{\mu\nu}$, finding two complete systems of solutions ψ_i^a and ψ_i^b, respectively, where i labels the quantization index, analogous to $\vec{k}$ in the Minkowskian case. The two sets obey independently the commutation relations

$$\begin{aligned}
\left[\psi_i^{a,b}, \psi_j^{a,b}\right] &= 0 \ , \\
\left[\psi_i^{a,b}, (\psi_j^{a,b})^+\right] &= i\delta_{ij} \ , \\
\left[(\psi_i^{a,b})^+, (\psi_j^{a,b})^+\right] &= 0 \ .
\end{aligned} \tag{5.13}$$

Thus the quantization proceeds defining the two vacua $|0^a>$ and $|0^b>$, obeying the relations (5.7,5.8,5.9, 5.11) independently for the a and b systems. Since the two sets are both complete systems, each operator in one set is a linear combination of the other ones; thus one may write

$$\psi_i^b = \sum_j [\alpha_{ij}\psi_j^a + \beta_{ij}(\psi_j^a)^+] \ \ , \ \ (\psi_i^b)^+ = \sum_j]\alpha_{ij}^*(\psi_j^a)^+ + \beta_{ij}^*\psi_j^a] \ , \tag{5.14}$$

where the numbers α_{ij}, β_{ij} are known as Bogoliubov coefficients. The important consequence is that the vacuum of b is not empty as seen by a; indeed calculating the expectation of the number operator $N_i^a = a_i^+ a_i$ on $|0^b>$, one finds immediately

$$< 0^b|N_i^a|0^b >= \sum_j |\beta_{ij}|^2 \neq 0 \ . \tag{5.15}$$

This means that the quantum vacuum is not a relativistic invariant. Two observers, in arbitrary relevant motion with respect to each other, may see different quantum populations in their states. The evaporation of black holes works in this way. An observer at infinity with respect to the black hole sees the region nearby the event horizon as a not-empty state, whereas the obserer located there sees a vacuum. Due to the force of the gravitational field the quantum state is populated even at extreme energies, allowing some particle to escape from the horizon. Since this occurs at expenses of the gravity field, the hole looses energy

and mass, process which is described by analogy as an evaporation.
As we see in the following, the generation of cosmological perturbations is merely another example where the quantum vacuum is not invariant.

5.2 Inflation perturbations

As all quantities in cosmology, the inflation field ψ is supposed to be the sum of a background component plus perturbations:

$$\bar{\psi}(t,\vec{x}) = \psi(t) + \delta\psi(t,\vec{x}) \ . \tag{5.16}$$

As we have seen in the previous chapter, at a classical level any fluctuation is wiped out by the inflationary expansion. That allows us to analyze the system in the linear approximation, $\delta\psi \ll \psi$. Moreover, for simplicity, in this section we work in the limit of exact exponential inflation, e.g.

$$\psi_t = 0 \quad , \quad a(t) \propto \exp(Ht) \quad , \quad H_t = 0 \ , \tag{5.17}$$

and assume a chaotic inflation potential, $V = m^2\psi^2/2$.
The first thing to consider is the meaning of the field fluctuation. First of all, it may be expressed as a function of the perturbed scalar field stress energy tensor defined in (4.7), which has the following general expression:

$$\delta T^{\nu}_{\mu} = \delta\psi_{,\mu} g^{\nu\nu'} \psi_{,\nu'} + \psi_{,\mu} g^{\nu\nu'} \delta\psi_{,\nu'} +$$

$$+\psi_{,\mu}\psi_{,\nu}\delta g^{\nu\nu'} - \psi_{,\mu'} g^{\mu'\nu'} \delta\psi_{,\nu'} - \frac{1}{2}\psi_{,\mu'}\psi_{,\nu'}\delta g^{\mu'\nu'} \ . \tag{5.18}$$

Recalling the link between the stress energy tensor and the fluctuations in energy density and pressure, $-\delta T^0_0 = \delta\rho$, $\delta T^i_i = \delta p$, it is easy to verify that

$$\delta\psi = \frac{\delta\rho - \delta p}{2V_\psi} \ . \tag{5.19}$$

Gauge transformations in cosmology are coordinate changes where the coordinate shifts are of the same order of cosmological perturbations. The latter are not invariant under a gauge transformation and $\delta\rho$ and δp in particular; if $\delta\psi^0$ represents a gauge shift in the time coordinate, one has

$$\delta\rho \to \delta\rho - \rho_t \delta x^0 \quad , \quad \delta p \to \delta p - p_t \delta x^0 \ . \tag{5.20}$$

