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Four Lectures On The Gauge/Gravity Correspondence

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Abstract

We review in a pedagogical manner some of the efforts aiming to extend the gauge/gravity correspondence to non-conformal supersymmetric gauge theories in four dimensions. After giving a general overview, we discuss in detail two specific examples: fractional D-branes on orbifolds and D-branes wrapped on supersymmetric cycles of Calabi-Yau spaces. We explore in particular which gauge theory information can be extracted from the corresponding supergravity solutions, and what the remaining open problems are. We also briefly explain the connection between these and other approaches, such as fractional branes on conifolds, branes suspended between branes, M5-branes on Riemann surfaces and M-theory on G2-holonomy manifolds, and discuss the rôle played by geometric transitions in all that.

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Contents

Foreword **2**

1 Lecture I - Introduction And Overview **3**

2 Lecture II - Fractional Branes On Orbifolds **8**

2.1 Fractional Branes: Open String Perspective 9

2.2 Fractional Branes: Closed String Perspective 19

2.3 Gauge Theory From Gravity: Example 1 25

3 Lecture III - Branes wrapped on Calabi-Yau spaces **38**

3.1 Wrapping Branes: The Topological Twist 39

3.2 Wrapping Branes: The Rôle Of Gauged Supergravity 44

3.3 Gauge Theory From Gravity: Example 2 46

4 Lecture IV - Dualities and connections to other approaches **65**

4.1 Branes Suspended Between Branes 65

4.2 M5-branes Wrapped On Riemann Surfaces 68

4.3 Fractional Branes on Conifolds 69

4.4 Dualities As Geometric Transitions 71

4.5 M-theory On G_2-holonomy Manifolds 73

5 A (Biased And Incomplete) Guided Tour Through The Literature **75**

References **79**

Foreword

These lectures are based on an eight-hour course I gave at SISSA/ISAS, December 2002, on non-conformal extensions of the gauge/gravity correspondence. This is a vast subject and I could not discuss in detail all approaches that have been used to pursue such a program. The choice was to focus on cases where supersymmetry and conformal invariance are broken by the geometry and/or D-branes, hence looking directly for gravity duals of non-conformal theories. That is to say, we did not consider cases where conformal invariance is broken by mass deformations starting from conformal theories. We analyzed in particular two systems in some detail: fractional D-branes on orbifolds and D-branes wrapped on supersymmetric cycles of Calabi-Yau manifolds.

This written version contains somewhat more material than I was able to cover during the course. Some more details are provided for both fractional and wrapped branes (lecture II and III, respectively), while lecture IV contains a wider (but still qualitative) presentation of other approaches used to study the gauge/gravity correspondence in non-conformal cases, with an emphasis on the relations between them.

Throughout the text there are a few (simple) exercises that might help in getting a more concrete handle on the concepts and the tools discussed. Plus, I added some inserts when some further details were needed on topics I could not explain in detail in the main text.

There have been hundreds of papers on this subject in the last few years and it would be a daunting task to give credit to all of them. As this is not a review but a series of lectures notes for students, I will not provide a complete list of references but instead accompany the reader into a brief tour through some of the literature.

Hope you enjoy the reading!

1 Lecture I - Introduction And Overview

The goal of these lectures is to review some of the efforts that have been made in the past few years to extend the gauge/gravity correspondence to non-conformal and less supersymmetric settings as compared to the original duality proposed by Maldacena, which relates type IIB string theory on $\mathrm{AdS}_5 \times S^5$ to four-dimensional $\mathcal{N} = 4$ Super Yang-Mills (SYM), this being a conformal and maximally supersymmetric gauge theory. This goal has been pursued in many different ways in the recent literature, so let me first explain what my specific point of view will be.

The celebrated AdS/CFT correspondence is a conjectured equivalence between two apparently very different theories

Type IIB string theory on $\mathrm{AdS}_5 \times S^5$	$\mathcal{N} = 4$ SYM in 4D with gauge group $SU(N)$

As stated above, this correspondence does not include any kind of D-branes. However, its seeds reside in the *double* nature of D-branes, this being related to the old open/closed string duality. In particular we know that

- D-branes are hypersurfaces on which open strings can end. Their dynamics is described in general by a (supersymmetric) gauge theory at low energy, this being the low energy spectrum of the corresponding open strings.
- D-branes are non-perturbative states of the closed string spectrum (their tension going as $1/g_s$ where g_s is the string coupling) and at low energy they are described by soliton-like solutions of the corresponding supergravity theory.

This suggests to try and exploit the classical geometrical properties of D-branes to study the gauge theory living on them or use the quantum properties of the gauge theory to study the dynamics of non-perturbative extended objects. This idea applies to any kind of D-brane and was considered well before the work of Maldacena (essentially it is part of the discovery by Polchinski of D-branes being non-perturbative states of the closed string spectrum). In the case considered by Maldacena, namely a stack of N D3-branes in flat ten-dimensional space, however, it was possible to carry this correspondence further by taking the so-called near-horizon limit and observing that in this limit the bulk (closed) and gauge (open) degrees of freedom decouple. This decoupling is at the core of the AdS/CFT duality.

An obvious thing one can try is to see whether a similar approach can be used to study non-conformal and less supersymmetric theories, as for instance $\mathcal{N} = 1, 2$ SYM (and eventually $\mathcal{N} = 0$) in four dimensions, starting from bound states of D-branes in less supersymmetric backgrounds, breaking eventually conformal invariance. There is a number of problems that one usually encounters in pursuing this program. Let me briefly anticipate some of them.

- The dual supergravity solution of a non-conformal gauge theory does not display an AdS-like geometry. This means that, strictly speaking, holography is not at work in these cases, in general. What one can do, at best, is to use what one learned in the conformal case as a guiding principle but one cannot safely rely on all theorems that proved so fruitful in the original AdS/CFT duality.

- In general, the supergravity backgrounds one finds are singular. To give a meaning to the solution one has of course to cure the singularity and look for a singularity-free solution. At the same time one should understand what the rôle of the singularity is from the gauge theory point of view, as well as the meaning of its resolution. Hence, more than being a problem, this can be a source of interest after all. We will learn much more about this.

- As already pointed out, a basic aspect of the original AdS/CFT correspondence is the decoupling between open and closed degrees of freedom, this being the starting point to state the exact duality. The duality can be stated at different levels (see Insert 1) and in general relates a given regime of the $\mathcal{N} = 4$ gauge theory to a given regime of type IIB string theory on $\mathrm{AdS}_5 \times S^5$. In particular, there exists a limit in the parameter space (i.e. the large 't-Hooft coupling limit) where the gauge theory is supposed just to be dual to supergravity, without any addition of string states (this is actually the regime where the duality as been mostly checked). In non-conformal cases it turns out this is not the case. Roughly speaking, if one insists in retaining just supergravity modes, the dual gauge theory cannot be completely decoupled from the bulk. The way this manifests is very much case-dependent, in particular there is a qualitative difference between the $\mathcal{N} = 2$ and the $\mathcal{N} = 1$ cases, as we shall see in detail during these lectures. But it is a fact that within the supergravity regime (which is the one we will mainly investigate) a complete decoupling does not hold.

Insert 1: **Different Limits Of The AdS/CFT Correspondence**

The AdS/CFT correspondence is an equivalence between two theories, a string theory and a gauge theory. In its more strongest version the correspondence is supposed to hold for generic values of the parameters defining the regime of the two theories. By taking suitable limits the duality boils down to an equivalence between type IIB supergravity and gauge theory at strong coupling. This is the regime where the correspondence has been mainly tested, so far.

Exact equivalence

Type IIB string theory on $\mathrm{AdS}^5 \times \mathrm{S}^5$	$\mathcal{N}=4$ SYM in 4D
string coupling g_s and string tension T	YM coupling g_{YM} and number of colors N

where $T = R^2/(2\pi\alpha')$, R being the (common) radius of the AdS space and of the S^5. The dictionary is determined in terms of the two basic relations

$$4\pi g_s = g_{\mathrm{YM}}^2 \quad , \quad T = \frac{1}{2\pi}\sqrt{g_{\mathrm{YM}}^2 N} = \frac{1}{2\pi}\sqrt{\lambda}$$

It is very difficult to test the conjecture at this level as we do not know how to treat string theory for generic value of the string coupling. Better to take the weak coupling limit, $g_s \to 0$. In this limit we select the sector of the gauge theory surviving at large N.

Equivalence in the classical limit

$g_s \to 0$ with T-fixed	$g_{\mathrm{YM}} \to 0$ with $\lambda = g_{YM}^2 N$-fixed
Classical string theory (non-interacting strings)	Large N limit, planar diagrams only

As opposed to string theory in flat space, we do not even know how to study classical string theory in curved backgrounds with RR fluxes. Better to take the low energy limit. This leads to the weaker (though more tractable) version of the conjecture, where string theory reduces to type IIB supergravity.

Equivalence at low energy

$g_s \to 0$ with $T \to \infty$	$g_{\mathrm{YM}}^2 \to 0$ with $\lambda \to \infty$
Type IIB supergravity in $\mathrm{AdS}_5 \times \mathrm{S}^5$	$\mathcal{N}=4$ SYM in 4D at strong 't-Hooft coupling

The last equivalence means that in the original AdS/CFT correspondence there exists a regime in the gauge theory (the large 't-Hooft coupling regime) in which the dynamics of the gauge theory is supposed to be captured solely by supergravity modes, without any addition of string states.

For these reasons we are not yet at a point where we can really state a duality á la Maldacena for non-conformal theories, at least at the supergravity level. It is believed that a proper duality holds if one lets string states enter into the game, but as it is the case for the original AdS/CFT correspondence, it is much harder to go beyond the supergravity regime and check the duality at the stringy level. This is a crucial point which makes manifest a conceptual difference between conformal and non-conformal dualities, but we do not discuss this further, for the time being. Instead, we will take a more humble approach and elaborate on the idea illustrated previously, namely trying to exploit as much as possible the power of open/closed string duality and see what can we learn about the dynamics of non-conformal supersymmetric gauge theories from supergravity, and vice-versa. After all, the dynamics of these theories is richer than that of $\mathcal{N}=4$ and is worth give it a try. The program is then to *i*) study D-branes on non-maximally supersymmetric backgrounds *ii*) break conformal invariance *iii*) exploit open/closed string duality *iv*) ... see what happens. As we shall see, we can learn a lot and eventually shed also some light on what is the final goal, namely to find an exact duality. There are in general two conceptually different ways to pursue this program.

- Start from a conformal theory of which we know the supergravity dual, and deform it by means of relevant or marginal operators which break both supersymmetry and conformal invariance.

- Start from configurations where both supersymmetry and conformal invariance are broken from the very beginning by a specific target-space geometry, a D-brane configuration or a combination of the two, and search the corresponding dual.

In these lectures I will discuss examples belonging to the second class only. There are many of them. One can consider, for instance

- (Fractional) D-branes on orbifolds.

- (Fractional) D-branes on conifolds.

- D-branes wrapped of supersymmetric cycles of Calabi-Yau (CY) spaces.

- Branes suspended between branes.

- M5-branes wrapped on Riemann surfaces.

- M-theory on manifolds of G2 holonomy.

Most of these apparently different settings are related between each other (by dualities of various kind, geometric deformations, etc...) and in the last lecture I will outline what the precise connections are. Still, we would like to have some quantitative handling on all this and is the purpose of these lectures to discuss in some detail just a couple of the above approaches. In the next lecture we shall discuss fractional branes on orbifolds and in the subsequent one branes wrapped on smooth CY spaces. In both cases one can construct four-dimensional gauge theories with either $\mathcal{N}=1$ or $\mathcal{N}=2$ supersymmetry, with and without matter. However, in order to be specific and have the time to bring some computations until the very end, when discussing fractional branes on orbifolds we will focus on the case with 8 supercharges, while when discussing branes wrapped on supersymmetric cycles of CY spaces we will focus on the case with 4 supercharges. When it comes to write down explicit supergravity solutions and exploit them to study the gauge theory dual, we will use in both cases a key example, pure (i.e. without matter) $\mathcal{N}=2$ SYM for fractional branes and pure $\mathcal{N}=1$ SYM for wrapped branes.

As I will be rather pedagogical and somewhat lengthly, let me anticipate some of the features of gauge theories we will be able to recover from the dual supergravity backgrounds.

- The running of the gauge coupling and the corresponding perturbative β-function. For the $\mathcal{N}=1$ case we will also predict some (unexpected) non-perturbative contributions.

- The $U(1)_R$ anomaly and its corresponding symmetry breaking pattern.

- In the case of pure $\mathcal{N}=1$ the phenomenon of gaugino condensation and the further breaking of the chiral symmetry down to $\mathbb{Z}_2$ in the vacuum, as well as the appearance of N inequivalent vacua (where N is the number of colors).

- The correct action for the gauge theory instantons.

- For theory with moduli, as for instance $\mathcal{N}=2$, a general understanding of the moduli space in terms of D-branes dynamics.

- For confining theories an estimate of some more fancy things as the confining string tension, the domain wall tension, the glueballs mass, etc... These computations rely heavily on tools inherited from the original AdS/CFT correspondence and on the use of holography. Hence, as anticipated, they should be taken with some care and considered just as qualitative (but promising) results.

The gauge theory information listed above belongs either to the perturbative or to the non-perturbative regime of the gauge theory. The lesson one learns in studying these non-conformal gauge theory duals is that in general the perturbative dynamics of the gauge theory is precisely accounted for by the supergravity solutions. As non-conformal theories changes sensibly with the scale, the information one can get at the perturbative level are already non trivial, and rather interesting. Also non-perturbative properties get an holographic description and they are in general related to the way the (would be) singular solution is deformed into a smooth one. Some time it is possible to find quantitative predictions, some time just qualitative, some other time it is possible just to understand what the precise path to follow would be in order to recover them.

2 Lecture II - Fractional Branes On Orbifolds

D-branes in flat space are soliton-like states which break half the supersymmetries of the original 32 preserved by the $\mathbb{R}^{1,9}$ background. The low energy spectrum of open strings living on them is a supersymmetric gauge theory with 16 supercharges. When considering D-branes on orbifolds, which are backgrounds already breaking some supersymmetry, the amount of supersymmetry on the D-brane world volumes is also reduced. More precisely

- D-branes on orbifold limits of ALE spaces ($\mathbb{C}_2/\Gamma$ with Γ being a discrete subgroup of $SU(2)$) give rise to 1/2 supersymmetric backgrounds and their world volume theory is described by a SYM theory with 8 supercharges.
- D-branes on orbifold limits of CY spaces ($\mathbb{C}_3/\Gamma$ with Γ being now a discrete subgroup of $SU(3)$) give rise to 1/2 supersymmetric backgrounds and their world volume theory is described by a SYM theory with 4 supercharges.

As we shall see, these gauge theories are in general conformal: the vector multiplet is coupled to some matter multiplet just in the right way to give back a conformal theory. The reason for that is that the low energy spectrum is nothing else but the projection to lower supersymmetry representations of the maximally supersymmetric (and conformal) one enjoyed by D-branes in flat space. However, when considering string theory on orbifolds, there is a specific kind of D-branes in the spectrum, the so-called *fractional* branes, which also break conformal invariance. The study of these objects and their gravity duals is the subject of this lecture.

Playing with different orbifolds and different bound states of fractional D-branes one can build-up SYM theories with either $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry, with product gauge groups and with matter in different representations. As anticipated, we will focus here on the $\mathcal{N}=2$ case, leaving the $\mathcal{N}=1$ case to the wrapped branes setting. Moreover, in order to let the students see some concrete computations in full detail, I will mainly refer to a key example, i.e. pure $\mathcal{N}=2$ SYM. Once having become familiar with the technical tools needed and more generally with the logic underlying the computations we will carry on for this key example, the reader should be able to obtain the same results for any kind of $\mathcal{N}=1,2$ SYM theory.

2.1 Fractional Branes: Open String Perspective

D-branes are defined by open strings ending on them. An open string state is generically defined by

$$\lambda \otimes \textit{oscillators}\, |k\rangle \tag{2.1}$$

where k is the momentum along the brane (recall that open strings can have momentum along Neumann directions only) and λ is the Chan-Paton (CP) factor. For a single D-brane in flat space, λ is a number. For a D-brane on an orbifold this is not necessarily true: λ is in general a matrix. This means that in this case λ can transform under different representations of the orbifold group Γ. In general, for any element h of the orbifold group the CP factors transform as follows

$$\lambda \rightarrow \lambda' = \gamma(h)\, \lambda\, \gamma(h)^{-1} \tag{2.2}$$

According to eq. (2.1) this leads to the possibility of having different kinds of D-branes on orbifolds.

<u>Def</u>: A *regular* D-brane is a D-brane whose CP factors λ transform under the regular representation $\mathcal{R}$ of the orbifold group Γ (this is the representation whose dimension equals the order $|\Gamma|$ of Γ).

As the regular representation is reducible, it can be decomposed in irreducible representations. In particular

$$\mathcal{R} = \oplus n_I \mathcal{D}_I \quad \text{with} \quad \sum_{I=0}^{m-1} n_I = |\Gamma| \quad , \quad I = 0, 1, \ldots, m-1 \tag{2.3}$$

where $\mathcal{D}_I$ are the irreducible representations of Γ and n_I their dimension. Note that for abelian orbifolds, namely for $\Gamma = \mathbb{Z}_N$, $n_I = 1$ for any I. In this case the number of irreducible representations equals the number of elements of the orbifold group.

Def: A *fractional* D-brane (of type I) is a D-brane whose CP factors λ transform in the I-th irreducible representation D_I of Γ.

Well, here it is all the magic about fractional branes. Let us now try to have a more physical intuition of what these definitions really mean. When studying string theory on orbifolds the physical states are those which are (globally) left invariant under the orbifold projection. This applies also to D-branes, of course. Each physical D-brane which is free to move in the transverse space should have images in the covering space. In figure 1 the case of the most simple orbifold, $\Gamma = \mathbb{Z}_2$, is reported.

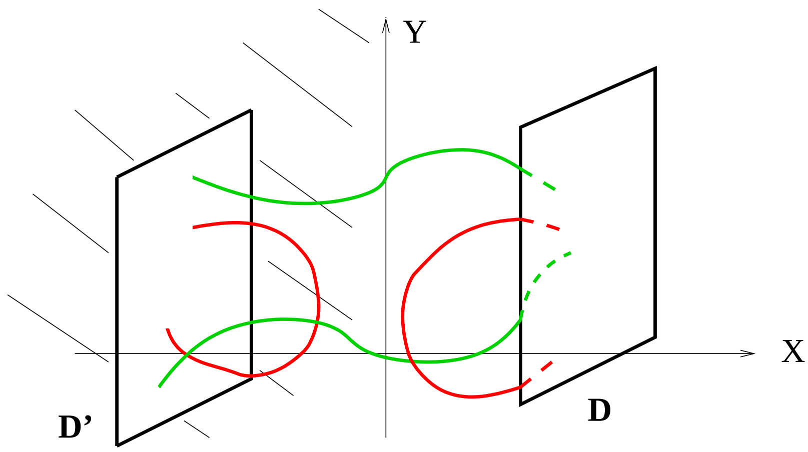


Figure 1: A freely moving D-brane on the most simple orbifold as it appears in the covering space. The brane D and its image D' are identified in the physical space. The X axis represents the orbifold directions (these being all transverse to the brane), the Y axis the flat longitudinal directions. To see the flat transverse directions... we would need an extra dimension! But they are there, of course.

From figure 1 one easily sees that in this case the CP factors are 2 by 2 matrices

$$\lambda = \begin{pmatrix} D-D & D-D' \\ D'-D & D'-D' \end{pmatrix} \tag{2.4}$$

This generalizes in a straightforward way to more complicated orbifolds. The corresponding matrix will have a number of entries which essentially equals the number of images in the covering space, these being one but the same D-brane in the physical space. The key point now is that the set of points including the brane and its images forms an invariant configuration of Γ and transforms in the *regular* representation! We will see this explicitly when discussing our example.

As it is (hopefully) clear from the figure, the maximal number of massless states arise when the D-brane is at the orbifold fixed point $X = 0$. There open strings of type, say, $D - D'$ contribute to the massless spectrum. On the contrary, out of the orbifold fixed point the space is essentially flat and consistently it turns out that the spectrum of a D-brane at a generic point is equivalent to (the Coulomb branch of) that of an usual D-brane in flat space.

What we learn then is that a regular D-brane can move in the full transverse space, as it is a D-brane with images.

