

A more rigorous derivation

- Drop weak-gravity assumption = assume geodesic motion in strongly curved spacetime when calculating source (ie quadrupole, octupole etc)

$$\bar{H}^{\mu\nu} \equiv \eta^{\mu\nu} - (-g)^{1/2} g^{\mu\nu} \quad \partial_\beta \bar{H}^{\alpha\beta} = 0 \quad \text{harmonic gauge}$$

$$\rightarrow \square_{\text{flat}} \bar{H}^{\alpha\beta} = -16\pi \tau^{\alpha\beta} \quad \tau^{\alpha\beta} = (-g)T^{\alpha\beta} + (16\pi)^{-1}\Lambda^{\alpha\beta}$$

Full Einstein equations!

$$\Lambda^{\alpha\beta} = 16\pi(-g)t_{\text{LL}}^{\alpha\beta} + (\bar{H}^{\alpha\mu},_\nu \bar{H}^{\beta\nu},_\mu - \bar{H}^{\alpha\beta},_{\mu\nu} \bar{H}^{\mu\nu})$$

$$\begin{aligned} 16\pi(-g)t_{\text{LL}}^{\alpha\beta} &\equiv g_{\lambda\mu}g^{\nu\rho} \bar{H}_{,\nu}^{\alpha\lambda} \bar{H}_{,\rho}^{\beta\mu} \\ &+ \frac{1}{2}g_{\lambda\mu}g^{\alpha\beta} \bar{H}_{,\rho}^{\lambda\nu} \bar{H}_{,\nu}^{\rho\mu} - 2g_{\mu\nu}g^{\lambda(\alpha} \bar{H}_{,\rho}^{\beta)\nu} \bar{H}_{,\lambda}^{\rho\mu} \\ &+ \frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau}) \bar{H}_{,\lambda}^{\nu\tau} \bar{H}_{,\mu}^{\rho\sigma} \end{aligned}$$

A more rigorous derivation

- Drop weak-gravity assumption = assume geodesic motion in strongly curved spacetime when calculating source (ie quadrupole, octupole etc)

$$\bar{H}^{\mu\nu} \equiv \eta^{\mu\nu} - (-g)^{1/2} g^{\mu\nu} \quad \partial_\beta \bar{H}^{\alpha\beta} = 0 \quad \text{harmonic gauge}$$

$$\rightarrow \square_{\text{flat}} \bar{H}^{\alpha\beta} = -16\pi \tau^{\alpha\beta} \quad \tau^{\alpha\beta} = (-g)T^{\alpha\beta} + (16\pi)^{-1}\Lambda^{\alpha\beta}$$

Full Einstein equations!

$$\Lambda^{\alpha\beta} = 16\pi(-g)t_{\text{LL}}^{\alpha\beta} + (\bar{H}^{\alpha\mu},_\nu \bar{H}^{\beta\nu},_\mu - \bar{H}^{\alpha\beta},_{\mu\nu} \bar{H}^{\mu\nu})$$

From gauge condition,

$$16\pi(-g)t_{\text{LL}}^{\alpha\beta} \equiv g_{\lambda\mu}g^{\nu\rho} \bar{H}_{,\nu}^{\alpha\lambda} \bar{H}_{,\rho}^{\beta\mu}$$

$$+ \frac{1}{2}g_{\lambda\mu}g^{\alpha\beta} \bar{H}_{,\rho}^{\lambda\nu} \bar{H}_{,\nu}^{\rho\mu} - 2g_{\mu\nu}g^{\lambda(\alpha} \bar{H}_{,\rho}^{\beta)\nu} \bar{H}_{,\lambda}^{\rho\mu}$$

$$+ \frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau}) \bar{H}_{,\lambda}^{\nu\tau} \bar{H}_{,\mu}^{\rho\sigma}$$

$$\tau^{\alpha\beta},_\beta = 0$$

= geodesic motion in
curved metric g

A more rigorous derivation

- Following same procedure as before, we re-obtain Green, quadrupole formula but with T replaced by τ and

$$\bar{H}^{\mu\nu} \approx \bar{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}h\eta^{\mu\nu}$$

$$g_{00} = -1 - 2\frac{\phi}{c^2} + O(1/c^4)$$

$$\tau^{00} = T^{00}(1 + O(1/c^2))$$

$$g_{0i} = O(1/c^3)$$

$$\tau^{0i} = T^{0i}(1 + O(1/c^2))$$

$$g_{ij} = \left(1 - 2\frac{\phi}{c^2}\right)\delta_{ij} + O(1/c^4) \quad \tau^{ij} = \left(T^{ij} + \frac{1}{4\pi G} \left(\partial^i\phi\partial^j\phi - \frac{1}{2}\delta^{ij}\partial_k\phi\partial^k\phi\right)\right)(1 + O(1/c^2))$$