From the expressions of the scalar field energy density and pressure (4.11), it is easy to see that under a gauge transformation $\delta\psi$ transforms as

$$\delta\psi \to \delta\psi - \psi_t \delta x^0 \ . \tag{5.21}$$

It follows that with a proper choice of the time coordinate, the field results homogeneous. One may argue then that the meaning of the perturbations in inflation and cosmology in general is ambiguous, as they can be shut down by a coordinate change; actually the latter operation is not possible for all of them at once, but for a sub-system only. Actually, it is possible to define a gauge invariant variable associated with the scalar field fluctuations, and perform the whole analysis with that, reaching the same conclusions outlined below. For our

purposes, it is enough to realize that the gauge shift in (5.21) is proportional to ψ_t, which is zero in our present approximation, making $\delta\psi$ gauge invariant in this limit.
Our aim is to perform a quantization of the field $\delta\psi$, in a way similar to the one in the previous section; therefore, let us expand it into the Fourier space, using again the sum notation for simplicity:

$$\delta\psi(t,\vec{x}) = \sum_{\vec{k}} a_{\vec{k}}\delta\psi_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} + a^*_{\vec{k}}\delta\psi^*_{\vec{k}}(t)e^{-i\vec{k}\cdot\vec{x}} \ . \tag{5.22}$$

The next step consists in writing down the Klein Gordon equation in the Fourier space. In general that is expressed as

$$\delta(\nabla\psi + V_\psi) = 0 = (\delta\nabla)\psi + \nabla(\delta\psi) + V_{\psi\psi}\delta\psi \ , \tag{5.23}$$

and one may notice that the first term in the right hand side contains perturbations in the metric tensor, $\delta g_{\mu\nu}$. But they are multiplied by ψ_t or ψ_{tt}, and therefore are not present in our limit. Then the Klein Gordon equation in the Fourier space may be written as

$$\delta\psi_{\vec{k}tt} + 3H\delta\psi_{\vec{k}t} + \frac{k^2}{a^2}\delta\psi_{\vec{k}} + m^2\delta\psi_{\vec{k}} = 0 \ . \tag{5.24}$$

This equation already expresses the relevant issues of how inflation generates perturbations. Well inside the horizon, $k/aH \gg 1$, the equation reduces to the one in the Minkowski space, (5.1), admitting plane waves as solutions. As the expansion proceeds, a grows and wavelength eventually get comparable to the horizon. At that point, the equation admits solutions different from the plane waves; the corresponding quantum operators applied to the original vacuum state, do not give solutions corresponding to the vacuum. That means that perturbations have been generated.
The next simplification consists in getting rid of the field mass m. Indeed, the condition $\eta \ll 1$ is equivalent to say that $m^2 \ll H^2$. Therefore, for wavelength inside the horizon such that $k/aH \gg 1$, the mass term may be neglected in the Klein Gordon equation. In conclusion, well inside the horizon and in the limit of a static field, the physics of $\delta\psi$ is that of a massless scalar field in Minkowski spacetime. It is easy to see that a solution to the Klein Gordon equation (5.24) is

$$\delta\psi_{\vec{k}}(t) = \frac{H}{\sqrt{2k^3}}\left(i + \frac{ka_{BB}}{aH}\right)e^{ika_{BB}/aH} \ , \tag{5.25}$$

where in the present approximation $a = a_{BB}e^{H(t-t_{BB})}$, and the factor in front of the solution is chosen for dimension reasons and convenience, as it will be clear in the following. These functions represent the eigenfunctions of the Klein Gordon equation in an exponentially expanding background. Since they are a complete set of solutions, the combination (5.22) with suitable numbers $a_{\vec{k}}$ and $a^*_{\vec{k}}$, describe all possible solutions. let us proceed to the quantization as we did for the scalar field in a Minkowski background; the amplitudes of each eigenfunction become operators, and we write

$$\delta\psi_{\vec{k}}(t,\vec{x}) = a_{\vec{k}}\delta\psi_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} + a^+_{\vec{k}}\delta\psi^*_{\vec{k}}(t)e^{-i\vec{k}\cdot\vec{x}} \ . \tag{5.26}$$