Fractional branes are D-branes whose CP factors transform in the irreducible representations of the orbifold group Γ. This implies they do not have images and are therefore stuck at the orbifold fixed point. The analogue of figure 1 is now

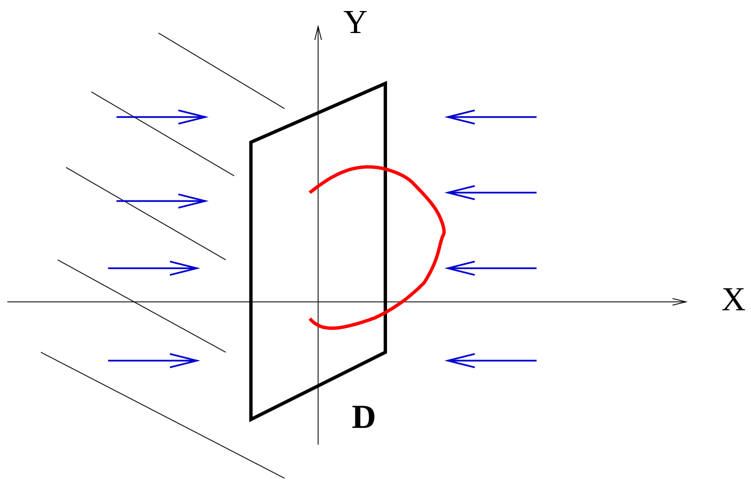


Figure 2: A fractional D-brane on the most simple orbifold as it appears in the covering space. The brane does not have any image in this case and it is stuck at the orbifold fixed plane.

Fractional D-branes cannot move along the orbifold but just along the flat transverse directions. This clearly implies their dynamics is described in terms of less degrees of freedom as compared to regular branes and, as we shall see, this is the basic reason why the gauge theory living on them is a non-conformal gauge theory. Note that now the CP factors are just numbers as there is only one possible kind of open strings, the $D-D$.

We would like now to compute the low energy (i.e. gauge theory) spectrum living on regular and fractional branes. In order to work this out explicitly, let us use our key example. Namely, we start considering D3-branes on the orbifold $\mathbb{R}^{1,5} \times \mathbb{C}^2/\mathbb{Z}_2$. The group $\mathbb{Z}_2$ consists of two generators: g acting on the coordinates of $\mathbb{C}^2$ as a reflection operator, $g : X \to -X$, and its square g^2 that is nothing but the identity, $g^2 \equiv e : X \to X$. Let us suppose the D3-brane extends along directions $x^0, ..., x^3$. This means that the transverse space consists of two flat directions, x^4, x^5, plus the four orbifold directions, $x^6, ..., x^9$ where the element g of the orbifold group acts non-trivially.

As already explained, to let a (regular) D3-brane be located at a generic point on the orbifold covering space, we must include its image (we just have one in this case) and consequently we have four kinds of open strings. Two corresponding to open strings having both their end-points on the brane or on its image and two other kinds corresponding to open strings having one endpoint on the brane and the other on its image, and vice-versa. These four kinds of open strings are described by two by two matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.5}$$

which altogether make-up the two by two matrix (2.4). The first question we would like to answer is under which representations of the orbifold group these CP factors transform. Using eq. (2.2) one easily sees that in this case

$$\gamma(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \gamma(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.6}$$

The matrix $\gamma(g)$ can be determined by requiring that it exchanges an open string ending on the D-brane with an open string ending on its image and vice-versa. In particular it should send the first (third) matrix in (2.5) into the second (fourth) and vice-versa.

The representation (2.6) is a 2-dimensional representation and is nothing but the regular representation of $\mathbb{Z}_2$. Hence we see that D-branes having images are in fact *regular* branes, as anticipated. The representation (2.6) is reducible and decomposes into two irreducible representations

$$\mathcal{D}_0 \qquad \gamma(e) = 1 \qquad , \qquad \gamma(g) = 1 \tag{2.7}$$

$$\mathcal{D}_1 \qquad \gamma(e) = 1 \qquad , \qquad \gamma(g) = -1 \tag{2.8}$$

As it can be easily seen, the regular representation is the direct sum of the above two irreducible representations

$$R = \oplus \mathcal{D}_I \quad , \quad I = 0, 1 \tag{2.9}$$

We thus have two kinds of *fractional* branes, according to the representations of their CP factors. The number of inequivalent fractional branes coincides with the order of the orbifold group, as the dimension of the irreducible representation is one, in this case. Note that the CP factors are now numbers and, as anticipated, the corresponding fractional branes are stuck at the orbifold fixed point as there are not the degrees of freedom (described by open strings of the form $D - D'$) to allow them to move along the orbifold directions.

We can now compute the massless spectrum of open strings living on regular and fractional branes, respectively. Let us start from regular D-branes. A massless state of the NS sector has the following form

$$\lambda \, \psi^M_{-1/2} \, |k\rangle \quad , \quad M = 0, 1 \ldots 9 \tag{2.10}$$

In order to keep world volume supersymmetry, $\mathbb{Z}_2$ must act on the fermionic coordinates in the same way as on the bosonic ones; thus the oscillator part of the state in eq. (2.10) transforms under g as follows

$$\psi^\mu_{-1/2}|k\rangle \;\; \rightarrow \;\; + \psi^\mu_{-1/2}|k\rangle \qquad \mu = 0, 1, 2, 3 \tag{2.11}$$

$$\psi^i_{-1/2}|k\rangle \;\; \rightarrow \;\; + \psi^i_{-1/2}|k\rangle \qquad i = 4, 5 \tag{2.12}$$

$$\psi^m_{-1/2}|k\rangle \;\; \rightarrow \;\; - \psi^m_{-1/2}|k\rangle \qquad m = 6, \ldots 9 \tag{2.13}$$

where we have denoted with μ the world volume directions of the D3-brane, with m the four directions along the orbifold and with i the transverse ones outside the orbifold.

Taking into account the action of the orbifold group on both the oscillators and the CP factors λ which can be easily inferred from eqs. (2.2) and (2.6), one gets the following globally invariant states surviving the orbifold projection

$$\frac{\mathbb{1} + \sigma_1}{2} \otimes \psi^{\mu,i}_{-1/2}|k\rangle \quad , \quad \frac{\mathbb{1} - \sigma_1}{2} \otimes \psi^{\mu,i}_{-1/2}|k\rangle \tag{2.14}$$

corresponding to two gauge fields living on the world volume of the D3-brane represented by the index μ and four real Higgs fields represented by the index i, and

$$\frac{\sigma_3 + i\sigma_2}{2} \otimes \psi^m_{-1/2}|k\rangle \quad , \quad \frac{\sigma_3 - i\sigma_2}{2} \otimes \psi^m_{-1/2}|k\rangle \quad m = 6, 7, 8, 9 \tag{2.15}$$

corresponding to 8 scalars. To these bosonic states one should add the fermionic ones which come from the R sector. At the orbifold fixed point all these fields are massless and are grouped together in two $\mathcal{N} = 2$ vector multiplets A_1 and A_2, containing a gauge field and two real Higgs fields each (plus fermions), and two hypermultiplets Φ_1 and Φ_2, containing 4 scalars each (plus fermions).

The upshot of this analysis is that the gauge theory living on a regular brane is a $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $U(1) \times U(1)$ coupled to two hypermultiplets in the bi-fundamental (see Exercise 1 below) representation. By piling N regular D3-branes on top of each other one then gets $\mathcal{N} = 2$ SYM with gauge group $U(N) \times U(N)$ plus two hypermultiplets in the $(N, \overline{N})$, $(\overline{N}, N)$. Note that

- The scalars in the vector multiplets are degrees of freedom associated to displacement along flat directions. The scalars in the hypermultiplets are degrees of freedom associated to displacement along the orbifold directions. This shows that regular branes have both *Coulomb* and *Higgs* phases.
- The gauge theory is conformal, i.e. the β-functions of both gauge groups vanish.
- Regular branes on orbifolds are pretty similar to usual branes in flat space: when moving along the orbifold directions only the diagonal $U(N)$ survives and the low energy effective theory is equivalent to the Coulomb branch of $U(N)$ $\mathcal{N} = 4$ SYM, as it is the case for D3-branes in flat space.

Exercise 1 - *Compute the charges of the hypermultiplets with respect to the two gauge factors, and show they transform in the bi-fundamental representation.* Hint: perform a change of basis in the space of CP factors with a matrix $A = (\mathbb{1} - i\sigma_2)/\sqrt{2}$.

We can easily repeat the above reasoning for the fractional branes, defined through eqs. (2.7) and (2.8). In this case, for either fractional branes of type 1 and 0, the massless open string states surviving the orbifold projection are easily found to be

$$a\, \psi^{\mu,i}_{-1/2}\,|k\rangle \tag{2.16}$$

where a is just a number. These states correspond to a gauge field, represented by the index μ, and two real scalar fields, represented by the index i, belonging to an $\mathcal{N}=2$ vector multiplet (again, fermions should be added by considering the R sector). The additional scalars belonging to the hypermultiplets are projected out by the orbifold projection (this implying that fractional branes are stuck on the orbifold fixed plane, as anticipated). By piling N fractional D3-branes on top of each other one then gets pure $\mathcal{N}=2$ SYM with gauge group $U(N)$. Note that

- Fractional branes have *Coulomb* phase only (the hypermultiplets are frozen).

- The gauge theory living on them is a non-conformal theory. This is why they are so interesting objects to be studied when looking for non-conformal extensions of the gauge/gravity correspondence.

All what we have been saying can be extended to any orbifold of the so-called ADE series, $\mathbb{C}^2/\Gamma$, as well as to less supersymmetric orbifolds, i.e. orbifold limits of CY three-folds, $\mathbb{C}^3/\Gamma$, where Γ is now a discrete subgroup of $SU(3)$. While for abelian orbifolds the dimension of the irreducible representations is one, for non abelian orbifolds this is not true anymore and the number of different fractional branes is less than the order of the group Γ. The analysis of the massless open string spectrum we performed for $\mathbb{Z}_2$ can be also generalized, of course, and the result one finds can be finally summarized as follows

N regular D3-branes on $\mathbb{C}^2/\Gamma$ or $\mathbb{C}^3/\Gamma$ are described by

- Conformal $\mathcal{N}=2$ or $\mathcal{N}=1$ SYM.
- Gauge group $U(n_0N)\times U(n_1N)\times \ldots \times U(n_{m-1}N)$ where m is the number of irreps of Γ and n_I their dimensions.
- Matter in the bi-fundamental (either hypermultiplets or chiral multiplets). The amount of matter is such to make the theory conformal.

N fractional D3-branes $\mathbb{C}^2/\Gamma$ or $\mathbb{C}^3/\Gamma$ of the I-th type are described by

- Pure $\mathcal{N}=2$ or $\mathcal{N}=1$ SYM.
- Gauge group $U(n_I N)$ where n_I is the dimension of the I-th irrep.
- There are m different types of fractional branes as $I=0,1,...,m-1$.

Note that fractional D3-branes on $\mathcal{N}=1$ orbifolds are completely stuck at the orbifold apex as there is not flat transverse space available for them to move. This is consistent with what one learns from the gauge theory living on them, which is pure $\mathcal{N}=1$ SYM and which does not have any scalar degrees of freedom.

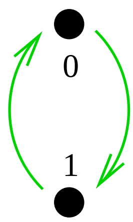
Exercise 2 - *Consider the orbifold $\mathbb{C}_2/\mathbb{Z}_3$. The orbifold group $\mathbb{Z}_3$ is generated by three elements now, $\mathbb{Z}_3=\{g,g^2,g^3\equiv e\}$. Compute the regular and the irreducible representations of $\mathbb{Z}_3$ and find the low energy gauge theory spectrum of open strings ending on regular and fractional D3-branes, respectively.* Hint: the group element g acts on the coordinates $z=x^6+ix^7$ and $w=x^8+ix^9$ as a two by two diagonal matrix with entries $\exp(2i\pi/3)$.

By considering bound states of different numbers and types of fractional branes one can construct SYM theories with product gauge group coupled to bi-fundamental matter (notably, in this case one can have bound states of fractional branes which can actually move along the orbifold). One can also introduce fundamental matter by adding D7-branes. For instance, by considering a bound state of N D3-branes plus M D7-branes on the $\mathcal{N}=1$ orbifold $\mathbb{C}_3/\mathbb{Z}_2\times\mathbb{Z}_2$ one ends up with $\mathcal{N}=1$ SYM coupled to M chiral multiplets in the fundamental, this resembling Super QCD. In fact, this way of brane-engineering gauge theories by means of fractional branes has been much used recently in the context of the bottom-up approach of string phenomenology.

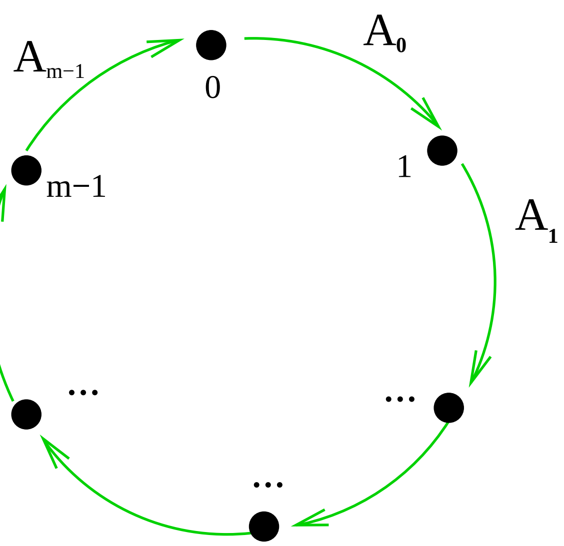
All previous considerations, which we have done for D3-branes, can be easily extended to regular and fractional Dp-branes, with $p\neq 3$. The only essential difference, at this level, is that the low energy effective theory living on them is in general a (p+1)-dimensional gauge theory. Since we are mainly interested in four dimensional gauge theories, we will not discuss Dp-branes further.

Insert 2: **Quiver Interlude**

All gauge theories one can obtain by considering D-branes on orbifolds can be organized in terms of *quiver* diagrams (hence the name "quiver gauge theories" often used in the literature). This originates from the one-to-one correspondence between the gauge theories of D-branes of $\mathbb{C}_2/\Gamma$ orbifolds and the (extended) simply laced Dynkin diagrams of the ADE series. For instance, abelian orbifolds $\Gamma = \mathbb{Z}_m$ are classified by the extended A_{m-1} Dynkin diagrams. Each dot corresponds to a gauge group factor while lines connecting two dots represent matter in the bi-fundamental representation of the two corresponding gauge groups. For instance, the gauge theory on N regular D3-branes of our key example, $\Gamma = \mathbb{Z}_2$, is represented by



We have two gauge factors, represented by the two dots (where the vector multiplets sit), and two bi-fundamental hypermultiplets, represented by the arrows going from one point to the other. More generally, the $\mathcal{N} = 2$ gauge theory living on N regular D3-branes on the orbifold $\mathbb{C}_2/\mathbb{Z}_m$ is represented by the following quiver diagram

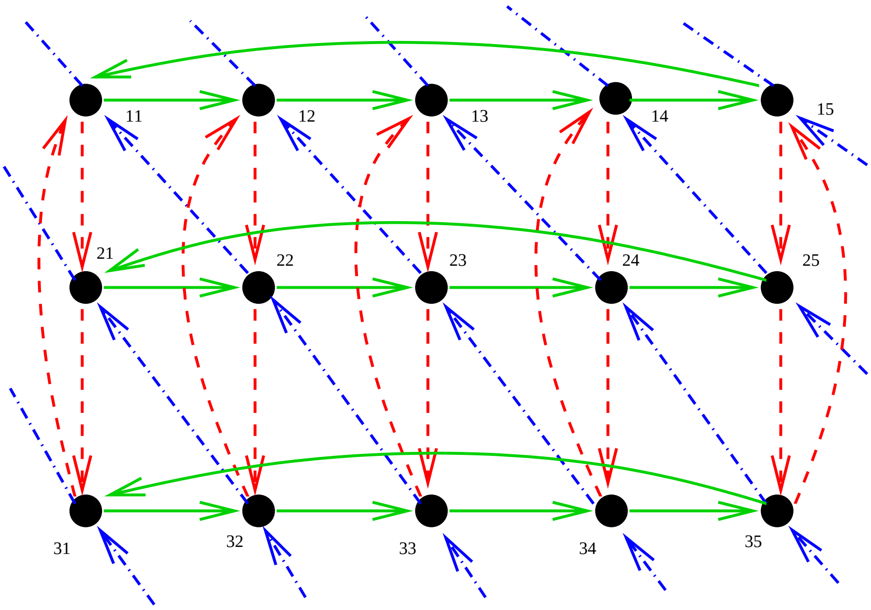


Continued...

Insert 2: *Continued...*

Again, each dot represents a $U(N)$ gauge factor while the lines between two dots correspond to a hypermultiplet A_I $(I = 0, ..., m-1)$ transforming in the bi-fundamental representation of the two groups belonging to the two dots. Arrows go from fundamental to anti-fundamental representations of the corresponding gauge groups. The fermionic partner of each bosonic field is implicit in the figure.

Although for $\mathcal{N} = 1$ orbifolds the mathematical correspondence just discussed does not hold, strictly speaking, in some cases all what we have been saying can be extended also to the $\mathcal{N} = 1$ case. As an example, we draw the quiver diagram for the $\mathcal{N} = 1$ quiver gauge theory on the orbifold $\mathbb{C}_3/\mathbb{Z}_5 \times \mathbb{Z}_3$, where matter is now represented by chiral multiplets.



Each dot represents a $U(N)$ gauge factor, in double index notation while the lines connecting dots correspond to bi-fundamental chiral multiplets. Again, arrows go from fundamental to anti-fundamental representations. We have been using different colors just to let the picture be more readable. The blue lines starting from the first row go back to the third one.

As a final remark, note how eqs.(2.3) and (2.9) seem to suggest that regular branes can be thought as 'made of' fractional branes. This idea is supported also by considering the gauge theory living on them. From what said above, it is clear that by considering a bound state made of N fractional brane of type 0, N of type 1, etc... up to N of type $m-1$ (i.e. $N_I = N$ for any $I = 0, ..., m-1$), one ends up with a gauge theory which is exactly that of a bound state of N regular branes. This näive identification of regular branes in terms of fractional branes turns out to be correct, as it will become apparent in the next section, when discussing fractional branes from the closed string perspective.

2.2 Fractional Branes: Closed String Perspective

D-branes are non-perturbative states of the closed string spectrum. As such, they couple to closed string states and to find the supergravity solutions describing them at low energy one should know the massless fields to which the D-branes couple. A usual Dp-brane in flat space couples to the graviton, the dilaton and a RR-form potential

$$\text{NSNS}: \; G_{MN} \;, \; \Phi \quad ; \quad \text{RR}: \; C_{(p+1)}$$

What do we get when considering D-branes on orbifolds? One should remind that now there are both untwisted and twisted sectors, the number of twisted sectors being equal to the number of non-trivial element of the orbifold group Γ. Recall that the strings belonging to the twisted sectors do not carry momentum along the orbifold directions and their center of mass is stuck at the orbifold fixed point. This means that the corresponding massless fields have a (10-n)-dimensional dynamics, where n is the spatial dimension of the orbifold, as opposed to massless fields in the untwisted sector, which of course have a ten-dimensional dynamics.

An efficient way to find out which fields couple to regular and fractional branes on orbifolds is to implement open/closed string duality and describe D-branes in terms of boundary states, $|D\rangle$, and compute the overlap $\langle D|\phi\rangle$, with ϕ being any possible field of the relevant supergravity spectrum. In doing so one finds that

- Regular Dp-branes couple to the untwisted sector only, right in the same way as usual D-branes in flat space

$$\text{NSNS}: \; G_{MN} \;, \; \Phi \quad ; \quad \text{RR}: \; C_{(p+1)}$$

- Fractional Dp-branes couple to both untwisted and twisted sectors. In particular, fractional Dp-branes of type I (recall previous section) couple to the fields belonging to the I-th twisted sector: a scalar field b_I in the NSNS sector and a (p+1)-form potential in the RR one

$$\begin{array}{lllll} \mathrm{NSNS}: & G_{MN}\ ,\ \Phi & ; & \mathrm{RR}: & C_{(p+1)} \\ \mathrm{NSNS}_T: & b_I & ; & \mathrm{RR}_T: & A^I_{(p+1)} \end{array}$$

The coupling of fractional branes to twisted sectors is nothing but the closed string counterpart of fractional branes not having dynamics along the orbifold.