- So Green formula gets corrected, but quadrupole formula is NOT
- Exercise: show that the extra terms in the Green formula for account for the factor 2 discrepancy with the quadrupole formula found for a circular, Keplerian binary

An example: a binary system

- Binary with total mass M , reduced mass μ , separation R , orbital frequency Ω ; orbit lies in xy plane
- Consider GWs along z axis at distance r

$$h_{ij}^{\text{TT}} = h \times \begin{bmatrix} \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & -\cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$h = \frac{4\mu\Omega^2 R^2}{r} = \frac{4\mu M^{2/3}\Omega^{2/3}}{r}$$

$$h \simeq 10^{-21} \left(\frac{M}{2M_\odot} \right)^{5/3} \left(\frac{1 \text{ hour}}{P} \right)^{2/3} \left(\frac{1 \text{ kiloparsec}}{r} \right)$$

$$\simeq 10^{-22} \left(\frac{M}{2.8M_\odot} \right)^{5/3} \left(\frac{0.01 \text{ sec}}{P} \right)^{2/3} \left(\frac{100 \text{ Megaparsecs}}{r} \right)$$

vs $h_{\text{Sun}} \sim G M_{\text{sun}} / (R_{\text{sun}} c^2)$
 $\sim 2 \times 10^{-6}$

Generalizing the quadrupole formula

- Why? Approximate because based on **slow-motion, weak gravity approximations**
- Drop slow-motion approximation = include mass octupole, current quadrupole and higher order terms

$$\bar{h}^{jk} = \frac{2}{r} \left[\ddot{I}^{jk} - 2n_i \ddot{S}^{ijk} + n_i \ddot{M}^{ijk} \right]_{t'=t-r},$$

$$I^{jk}(t') = \int x'^j x'^k T^{00}(t', \mathbf{x}') d^3x'$$

mass quadrupole

$$S^{ijk}(t') = \int x'^j x'^k T^{0i}(t', \mathbf{x}') d^3x',$$
$$M^{ijk}(t') = \int x'^i x'^j x'^k T^{00}(t', \mathbf{x}') d^3x'.$$

current quadrupole
mass octupole

$$\bar{h}^{jk}(t, \mathbf{x}) = \frac{2}{r} \frac{d^2}{dt^2} \int \left[(\mathcal{T}^{00} - 2\mathcal{T}^{0l}n_l + \mathcal{T}^{lm}n_l n_m) x'^j x'^k \right]_{t'=t-|\mathbf{x}-\mathbf{x}'|} d^3x',$$

all multipole moments (Press 1977)

A potentially complicated waveform structure

Quadrupole (or quadrupole + octupole + higher moments) formula + geodesic motion is often decent approximation,
eg for particle around Kerr BH ("kludge" waveforms)