For spacetime regions lying inside the horizon, $ka_{BB}/aH \gg 1$ and $H(t-t_{BB}) \ll 1$, it is easy to see that $ka_{BB}/aH \simeq k/H - k(t-t_{BB})$. The initial evolution, i.e. for a time interval much smaller than H^{-1}, the solution reduces to

$$\delta\psi_{\vec{k}} = \frac{1}{2\sqrt{k}} e^{-ikt} \ , \tag{5.27}$$

which indeed corresponds to the plane wave solution in the Minkowski space for a massless particle. As the time goes however, the cosmological scale associated to k approaches the horizon, and the fact of being in an expanding background becomes important; since ka_{BB}/aH becomes smaller and smaller, eventually $\delta\psi$ converges to the value $iH/\sqrt{2k^3}$. As a consequence of this, an observer at rest in comoving coordinates would say that the initial vacuum is changing, i.e. the expansion from sub-horizon scales to the horizon crossing caused the creation of particles. In particular, the variance of $\delta\psi$ over the vacuum changes in time as

$$< 0||\delta\psi_{\vec{k}}|^2|0> = < 0|a_{\vec{k}} a^+_{\vec{k}}|0> |\delta\psi_{\vec{k}}(t)|^2 = \frac{1}{2k} \to \frac{H^2}{2k^3} \ , \tag{5.28}$$

where in the second expression we still neglect the role of the mass m, which however would be relevant when the cosmological scale associated with k crosses the horizon, although its inclusion would not change the phenomenology outlined here. Notice that the strength of the perturbations in $\psi_{\vec{k}}$ as seen at the horizon crossing are proportional to the Hubble expansion rate. Again, all this means that the initial vacuum state deep inside the horizon is not seen empty anymore at the horizon crossing.

Before concluding this section, let us define the power spectrum of the inflaton fluctuations. That is the distribution in the Fourier space of the variance of the field in a given space point $\vec{x}$, given by $< 0||\delta\psi(t,\vec{x})|^2|0 >$. It is Expanding $\delta\psi(t,\vec{x})$ in the Fourier space, using (5.26), one gets

$$< 0||\delta\psi(t,\vec{x})|^2|0> = \int d^3k d^3k' < 0|\delta\psi(t,\vec{x})_{\vec{k}} \delta\psi^+(t,\vec{x})_{\vec{k}'}|0> =$$

$$= \int d^3k d^3k' < 0|a_{\vec{k}} a^+_{\vec{k}'}|0> \delta\psi_{\vec{k}}(t) \delta\psi_{\vec{k}'}(t) e^{i\vec{x}(\vec{k}-\vec{k}')} \ . \tag{5.29}$$

From this, using the commutation relations in a continuum, $[a_{\vec{k}}, a^+_{\vec{k}'}] = \delta(\vec{k}-\vec{k}')$, the final result is

$$< 0||\delta\psi(t,\vec{x})|^2|0> = \int d^3k |\delta\psi_{\vec{k}}(t)|^2 = \int \frac{dk}{k} 4\pi k^3 |\delta\psi_k(t)|^2 \tag{5.30}$$

where in the last passage we used the fact that in (5.27) and (5.28) there is no dependence on the direction of the wavevector. The power spectrum of the inflaton field fluctuation is defined as

$$P_{\delta\psi} = 4\pi k^3 |\delta\psi_k(t)|^2 \ , \tag{5.31}$$

and from (5.28), at the inflationary horizon crossing (IHC) it has the value

$$P^{IHC}_{\delta\psi} = 2\pi H^2 \ . \tag{5.32}$$

the same for all scales. From this computation we almost learn another fundamental aspect of inflation fluctuations. The fact that the commutation relation correlate only a wavevector with itself is deeply rooted in the quantum nature of the present analysis. This means that in making the calculus above, no information is lost during the integration, and the field variance is merely a counting of the power on each wavevector. In other words, the script

$$< 0|\delta\psi_{\vec{k}}\delta\psi^{+}_{\vec{k}'}|0 >= |\delta\psi_k|^2\delta(\vec{k}-\vec{k}') \tag{5.33}$$

defines the statistics of inflationary perturbations. There is complete uncorrelation between modes different Fourier modes. The auto-correlation is independent on the direction of $\vec{k}$. This is known as Gaussian statistics, and is a prediction of the inflationary scenario for structure formation, at least at a linear level.