Using boundary state language one can see that the analogue of relation (2.3) is now

$$|D\rangle = \sum_I |D\rangle_I \qquad \text{with} \quad I = 0, 1, ..., m-1 \tag{2.17}$$

where $|D\rangle$ is the boundary state representing a regular brane and $|D\rangle_I$ the boundary state representing a fractional brane of type I. Consistently, one can check that by summing up the boundary states of all kinds of fractional branes as dictated by Eq. (2.17), the overall twisted coupling cancel, so that the boundary state couples only to the untwisted sector, as it should be the case for a regular D-brane. Note that I runs on the irreps of Γ which equals the number of twisted sector *plus* one. While this turns out to be crucial in the cancellation procedure just mentioned, it can be rather confusing as it seems to spoil the on-to-one correspondence between fractional branes and twisted sectors outlined above. Actually, the number of twisted sectors is indeed one less than the number of different fractional branes. In order to understand this point we should make a digression. This is worth doing though, as it will pay out a lot for the understanding of the whole picture.

There is another point of view to look at D-branes on orbifolds which is rather illuminating in that it helps to understand better some of the issues raised so far in comparing regular and fractional branes, orbifold group representations, twisted sectors and so on. In what follows, we concentrate on $\mathcal{N}=2$ orbifolds, namely those of the form $\mathbb{C}_2/\Gamma$, Γ being a discrete subgroup of $SU(2)$. These orbifolds are singular limits of ALE spaces, the latter being characterized by compact two-cycles $\mathcal{C}_I$ which are topologically spheres and shrink to zero size in the orbifold limit. ALE spaces are classified by the ADE Dynkin diagrams and through a precise mathematical construction due to McKay and Kronheimer the following one-to-one correspondences hold

$$\alpha_I \longleftrightarrow \mathcal{C}_I \longleftrightarrow \mathcal{D}_I \tag{2.18}$$

α_I being the simple roots of the corresponding Dynkin diagram, $\mathcal{C}_I$ the 2-cycles characterizing the ALE space and $\mathcal{D}_I$ the m irreducible representations of the orbifold group Γ. The trivial representation $\mathcal{D}_0$ is associated to the *non-independent* 2-cycle $\mathcal{C}_0 = -\sum_{I\neq 0} d_I\mathcal{C}_I$ and the corresponding root α_0 is nothing but the extra root in the Dynkin diagram, which is in fact an *extended* Dynkin diagram. The quiver diagrams discussed in Insert 2 are just examples of the above correspondence.

The twisted fields can be seen as the zero modes of the Kaluza-Klein (KK) states of higher forms of the ten-dimensional supergravity spectrum over the cycles $\mathcal{C}_I$ in the orbifold limit

$$b_I \equiv \int_{\mathcal{C}_I} B_{(2)} \qquad\qquad A^I_{(p+1)} \equiv \int_{\mathcal{C}_I} C_{(p+3)} \tag{2.19}$$

where $B_{(2)}$ is the NSNS 2-form potential. From this point of view the reason why twisted fields have a (10-n=6)-dimensional dynamics is simply due to the fact that the cycles $\mathcal{C}_I$ are localized at the orbifold fixed point, in the orbifold limit. Collecting the following three facts

- Fractional branes of type I couple to twisted fields of the I-th twisted sector
- Fractional branes of type I are associated to the irrep $\mathcal{D}_I$ of Γ and this is in one-to-one correspondence with the cycle $\mathcal{C}_I$
- Twisted fields of rank q are the zero modes of KK of (q+2)-forms on $\mathcal{C}_I$

one is tempted to conclude that

A fractional Dp-brane (of type I) is a D(p+2)-brane
wrapped on the cycle $\mathcal{C}_I$ in the orbifold limit.

This is indeed the case and can be proved rigorously following the mathematical construction exemplified in the relation (2.18). We are not going to do this explicitly as it is beyond the purpose of this course, but I will soon give a consistency check for that. Before doing so it is worth making a couple of comments.

- If fractional branes branes are branes wrapped on vanishing cycles, how can they have a non-vanishing tension? The point is that in the orbifold limit the cycles $\mathcal{C}_I$ vanish but a non-zero $B_{(2)}$-flux persists on them giving a non-vanishing *background* value to the field b_I

$$b_I \equiv \int_{\mathcal{C}_I} B_{(2)} \; = \; \frac{d_I}{|\Gamma|} \neq 0 \qquad (2.20)$$
$$= \; \frac{1}{2} \quad \text{for} \quad \mathbb{C}_2/\mathbb{Z}_2$$

where we are using here units where $4\pi\alpha'^2 = 1$. This means that, roughly speaking, the *stringy* volume of the cycle is not vanishing. We will show this explicitly later, when discussing in detail our usual key example. The above statement can be also justified from a CFT point of view. One can show that a non-singular CFT on $\mathbb{C}_2/\Gamma$, as it is the string sigma model, is obtained if and only if there is a non-vanishing background value for b_I. The precise value in the above equation turns out to be the value for which the sigma model is free.

- The fractional brane associated to $\mathcal{D}_0$ corresponds to a brane wrapped on the cycle $\mathcal{C}_0$. This is not an independent cycle so it could seem the corresponding fractional brane is not independent either. In fact, this is not the case. The point is that the fractional brane associated to $\mathcal{D}_0$ corresponds to a brane wrapped on $\mathcal{C}_0$ with an additional background flux of the world volume field strength $\mathcal{F}$ switched on, and normalized as $\int_{\mathcal{C}_0} \mathcal{F} = 2\pi$. As it will become clear shortly, this ensures that the fractional brane of type 0 has the same untwisted charge of other fractional branes (so it is not an anti-brane) and is indeed independent on them.

The identification between fractional and wrapped branes can be checked, for instance, by comparing the action of a fractional Dp-brane (obtained for example using boundary state techniques) with that of a wrapped D(p+2)-brane. If one consider a fractional Dp-brane of type I, the structure of its world volume action (which can be derived for instance using boundary state techniques) reads

$$\mathcal{S}_I \; = \; -\frac{T_p}{\kappa} \int d^{p+1}\xi \; \mathrm{e}^{\frac{p-3}{4}\Phi} \sqrt{-\det G} \, b_I \, + \, \frac{T_p}{\kappa} \left[\int C_{(p+1)} \, b_I \, + \, \int A_{(p+1)} \right] \qquad (2.21)$$

where b_I is now the complete twisted scalar field (namely its background value (2.20) plus the fluctuation, $\widetilde{b}_I$), $T_p = \sqrt{\pi}\,(2\pi\sqrt{\alpha'})^{3-p}$ and $\kappa = 8\pi^{7/2}\alpha'^2 g_s$. By summing up the world volume action of all possible fractional branes in the given orbifold theory one gets the world volume action for the corresponding regular brane, $\mathcal{S} = \sum_I \mathcal{S}_I$. This is the analogue of relations (2.3) and (2.17). From eq. (2.21) we can see why the background value of the scalar twisted field b_I, eq. (2.20), is crucial to make the fractional branes tension-full. Moreover, we also see why they are called fractional: their RR untwisted charge (i.e. the coefficient multiplying the WZ term for $C_{(p+1)}$) is a *fraction* of that of a regular brane. What fraction this is, it is determined by the orbifold at hand, as it can be seen from eq. (2.20).

Let us now see how one can get eq. (2.21) following the wrapped brane program previously outlined. We concentrate for simplicity on our key example, i.e. the orbifold $\mathbb{C}_2/\mathbb{Z}_2$. According to what we have learnt, in this case we have two different kinds of fractional D-branes, as we have two irreducible representations of the orbifold group, and just one shrinking cycle, $\mathcal{C}_1$. The fractional Dp-brane of type 1 should correspond to a D(p+2)-brane wrapped on $\mathcal{C}_1$. The fractional Dp-brane of type 0 to a D(p+2)-brane wrapped on $\mathcal{C}_0 = -\mathcal{C}_1$ with a non vanishing $\mathcal{F}$-flux on it. In the following we will be pedantic with all factors of π, α', etc... In the Einstein frame the world volume action of D(p+2)-brane has in general the form

$$\begin{aligned}\mathcal{S} = & -\frac{T_{p+2}}{\kappa}\int d^{p+3}\xi\ \mathrm{e}^{\frac{p-1}{4}\Phi}\sqrt{-\det\left[G + \mathrm{e}^{-\phi/2}\left(B + 2\pi\alpha'\mathcal{F}\right)\right]} + \\ & + \frac{T_{p+2}}{\kappa}\int\left[C\wedge \mathrm{e}^{B+2\pi\alpha'\mathcal{F}}\right]_{(p+3)}\end{aligned} \tag{2.22}$$

Our D(p+2)-brane is not generic, but actually a very specific one. The smooth limit of the $\mathbb{Z}_2$ orbifold is the well known Eguchi-Hanson space, which has an antiself-dual two-form ω_2 which is associated to the compact 2-sphere $\mathcal{C}_1$. The ω_2 satisfies the following properties

$$\omega_2 = -{}^*\omega_2 \quad , \quad \int_{\mathcal{C}_1}\omega_2 = 1 \quad , \quad \int {}^*\omega_2\wedge\omega_2 = \frac{1}{2} \tag{2.23}$$

The compact cycle vanishes in the orbifold limit but, as already said, a non-zero $B_{(2)}$-flux persists on it. In order to obtain from the action in eq.(2.22) the world volume actions of the two fractional Dp-branes, we should start from an action with no world volume fields switched-on along the p+1 non-compact directions of the world volume. That is to say, both $B_{(2)}$ and $\mathcal{F}$ are non-vanishing only on the cycle $\mathcal{C}_1$. Moreover, to make the action in eq. (2.22) describing a brane wrapped on $\mathcal{C}_1$, we should consider its (p+3)-dimensional world volume $\mathcal{V}_{p+3}$ as a product of a flat (p+1)-dimensional volume $\mathcal{V}_{p+1}$ times the volume of the cycle $\mathcal{C}_1$ and keep only those fields that are left in the orbifold limit

$$\mathcal{V}_{p+3} = \mathcal{V}_{p+1}\times\mathcal{C}_1 \quad , \quad B_{(2)} = b_1\,\omega_2 \quad , \quad C_{(p+3)} = A_{(p+1)}\wedge\omega_2 \tag{2.24}$$

By noticing that the metric has no support on the vanishing cycle, one can easily factorize the matrix in the determinant in the action (2.22) as a direct product of a $(p+1)\times(p+1)$

matrix G times a 2×2 matrix where only $B_{(2)}$ and $\mathcal{F}$ are present

$$\begin{aligned} & \sqrt{-\det\left[G+\mathrm{e}^{-\Phi/2}\left(B+2\pi\alpha'\mathcal{F}\right)\right]_{p+3\times p+3}} = \\ = & \sqrt{-\det\left[G\right]_{p+1\times p+1}}\;\mathrm{e}^{-\Phi/2}\int_{\mathcal{C}_1}\left(B_{(2)}+2\pi\alpha'\mathcal{F}\right) \end{aligned} \tag{2.25}$$

We now have all the ingredients to get the desired result. Let us consider first the case of a fractional brane of type 1. We want to show that it corresponds to a D(p+2)-brane wrapped on $\mathcal{C}_1$ with no $\mathcal{F}$-flux. Inserting the expressions (2.24) into eq. (2.22) one gets

$$\begin{aligned} \mathcal{S}_1 = & -\frac{T_{p+2}}{\kappa}\int d^{p+1}\xi\;\mathrm{e}^{\frac{p-3}{4}\Phi}\sqrt{-\det G}\int_{\mathcal{C}_1}B_{(2)} + \\ & + \frac{T_{p+2}}{\kappa}\left[\int C_{(p+1)}\int_{\mathcal{C}_1}B_{(2)}+\int A_{(p+1)}\right] \end{aligned} \tag{2.26}$$

Recalling now that in this case

$$\int_{\mathcal{C}_1}B_{(2)} = 4\pi\alpha'^2\left(\frac{1}{2}+\frac{1}{4\pi\alpha'^2}\,\widetilde{b}_1\right) \tag{2.27}$$

and $T_p = 4\pi\alpha'^2\,T_{p+2}$ we finally get

$$\begin{aligned} \mathcal{S}_1 = & -\frac{T_p}{\kappa}\int d^{p+1}\xi\;\mathrm{e}^{\frac{p-3}{4}\Phi}\sqrt{-\det G}\left(\frac{1}{2}+\frac{1}{4\,\pi^2\alpha'}\,\widetilde{b}_1\right) + \\ & + \frac{T_p}{\kappa}\left[\int C_{(p+1)}\left(\frac{1}{2}+\frac{1}{4\,\pi^2\alpha'}\,\widetilde{b}_1\right)+\frac{1}{4\,\pi^2\alpha'}\int A_{(p+1)}\right] \end{aligned} \tag{2.28}$$

which is the desired result, see eq. (2.21). By repeating the same reasoning for a D(p+2)-brane which is now wrapped on $\mathcal{C}_0 = -\mathcal{C}_1$ but with an additional $\mathcal{F}$-flux normalized as $\int_{\mathcal{C}_0}\mathcal{F} = 2\pi$ one gets

$$\begin{aligned} \mathcal{S}_0 = & -\frac{T_{p+2}}{\kappa}\int d^{p+1}\xi\;\mathrm{e}^{\frac{p-3}{4}\Phi}\sqrt{-\det G}\int_{\mathcal{C}_0}\left(B_{(2)}+2\pi\alpha'\mathcal{F}\right) + \\ & + \frac{T_{p+2}}{\kappa}\left[\int C_{(p+1)}\int_{\mathcal{C}_0}\left(B_{(2)}+2\pi\alpha'\mathcal{F}\right)-\int A_{(p+1)}\right] \end{aligned} \tag{2.29}$$

Using now that

$$\int_{\mathcal{C}_0}\left(B_{(2)}\,+\,2\pi\alpha'\mathcal{F}\right) = -4\pi^2\alpha'\left(\frac{1}{2}+\frac{1}{4\pi^2\alpha'}\,\widetilde{b}_1\right)+4\pi^2\alpha' = 4\pi^2\alpha'\left(\frac{1}{2}-\frac{1}{4\pi^2\alpha'}\,\widetilde{b}_1\right) \tag{2.30}$$

we finally get

$$\begin{aligned} \mathcal{S}_0 = & -\frac{T_p}{\kappa}\int d^{p+1}\xi\;\mathrm{e}^{\frac{p-3}{4}\Phi}\sqrt{-\det G}\left(\frac{1}{2}-\frac{1}{4\pi^2\alpha'}\,\widetilde{b}_1\right) + \\ & + \frac{T_p}{\kappa}\left[\int C_{(p+1)}\left(\frac{1}{2}-\frac{1}{4\pi^2\alpha'}\,\widetilde{b}_1\right)-\frac{1}{4\pi^2\alpha'}\int A_{(p+1)}\right] \end{aligned} \tag{2.31}$$

which represents the world volume action of the fractional Dp-brane of type 0. We can see from eq. (2.30) how the presence of the $\mathcal{F}$-flux makes the asymptotic value of the *untwisted* charge to be unchanged, as anticipated. The difference between the actions (2.28) and (2.31) amounts to a different sign in the couplings to the fields of the twisted sector. This is a trivial case since we have just one twisted sector. In general, the fractional brane of type 0 couples to all twisted sectors with a minus sign with respect to the other fractional branes. Still, its coupling to the untwisted sector is the same as the other branes. This ensures it is a brane and not an anti-brane. Note that by summing up the two actions (2.28) and (2.31) the coupling to the twisted sector cancels out and one gets back the world volume action of a regular brane, as expected. This agrees with the idea that regular branes can be thought of as bound states of fractional branes of different kinds. The same holds, of course, for a more general orbifold: the sum of the world volume actions of all fractional branes give the action of the regular brane.

Exercise 3 - *Consider the orbifold* $\mathbb{C}_2/\mathbb{Z}_3$*. There are two non trivial cycles now,* $\mathcal{C}_1$, $\mathcal{C}_2$*, two twisted sectors and three different kinds of fractional branes. Derive the world volume action of the three different D3-branes and show they sum-up giving the world volume action of the corresponding regular D3-brane, the latter not coupling to the twisted sectors.* Hint: the background value of the b_I fields ($I = 1, 2$) is now 1/3.

2.3 Gauge Theory From Gravity: Example 1

We have now all the information we need to derive the supergravity solutions describing (bound states of) fractional branes of various kind as we know what fields of the supergravity spectrum the branes couple to. Moreover, for any case we like to study, we know how to determine the corresponding dual supersymmetric gauge theory should be. Once we find the supergravity solutions, we can then exploit them to see what can we learn about the corresponding dual gauge theories.

To be specific, we will go on focusing on our key example. This amounts to study the supergravity solution describing a bound state of N fractional D3-branes on the orbifold $\mathbb{C}_2/\mathbb{Z}_2$ whose low energy effective dynamics is four dimensional $\mathcal{N} = 2$ pure SYM. Still, the underlying philosophy is all the same, and the logical procedure and technical tools developed in this case should allow the reader to understand (and work out) any other case without much effort.

From what we learned in the previous section fractional D3-branes on the orbifold $\mathbb{C}_2/\mathbb{Z}_2$ couple to the graviton, a twisted scalar b (as we have just one twisted sector we omit the index I) and two RR potentials, one in the untwisted sector, $C_{(4)}$, and one in the twisted sector, $A_{(4)}$. Hence we expect that the other fields of the supergravity spectrum are trivial in the solution.

Let us start from the type IIB supergravity action in 10 dimensions which reads

$$\mathcal{S}_{\rm IIB} = \frac{1}{2\kappa^2}\Bigg\{\int d^{10}x\,\sqrt{-\det G}\,\mathcal{R} - \frac{1}{2}\int\Big[d\Phi\wedge{}^*d\Phi + \mathrm{e}^{-\Phi}H_{(3)}\wedge{}^*H_{(3)} + \mathrm{e}^{2\Phi}\,F_{(1)}\wedge{}^*F_{(1)}$$

$$+\,\mathrm{e}^{\Phi}\,\widetilde{F}_{(3)}\wedge{}^*\widetilde{F}_{(3)} + \frac{1}{2}\,\widetilde{F}_{(5)}\wedge{}^*\widetilde{F}_{(5)} + C_{(4)}\wedge H_{(3)}\wedge F_{(3)}\Big]\Bigg\} \tag{2.32}$$

where

$$H_{(3)} = dB_{(2)}\ ,\ F_{(1)} = dC_{(0)}\ ,\ F_{(3)} = dC_{(2)}\ ,\ F_{(5)} = dC_{(4)} \tag{2.33}$$

are, respectively, the field strengths of the NS-NS 2-form and the 0-, 2- and 4-form potentials of the R-R sector, and

$$\widetilde{F}_{(3)} = F_{(3)} - C_{(0)}\wedge H_{(3)}\ ,\ \widetilde{F}_{(5)} = F_{(5)} - C_{(2)}\wedge H_{(3)} \tag{2.34}$$

As usual, the self-duality constraint ${}^*\widetilde{F}_{(5)} = \widetilde{F}_{(5)}$ has to be implemented on shell. In order to find the classical solution corresponding to a bound state of N fractional D3-branes of, say, type 1, we have to add to the previous bulk action (N times) the corresponding world volume action $\mathcal{S}_1$ previously found, eq. (2.28). By varying the sum of the bulk and boundary actions one can derive the equations of motion for the various fields and eventually find the desired solution. As we are choosing a static gauge and the D3-branes extend along directions $x^0,...,x^3$, the fields in the solution will depend only on the transverse space coordinates, the flat ones, x^4,x^5, and the four orbifold directions $x^6,...,x^9$.

On the action (2.32) we have of course to implement the suitable ansatz for the relevant fields, in particular (recall the previous section) we have

$$B_{(2)} = b\,\omega_2 \quad ,\qquad C_{(6)} = A_{(4)}\wedge\omega_2 \tag{2.35}$$

where b and $A_{(4)}$ only depend on the flat transverse coordinates, x^4,x^5. Actually, we are going to write the solution in terms of the metric, the 5-form field strength and a twisted *complex scalar*, γ, defined as $\gamma = c + \mathrm{i}\,b$ where c is nothing but the Hodge dual (in the six dimensional sense) of $A_{(4)}$. We are forced to do that. Indeed, this Hodge duality is inherited from that between $C_{(6)}$ and $C_{(2)}$ in ten dimensions (hint: to prove this use the relations (2.23) and the ansatz (2.24)). As well known, the supergravity action (2.32) is expressed in terms of lower form potentials, hence $C_{(6)}$ does not enter there but rather its dual $C_{(2)}$ on which we can then implement the fractional brane ansatz, $C_{(2)} = c\,\omega_2$. As the fractional D3-branes couple electrically to $A_{(4)}$, we expect a magnetic-like solution for the field c.