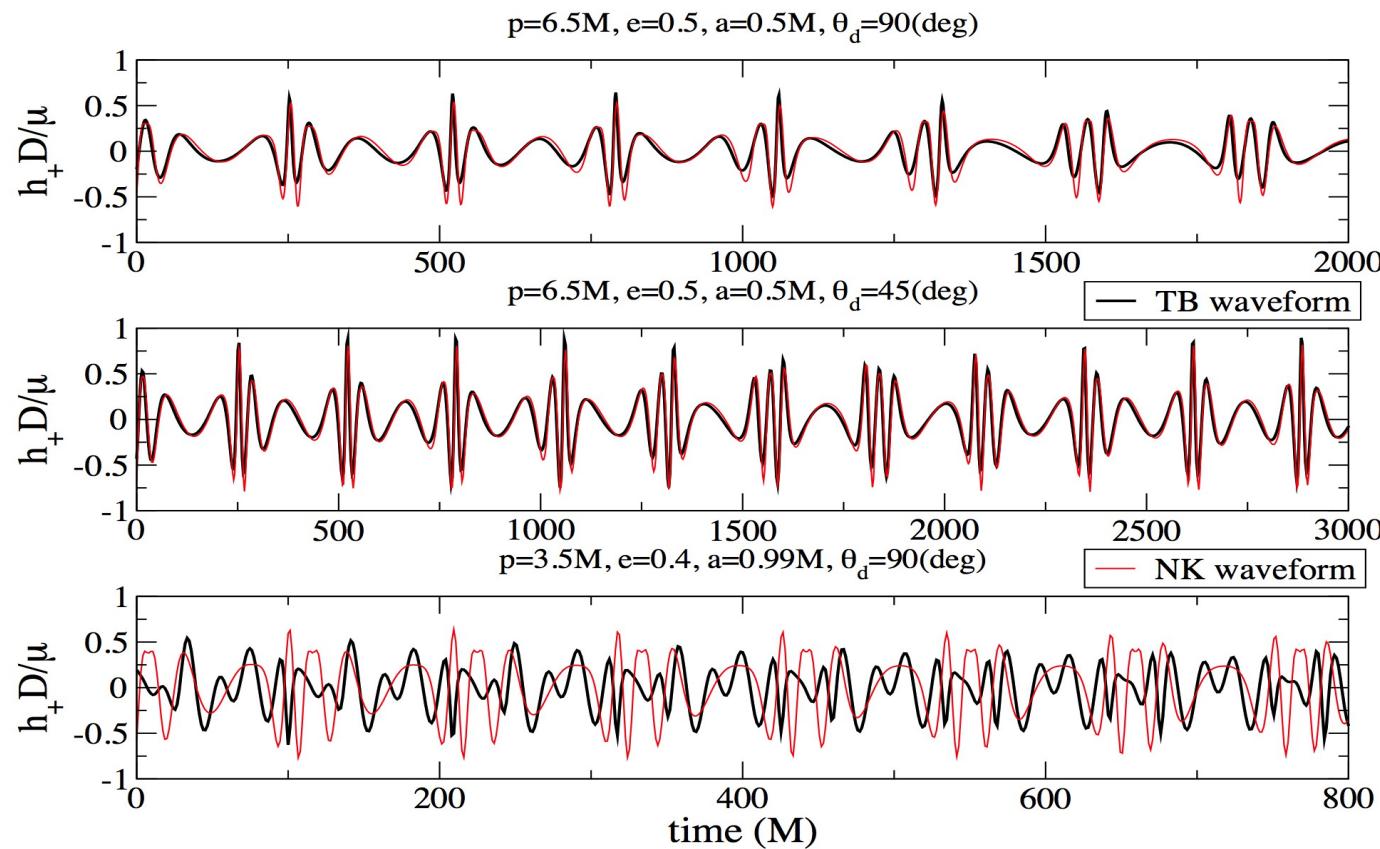
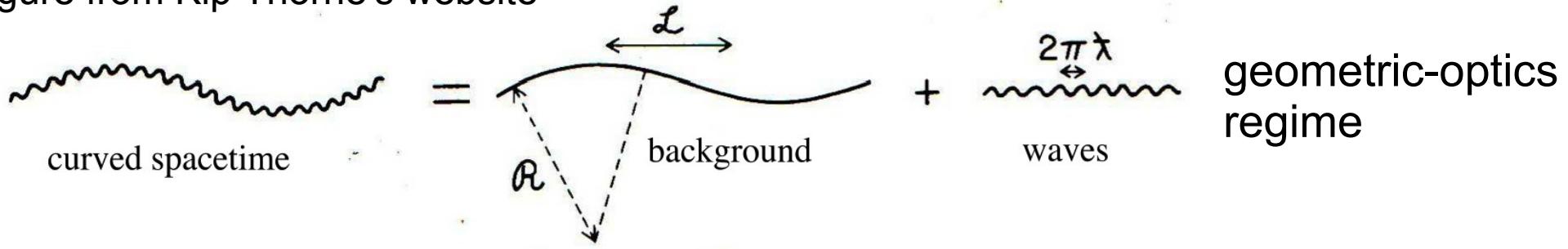


Figure from Babak et al Phys. Rev. D 75, 024005 (2007)

The stress energy tensor of GWs

Figure from Kip Thorne's website



$$g_{\alpha\beta}^B \equiv \langle g_{\alpha\beta} \rangle \quad g_{\alpha\beta} = g_{\alpha\beta}^B + \varepsilon h_{\alpha\beta} + \varepsilon^2 j_{\alpha\beta} + O(\varepsilon^3)$$

$$\begin{aligned} 0 &= G_{\alpha\beta} \\ &= G_{\alpha\beta}[g_{cd}^B] + \varepsilon G_{\alpha\beta}^{(1)}[h_{cd}; g_{ef}^B] + \varepsilon^2 G_{\alpha\beta}^{(1)}[j_{cd}; g_{ef}^B] + \varepsilon^2 G_{\alpha\beta}^{(2)}[h_{cd}; g_{ef}^B] \\ &\quad + O(\varepsilon^3). \end{aligned}$$

$$G_{\alpha\beta}[g_{cd}^B] = 0, \quad G_{\alpha\beta}^{(1)}[h_{cd}; g_{ef}^B] = 0, \quad G_{\alpha\beta}^{(1)}[j_{cd}; g_{ef}^B] = -G_{\alpha\beta}^{(2)}[h_{cd}; g_{ef}^B].$$