5.3 Cosmological perturbations from inflation

As we see now, fluctuations in the metric $\delta g_{\mu\nu}$ and energy density $\delta\rho/\rho$ arise correspondingly to the ones in the scalar field. From now on, we drop the simplifying assumption (5.17), and also omit the Fourier wavevector subscript, $()_{\vec{k}}$. It is convenient to use the Poisson equation relating the gravitational potential Φ to any source of fluctuation in the stress energy tensor. In the Newtonian gauge, and in the Fourier space, it is

$$\frac{k^2}{a^2}\Phi = 4\pi G\left[\rho\delta + 3H\frac{a}{k}(\rho+p)\,v\right] \ , \tag{5.34}$$

where $\rho\delta$ and $(\rho+p)v$ are the Fourier amplitudes of $-\delta T^0_0$ and δT^0_j, respectively. From (5.18) it is easy to see that

$$\rho\delta = \psi_t\delta\psi_t + V_\psi\delta\psi - \psi_t^2\Phi \ , \tag{5.35}$$

$$(\rho+p)v = \frac{k}{a}\psi_t\delta\psi \ . \tag{5.36}$$

Exploiting these relations, the Poisson equation becomes

$$\left(\frac{k^2}{a^2} - 4\pi G\psi_t^2\right)\Phi = 4\pi G\left(\psi_t\delta\psi_t + V_\psi\delta\psi + 3H\psi_t\delta\psi\right) \ . \tag{5.37}$$

As it can be seen from (5.25), $\delta\psi_t$ is damped as $1/a$ during the inflationary expansion, and in particular after the horizon crossing, thus we neglect it. Moreover, using (4.16) it is easy to see that the term in parenthesis in (5.37) is

$$4\pi G\psi_t^2\left(\frac{k^2}{4a^2H^2\epsilon} - 1\right) \ . \tag{5.38}$$

Even if ϵ is a small number during slow rolling, outside the horizon the first term above is negligible with respect to the second. Thus we may write

$$\Phi = \frac{\psi_{tt}}{H\psi_t}\frac{H}{\psi_t}\delta\psi \ , \tag{5.39}$$

where we have used the Klein Gordon equation $3H\psi_t + V_\psi = -\psi_{tt}$. Differentiating with respect to time, and using (4.16), one gets

$$\Phi = (4\epsilon - 3\eta)\frac{H}{\psi_t}\delta\psi \ , \tag{5.40}$$

where we notice that we used a slightly different definition of η with respect to (4.16), i.e. keeping its sign. In terms of power spectrum, using (5.25) and (5.32), at the horizon crossing one gets

$$P_\Phi^{IHC} = 4\pi k^3 \Phi^2 = 2\pi(4\epsilon - 3\eta)^2 \frac{H^4}{\psi_t^2} =$$

$$= 18\pi \left(\frac{8\pi G}{3}\right)^3 (4\epsilon - 3\eta)^2 \frac{V^3}{V_\psi^2} = \left(\frac{8\pi G}{3}\right)^2 \frac{(4\epsilon - 3\eta)^2}{\epsilon/\pi} V \ , \tag{5.41}$$

where again we used the slow rolling equations. In the limit of inflation as a stationary process, i.e. constant ψ and H, the spectrum presents no dependence on the scale considered, k. This spectrum is known as Harrison Zel-dovich. To find the correction to the scale independence to first order in the slow rolling approximation, let us define the spectal index

$$n = 1 + \frac{k}{P_\Phi^{IHC}} \frac{dP_\Phi^{IHC}}{dk} \ , \tag{5.42}$$

parametrizing the deviation from scale independence as $P_\Phi^{IHC} \propto k^{n-1}$ if n is constant. Since at the horizon crossing one has $k = aH$, and the rate of variation of a is much large than the one for H, one may see that $dk/k = Hdt$. Moreover, since $Hdt = Hd\psi/\psi_t = -3H^2 d\psi/V_\psi$ if the field motion is monotonic in time, one has

$$k\frac{d}{dk} \equiv -\frac{V_\psi}{8\pi G V}\frac{d}{d\psi} \ . \tag{5.43}$$

From these expressions, it is easy to get the general expression for n. We concentrate on the two inflationary models considered in section (4.2.2).