By working pretty hard with the equations obtained following the outlined procedure (this is rather lengthly!) one finally gets the following solution

$$ds^2 = H^{-1/2}\,\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}\,\left[d\rho^2 + \rho^2 d\theta^2 + \delta_{mn}\,dx^m dx^n\right] \tag{2.36}$$

$$\widetilde{F}_{(5)} = d\left(H^{-1}dx^0 \wedge ...dx^3\right) + {}^*d\left(H^{-1}dx^0 \wedge ...dx^3\right) \tag{2.37}$$

$$\gamma = c + \mathrm{i}\,b = 4\pi\alpha' g_s N\,\log\frac{z}{\rho_E} \tag{2.38}$$

where $\rho = (x_4^2 + x_5^2)^{1/2}$, $z = \rho\, e^{\mathrm{i}\,\theta}$, $r = (x_6^2 + ... + x_9^2)^{1/2}$, ρ_E a short distance regulator for the logarithm, and H a function of the radial coordinates ρ and r, whose explicit expression we do not need, for the time being.

Now that we have the supergravity solution, we should exploit it and see what properties of the gauge theory we can reproduce. As this is the core of the gauge/gravity correspondence, let pause a bit and make a couple of comments. The basic point of the duality consists in providing a precise mapping between supergravity quantities and gauge theories operators. If one were able to give a prove of the duality, this should come out for free. In fact, this is not the case. As far as our present understanding of the gauge/gravity correspondence is concerned, what has been done, so far, was to check its validity (both in conformal as well as in non-conformal cases), but to prove it from first principles is something different. This is indeed a very challenging task which is far from being reached, yet. Rather than giving a prove, what we can do, at best, is to make a proposal, based on some reasonable assumptions, and check it. What follows is then nothing but a (well based) recipe to get such a dictionary. The dictionary can then be put under test. The way we derive the dictionary is ultimately related with open/closed string duality but there could possibly be other equivalent ways to arrive to the same result. The other one we are aware of is based on purely geometric reasoning and it would be interesting to find out the relations (which are there, of course) between these two apparently different approaches. This is part of present research and could actually also shed some light on the actual prove of the correspondence.

Let us summarize our recipe. The logic behind it is rather simple. Essentially, what one is doing when dealing with D-branes is studying their dynamics, which is given in terms of open strings, in the background of the closed strings. At low energy this amounts to study the back-reaction of the supergravity background on the D-branes and this, in the end, let one express various gauge theory quantities, which are those governing the dynamics of D-branes at low energy, in terms of supergravity fields.

The dynamics of D-branes is described by the non-abelian Born-Infeld (BI) action, and this we do not really know. What we know is that at low energy this is nothing but the action of a given non-abelian gauge theory. The procedure is then to start considering the abelian BI action (that we know) with abelian world volume fields switched-on, take the $\alpha' \to 0$ limit and eventually promote these fields to the adjoint representation of the gauge group and replace all derivatives with covariant ones.

It is not difficult to show that the world volume action of a fractional D3-brane on our orbifold with world volume fields switched-on reads (we are again pedantic about numerical factors)

$$\begin{aligned} \mathcal{S} = \; & - \; \frac{T_3}{\kappa} \int d^{p+1}\xi \, \sqrt{-\det\left[G + 2\pi\alpha' F\right]} \; \frac{b}{4\,\pi^2\alpha'} \\ & + \; \frac{T_3}{\kappa} \left[\int C_{(4)} \, \frac{b}{4\,\pi^2\alpha'} + \frac{1}{4\,\pi^2\alpha'} \int A_{(4)} + \frac{\alpha'}{8} \int d^4\xi \; F_{\mu\nu} \widetilde{F}^{\mu\nu} \right] \end{aligned} \tag{2.39}$$

Starting from this action one should

1. Choose the static gauge, i.e. $X^\mu = x^\mu$, $X^i = X^i(x^\mu)$ with $\mu = 0, ..., 3$ and $i = 4, ..., 9$.
2. Expand the above action up to terms quadratic in the derivatives of the fields.
3. Take the $\alpha' \to 0$ limit keeping the combination $\phi = (2\pi\alpha')^{-1} z$ fixed.
4. Promote $F \to F_A T^A$ where T^A are the generators of $SU(N)$ normalized as $Tr\left(T^A T^B\right) = \frac{1}{2}\,\delta_{AB}$, A, B being adjoint indexes.
5. Evaluate the action on the classical solution, call it $\mathcal{S}_{YM}$ ... and see what happens.

If doing so, after some algebra (this is left as a non-trivial exercise) one gets

$$\mathcal{S}_{YM} = -\frac{1}{g^2_{\rm YM}} \int d^4\xi \left[\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A + \frac{1}{2}\, D_\mu \overline{\phi}^A D^\mu \phi_A \right] + \frac{\theta_{\rm YM}}{32\pi^2} \int d^4\xi \; F^A_{\mu\nu} \widetilde{F}^{\mu\nu}_A \; + \; \text{ferm.} \tag{2.40}$$

Insert 3: **Some Facts About $\mathcal{N} = 2$ Super Yang-Mills**

Let us briefly recall some features of $\mathcal{N} = 2$ Super Yang-Mills with gauge group $SU(N)$. This is a supersymmetric gauge theory whose field content (auxiliary fields are not included) is described by the $\mathcal{N} = 2$ vector supermultiplet

$$\left(A_\mu \ , \ \psi^\pm \ , \ \phi\right)$$

corresponding to a vector field, two Majorana spinors and a complex scalar, all transforming in the adjoint representation of the gauge group. On shell this corresponds to 4 bosonic and 4 fermionic degrees of freedom.

The theory has both a scale anomaly and a $U(1)_R$ anomaly, at the quantum level. The scale anomaly is accounted for by the β-function which is one-loop exact perturbatively and reads

$$\beta = \frac{g^3_{\rm YM}}{16\pi^2}\left(-\frac{11}{3}\,c_v + \frac{1}{6}\,n_s\,c_s + \frac{2}{3}\,n_f\,c_f\right) = -\frac{2N}{16\pi^2}\,g^3_{\rm YM}$$

where c_v, c_s and c_f are the quadratic Casimir of the adjoint representation, $c_v = c_s = c_f = N$, $n_s = 2$ is the number of real scalars and $n_f = 2$ that of Majorana fermions. The corresponding running coupling constant reads

$$\frac{1}{g^2_{\rm YM}(\mu)} = \frac{1}{g^2_{\rm YM}(\Lambda_0)} + \frac{2N}{8\pi^2}\log\frac{\mu}{\Lambda_0}$$

where Λ_0 is some UV cut-off and μ the subtraction scale. The theory possesses a $U(2) \simeq SU(2) \times U(1)_R$ R-symmetry. The $U(1)_R$ is anomalous and gets broken to $\mathbb{Z}_{4N}$ at the quantum level (this can be seen by computing the triangular one-loop diagram with one global current and two gauge currents). We have

$$\partial_\mu J^\mu_R \to \partial_\mu J^\mu_R = n_f\,c_f\,R\,\frac{1}{16\pi^2}\,F^A_{\mu\nu}\,\widetilde{F}^{\mu\nu}_A = 4\,N\,q(x)$$

where $q(x) \equiv \frac{1}{32\pi^2}\,F^A_{\mu\nu}\widetilde{F}^{\mu\nu}_A$ while $R = 1$ is the R-charge of the gauginos (this means that ψ transform as $e^{{\rm i}\,\epsilon}\psi$ under a $U(1)_R$ transformation with parameter ϵ).

Continued...

Insert 3: *Continued...*

The effect of the anomaly is equivalent to assigning the $\theta_{\rm YM}$-angle transformation properties under $U(1)_R$ as

$$\theta_{\rm YM} \to \theta_{\rm YM} - 4N\epsilon$$

The theory is invariant under shifts $\theta_{\rm YM} \to \theta_{\rm YM} + 2\pi k$. So, if $\epsilon = \frac{\pi k}{2N}$ the theory is unchanged even at the quantum level. This shows that the full quantum theory is invariant under $\mathbb{Z}_{4N}$ transformations only. In the $\mathcal{N}=2$ theory the scale and the $U(1)_R$ anomaly sit in the same supermultiplet and one can construct a complex coupling τ out of $g_{\rm YM}$ and $\theta_{\rm YM}$

$$\tau = \frac{\theta_{\rm YM}}{2\pi} + \frac{4\pi {\rm i}}{g^2_{\rm YM}} = {\rm i}\,\frac{2N}{4\pi}\,\log\frac{\phi^2}{\Lambda^2}$$

where the last equality can be proved to hold exactly, at the perturbative level (Λ is the dynamically generated scale). This last equation implies that the anomalies can be read from the response of τ to a re-scaling of the energy by a parameter s and to a $U(1)_R$ transformation with parameter ϵ. Indeed under these re-scalings the complex scalar ϕ (which has dimension of energy and R-charge $R=2$) transforms as

$$\phi \longrightarrow s\, e^{2\,{\rm i}\,\epsilon}\,\phi$$

and one gets

$$\tau \longrightarrow \tau + {\rm i}\,\frac{2N}{2\pi}\,(\log s + 2{\rm i}\epsilon) \qquad \left\{ \begin{array}{l} \frac{1}{g^2_{\rm YM}} \to \frac{1}{g^2_{\rm YM}} + \frac{2N}{8\pi^2}\log s \\ \theta_{\rm YM} \to \theta_{\rm YM} - 4N\epsilon \end{array} \right.$$

These are all perturbative features of the theory. However, $\mathcal{N}=2$ SYM possesses a moduli space and the metric of this moduli space, which is proportional to the inverse of the gauge coupling squared, is always positive definite. This means we do not expect a Landau pole as instead predicted by the perturbative running of the gauge coupling. As shown by Seiberg and Witten time ago, at scales of order Λ instantons become relevant and change the (otherwise divergent) perturbative running of the gauge coupling.

where

$$\frac{1}{g_{\rm YM}^2} = \frac{1}{16\pi^3\alpha' g_s}\int B_{(2)} = \frac{N}{4\pi^2}\log\frac{\rho}{\rho_E} \tag{2.41}$$

$$\theta_{\rm YM} = \frac{1}{2\pi\alpha' g_s}\int C_{(2)} = -2N\theta \tag{2.42}$$

Equations (2.41) and (2.42) are the supergravity predictions for the running coupling constant and the $\theta_{\rm YM}$-angle of the dual $\mathcal{N}=2$ SYM theory. Let us see what we learn from them.

– *The perturbative predictions*

From the identification $\phi = (2\pi\alpha')^{-1}z$ and recalling that $z = \rho\, e^{{\rm i}\,\theta}$ and that ϕ has R-charge 2 we get that under a chiral transformation the *physical* angle θ shifts as

$$\theta \to \theta + 2\pi \tag{2.43}$$

Since $(1/4\pi^2\alpha' g_s)\int C_{(2)}$ is allowed to change by integer values we get that

$$\theta \to \theta + \frac{\pi}{N}\,k \tag{2.44}$$

are true symmetries of the supergravity solution. On the gauge theory side this corresponds, see the holographic relation (2.42), to a R-transformation with parameter

$$\epsilon = \frac{\pi\,k}{2\,N} \tag{2.45}$$

which means that every time ϵ changes by these discrete values the corresponding gauge theory, as determined from the supergravity dual, is not changed. This is exactly the expected $\mathbb{Z}_{4N}$ symmetry of $\mathcal{N}=2$ SYM. So, the classical supergravity solution knows about a quantum phenomenon, the breaking of the $U(1)_R$ down to $\mathbb{Z}_{4N}$! This is an encouraging result.

Let us now consider the other holographic relation, eq. (2.41). To this end, there is in fact a lacking element. We need to establish a precise energy/radius relation which is crucial to interpret correctly (2.41). What is the relation between the radial distance ρ, which is the unique dimensional quantities in the supergravity solution, and the energy μ where the gauge theory is defined? This is a very deep and important issue in any well established gauge/gravity correspondence and for non-conformal theories there is not an unique answer. We will appreciate this point much more in the next lecture, when discussing the case of $\mathcal{N}=1$ SYM, as in the present case the answer will be rather simple. Still, we want to emphasize this point, already.

The general recipe is to find the gravity dual of some protected operator of the gauge theory and exploit the corresponding equivalence to extract a relation between ρ and μ. Here things are pretty simple as the protected operator is nothing but ϕ, the complex scalar. Under a scale transformation with parameter s the mass scales as $m \to sm$ and then we get $\phi \to s\phi$. Recalling now that $\phi = (2\pi\alpha')^{-1} z$ and that $z = \rho\, e^{\mathrm{i}\,\theta}$, we easily extract the energy/radius relation

$$\rho = 2\,\pi\alpha'\mu \quad (\rho_E = 2\,\pi\alpha'\Lambda) \quad \to \frac{\rho}{\rho_E} = \frac{\mu}{\Lambda} \tag{2.46}$$

Inserting this result into eq. (2.41) we get

$$\frac{1}{g^2_{\mathrm{YM}}} = \frac{2\,N}{8\,\pi^2}\,\log\frac{\mu}{\Lambda} \tag{2.47}$$

which is indeed the correct result for the (perturbative) running coupling constant of $\mathcal{N}=2$ SYM. From the above equation one obtains also the expected perturbative β-function, of course. From the identification (2.46) we also understand what the meaning of the regulator ρ_E is at the gauge theory level: this is nothing but the dynamically generated scale, where the perturbative running coupling is expected to diverge. Note that we can rewrite eq. (2.41) by implementing the value of the $B_{(2)}$ background flux (use eq. (2.27)) and get

$$\begin{aligned} \frac{1}{g^2_{\mathrm{YM}}(\mu)} &= \frac{N}{4\pi^2}\log\frac{\rho}{\rho_E} \equiv \frac{1}{16\pi^3\alpha' g_s}\int B_{(2)} = \\ &= \frac{1}{4\pi g_s}\left(\frac{1}{2}+\frac{N g_s}{\pi}\log\frac{\rho}{\rho_0}\right) = \frac{1}{g^2_{\mathrm{YM}}(\Lambda_0)} + \frac{N}{4\pi^2}\log\frac{\mu}{\Lambda_0} \end{aligned} \tag{2.48}$$

where ρ_0 is now a long distance regulator for the logarithm related to ρ_E by $\rho_E = \rho_0 e^{-\pi/N g_s}$. Upon the corresponding identification $\rho_0 = 2\pi\alpha'\Lambda_0$, with Λ_0 being the scale where the bare coupling is defined, we finally get the expected relation between the dynamically generated scale and the UV cut-off in the gauge theory, $\Lambda = \Lambda_0\, e^{-8\pi^2/N g^2_{\mathrm{YM}}}$. Putting together all what we have learnt, we can summarize our findings saying that the twisted complex scalar γ is nothing but the holographic dual of the complex gauge coupling of the gauge theory, τ (modulo $4\pi^2\alpha' g_s$).

All the results we have been found for our working example have in fact a much wider validity. In particular

- Adding D7-branes amounts to add fundamental matter. Considering the bound state of N fractional D3-branes with M D7-branes, one ends up with $\mathcal{N}=2$ SYM with gauge group $SU(N)$ coupled to M fundamental hypermultiplets. The solution is more complicated as the D7-branes couple to the dilaton Φ and the axion C_0 but also in this case the predicted running coupling constant and $\theta_{\rm YM}$-angle turn out to be the correct ones

$$\begin{aligned}\frac{1}{g_{\rm YM}^2} &= \frac{1}{16\pi^3\alpha' g_s}\, e^{-\Phi}\int B_{(2)} = \frac{2N-M}{8\pi^2}\log\frac{\rho}{\rho_E}\\ \theta_{\rm YM} &= \frac{1}{2\pi\alpha' g_s}\left[\int C_{(2)} + C_{(0)}\int B_{(2)}\right] = (2N-M)\,\theta\end{aligned}$$

- Considering more complicated orbifolds of the ADE series one gets product gauge groups $U(N) \to U(N_0)\times U(N_1)\times ...\times U(N_{m-1}) +$ bi-fundamental matter. Also in this case the perturbative features recovered for the pure case are reproduced correctly.

- Considering orbifold limit of CY three-folds one gets theories with $\mathcal{N}=1$ supersymmetry. As explained previously, playing with different orbifolds and different bound states of branes, one can construct a zoology of gauge theories with matter, study the corresponding supergravity solutions and finally recover gauge theory information. A particularly interesting case is to consider a bound state of N fractional D3-branes with M D7-branes. This corresponds to $\mathcal{N}=1$ SYM coupled to chiral matter, resembling very much Super QCD. Again, also for these cases all perturbative gauge theory properties are correctly reproduced by the dual supergravity backgrounds.

– *The enhançon and the non-perturbative corrections*

All what we have been found so far are perturbative information. A question naturally arises at this point: what about non-perturbative contributions? Are these supergravity duals able to recover them, too?

To start with, a detailed analysis of the supergravity solution (2.36)-(2.38) shows that it is actually singular: the explicit form of the warp factor $H = H(\rho, r)$ shows that the metric has a *naked* singularity for some $r = r_s$ (i.e. $H(r_s) = 0$). This singularity should be cured, somehow, for the all story to make sense.

Let us start from the end. There is a general a mechanism, known as the *enhançon* mechanism, which excises the singularity rendering a singularity-free solution: at a distance $\rho = \rho_E$, the *enhançon*, new light degrees of freedom not accomplished by the supergravity approximation come into play and modify the background. This means that the physical description of the system at distances $\rho < \rho_E$ should include these states which change the form of the solution in the interior. The important point is that the singularity is cloaked behind ρ_E and is therefore excised from the solution. This means that the singularity we have found was a fake one. No matter what the "true" solution looks like, the one we have found has a meaning as a low energy description of our system of fractional branes only at distances bigger than the enhançon, and so the singularity was never really there, after all.

A number of questions naturally arise at this point. The first one is of course: what is the enhançon, in more quantitative terms? This is simple to answer. The enhançon is the locus where the $B_{(2)}$-flux vanishes, namely where the fluctuations of the b fields cancels its background value

$$\int B_{(2)} = b \sim 4\pi^2\alpha' \frac{1}{2} + \widetilde{b}|_{\rho=\rho_E} \equiv 0 \tag{2.49}$$

Given the above definition, we know what is happening at the enhançon: fractional brane probes become tensionless as their world volume action (recall what we learned in section 2.2) is proportional to the $B_{(2)}$-flux. To be more precise, by probing the geometry generated by our stack of fractional D3-branes by a fractional Dq-brane probe one finds that the probe tension is ρ-dependent and vanishes at $\rho = \rho_E$.

Recalling the discussion after eq.(2.47), the enhançon is then just the scale where the dual gauge theory becomes strongly coupled since

$$\frac{1}{g^2_{\rm YM}} \sim \int B_{(2)} \tag{2.50}$$

The final lesson is then that in order to incorporate the non-perturbative contributions in the gauge theory we should go beyond the supergravity approximation. This looks quite unfortunate but we could already have guessed it from eq. (2.41). It is well known that the metric of the moduli space of $\mathcal{N} = 2$ SYM, which is proportional to the inverse gauge coupling squared, is positive definite. This means the gauge coupling can never diverge in the exact theory, in disagreement with the perturbative prediction (2.41).

Now that we have learned what the enhançon is, the second question we should try to answer is: what are these new light degrees of freedom? This depends on the supergravity solution (or equivalently the dual gauge theory) one is considering. Again, let us start from the end. The bound state of fractional Dp-branes generating the background (let us be general here, and let p be also different from 3) couples to

$$\text{NSNS}: \; G_{MN} \;,\; \Phi \;,\; b \quad ; \quad \text{RR}: \; C_{(p+1)} \;,\; A_{(p+1)} \quad (\leftrightarrow \; A_{(3-p)})$$

where the solution is written in general in terms of the magnetic dual of the RR twisted potential $A_{(p+1)}$, i.e. $A_{(3-p)}$, to which the branes couple magnetically. The light degrees of freedom becoming relevant at the enhançon come from fractional D-branes carrying an *electric* charge with respect to $A_{(3-p)}$.

As this could seem just an abstract recipe, let us see how it works in our example. In the fractional D3-brane solution we have discussed, the rôle of $A_{(p+1)}$ is played by $A_{(4)}$ while that of $A_{(3-p)}$ by the RR scalar $c = \int C_{(2)}$. The corresponding electric state is then a fractional D(-1)-brane whose world volume action looks like

$$\mathcal{S}_{D(-1)} = \frac{T_{-1}}{\kappa}\left[\left(C_{(0)} + e^{-\phi}\right)\int B_{(2)} + \mathrm{i}\int C_{(2)}\right] \tag{2.51}$$

By evaluating it in the background of our solution we get

$$\mathcal{S}_{D(-1)} = \frac{1}{2\pi\alpha' g_s}\left(\int B_{(2)} + \mathrm{i}\int C_{(2)}\right) = -\,\frac{\mathrm{i}}{2\pi\alpha' g_s}\gamma = -\,2\pi\mathrm{i}\,\tau \tag{2.52}$$

where in the last step we have used the holographic identification between γ and τ.