The stress energy tensor of GWs

Average Einstein equations on scale $\gg \lambda$ and $\ll L$

$$\Delta j_{\alpha\beta} = j_{\alpha\beta} - \langle j_{\alpha\beta} \rangle$$

$$G_{\alpha\beta}^{(1)}[\langle j_{cd} \rangle; g_{ef}^B] = -\langle G_{\alpha\beta}^{(2)}[h_{cd}; g_{ef}^B] \rangle$$

$$G_{\alpha\beta}^{(1)}[\Delta j_{cd}] = -G_{\alpha\beta}^{(2)}[h_{cd}; g_{ef}^B] + \langle G_{\alpha\beta}^{(2)}[h_{cd}; g_{ef}^B] \rangle$$

$$G_{\alpha\beta}[g_{cd}^B + \varepsilon^2 \langle j_{cd} \rangle] = 8\pi G T_{\alpha\beta}^{\text{GW,eff}} + O(\varepsilon^3) \quad T_{\alpha\beta}^{\text{GW,eff}} = -\frac{1}{8\pi G} \langle G_{\alpha\beta}^{(2)}[h_{cd}; g_{ef}^B] \rangle$$

Commuting derivatives and using $\lambda \ll L$

$$T_{\alpha\beta}^{\text{GW,eff}} = \frac{1}{32\pi G} \langle \nabla_\alpha^B h_{\rho\sigma}^{\text{TT}} \nabla_\beta^B h_{\rho\sigma}^{\text{TT}} \rangle$$

The GW luminosity

Quadrupole formula + GW stress energy tensor

$$L_{\text{mass quadrupole}} \equiv \frac{1}{5} \frac{G}{c^5} \langle \ddot{\mathbf{T}} \rangle^2 = \frac{1}{5} \frac{G}{c^5} \langle \ddot{\mathcal{T}}_{jk} \ddot{\mathcal{T}}_{jk} \rangle^2$$

$$\ddot{\mathcal{T}}_{jk} \sim \frac{(\text{mass of the system in motion}) \times (\text{size of the system})^2}{(\text{time scale})^3} \sim \frac{MR^2}{\tau^3} \sim \frac{Mv^2}{\tau}$$

$$L_{\text{mass quadrupole}} \sim \frac{G}{c^5} \left(\frac{Mv^2}{\tau} \right)^2 G/c^5 \sim 10^{-59} \text{ (in CGS units)}$$

Conversion of any type of energy into GWs is inefficient, unless large masses and/or $v \sim c$

Propagation of GWs

GW propagating in z-direction

$$\square h_{ij}^{\text{TT}} = 0 \quad \longrightarrow \quad h_{ij}^{\text{TT}} = h_{ij}^{\text{TT}}(t - z)$$

$$\partial_z h_{zj}^{\text{TT}} = 0 \quad \longrightarrow \quad h_{zj}^{\text{TT}} = 0$$

$$h_{ii}^{\text{TT}} = 0 \quad \longrightarrow \quad \begin{aligned} h_{xx}^{\text{TT}} &= -h_{yy}^{\text{TT}} \equiv h_+(t - z) ; \\ h_{xy}^{\text{TT}} &= h_{yx}^{\text{TT}} \equiv h_\times(t - z) . \end{aligned}$$

Propagation of GWs

$$\boldsymbol{h}^{\text{TT}} = h^+(t-z)\boldsymbol{e}^+ + h^\times(t-z)\boldsymbol{e}^\times \quad \begin{aligned}\boldsymbol{e}^+ &\equiv \boldsymbol{e}_x \otimes \boldsymbol{e}_x - \boldsymbol{e}_y \otimes \boldsymbol{e}_y, \\ \boldsymbol{e}^\times &\equiv \boldsymbol{e}_x \otimes \boldsymbol{e}_y + \boldsymbol{e}_y \otimes \boldsymbol{e}_x.\end{aligned}$$

Linear polarization

$$\begin{aligned}h^+(t-z) &= h(t-z) \cos 2\lambda & h^\times(t-z) &= \pm i h(t-z) \\ h^\times(t-z) &= h(t-z) \sin 2\lambda & h^+(t-z) &= h(t-z)\end{aligned}$$

Circular polarization

Elliptic polarization=
other phase differences

Binary with masses m_1 and m_2 , separation R , orbital frequency Ω , distance r ;

θ = angle between orbital angular momentum and direction to observer
($\theta = 0$ or 180 deg: face-on; $\theta = 90$: edge on)

$$h^+ = \frac{2m_1 m_2}{rR} (1 + \cos^2 \theta) \cos[2\Omega(t-r) + 2\Delta\phi],$$

$$h^\times = -\frac{4m_1 m_2}{rR} \cos \theta \sin[2\Omega(t-r) + 2\Delta\phi],$$