5.3.1 Spectral index for chaotic inflation

As we have seen already, a simple mass potential $V = m^2\psi^2/2$ implies $\epsilon = \eta = 1/12\pi G\psi^2$. Then it easy to see from (5.41) that

$$P_\Phi^{IHC} = \left(\frac{8\pi G}{3}\right)^2 \frac{m^2}{24G} \ , \tag{5.44}$$

which is constant and implies the exact scale invariance, $n = 1$.

5.3.2 Spectral index for exponential inflation

In this case, $V = We^{\lambda\psi}$, one has $\epsilon = \eta/2 = \lambda^2/48\pi G$ so that

$$P_\Phi^{IHC} = \left(\frac{8\pi G}{3}\right)^2 \frac{4\pi\lambda^2 We^{\lambda\psi}}{48\pi G} \ , \tag{5.45}$$

which implies

$$n = 1 - \frac{\lambda^2}{8\pi G} \ , \tag{5.46}$$

corresponding to a spectrum with amplitude increasing with the scale considered.

Chapter 6

Inflationary observables

In this chapter we derive the most important observational consequences for inflationary perturbations. We first take advantage from the work performed in the previous chapter in order to outline the main idea.
First of all, the use of the variable Φ for quantifying the power of perturbations is convenient as the latter has no dynamics in inflation as well as during the following eras. Indeed, as we have seen during inflation the power is constant after horizon crossing, apart from the residual time dependence of H. Moreover, it is possible to show that Φ is also constant during the radiation and matter dominated eras. Indeed, the Poisson equation relating Φ with the density contrast δ in the Fourier space may be written as

$$k^2\Phi = 4\pi G a^2 \rho \delta \ , \tag{6.1}$$

where ρ represents the cosmological critical density. It is possible to show that δ as a function of a grows like a^2 and a respectively in the radiation and matter dominated eras, making Φ constant. The latter does change during the transitions between different eras, but those changes are not substantial and do not affect the validity of the present qualitative discussion. Therefore, the k dependence of Φ, parametrized as $P_\Phi \propto k^{n-1}$ for constant n, may be considered valid anytime for all scales larger than the horizon. Then, from the Poisson equation (6.1), getting rid of all time dependences one is left with

$$\delta^2 \propto k^n \ , \tag{6.2}$$

which is valid at any given time and for all scales larger than the horizon.
At decoupling, the anisotropies are essentially given by the fluctuations in the radiation black body. If the fluctuation maintain the thermal equilibrium, one may use the Stephan Boltzmann law to say that the fluctuations in the thermodynamical temperature T are simply related to the ones in the density as

$$\frac{\delta T}{T} \simeq \delta \ . \tag{6.3}$$

This means that the temperature fluctuations essentially map the density ones at decoupling. If those are due to some inflationary process, that means that by looking at the anisotropies in the cosmic microwave background at decoupling on super-horizon scales we have access to the inflationary perturbations. The phenomenologies of this is sketched in figure 6.1.

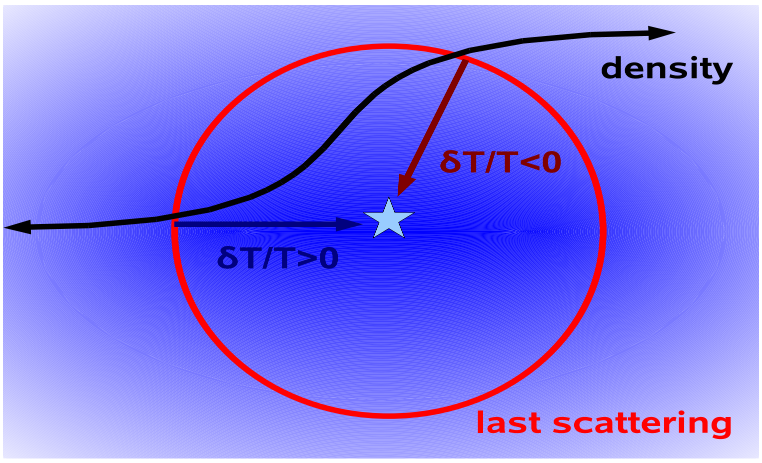

Figure 6.1: A representation of the relation between temperature fluctuations in the cosmic microwave background and large scale density perturbations.