From the $\mathcal{N} = 2$ point of view we expect instantons to become relevant at strong coupling and de-singularize the moduli space. The instanton contribution enters the partition function with an action as

$$\mathcal{S}_{ins} = \frac{8\pi^2}{g^2_{\mathrm{YM}}} - \mathrm{i}\,\theta_{\mathrm{YM}} \equiv -2\pi\mathrm{i}\,\tau \tag{2.53}$$

in perfect agreement with equation (2.52). Hence, we are on the right path. Can we do better? In order to completely solve the theory one should include these extra states (the fractional D(-1)-branes) in the analysis and look for a solution which is asymptotically identical to the one we have found but changing sensibly at short distances. This should let one reproduce the full moduli space of $\mathcal{N} = 2$ from the supergravity (or better the string) dual. Unfortunately, this goal has not been achieved, yet.

As we just said, the enhançon phenomenon is not specific to the D3-brane case, but has a much wider validity. Let us consider for instance a gauge/gravity correspondence testing SYM with 8 supercharges in 3 space-time dimensions (in fact, this has been the first case where the enhançon phenomenon was observed). The fractional branes one has to consider in this case are the fractional D2-branes of type IIA string theory. The all story goes on pretty much the same. The branes couple to the graviton, the dilaton (this is non-trivial now) and the RR 3-form potential $C_{(3)}$ in the untwisted sector, and to the scalar field b and the RR 3-form potential $A_{(3)}$ in the twisted sector (we take for simplicity the orbifold $\mathbb{C}_2/\mathbb{Z}_2$ so to have just one twisted sector). The solution is given in terms of the magnetic dual of $A_{(3)}$, i.e. the 1-from potential A_1, to which the D2-branes couple magnetically. Following the recipe outlined above, we expect the degrees of freedom associated to fractional D0-branes to become relevant at the enhançon. The world volume action of fractional D0-branes is

$$\mathcal{S}_{D0} \sim \mp \frac{T_0}{\kappa} \int d^3\xi \sqrt{-\det G}\, e^{-\frac{1}{4}\Phi} \int_{\mathcal{C}_1} B_{(2)} \pm \frac{T_0}{\kappa} \left[\int C_{(1)} \int_{\mathcal{C}_1} B_{(2)} + \int A_{(1)} \right] \tag{2.54}$$

and at the enhançon they become tensionless. The $\pm$ sign is related to which type of fractional brane we are speaking about (they both contribute). The rôle of these states is similar to that of $W^{\pm}$ boson as they couple electrically to the "gauge" field $A_{(1)}$, enhancing the $U(1)$ symmetry to $SU(2)$ at the enhançon, where they become massless (hence the name enhançon for this phenomenon, which however applies, as we have seen, also to other situations)

$$A_{(1)} \longrightarrow \begin{pmatrix} A^{+}_{(1)} \\ A_{(1)} \\ A^{-}_{(1)} \end{pmatrix} \qquad\qquad U(1) \longrightarrow SU(2) \tag{2.55}$$

The upshot is that in the interior, namely at distances smaller than the enhançon radius, the solution should be described by an N-charge $SU(2)$ monopole. On the contrary, at large distances the deformations due to these extra states is negligible and the solution is the same as the singular one. It turns out this perfectly agrees with gauge theory expectations. Indeed it is known that the moduli space of $SU(2)$ monopole is smooth and is equivalent to that of $2+1$ SYM with 8 supercharges, automatically including the instanton corrections. Again, the supergravity/string background gives the correct prediction. We will come back to this issue in the last lecture, when discussing dualities, as we will have a different (but dual-equivalent) description of the enhançon phenomenon.

Before ending this section a last comment is in order. There is in fact another interesting (and related) feature of our solution, eqs. (2.36)-(2.38). If one computes the flux of the

5-form $\widetilde{F}_{(5)}$ along a closed surface intersecting the z-plane on some curve Σ one gets

$$\Phi(\widetilde{F}_{(5)}) = 4\pi^2 g_s N \left(\frac{1}{2} + \frac{N g_s}{\pi} \log \frac{\rho}{\rho_0} \right) \tag{2.56}$$

Differently from what happens in the original AdS/CFT correspondence where this integral gives just N (the number of constituent branes which represents the number of colors of the dual gauge theory), we get here a scale dependent result! Well, this is correct. In a non-conformal theory the number of effective degrees of freedom decreases through the infrared and the above formula predicts that (recall the relation between ρ and μ). This is just qualitative agreement and one would like to have a more quantitative one. To start with, note that the value of the flux goes to zero at the enhançon (this is due to the fact that the $\widetilde{F}_{(5)}$-flux turns out to be proportional to the $B_{(2)}$-flux). What is the meaning of all that? There has been some debate in the literature about the precise field theory interpretation of this phenomenon and its relation with the enhançon.

The safer thing we can say, so far, is as follows. One can think to build-up the configuration giving raise to the supergravity solution we have been discussing as a step-by-step process. Starting from a single (fractional) D3-brane positioned at the origin, one can take a second one far at infinity and move it toward the origin (this can be done at no cost as the system is BPS), then a third one and so on. In doing so, for the reasons we have been just discussing, one sees it is not possible to build up a source made of fractional D-branes located at the origin, $\rho = 0$: at the enhançon, whose radius is proportional to the number of constituent branes, the branes one is taking from infinity becomes tensionless. It evaporates, in a sense. Moreover, one can see that the Newton-like force experienced by a brane probe in the supergravity background under consideration ($F \sim -\nabla G_{00} = \nabla H^{-1/2}$) becomes repulsive at short distances, thus indicating that the branes tends to repel each other, gravitationally. The final picture is then that the branes making-up the configuration are not at the origin but rather expand to form a ring-like shell (the enhançon in fact) around a region which appears singular in the original supergravity solution. The enhançon is just the distance where the tension of the branes drops to zero and that is the last radius where there is a meaning to the constituent branes as localized sources. In this way it is clear that, while the exterior solution, due to Gauss' theorem, is of course unchanged (and it is nothing but the supergravity solution we have found), the interior one could look completely different, and is actually flat at leading order in $1/N$, as there is no point-like source in the inside. This is the classical stable solution one should start with, and which reproduces (is dual to) the perturbative part of the gauge theory moduli space. To take into account non-perturbative corrections, one should include the already mentioned new light degrees of freedom which become relevant at the enhançon. These would take care of higher order corrections in $1/N$ and would modify the interior region. This is what string theory tell us, at large N. This picture has been perfectly confirmed by a dual field theory analysis: studying the gauge theory side of the duality by means of the corresponding Seiberg-Witten curve, it has been shown the results agree quantitatively with the above picture at large N. Indeed, for large N, instanton corrections are implemented by assuming there is no running of the gauge coupling below the scale Λ and this is automatically taken into account by imaging the fractional branes being smeared at the enhançon.

Still, it has not been possible to find the complete answer including higher order corrections. This would imply one was able to "resolve" the enhançon by really including the new light degrees of freedom in the low energy closed string analysis. But, as already stressed, this has not been done, yet.

3 Lecture III - Branes wrapped on Calabi-Yau spaces

Another possibility to reduce the number of supersymmetries is to consider D-branes whose world volume is (partially) wrapped on topologically non-trivial supersymmetric cycles of a CY manifold. This is the natural counterpart in smooth spaces of fractional branes on orbifolds (in fact, these configurations are related by T-duality, as we will see later). The logic behind the construction of the gauge theory out of these D-branes can be summarized as follows. Consider a Dp-brane wrapped on a q-cycle inside a CY space (when we say CY space we really mean a *non-compact* CY space; this will always be understood in the following).

- The unwrapped part of the brane world volume remains flat and supports a (p+1-q)-dimensional effective theory.
- In order to preserve some supersymmetry the q-cycle normal bundle has to be partially *twisted*. As we shall see, this implies that some (would be massless) fields become massive and decouple, at low-energy.
- Taking a low energy limit where both the string states and the KK excitations on the q-cycle decouple, one ends up with a supersymmetric gauge theory in (p+1-

q)-dimensions. The amount of preserved supersymmetry depends on the way the q-cycle is embedded in the ambient space, a CY three-fold or a CY two-fold.

This is how one engineers the gauge theory. The general idea is that the supergravity solution generated by the bound state of D-branes one is considering should be dual, in some limit, to the (p+1-q)-dimensional supersymmetric gauge theory living on them.

This procedure was first used by Maldacena and Nuñez (MN) to study pure $\mathcal{N} = 1$ SYM in four dimensions and later generalized to other cases with different space-time dimensions and/or amount of preserved supersymmetry. We will focus in what follows on the case discussed by MN as we want to study the basic case of pure $\mathcal{N} = 1$ SYM. As it was the case for fractional branes, the logic procedure and technical tools we will develop should enable the reader to address any other case.

The MN setting corresponds to consider N D5-branes wrapped on a supersymmetric two-cycle, which is topologically a two-sphere, inside a CY three-fold. A CY three-fold preserves 1/4 supersymmetries and the D-branes break 1/2 more so the gauge theory living on the D-branes preserves 4 supercharges. We have in this case p=5 and q=2 so, according to the discussion above, we end up with a four-dimensional $\mathcal{N} = 1$ gauge theory with gauge group $SU(N)$, at low energy. As we are going to show, it turns out there is no matter coupled to it so the theory at hand is described by vector multiplets only, i.e. is pure. Once the supergravity solution generated by this bound state of D-branes is derived, one can exploit it and study the perturbative and the non-perturbative properties of $\mathcal{N} = 1$ SYM theory.

3.1 Wrapping Branes: The Topological Twist

The first thing we have to understand is to what extent we can wrap a D-brane on a topologically non-trivial cycle of a CY space and at the same time preserve supersymmetry. In fact, supersymmetry gets all broken, in general. This is a non-trivial geometric problem which would require a careful treatment. In what follows we are just going to give a rough idea on how things work, this being sufficient for our purpose. In order to preserve some supersymmetry on the D-brane world volume one should solve an equation like

$$D_M \, \epsilon = (\partial_M + \omega_M) \; \epsilon = 0 \qquad\qquad M = 0, 1, ..., p \tag{3.1}$$

where ω_M is a short cut for $\omega_M^{NP}\gamma_{NP}$, ω_M^{NP} being the spin connection on the q-cycle under consideration, and ϵ is a killing spinor. This equation does not admit any solution as in general there are no covariantly constant spinors on a topologically non-trivial cycle.

Here it is where the twist comes about. If the theory has some global R-symmetry group, we can couple the theory to an external "gauge" field A_M that couples to the R-symmetry current. In doing so, eq. (3.1) becomes

$$(\partial_M + \omega_M - A_M)\,\epsilon = 0 \tag{3.2}$$

This is not done by hand as it could seem from this discussion, and has a simple geometrical explanation. The coupling to the external field A_M is just what takes into account the fact that the q-cycle is non-trivially fibered within the CY space: the directions normal to the q-cycle form a non-trivial bundle, the so-called normal bundle, and A_M is nothing but the connection on this normal bundle.

Upon the identification $\omega_M = A_M$ one can easily preserve supersymmetry as now eq. (3.2) becomes

$$\partial_M\,\epsilon = 0 \tag{3.3}$$

and this amounts to have just a constant spinor, which we certainly have. This is of course not the only way to solve eq. (3.2), but it turns out it is the way things work when dealing with D-branes.

The above operation has a striking effect: the Lorentz assignment (i.e. the spin) of the various fields gets changed (for instance, as it can be seen from eq. (3.3), the supersymmetry parameter becomes a scalar!). This is why we say the resulting theory is *twisted*. The crucial point now is that although the theory on the (p+1)-dimensional world volume is indeed twisted, that on the flat (p+1-q)-dimensional part, which is the one we are finally interested in, is not. That is just an ordinary field theory.

Let us see how this works in the case we want to study, a bunch of D5-branes wrapped on a two-sphere inside a CY space. Let us start considering a D5-brane in flat space. The presence of the brane breaks the ten-dimensional Lorentz group as follows

$$D5 \quad : \quad SO(1,9) \longrightarrow SO(1,5) \times SO(4) \tag{3.4}$$

where SO(4) is nothing but the R-symmetry group of the six-dimensional gauge theory living on the world volume of the D5-brane. If considering a D5-brane wrapped on an

S^2 we have instead

$$D5_{S^2} \quad : \quad SO(1,9) \longrightarrow SO(1,3) \times SO(2) \times SO(4) \tag{3.5}$$

where $SO(2) \simeq U(1)_J$ is the tangent bundle of the two-sphere. The twist is introduced by identifying $U(1)_J$ with some $U(1) \subset SO(4) \simeq SU(2)_L \times SU(2)_R$. Let us see what does this imply for the world volume fields living on the D5-brane. First we have to see how the fields transform with respect to the broken ten-dimensional Lorentz group $SO(1,3) \times U(1)_J \times SU(2)_L \times SU(2)_R$. Then we impose the twist, i.e. the identification between $U(1)_J$ and a $U(1) \subset SU(2)_L \times SU(2)_R$. This is just an operative procedure since we are actually breaking supersymmetry on the way, when we perform the twist. This is done for pedagogical purpose, but one should bare in mind that the actual situation for a wrapped brane world volume theory is directly what we will get in the end, of course.

The (flat) D5-brane field content amounts to a vector V_M ($M = 0, 1, ..., 5$), four scalars ϕ^A ($A = 1, ..., 4$), and two complex Weyl spinors with opposite chirality $\psi^{\pm}$. Playing a bit with representation theory we easily get

	$SO(1,5) \times SO(4)$	$SO(1,3) \times U(1)_J \times SU(2)_L \times SU(2)_R$
V_M	$(6,1)$	$(4_0,1,1) \oplus (1_{\pm},1,1)$
ϕ^A	$(1,4)$	$(1_0,2,2)$
ψ^+	$(4,2)$	$(2_+,2,1) \oplus (\overline{2}_-,2,1)$
ψ^-	$(4',2')$	$(\overline{2}_+,1,2) \oplus (2_-,1,2)$

where the subscript $0, \pm$ in the second column represents the $U(1)_J$ charge. This is the field content in six dimensions as seen from a four-dimensional point of view. We have a vector, six scalars and four fermions. So far this is just representation theory, the supersymmetry is all there (16 supercharges).

Let us now implement the twist. This can be done in different ways and the number of preserved supersymmetries on the D-brane world volume changes accordingly. To our purpose we have two inequivalent choices.

- $U(1)_J \equiv U(1)_D \subset SU(2)_D \equiv D\left(SU(2)_R \times SU(2)_L\right)$

Let us rewrite the table above taking into account the identification between $U(1)_J$ and $U(1)_D$ and let us call the corresponding group $U(1)$. We have to consider the charge of the various fields with respect to this $U(1)$ since the massless fields in four dimensions are those fields which are singlets under $U(1)$ (this can be understood as a charge under the overall $U(1)$ acts as a mass term from the four-dimensional point of view). Moreover, recall we are at energies such that both string states as well as KK modes on the S^2 are decoupled so we should care only of the zero modes in the expansion of six-dimensional massless fields into S^2 harmonics. Implementing the twist we easily get the following decomposition

	$SO(1,3) \times U(1)$
V_M	$4_0 \oplus 1_\pm$
ϕ^A	$2 \times 1_0 \oplus 1_+ \oplus 1_-$
$\psi^\pm$	$2 \times 2_0 \oplus 2 \times 2_{++}$ $2 \times \overline{2}_0 \oplus 2 \times \overline{2}_{--}$

From the above table we see that the *massless* field content in four dimensions amounts to a vector $A_\mu \simeq 4_0$, two scalars $\phi^a \simeq 2 \times 1_0$ with $a = 1, 2$, and two Majorana spinors, $\psi^a \simeq 2 \times \left(2_0 \oplus \overline{2}_0\right)$ (one could equivalently arrange the fermionic degrees of freedom into two Weyl spinors of opposite chirality). This is nothing but the field content of a $\mathcal{N} = 2$ vector multiplet in four dimensions. Hence the twist we have been considering leaves a theory with 8 supercharges only (of which we consider just those supermultiplets which appear massless from the four dimensional point of view). Note that all these four dimensional fields have ordinary statistic, as anticipated: the four-dimensional theory living on the flat part of the world volume of the D5-brane is not twisted. The supergravity solution generated by a bound state of N such D5-branes is thus relevant to study pure $\mathcal{N} = 2$ SYM theory with gauge group $SU(N)$ in four dimensions.

One can show that the geometric counterpart of this operation amounts to make the S^2 non-trivially fibered on a 2-dimensional sub-manifold of the overall transverse space. In other words, the geometry of the CY three-fold is $K3 \otimes \mathbb{R}^2$. This is a degenerate CY manifold as the number of preserved supersymmetries is 1/2 rather than 1/4 as for an ordinary CY with $SU(3)$ holonomy (the holonomy in this case is just $SU(2)$, in fact). As the brane breaks further 1/2 supersymmetry, one would expect a world volume theory with 8 preserved supercharges, i.e. $\mathcal{N}=2$ in four dimensions, consistently with what we have just found.

- $U(1)_J \equiv U(1)_L \subset SU(2)_L$

This choice is of course equivalent with the identification of $U(1)_J$ with $U(1)_R \subset SU(2)_R$ (there is a left-right symmetry here). We should proceed right in the same way as we did before. Decomposing the six-dimensional fields with respect to the twisted Lorentz group we have in this case

	$SO(1,3) \times U(1)$
V_M	$4_0 \oplus 1_\pm$
ϕ^A	$2 \times 1_+ \oplus 2 \times 1_-$
$\psi^\pm$	$2_0 \oplus 2_{++} \oplus 2 \times 2_-$ $\overline{2}_0 \oplus \overline{2}_{--} \oplus 2 \times \overline{2}_+$

We see that the *massless* field content in four dimensions amounts now to a vector $A_\mu \simeq 4_0$ and a Majorana spinor, $\psi \simeq 2_0 \oplus \overline{2}_0$. This is nothing but the field content of a $\mathcal{N}=1$ vector multiplet in four dimensions. Hence the twist leaves now a theory with 4 supercharges. As in the previous case, all the fields we are considering in four dimensions are ordinary ones, i.e. the four-dimensional field theory is not twisted. The supergravity solution generated by a bound state of N such D5-branes is thus relevant to study pure $\mathcal{N}=1$ SYM theory with gauge group $SU(N)$ in four dimensions.

Also in this case there is a geometric counterpart of all that. The S^2 turns out to be non-trivially fibered on the full transverse space. In other words, the target space geometry is now that of an ordinary CY three-fold with $SU(3)$ holonomy which preserves 1/4 supersymmetry. As the brane breaks further 1/2 supersymmetry, one would expect a world volume theory with 4 preserved supercharges, i.e. $\mathcal{N}=1$ SYM in four dimensions, consistently with what we have just found.

This is the case we are going to deal with in the reminder of this lecture. Our task will then be to find the supergravity solution describing a bound state of N D5-branes wrapped on a two-sphere inside a CY three-fold. As just seen, this is the configuration to exploit in order to study pure $\mathcal{N}=1$ SYM in four dimensions.

All what we have been doing so far can of course be extended to other cases. Considering for instance D6-branes wrapped on non-trivial three-cycles of a CY manifold, one would again obtain $\mathcal{N}=1$ SYM in four dimensions. Alternatively, one could consider D5-branes wrapped on three-cycles or D4-branes wrapped on two-cycles, getting in this case three-dimensional supersymmetric gauge theories at low energy. In the latter case, if the CY space has $SU(2)$ holonomy, one would get three-dimensional SYM with 8 supercharges. The logic in getting the spectrum, starting from the flat brane case and then performing the twist, goes on pretty much the same.

Exercise 4 - *Consider $\mathcal{N}=2$ SYM theory in four dimensions. This theory admits an internal symmetry $SU(2)_I$ while the Lorentz group is $L = SU(2)_L \times SU(2)_R$. Define a new Lorentz group $L' = SU(2)_L \times SU(2)'_R$ where $SU(2)'_R \equiv D\left(SU(2)_R \times SU(2)_I\right)$ (this is a twist) and compute how the field statistic (i.e. the spin) changes.* Hint: recall that the (original) field content consists of a gauge field A_μ, two Majorana spinors $\psi^{1,2}$ and two real scalars $\phi^{1,2}$.

3.2 Wrapping Branes: The Rôle Of Gauged Supergravity

To find the supergravity solution generated by a bound state of wrapped Dp-branes is not in general an easy task. In principle, one should just come out with the right ansatz in ten dimensions and solve the relevant equations of motion. But this turns out to be hard to do, as compared to the flat brane case.

As originally proposed by MN, an efficient way to circumvent these difficulties is to find a (wrapped) domain wall solution in the lower dimensional (p+2)-dimensional gauged supergravity and then lift the solution up in ten dimensions on a (8-p)-sphere. In order to understand how and why this works, let us consider flat branes, to start with. The near horizon geometry of a bound state of D3-branes in flat space is $\mathrm{AdS}_5 \times \mathrm{S}^5$. This is the geometry which is expected to be dual to four-dimensional $\mathcal{N}=4$ SYM, the theory describing D3-brane dynamics, at low energy. This ten-dimensional geometry can be thought of as a domain wall solution of five-dimensional gauged supergravity, which is then up-lifted to ten dimensions on a five-sphere. This is a simple case though, since in the solution none of the gauge fields of the five-dimensional gauged supergravity are switched-on, nor any scalar. Notice that the isometry group of S^5 is $SO(6)$, this being the R-symmetry group of $\mathcal{N}=4$ SYM theory, the theory living on the D3-branes world volume, as well as the group one could gauge, in fact, in the five-dimensional effective supergravity theory.

Let us now consider the case of (still flat) Dp-branes with p$\neq$ 3. In this case one can show there always exists a frame, the so-called dual frame, where the near horizon geometry looks (locally) like $\mathrm{AdS}_{p+2} \times \mathrm{S}^{8-p}$. As it is the case for the D3-branes, this solution can be thought of as a domain wall in the (p+2)-dimensional supergravity theory up-lifted to ten dimensions on a (8-p)-sphere (in fact this is consistent in ten dimensions only for p=2,3,5,7, but this is enough for us). In this case, however, we do not really have an AdS-like geometry, since the metric is warped now, i.e. there is an r-dependent function multiplying it: the space is only conformally equivalent to AdS. This makes the presence of some non-trivial scalar necessary for the consistency of the solution, while also in this case all gauge fields of the (p+2)-dimensional effective theory are trivial in the solution. The isometry group of S^{8-p} is now $SO(9-p)$, this being the R-symmetry group of the corresponding maximally supersymmetric gauge theory living on the Dp-branes world volume, as well as the group one could gauge in the lower dimensional supergravity theory.

What we have learnt in the previous section is that in order to describe a curved Dp-brane we should have some non-trivial external field A coupled to the world volume theory. This field, which is the one responsible for the twist, gauges the corresponding generator of the R-symmetry group. It is clear then that if we follow the same logical procedure as before, we should just look now for a slightly more involved (p+2)-dimensional domain wall solution where the corresponding gauge field A of the (p+2)-dimensional gauged supergravity has a non-trivial profile. Note that once the coordinates parameterizing the (p+2)-dimensional space are chosen, the identification between the spin connection of the q-cycle and the gauge field of the R-symmetry group also suggests what the suitable ansatz for this gauge field should be.

The lesson is then that what is twisting in the world volume theory (open string side) translates into having some non-trivial gauge fields in the (p+2)-dimensional supergravity solution (closed string side). These gauge fields are precisely those fields gauging the R-symmetry group of the world volume theory of the branes. This is the guiding principle one should follow in searching for any kind of supergravity solution describing (the near-horizon geometry of) wrapped branes.

Let us see how all this works in the case we are interested in, i.e. that of N D5-branes wrapped on a two-cycle inside a CY three-fold. We have in this case a domain wall solution of seven-dimensional gauged supergravity which should be up-lifted to ten dimensions on a three-sphere. The isometry group of the three-sphere is $SO(4)$, this being the R-symmetry group of the six-dimensional gauge theory living on the D5-brane. What we have learnt in the previous section is that in order to describe such brane we should have some external field gauging a $U(1) \subset SO(4)$. This has its natural seven-dimensional gauged supergravity counterpart: this same gauge field is just the one we should make a non-trivial ansatz for, in searching for the seven-dimensional domain wall solution corresponding to our wrapped brane. To say it in other words, a non-trivial gauge field $A \subset SU(2)_D$ or $A \subset SU(2)_L$ should be present in the seven-dimensional supergravity ansatz in order to find a solution corresponding to a D5-brane wrapped on an S^2 inside a K3 or a CY manifold and being dual to pure $\mathcal{N} = 2$ and $\mathcal{N} = 1$ SYM, respectively.

As the $\mathcal{N} = 2$ case has already been discussed in the previous lecture, in the following we will concentrate on $\mathcal{N} = 1$. We will derive the (near-horizon limit of the) supergravity solution describing N D5-branes wrapped on an S^2 inside a CY three-fold and exploit it to learn something about pure $\mathcal{N} = 1$ SYM.

3.3 Gauge Theory From Gravity: Example 2

Let us start by briefly mentioning the structure of the domain-wall solution of the seven-dimensional $SO(4)$ gauged supergravity theory we should start with. The fields present in this theory are the metric, six gauge fields transforming in the adjoint of $SO(4)$, ten scalar fields organized in a symmetric matrix T_{ij} (i, j being vector indexes of $SO(4)$), and a two-form potential. In order to obtain the solution we are looking for, we must first truncate the $SO(4)$ gauge group to its $SU(2)_L$ part and then identify a $U(1) \subset SU(2)_L$ with the spin connection, as dictated by the $\mathcal{N} = 1$ twist discussed in section 3.1. One can show this truncation requires the scalar matrix T_{ij} be proportional to a δ-function, while the two-form can be consistently put to zero. By choosing the seven-dimensional coordinates as $x_0, \ldots, x_3, \rho, \theta_1, \phi_1$, where the angles θ_1 and ϕ_1 parameterize the two-sphere, the solution looks like

$$
\begin{aligned}
ds^2 &\sim f(r)\left(dx_{1,3}^2+dr^2\right)+\frac{1}{\lambda^2}\,g(r)\left(d\theta_1^2+\sin^2\theta_1 d\phi_1^2\right) && (3.6)\\
T_{ij} &\sim e^{y(r)}\,\delta_{ij} && (3.7)\\
A &\sim \cos\theta_1\,d\phi_1 && (3.8)
\end{aligned}
$$

where λ is some dimension-full parameter and $f(r)$, $g(r)$ and $y(r)$ are functions of the radial coordinate r (we have not being precise with numerical coefficients as this is not important for our purposes, for the time being). Note how the actual form of A makes explicit its identification with the S^2 spin connection.

It turns out that this solution is singular (as it was the case for fractional branes), but remarkably the singularity, which is at $r=0$, can be easily removed, this time! This can be done starting from a more general ansatz where all the three one-forms belonging to $SU(2)_L$ are switched-on

$$
A^1\sim a(r)\,d\theta_1 \quad , \quad A^2\sim a(r)\sin\theta_1\,d\phi_1 \quad , \quad A^3\sim\cos\theta_1\,d\phi_1 \tag{3.9}
$$

where $a(r)$ is again a function of the radial coordinate r. Inserting the ansatz (3.6),(3.7) and (3.9) in the effective seven-dimensional supergravity lagrangian one can determine the functions $f(r)$, $g(r)$ and $a(r)$ and verify the new solution is indeed free of singularity.

As we are not going to discuss the $\mathcal{N}=2$ case explicitly, it is probably worth making a comment at this point. If repeating the above reasoning for the $\mathcal{N}=2$ case, considering now a domain wall solution with a $U(1)$ field belonging to $SU(2)_D$ (see section 3.1), one ends up again with a singular solution. However, differently from the present case, it turns out there is no way to smooth out the singularity. The solution is singular in seven dimensions and remains so also after the up-lifting to ten dimensions. The nature of the singularity is the same as the one discussed in the previous lecture, namely is a repulson-like singularity, and can be cured. This goes along the same lines as for the fractional brane case, i.e. the enhançon phenomenon is at work here, too. There is a distance where new light degrees of freedom come into play and change the low energy effective theory, and correspondingly the form of the solution, at short distances. How this can be done, we all know by now, but explicit results have not been really achieved, yet. This same thing happens for all situations in which one is studying supergravity duals of supersymmetric gauge theories with eight supercharges. No matter the theory under consideration is pure or with matter, no matter if it is in four, three or any number of dimensions. The enhançon phenomenon and its gauge theory meaning is a common feature of non-conformal supersymmetric gauge theories with eight supercharges and does not depend on the way (fractional branes, wrapped branes or anything else) one is using to study the supergravity dual. In fact, the rôle of the singularity looks rather different between the case of four supercharges and that of eight. In both cases the singularity can be cured. However, in the first case this can be done within the supergravity framework, in the second case one needs to include string states. This difference resides in the different properties of the corresponding gauge theories, of course.

Let us now come back to our solution. The non-singular seven-dimensional solution (3.6)-(3.9) has to be lifted up in ten dimensions on a three-sphere which we parameterize with coordinates (ψ, θ_2, ϕ_2). An important point to notice is that the effect of the twist is to make the seven-dimensional geometry non-trivially embedded in the ten-dimensional one. This means that there is a non-trivial mixing between the three-sphere used for the up-lift and the two-sphere of the seven-dimensional solution. This is related to the two-sphere being non-trivially fibered in the CY space. As we shall see, this fact has important consequences.

By following the inverse path which led to the consistent Kaluza-Klein truncation of ten-dimensional supergravity theory down to the $SO(4)$ gauged seven-dimensional supergravity, one can easily obtain the ten-dimensional solution from the seven-dimensional one (there are simple up-lift formulæ one can use). The ten-dimensional solution is expected to be described by a metric, a dilaton Φ, and a RR three-form magnetic flux $F_{(3)}$ (this is because D5-branes couple magnetically to the corresponding RR two-form potential $C_{(2)}$). This is exactly what one gets performing the up-lift. The final result (we are now precise with all numerical factors) turns out to be, in the string frame

$$\begin{aligned} ds^2 &= e^{\Phi} dx_{1,3}^2 + e^{\Phi} \alpha' g_s N \left[e^{2h} \left(d\theta_1^2 + \sin^2\theta_1 \, d\phi_1^2 \right) + d\rho^2 + \sum_{a=1}^{3} (\omega^a - A^a)^2 \right] && (3.10) \\ e^{2\Phi} &= \frac{\sinh 2\rho}{2\, e^h} && (3.11) \\ F_{(3)} &= 2\, \alpha' g_s N \prod_{a=1}^{3} (\omega^a - A^a) - \alpha' g_s N \sum_{a=1}^{3} F^a \wedge \omega^a && (3.12) \end{aligned}$$

where

$$\begin{aligned} A^1 &= -\frac{1}{2}\, a(\rho)\, d\theta_1 \quad , \quad A^2 = \frac{1}{2}\, a(\rho) \sin\theta_1 \, d\phi_1 \quad , \quad A^3 = -\frac{1}{2} \cos\theta_1 \, d\phi_1 && (3.13) \\ e^{2h} &= \rho \coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4} \quad , \quad a(\rho) = \frac{2\rho}{\sinh 2\rho} && (3.14) \end{aligned}$$

and $F^a = \nabla A^a \equiv dA^a + \epsilon^{abc} A^b \wedge A^c$. The ω^a are the left-invariant one-forms parameterizing the three-sphere

$$\begin{aligned}
\omega^1 &= \frac{1}{2}\left(\cos\psi\, d\theta_2 + \sin\psi \sin\theta_2 d\phi_2\right) \quad , \quad \omega^2 = -\frac{1}{2}\left(\sin\psi\, d\theta_2 - \cos\psi \sin\theta_2 d\phi_2\right) \\
\omega^3 &= \frac{1}{2}\left(d\psi + \cos\theta_2 d\phi_2\right)
\end{aligned} \tag{3.15}$$

and the five angles are defined as $0 \leq \theta_{1,2} \leq \pi$, $0 \leq \phi_{1,2} \leq 2\pi$ and $0 \leq \psi \leq 4\pi$. Finally, ρ is a dimensionless quantity and is related to the actual radial distance r as $\rho = r/\sqrt{\alpha' g_s N}$. For later convenience, let us also report the asymptotic behavior of the functions $a(\rho)$, $h(\rho)$ and of the dilaton

$$a(\rho) \underset{\rho\to\infty}{\sim} \rho\, e^{-2\rho} \to 0 \quad , \quad a(\rho) \underset{\rho\to 0}{\sim} 1 \tag{3.16}$$

$$e^{2h(\rho)} \underset{\rho\to\infty}{\sim} \rho \quad , \quad e^{2h(\rho)} \underset{\rho\to 0}{\sim} \rho^2 \tag{3.17}$$

$$e^{2\Phi} \underset{\rho\to\infty}{\sim} \rho^{-1/2}\, e^{2\rho} \quad , \quad e^{2\Phi} \underset{\rho\to 0}{\sim} 1 \tag{3.18}$$

as well as the explicit expression for the RR two-form potential $C_{(2)}$ one can get out of $F_{(3)}$. A straightforward computation gives

$$\begin{aligned}
C_{(2)} &= \frac{1}{4}\alpha' g_s N \left[(\psi + \psi_0)\left(\sin\theta_2\, d\theta_2 \wedge d\phi_2 - \sin\theta_1\, d\theta_1 \wedge d\phi_1\right) + \cos\theta_1 \cos\theta_2\, d\phi_1 \wedge d\phi_2\right] \\
&+ \frac{1}{2}\alpha' g_s N a(\rho)\left[d\theta_1 \wedge \omega^1 - \sin\theta_1\, d\phi_1 \wedge \omega^2\right]
\end{aligned} \tag{3.19}$$

Note that integrating $F_{(3)}$ the angular variable ψ does not get fixed and is defined modulo an arbitrary constant, ψ_0. However, the specific value of this constant is not really relevant as the physics sees $F_{(3)}$ rather than $C_{(2)}$, of course.

A couple of comments are in order at this point.

- As it can be seen from eq. (3.9), the function $a(\rho)$ is responsible for the non-abelian structure of the seven-dimensional solution and, as already pointed out, this is what makes the solution smooth and free of singularities. From the asymptotic behavior (3.16) one sees that $a(\rho)$ plays a relevant rôle at short distances only: at large distances the solution is not sensibly different from the singular one, as the function $a(\rho)$ vanishes and so do A^1 and A^2, while it displays its non-abelian structure at short distances. Another important point to notice is that the function $a(\rho)$ does not preserve the twist. This is not a problem though, since the twist should be preserved only at large distances. Indeed (this will be shown later), this is the region corresponding to the UV of the gauge theory, where the perturbative D5-brane spectrum has been computed and the twist performed. As soon as the theory becomes strongly coupled, and eventually confines, it looks completely different and the relevant degrees of freedom do not coincide with those in the UV. As we shall see, the function $a(\rho)$ plays a crucial rôle in obtaining the precise gauge/gravity dictionary.

- We said the solution is smooth, i.e. no singularity at $\rho = 0$ is present. Let us see this. The singularity could arise from the transverse part of the metric only, as the dilaton smoothly goes to zero for small ρ (see eq. (3.18)), so no problem there. By computing the six-dimensional transverse part of the metric at $\rho = 0$ one easily gets

$$
\begin{aligned}
ds_5^2 \sim &\ \frac{1}{4}\left(\cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2 - \sin\theta_1 d\phi_1\right)^2 + \\
&+ \frac{1}{4}\left(\sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2 + d\theta_1\right)^2 + \frac{1}{4}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2
\end{aligned}
\qquad (3.20)
$$

  which is finite (in fact, this is topologically a three-sphere with constant radius, as it will become apparent shortly). Hence there is indeed no singularity.

- As we will explicitly show later, what was the rôle of the $B_{(2)}$-flux over the shrinking cycles in the fractional brane case, is now played by the volume of the cycle the branes are wrapped on. It is then crucial to identify exactly what the actual cycle is, in the ten-dimensional geometry (3.10). This is not as trivial as it could seem at first sight. Naively, one would say that this cycle is the cycle parameterized by the two coordinates (θ_1, ϕ_1). This is the original cycle characterizing the seven-dimensional solution one has started with. In fact, the seven-dimensional solution is non-trivially embedded in ten dimensions. As a result of this, as we have already pointed out, there is a non-trivial mixing between the three coordinates of the three-sphere along which one up-lifts the solution (θ_2, ϕ_2, ψ) and those of the two-sphere along which the original seven-dimensional domain wall is wrapped (θ_1, ϕ_1). This mixing can be seen explicitly by the appearance of the seven-dimensional gauge connection in the ten-dimensional metric (3.10). We can say that the seven-dimensional domain wall already knows about the ten-dimensional geometry via the twist. From a seven-dimensional point of view this actually mixes space-time degrees of freedom with internal ones (note that in ten dimensions all these degrees of freedom are relative to space-time). For this reason, it turns out that the correct two-cycle is different from that suggested by the naive intuition.

To identify the relevant two-cycle we should focus on the five-dimensional angular part of the metric (3.10). Let us consider the limit $\rho \to \infty$. At large ρ, from the solution (3.10) we easily get

$$ds_5^2 \sim \rho\,(d\theta_1^2+\sin^2\theta_1\,d\phi_1^2)+\frac{1}{4}(d\theta_2^2+\sin^2\theta_2\,d\phi_2^2)+\frac{1}{4}(d\psi+\cos\theta_1\,d\phi_1+\cos\theta_2\,d\phi_2)^2 \quad (3.21)$$

It is easy to see that this is the metric of a $T^{1,1}$ manifold. This is a coset manifold, $T^{1,1} = [SU(2)\times SU(2)]\,/U(1)$, which can be seen as a $U(1)$ bundle over two S^2 (the explicit form of the metric above makes this structure manifest). Still, topologically $T^{1,1}$ can be seen as a two-sphere fibered over a three-sphere. This is the more convenient way we should think about it, in the present context. Indeed, together with the coordinate ρ, this manifold makes-up the CY three-fold characterizing our target space, and this is characterized by topologically non-trivial two and three-cycles. Even if the metric (3.21) differs from the standard $T^{1,1}$ one, as it is re-scaled in a way it is no longer an Einstein space, we can anyhow determine the non-trivial cycles. They are those of the standard $T^{1,1}$, since the only difference with the above manifold is just a metric difference. In our set of coordinates the (topologically non-trivial) two-cycle is not uniquely defined and one can show there are two possible choices

$$S^2 \quad : \quad \theta_1 = \pm\theta_2 \quad , \quad \phi_1 = -\phi_2 \quad , \quad \psi = \text{any} \quad (3.22)$$

where the value of ψ does not get fixed to any specific value, by topological arguments. The three-cycle is instead parameterized by

$$S^3 \quad : \quad \theta_1 = \phi_1 = 0 \quad (3.23)$$

The angle ψ in eq. (3.22) gets fixed by the physical requirement that the cycle is that of minimal volume, which means minimal energy, since the volume of the cycle is proportional to the tension of the wrapped D5-branes and we are looking for classically stable configurations. By computing the volume of the S^2 using eqs. (3.22) and (3.10), it is easy to show (this is left as an exercise) that the following holds

$$S^2 \quad : \quad \theta_1 = -\theta_2 \quad , \quad \phi_1 = -\phi_2 \quad , \quad \psi = 0 \mod 2\pi \quad (3.24)$$

$$S^2 \quad : \quad \theta_1 = \theta_2 \quad , \quad \phi_1 = -\phi_2 \quad , \quad \psi = \pi \mod 2\pi \quad (3.25)$$

The two cycles (3.24) and (3.25) are physically equivalent from the dual gauge theory point of view. This is due to the fact that the corresponding volumes are equal, namely

$$\text{Vol}(S^2) = \int_{S^2} \mathrm{e}^{-\Phi}\,\sqrt{\det G} \;\sim\; 4\,\mathrm{e}^{2\,h(\rho)} + (a(\rho)-1)^2 \quad (3.26)$$

where we set for simplicity $\theta \equiv \theta_1$ and $\phi \equiv \phi_1$ in both cases. Note that in eqs. (3.24) and (3.25) ψ is defined modulo 2π, this *not* being its period, which is instead 4π. This has a precise (and very nice) gauge theory interpretation, as we shall see.

Let us now consider the metric at the origin. This is just eq.(3.20) and is nothing but the metric of a deformed conifold (a cone in the radial coordinate ρ with base $T^{1,1}$) at the apex. The parameterization of the topologically non-trivial two and three-cycle is known for this metric, and is consistent with the one found above. By implementing eq. (3.24) (or equivalently eq. (3.25)) and eq. (3.23) in the metric (3.20) one finds a vanishing radius for the two-sphere while a finite one for the three-sphere, as expected for a *deformed* conifold at $\rho = 0$. In fact, the size of the basis of an ordinary cone shrinks to zero at the apex. This cone is deformed as a non-vanishing volume for the three-sphere persists at $\rho = 0$. The blown-up volume is what makes the metric (3.10) non-singular. Note that this is due to the presence of the function $a(\rho)$, which goes to one at $\rho = 0$, see eq.(3.16). For future reference, remind that a *resolved* conifold would be instead a conifold where the singularity at the apex is removed by the two-sphere being blown-up. As our manifold is not topologically a conifold for any ρ but just looks like a conifold in the two limits $\rho \to 0$ and $\rho \to \infty$, the above terminology gets extended to deformed CY and resolved CY, meaning a smooth CY three-fold which is topologically a three-sphere or a two-sphere at $\rho = 0$, respectively.

Let us summarize the lesson. We start with a CY manifold being characterized by a topologically non-trivial two-cycle where the D5-branes are wrapped and the gauge theory is being engineered. The back-reaction of the D-branes deforms the original background and changes its topology: in the resulting geometry the two-cycle shrinks to zero size at $\rho = 0$ while a three-cycle has blown-up. The manifold has undergone a *geometric transition*. We will further discuss this point in the last lecture.

We have now all the necessary ingredients to finally investigate the gauge/gravity correspondence for the system we have been studying. The procedure to get the proper gauge/gravity dictionary is essentially the same as the one we pursued for the fractional brane case. By expanding the action of the (wrapped) D5-branes up to terms quadratic in the world volume fields (which, in the low energy limit we are considering, we let depend on the flat space directions only) and evaluating it in the background (3.10)-(3.12) we get in this case the action for pure $\mathcal{N} = 1$ SYM (this is left as an exercise). The

Insert 4: **Some Facts About $\mathcal{N} = 1$ Super Yang-Mills**

Let us briefly recall some features of $\mathcal{N} = 1$ Super Yang-Mills with gauge group $SU(N)$. This is a supersymmetric gauge theory whose field content (auxiliary fields are not included) is described by the $\mathcal{N} = 1$ vector supermultiplet

$$(A_\mu \ , \ \lambda)$$

corresponding to a vector field and a Majorana spinor, the gaugino, both transforming in the adjoint representation of the gauge group. On shell this corresponds to 2 bosonic and 2 fermionic degrees of freedom.

This theory has many similarities with QCD, the only difference, at a superficial level, being that fermions transform in the adjoint representation rather than in the fundamental representation. In fact, it shares with QCD many physical properties. The theory is asymptotically free, only colorless asymptotic states exist, it is expected to confine and that a mass gap is dynamically generated, i.e. all particles in the spectrum are massive. The theory has both a scale anomaly and a $U(1)_R$ anomaly, at the quantum level. The gaugino has R-charge $R = 1$, i.e. under a $U(1)_R$ transformation it transforms as $\lambda \rightarrow e^{i\epsilon}\lambda$, while the vector field has R-charge $R = 0$. The scale anomaly is accounted for by the β-function which in general receives corrections at all loops and in the Pauli-Villars renormalization scheme reads

$$\beta = -3\,\frac{N g_{\mathrm{YM}}^3}{16\pi^2}\left(1 - \frac{N g_{\mathrm{YM}}^2}{8\pi^2}\right)^{-1}$$

This is known as the Novikov-Shifman-Vainstein-Zakharov (NSVZ) β-function. By analytic transformations in the gauge coupling the above result changes, corresponding to a different renormalization scheme, but universality of the two-loop coefficient is maintained (i.e. changes enter beyond two-loops only). In the Wilsonian scheme, which is related to the above by a *singular* transformation, the β-function becomes exact at one-loop. This is useful, as this is the scheme where holomorphicity is made manifest.

Continued...

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The $U(1)_R$ symmetry which is conserved at the classical level, is in fact anomalous and gets broken to $\mathbb{Z}_{2N}$ by quantum effects (this can be seen by computing the triangular one-loop diagram with one global current and two gauge currents). The effect of the anomaly is equivalent to assigning the $\theta_{\rm YM}$-angle transformation properties under a global $U(1)_R$ transformation with parameter ϵ as

$$\theta_{\rm YM} \to \theta_{\rm YM} - 2N\epsilon$$

The theory is invariant under shifts $\theta_{\rm YM} \to \theta_{\rm YM} + 2\pi k$. So if $\epsilon = \frac{\pi k}{N}$ the theory is unchanged even at the quantum level. This shows that the full quantum theory is invariant under $\mathbb{Z}_{2N}$ transformations only.

This is not the full story. The $\mathbb{Z}_{2N}$ is spontaneously broken down to $\mathbb{Z}_2$ in the IR and there exist N degenerate vacua where only a $\mathbb{Z}_2$ invariance is conserved. This further breaking is accompanied by the phenomenon of gaugino condensation, i.e. the fermion bilinear $\mathrm{Tr}\lambda^2$ acquires a vacuum expectation value

$$S \equiv \langle\, \mathrm{Tr}\lambda^2 \,\rangle = \Lambda^3\, e^{2\pi \mathrm{i} k/N} \quad , \quad k = 0, 1, ..., N-1$$

where Λ is the dynamically generated scale and the vacuum angle $\theta_{\rm YM}$ has been set equal to zero. Under a chiral transformation (we choose the gaugino to have R-charge equal 1) we have $\langle \mathrm{Tr}\lambda^2 \rangle \to e^{2\mathrm{i}\alpha} \langle \mathrm{Tr}\lambda^2 \rangle$ so in the vacuum we see that only a $\mathbb{Z}_2$ invariance is preserved, as anticipated. Choosing $\theta_{\rm YM} = 0$, as we did above, the phases of the gaugino condensate label the different vacua. However, one can show that the $\theta_{\rm YM}$ dependence of the gaugino condensate is

$$\langle\, \mathrm{Tr}\lambda^2 \,\rangle_{\theta_{\rm YM}} \;=\; \langle\, \mathrm{Tr}\lambda^2 \,\rangle_{\theta_{\rm YM}=0}\; e^{\mathrm{i}\,\theta_{\rm YM}/N}$$

showing that the N vacua are intertwined as far as the $\theta_{\rm YM}$ evolution is concerned: since $\theta_{\rm YM} \simeq \theta_{\rm YM} + 2\pi k$ for physical purpose, doing chiral transformations out of $\mathbb{Z}_2$ one can generate all vacua starting from a given one. In particular it is possible to make the gaugino condensate to be real in each vacuum, the different vacua being equivalently well labeled by the compensating value of the $\theta_{\rm YM}$-angle.

gauge coupling and the $\theta_{\rm YM}$-angle are expressed in terms of supergravity quantities as

$$\frac{1}{g^2_{\rm YM}} = \frac{1}{2(2\pi)^3\alpha' g_s}\int_{S^2} {\rm e}^{-\Phi}\sqrt{{\rm det}G} = \frac{N}{16\pi^2}Y(\rho) \sim {\rm Vol}(S^2) \tag{3.27}$$

$$\theta_{\rm YM} = -\frac{1}{2\pi\alpha' g_s}\int_{S^2} C_{(2)} = -N\,\psi_0 \tag{3.28}$$

where

$$Y(\rho) = 4\,e^{2h(\rho)} + (a(\rho)-1)^2 = 4\,\rho\,\tanh\rho \tag{3.29}$$

and where we have considered, for definitiveness, the two-cycle (3.24). The gauge coupling is related to the volume of the two-cycle, as anticipated. From eq. (3.27) we easily see that

$$\frac{1}{g^2_{\rm YM}} \simeq \frac{N\rho}{4\pi^2} \quad \text{for} \quad \rho\to\infty \tag{3.30}$$

$$\frac{1}{g^2_{\rm YM}} \simeq 0 \quad \text{for} \quad \rho\to 0 \tag{3.31}$$

Therefore large distances in supergravity correspond to the UV of the gauge theory, since for large ρ the gauge coupling happens to become smaller and smaller (this is what expected at high energy since $\mathcal{N}=1$ SYM is an asymptotically-free theory). At short distances the gauge coupling becomes instead bigger and bigger (in fact, it seems there is a Landau pole at $\rho=0$). Hence, short distances in supergravity correspond to the IR of the gauge theory, where $\mathcal{N}=1$ SYM becomes strongly coupled. This qualitative behavior is pretty much the same as the $\mathcal{N}=2$ case we discussed in the previous lecture. Still, to make it more precise we need to pint-point the proper radius/energy relation, which is missing so far. Before dealing with this important point, let us turn our attention to the $\theta_{\rm YM}$-angle.

From the identification (3.28) it is clear that $U(1)_R$ transformations should be realized as shifts in the angular variable ψ, in the dual supergravity background. The only question is what is the relation between a rotation in ψ of angle, say, α, and a $U(1)_R$ transformation on the gauge theory side with parameter ϵ, under which the $\theta_{\rm YM}$-angle transforms as $\theta_{\rm YM}-2N\epsilon$ (see Insert 4). From eq. (3.28) it follows that $\alpha=2\epsilon$, which means that under a $U(1)_R$ transformation with parameter ϵ the dual variable ψ transforms as

$$\psi\to\psi+2\,\epsilon \tag{3.32}$$

On the other hand eq. (3.28) is also saying that the MN solution is describing the gauge theory in a fixed $\theta_{\rm YM}$-vacuum. This could look unfortunate at first sight, as it seems to tell us we are not able to see anything interesting here. Actually this is not true and the prediction (3.28) is what it should be. Let us see why. We know that the overall $U(1)$ R-symmetry, enjoyed by $\mathcal{N}=1$ SYM at the classical level, is anomalous, and is broken to $\mathbb{Z}_{2N}$ by quantum corrections in the UV. This perturbative symmetry of the quantum theory is spontaneously broken to $\mathbb{Z}_2$ in the IR. So, the "true" theory itself lives in a fixed (up to $\mathbb{Z}_2$ symmetry) $\theta_{\rm YM}$-vacuum. We expect the MN solution to describe the quantum properties of $\mathcal{N}=1$ SYM, so it is only the $\mathbb{Z}_2$ invariance left-over in the vacuum we would like our *complete* solution being able to display. As we shall see in a moment, this symmetry is actually there, and is related to the ψ angle being fixed to be 0 *or* 2π, see eq. (3.24). And in the UV we can also see the $\mathbb{Z}_{2N}$ symmetry, in fact.

As it can be seen from eqs. (3.10)-(3.12), shifts in ψ are not symmetries of the MN solution neither isometries of the metric (this is due to mixed terms coming from the square of $\omega^{1,2}-A^{1,2}$). However, for $\rho\to\infty$ we know that $a(\rho)$ vanishes and so do A^1 and A^2. So, shifts in ψ are isometry of the metric at large ρ and the non-invariant term comes from $C_{(2)}$ only. The point now is that since the explicit ψ dependence drops out from the metric, the relevant S^2 obtained by minimizing the corresponding D5-brane tension does *not* fix ψ anymore. That is to say, what supergravity tells us about the two-cycle at large ρ is just eq. (3.22) rather then eqs. (3.24)-(3.25). By computing the integral of $C_{(2)}$ over the cycle (3.22) we get now

$$\frac{1}{4\pi^2\alpha' g_s}\int_{S^2} C_{(2)}=\frac{N}{2\pi}\,(\psi+\psi_0) \tag{3.33}$$

The above integral is allowed to change by integer values so shifts in ψ such that

$$\psi\to\psi+2\frac{\pi}{N}\,k \tag{3.34}$$

are symmetries of the solution, at large ρ. Recalling now eq. (3.32), we see that this corresponds to $U(1)_R$ transformations with parameter $\epsilon=\frac{\pi}{N}\,k$, which means $\mathbb{Z}_{2N}$. Supergravity predicts that at large ρ, which means UV, these should be symmetries of the gauge theory. Hence the symmetry (3.34) is nothing but the supergravity counterpart of the non-anomalous $\mathbb{Z}_{2N}$ symmetry of the gauge theory!

This is a nice result, of course, but we can do more. As already pointed out, the full solution is not invariant under transformations like (3.34) as now $a(\rho)$ is there, and is more and more relevant as we go to short distances (these corresponding to the IR of the dual gauge theory). Both in the metric and in $C_{(2)}$, see eq. (3.19), $a(\rho)$ multiplies terms proportional to $\cos\psi$ or $\sin\psi$ which are invariant only for

$$\psi \to \psi + 2\pi k \quad \text{which means} \quad \epsilon = k\,\pi \tag{3.35}$$

So this is the only symmetry in ψ enjoyed by the complete solution. Well, this is nothing but the gravitational counterpart of the spontaneous breaking of the R-symmetry from $\mathbb{Z}_{2N}$ down to $\mathbb{Z}_2$ in the IR. So, the full supergravity solution, which is expected to be dual to $\mathcal{N}=1$ SYM with gauge group $SU(N)$, is $\mathbb{Z}_2$ invariant, as it should be. We also see now what the meaning of the uncertainness in the exact value of ψ was, in eq. (3.24). Supergravity was just telling us that the vacuum of $\mathcal{N}=1$ SYM is $\mathbb{Z}_2$ symmetric!

Let us come to the second holographic relation, eq. (3.27), which involves the gauge coupling. Similarly to the case studied in the previous lecture, in order to make the correspondence concrete we should first find out what the radius/energy relation is. Curiously, the previous analysis of the (apparently unrelated) $\theta_{\rm YM}$-angle correspondence gives us the answer. What we have just seen is that $a(\rho)$ is the quantity responsible for the breaking of the $\mathbb{Z}_{2N}$ R-symmetry down to $\mathbb{Z}_2$ in the IR. On the gauge theory side this breaking is accompanied by the phenomenon of the gaugino condensation, i.e. the operator $\langle\lambda^2\rangle$ acquires a vacuum expectation value in the IR, $\langle\lambda^2\rangle = \Lambda^3$. Something similar happens to $a(\rho)$ which is sensibly different from zero only at short distances, while it vanishes at large distances. In other words, the function $a(\rho)$ plays the same rôle in supergravity has the gaugino condensate in the gauge theory. It is then natural to conjecture that $a(\rho)$ is the supergravity dual of the gaugino condensate

$$\langle\lambda^2\rangle \longleftrightarrow a(\rho) \tag{3.36}$$

The gaugino condensate has dimension three while the function $a(\rho)$ is of course dimensionless so the precise identification reads

$$\frac{\Lambda^3}{\mu^3} = \frac{2\rho}{\sinh 2\rho} \tag{3.37}$$

where μ is the subtraction scale where the gauge theory is defined and we have substituted for $a(\rho)$ its explicit form. Notice that as it was the case in the previous lecture, where we identified the radial coordinate of space-time with the scalar field of the $\mathcal{N}=2$ vector multiplet, we have here an identification between a supergravity field and a protected operator of the gauge theory (i.e. an operator whose dimension does not get changed by quantum corrections). This is necessary, as supergravity fields do not change their physical dimension with ρ, of course. The above identification is also remarkable since it says that the field which de-singularizes the solution, $a(\rho)$, gets associated with the non-trivial IR dynamics of the dual gauge theory. This is something we have already experienced, when discussing the enhançon mechanism for theories with 8 supercharges.

From eq. (3.37) we can extract the desired radius/energy relation which we can then use in the holographic expression (3.27) and get an exact expression for the running of the gauge coupling. To invert eq. (3.37) is actually a difficult task, unfortunately. Instead, it is rather simple to get the β-function. We can write

$$\beta(g_{\mathrm{YM}}) = \frac{\partial g_{\mathrm{YM}}}{\partial \ln(\mu/\Lambda)} = \frac{\partial g_{\mathrm{YM}}}{\partial \rho} \frac{\partial \rho}{\partial \ln(\mu/\Lambda)} \tag{3.38}$$

and compute the two derivative contributions from eqs. (3.27) and (3.37), respectively. In doing so, let us first disregard the exponential corrections, which are sub-leading at large ρ, this being the region where perturbative computations make sense in the gauge theory. Expanding the function $Y(\rho)$ appearing in eq. (3.27) one gets $Y(\rho) = 4\rho + \mathcal{O}(e^{-\rho})$. Similarly we get $a(\rho) = 2\rho\, e^{-2\rho} + \mathcal{O}(e^{-6\rho})$. From these expressions we easily get

$$\frac{\partial g_{\mathrm{YM}}}{\partial \rho} = \frac{\pi}{N^{1/2}} \rho^{-3/2} = -\frac{N g_{\mathrm{YM}}^3}{8\pi^2} \tag{3.39}$$

$$\frac{\partial \rho}{\partial \ln(\mu/\Lambda)} = \frac{3}{2}\left(1 - \frac{1}{2\rho}\right)^{-1} = \frac{3}{2}\left(1 - \frac{N g_{\mathrm{YM}}^2}{8\pi^2}\right)^{-1} \tag{3.40}$$

where in the second step of both equations we have used again eq. (3.30). The final result is then

$$\beta(g_{\mathrm{YM}}) = -3\,\frac{N g_{\mathrm{YM}}^3}{16\pi^2}\left(1 - \frac{N g_{\mathrm{YM}}^2}{8\pi^2}\right)^{-1} \tag{3.41}$$

which is the NSVZ β-function. This means that supergravity predicts the exact β-function of $\mathcal{N} = 1$ SYM at all loops!

A couple of remarks are in order at this point.

- The first question which naturally arises is why supergravity is giving the NSVZ β-function, which is in fact the β-function obtained in a particular scheme, the Pauli-Villars scheme. The only thing we can say in this respect is that this depends on the explicit form of the gauge/gravity relation (3.37) which supergravity does not fix uniquely. The geometric considerations leading to the identification of the gaugino condensate with the function $a(\rho)$ are insensible to a redefinition of the holographic relation (3.37) by means of an analytic function of the gauge coupling. If doing so, one can easily see that the result we have obtained, eq. (3.41), changes.

Still, this change enters beyond two loops only, meaning that supergravity respects the universality of the two-loop coefficient of the β-function.

- At short distances, which correspond to the IR of the gauge theory, one should wonder what the meaning of the β-function is, of course. But let us forget this for the time being and compare the β-function one gets for very large ρ and for very small ρ. In the first case we have the usual one-loop result, $\beta = -3/(16\pi^2) g_{\rm YM}^3 N$. Recalling that the dynamically generated scale Λ is defined as the *scale invariant* quantity $\Lambda = \mu \exp[-\int d\, g_{\rm YM}/\beta]$ we get

$$\Lambda^3 = \mu^3 \exp\left[-\frac{8\pi^2}{g_{\rm YM}(\mu)^2 N}\right] \tag{3.42}$$

where $\mu >> \Lambda$. Expanding eqs. (3.27) and (3.37) at very small ρ in powers of $1/\rho$ (we are now at low energy), one obtains instead $\beta = -9/(16\pi^2) g_{\rm YM}^3 N$. This leads to

$$\Lambda^3 = \mu'^3 \exp\left[-\frac{8\pi^2}{3\, g_{\rm YM}(\mu')^2 N}\right] \tag{3.43}$$

where now $\mu' \geq \Lambda$. The behavior (3.42) is usually interpreted as a non-perturbative effect due to fractional instantons of charge $1/N$. These should become more and more relevant at low energy, where the theory becomes strongly coupled. However, the IR estimate (3.43) seems to say there is something different there, as the coefficient is different of a factor 3. That is to say, if we take eqs. (3.27) and (3.37) seriously all the way down to the IR, $\langle \lambda^2 \rangle = \Lambda^3$ seems not to have a pure fractional instanton behavior. (I thank P. Olesen for pointing this out to me).

- In deriving eq.(3.41) we disregarded, both in $Y(\rho)$ and in $a(\rho)$, exponentially suppressed corrections. Still, supergravity seems to say they are there, no matter how small they are. What is their meaning? From the relation between ρ and $g_{\rm YM}$ it is clear that these correspond to (unexpected) non-perturbative contributions. In order to compute them one should consider the full expression for $Y(\rho)$ and $a(\rho)$ in eq.s (3.27) and (3.37) and re-derive eq. (3.41). This can easily be done and one can get an exact expression in ρ for the β-function. What is difficult is to re-express the result in terms of the gauge coupling. In fact, there is some debate in the literature on how to treat these non-perturbative corrections since it can also be the case they mix with KK degrees of freedom not belonging to the gauge theory and which are not decoupled in the supergravity limit (see below). If this is the case, without further insights it is very difficult to disentangle the two contributions and give a precise mathematical recipe on how to re-derive eq. (3.41). The only thing we can say, so far, is that supergravity seems to suggest some non-perturbative corrections are there, after all. It is also worth pointing out that these corrections would remove the pole of the NSVZ β-function, the physical interpretation of which is not completely clear. It would be nice to check this (unexpected) prediction by doing some computations in the field theory.

As it was the case for theories with 8 supercharges, also in this case one can obtain the correct action for the gauge theory instantons. It is well known that a system of N D3-branes and k D(-1)-branes in flat space describes the k-instanton sector of $\mathcal{N}=4$ $SU(N)$ SYM in four dimensions. It is also known that D1-branes are instantons within D5-branes. It is then natural to consider a system of N D5-branes and k (euclidean) D1-branes wrapped on the S^2 and see if the k wrapped D1-branes account for the k-instanton contribution (these are nothing but the analogue of the fractional D(-1)-branes we considered in the previous section, of course). The action of an Euclidean D1-brane (we choose for simplicity $k=1$) is

$$\mathcal{S}_{D1} = \frac{T_1}{\kappa}\int_{S^2} e^{-\Phi}\sqrt{\det G} - \mathrm{i}\frac{T_1}{\kappa}\int_{S^2} C_{(2)} \tag{3.44}$$

where all supergravity fields are pulled-back onto the brane world volume and $T_1/\kappa = 1/(2\pi g_s\alpha')$. Evaluating the above action in the background of the MN solution we get after some simple algebra

$$\mathcal{S}_{D1} = \frac{8\pi^2}{g^2_{\mathrm{YM}}} - \mathrm{i}\,\theta_{\mathrm{YM}} \tag{3.45}$$

which is the correct form of the instanton action.

There is one last point that may seem missing. It has to do, again, with the θ_{YM}-angle. The MN solution describes $SU(N)$ $\mathcal{N}=1$ SYM in a fixed vacuum. What about the other $N-1$ vacua? How to generate them? From what we have learned so far, it is not difficult to answer this question. Here it is where the rôle of gauge transformations in the seven-dimensional gauged supergravity comes into play. In the seven-dimensional solution (3.6)-(3.9) we are free to make $SU(2)_L$ gauge transformations

$$A \to A' = g^{-1}A\,g + \mathrm{i}\,g^{-1}dg \tag{3.46}$$

where g is an element of the $SU(2)_L$ group and A is the $SU(2)_L$ gauge connection. This means that the ten-dimensional solution is not univocally determined, even if all solutions should be physically equivalent, of course. It turns out that different seven dimensional gauge choices correspond just to different parameterizations of the relevant ten-dimensional geometry. In particular, if choosing $g = \exp(\mathrm{i}\epsilon\sigma_3)$ one gets

$$A' \;=\; e^{\mathrm{i}\epsilon\sigma_3}\, A\, e^{-\mathrm{i}\epsilon\sigma_3} \tag{3.47}$$

By repeating the reasoning which led to eq.(3.24) one finds now

$$S^2 \quad : \quad \theta_1 = -\theta_2 \quad , \quad \phi_1 = -\phi_2 \quad , \quad \psi = 2\epsilon \mod 2\pi \tag{3.48}$$

Re-deriving the expression for the gauge coupling and the θ_{YM}-angle using this modified cycle, it is easy to see that we are now describing the *same* running but we are studying the theory in a *different* θ_{YM}-vacuum, namely

$$\theta_{\mathrm{YM}} = -N\,(\psi_0 + 2\epsilon) \tag{3.49}$$

The integral of the RR two-form potential is indeed in this case

$$\frac{1}{4\pi^2\alpha' g_s}\int_{S^2} C_{(2)} = \frac{N}{2\pi}\,(\psi_0 + 2\epsilon) \tag{3.50}$$

For values of ϵ such that $2\epsilon = 2\pi k/N$ with $k = 0, ..., N-1$ we are just jumping between one vacuum to another, describing all the N vacua of the gauge theory! Each vacuum, see eq. (3.48), has its own $\mathbb{Z}_2$ symmetry. Summarizing, modifying the MN solution by means of the gauge transformations (3.47), we can indeed describe all vacua of the SYM theory. Notice that the new form of the solution seems to suggest that under the transformation (3.47) the gaugino condensate, which we identified with $a(\rho)$, picks-up a phase of $2\mathrm{i}\epsilon$, consistently with gauge theory expectations. This is a non-trivial confirmation of the transformation rule (3.32).

Exercise 5 - *Consider the gauge transformation generated by the $SU(2)$ element*

$$g \;=\; \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta_1\sigma_1}\mathrm{e}^{-\frac{\mathrm{i}}{2}\phi_1\sigma_3}$$

and compute the corresponding gauge connection A' using eq. (3.46). What is the expression for the ten-dimensional metric (3.10) now? Compute the new parametrization for the two-cycle, the corresponding running of the gauge coupling and the θ_{YM}-angle. Are there any changes in the physics? Hint: in order to identify the parametrization of the two-cycle look at the behavior of A' as $\rho \to 0$ and recall that the mixing between the three-sphere one up-lifts the seven-dimensional domain wall solution and the two-cycle depends on that behavior.

After all these nice findings, let us end this lecture pointing out some subtleties and open problems hidden in the correspondence we have investigated. These have to do with the decoupling limit issue we mentioned in the first lecture. One could wonder whether there is decoupling between (four-dimensional) gauge degrees of freedom and gravity degrees of freedom within the supergravity regime we have been considering. As explained in our first lecture it turns out this is not the case. This is a general problem one always has to deal with in non-conformal situations, and is the main obstruction in stating exact dualities in the regime where explicit computations are doable. Although we cannot treat this problem in detail here, there are a couple of simple remarks that could help in making what we just said more concrete.

- It is easy to see that the curvature of the MN background, eq.(3.10), goes like
$$\mathcal{R} \sim \frac{1}{\alpha' g_s N} \tag{3.51}$$
Therefore, in order to keep the curvature small in string units and make the supergravity approximation meaningful, one should take the limit $g_s N \to \infty$.

- We already pointed out, see the discussion after eq. (3.31), that the MN solution predicts a Landau pole at small ρ, suggesting that the gauge coupling diverges in the IR. In fact, this is not necessarily true. As just noticed, the regime in which the supergravity approximation is reliable is for large N. In this regime a Landau pole can indeed be present even if the gauge coupling remains finite at the scale Λ, since in eq. (3.31) it is really $g^2_{\rm YM}N$ which is going to infinity and not the gauge coupling itself. To discuss the duality in the deep IR at finite N, one has to go beyond the supergravity approximation.

- By considering the metric (3.10) at small ρ one sees there are KK states coming from the three-sphere (and not belonging to the gauge theory) whose mass spectrum is roughly
$$M^2_{KK} \sim \frac{1}{R^2_{S^3}} \sim \frac{1}{\alpha' g_s N} \tag{3.52}$$
At $\rho < 1$ we are probing distances of this same order since $\rho = r/\sqrt{\alpha' g_s N}$. This means that predictions in the deep IR of the gauge theory, not taking into account these states, should be taken with some care.

- Let us try to be a bit more specific on the point above. As a common feature of non-conformal gauge/gravity dualities, it turns out that the KK states found above cannot be decoupled from gauge theory states, within the supergravity regime. A way to see this is as follows. Using some tools inherited from the AdS/CFT correspondence (as such this should be done with some care and the results we get should be taken just as qualitative ones), we can give an estimate of the glueball mass spectrum as predicted from the MN background. We consider here the simplest possible case. The scalar glueball 0^{++} (0 refers to the spin while the two upper indexes refer to the parity and charge conjugation quantum numbers, respectively) couples to $\mathrm{Tr}F^2$, this operator being dual to the dilaton field. Hence, using the AdS/CFT prescription, the masses of such glueballs are in correspondence to the eigenvalues of the dilaton bulk equation

$$\partial_\mu \left(\sqrt{-\mathrm{det}G}\, G^{\mu\nu} \partial_\nu \Phi \right) = 0 \tag{3.53}$$

Computing the above equation for small ρ, this being the region where confining effects occur, we can forget about the S^2, which shrink to zero size in the MN solution, and expand the dilaton field in KK modes over the S^3, on which the dilaton in fact does not depend. This simplifies the above equation considerably (i.e. the indexes μ, ν run just along $0, 1, 2, 3$ and ρ) and expanding Φ in plane waves, $\Phi = \Phi(\rho)\, e^{ikx}$ (where x stands for x_0, x_1, x_2, x_3), so that a mode of momentum k has a mass $m^2 = -k^2$ in four dimensions, one easily finds

$$M_{gb}^2 \sim \frac{1}{\alpha' g_s N} \tag{3.54}$$

which says the glueball masses are of the same order of the KK masses we computed before, see eq. (3.52)! This shows (in a qualitative way, of course) the mixing phenomenon we already mentioned and makes it more evident why one should go beyond the supergravity approximation if looking for an exact duality. For instance, it can be that *part* of the non-perturbative contributions to the β-function we have found are in fact coming from these non gauge theory degrees of freedom.

- Using again some tools inherited from the AdS/CFT correspondence, we can also give an estimate of the confining string tension. The Nambu-Goto action of a fundamental string in the MN background is

$$S = \frac{1}{2\pi\alpha'} \int dx\, dt \sqrt{f(\rho)^2 + g(\rho)^2 (\frac{d\rho}{dx})^2} = \int dt\, E \tag{3.55}$$

where $f(\rho)^2 = G_{tt}G_{xx}$ and $g(\rho)^2 = G_{tt}G_{\rho\rho}$. From eq. (3.10) we get

$$f(\rho)^2 = \frac{\sinh 2\rho}{2\, e^{h(\rho)}} \quad , \quad g(\rho)^2 = \alpha' g_s N\, f(\rho)^2 \tag{3.56}$$

The area law is a consequence of the fact that $f(\rho)$ has a minimum at $\rho = 0$ so a fundamental string prefers to lie on the hyper-surface $\rho = 0$. From the above action one can read-off the string tension. The energy reads

$$E = \frac{1}{2\pi\alpha'} f(0)\, L + \text{corrections} \tag{3.57}$$

Since $f(0) = 1$ one gets for the tension T_s of the confining string

$$T_s = \frac{1}{2\pi\alpha'} \tag{3.58}$$

which equals the fundamental string tension. Recalling eq. (3.52), this means that $M^2_{KK} \sim T_s/g_sN$ which shows again that in order to keep the KK states decoupled from the gauge theory (the string tension sets the scale for gauge theory states), one should require $g_sN << 1$, which is the opposite regime for the validity of the supergravity approximation. This is a further indication that a complete string analysis would be needed in order to distil an exact duality.

- The MN solution has some isometries, as for instance shifts in ϕ_1 and ϕ_2, which do not have a gauge theory interpretation, as opposite to the case of the $\mathrm{AdS}_5 \times S^5$ correspondence where the isometries of the solution are in one-to-one correspondence with the R-symmetry group of the dual gauge theory. The only global (and broken) symmetry that has a dual interpretation in the present case is that in ψ, which is related to the $U(1)$ R-symmetry of $\mathcal{N} = 1$ SYM. Is this weird abundance of symmetries related to the fact that to find an exact duality we should go to the string regime? Are these extra isometries related to symmetries of the KK spectrum and should not be there in the end? If we believe an exact duality is there, this should probably be the case. Unfortunately, we do not have a definitive answer to this problem, yet.

This is all we wanted to say about $\mathcal{N} = 1$ SYM using the MN dual. In our last lecture we are going to reconsider the whole picture of non-conformal versions of the gauge/gravity correspondence and briefly discuss the connections between (some of) the different approaches that have been used in the literature. Some of the open problems we have discussed both in this lecture as well as in the previous one in establishing an exact duality between supergravity (or better, string theory) and gauge theories, may found easier answers using different approaches. What follows is just an overview aiming to let the reader having a rough idea on how alternative paths can be pursued.

4 Lecture IV - Dualities and connections to other approaches

This last lecture is a brief overview on the other approaches that have been used recently to construct four-dimensional supersymmetric gauge theories by means of string (or M) theory and which are related, in a way or another, to the ones we have been discussed so far. What follows is nothing more than a bird-eyes view. That is to say, if you are looking for explicit computations, skip this, and go directly to the reference list.

4.1 Branes Suspended Between Branes

Let us suppose to be in ten-dimensional flat space and to have two parallel NS5-branes extending along directions $x_0, ..., x_5$ but which are at a finite distance L in one of the transverse directions, say x_6 . Suppose this direction is compact, with radius R. Let us now stretch N parallel D4-branes between the two NS5-branes, with world volume directions along $x_0, ..., x_3$ and x_6. At weak string coupling the NS5-branes are much heavier than the D4-branes (there is a factor $1/g_s$ between the tension of NS5-branes and D-branes) and therefore at low energy, much lower than the inverse of L, the effective theory of this system is simply the gauge theory living on the non-compact part of the D4-brane world volume, which is actually four-dimensional. This configuration is depicted in figure 3.

Perform now a T-duality along the compact direction along which the D4-branes are stretched. What happens? This is something interesting. It turns out that the T-dual configuration is a bunch of N fractional D3-branes at the singularity of a $\mathbb{C}_2/\mathbb{Z}_2$ orbifold, the configuration we have been studied in our second lecture! This can be proved by implementing T-duality rules correctly, but we do not do it here. Let us just say what the crucial point is: the T-dual of NS-branes in flat space are orbifold-like singularities. Starting from a configuration of NS5-branes as that depicted in figure 3 (forget for a while about the D4-branes) and making a T-duality along x_6 one ends up in the orbifold $\mathbb{C}_2/\mathbb{Z}_2$ and no branes left. On the other hand, it is rather easy to understand that under T-duality D4-branes become D3-branes. The fact that these D4-branes are peculiar, i.e.

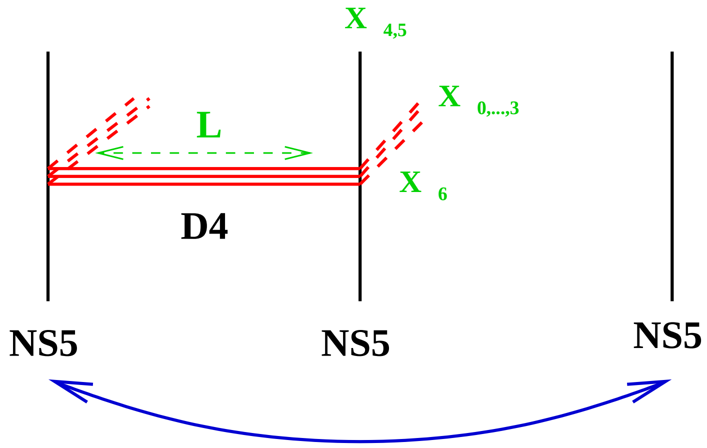

Figure 3: A configuration of D4-branes stretched between two parallel NS5-branes. The x_6 direction is compact, this meaning that the first and the third NS5-brane in the picture are identified. One can have as well D4-branes stretched from the second NS5-brane two the third. These would couple to the NS5-brane world volume fields with equal strength as the former, but with opposite sign.

they have a finite extension along the direction the T-duality is performed, makes the resulting D3-branes being equally peculiar, i.e. fractional branes.

Everything we learned in our second lecture can be translated into this branes-suspended-between-branes picture language. Geometrically, at least, this can be convenient. To start with, it is clear what the twisted fields really are: they are the world volume fields of the NS5-branes. This is why they have six-dimensional dynamics, only. The $B_{(2)}$-flux along the vanishing two-cycle, which is so relevant in the gauge/gravity dictionary, is related to the distance L between the NS5-branes. This should not be surprising, too. Under a T-duality we know that $B_{(2)}$ components and metric components get exchanged and this is what happens here. In order for string theory to be well-behaved we know that the background value of the $B_{(2)}$-flux should be equal to $1/2$, in string units (see eq. (2.20) and the subsequent discussion). This implies that in the T-dual configuration, figure 3, L should be just half the length of the compact direction x_6. The enhançon mechanism has a simple geometrical interpretation in this T-dual set-up. Starting from the configuration depicted in figure 3 and switching-on interactions, the D4-branes tend to pull the NS5-branes, bending them. The final stable configuration, once interactions are take into account, is then described by NS5-branes which are bent along the x_6 direction, i.e. L is a function of ρ. At a distance $\rho = \rho_E$ (recall ρ is the radial distance in the x_4, x_5 plane which is transverse to the D4-branes but lyes on the world volume of the NS5-branes) $L \to 0$ and the branes touch. The new light fields come from world volume field supermultiplets becoming massless and charged under both the NS5-branes. Recall that the world volume theory of NS5-branes of type IIA is described by (2,0) theory and we have scalars and self-dual two-forms (plus fermions) as light fields, so we cannot speak of an enhancement of gauge symmetry in this case, consistently with what we found in the fractional D3-brane scenario. At the enhançon the solution should be described by a 1/2 BPS A_1 (2,0) theory (A_1 is the first element of the A series of Dynkin diagrams and is related to the fact that we have two NS5-branes in this case). But this is hard to do explicitly, in practice.

We have learned that there are two different types of fractional D3-branes on the orbifold $\mathbb{C}_2/\mathbb{Z}_2$ and we described here the T-dual of just one type. What about the other? This is just a D4-brane stretched from the second NS5-branes to the third one (recall that x_6 is periodic). In this T-dual picture it is easy to see the opposite coupling to the twisted fields the two types of fractional branes have: since the corresponding D4-branes end on opposite sides of the NS5-branes, the coupling has of course opposite sign. A bound state of the two different D4-branes corresponds to a D4-brane which is not tied anymore to the NS5-branes (this can be seen as the coupling to the NS5-branes world volume fields cancel). This brane does not end on the NS5-branes and therefore can move freely in the transverse space. This is nothing but the T-dual description of a regular D3-brane, of course. Note how in this picture it is straightforward to understand why fractional branes of the $\mathbb{C}_2/\mathbb{Z}_2$ orbifold have half the tension (and charge) of regular ones: the tension of (both type of) fractional D3-branes is $T_3^f = T_4\, L$ (recall L is half the length of the compact direction), T_4 being the tension of the D4-brane. The tension of a regular D3-brane is instead $T_3 = 2\, T_4 L$, which is twice T_3^f. The T-dual of the fractional D(-1)-branes, which we learned are related to the instantons of the gauge theory, are now Euclidean D0-branes stretched between the NS5-branes (it is known that D0-branes are instantons within D4-branes, in fact).

All what we have been saying above is general. One can consider k parallel NS5-branes and in this case, after T-duality, one ends up with the orbifold $\mathbb{C}_2/\mathbb{Z}_k$, the rank of the orbifold being related to the number of NS5-branes. Similarly, one can start from a type IIB configuration of N D3-branes stretched between a couple of NS5-branes. Under T-duality this is the system of fractional D2-branes on $\mathbb{C}_2/\mathbb{Z}_2$ we discussed at the end of our second lecture which couples non-trivially to the twisted vector field $A_{(1)}$ and which describes three-dimensional SYM with eight supercharges. In this case we really have an enhancement of gauge symmetry at the enhançon, see (2.55). Indeed, the six-dimensional theory living on the NS5-branes of type IIB is $(1,1)$ theory (this is the low energy theory of the D1-branes living on them) and is made of vectors and scalars (plus fermions). Once the NS5-branes touch, the $U(1)$ gauge symmetry related to the vector field the D3-branes couple to, gets enhanced to $SU(2)$. This is exactly the same phenomenon one encounters when putting usual D-branes on top of each other: the gauge theory describing their dynamics at low energy gets enhanced.

One can also construct configurations preserving four supercharges, only, by just tilting the NS5-branes. This breaks further supersymmetry and represent the T-dual version of fractional branes on singular version (orbifolds or conifolds) of CY three-folds.

One can build-up a zoology of gauge theories playing with these basic tools, and this approach has been widely used in the past years to engineer the most different kind of gauge theories in terms of brane intersections. Here, we just pointed out what was useful in order to get the connection with the fractional brane construction.

In fact, there is another piece of information that might be interesting for us. Start again from the type IIA configuration of figure 3 and perform now a T-duality along a transverse direction which is transverse *both* to the NS5-branes and to the D4-branes. In this case one does not generate an orbifold anymore but the corresponding smooth ALE space (the Eguchi-Hanson space, in this case). The D4-branes become D5-branes (now the T-duality has been performed transverse to their world volume so this is again reasonable) wrapped on a topologically non-trivial two-cycle inside the ALE space. This is nothing but the ($\mathcal{N}=2$ version of the) configuration we studied in the last lecture: a wrapped brane on a CY manifold! So, we see that through the brane-suspended-between-branes picture we make contact between fractional and wrapped branes. This also show, once more, that fractional branes can be seen just as a particular kind of wrapped branes (similar conclusions hold for the $\mathcal{N}=1$ case).

4.2 M5-branes Wrapped On Riemann Surfaces

The branes-suspended-between-branes picture is also an intermediate step for embedding fractional branes into M-theory. Indeed, the configuration of figure 3 can be lifted-up in eleven dimensions. It turns out that the all system of NS5-branes and D4-branes becomes a single M5-brane wrapped on a complicated (but smooth) Riemann surfaces

Σ. The singular points of the configuration of figure 3, i.e. the points on the NS5-brane world volumes where the D4-branes end, get smoothed out by the D4-brane becoming an M5, as they get extended in the eleventh direction, x_{10}.

The remarkable point is that the condition for preserving $\mathcal{N} = 2$ supersymmetry restricts the embedding of the M5-branes world volume, and the function describing this embedding are precisely the corresponding Seiberg-Witten curves (the instantons are automatically taken care of as D0-branes are simply KK momentum modes in M-theory)! So the M-theory way to realize $\mathcal{N} = 2$ SYM is where the appearance of the quantum moduli space of the theory shows-up more naturally, and is therefore in principle the best place where to study the quantum property of supersymmetric gauge theories in terms of supergravity duals. On the other hand, we know little about the very structure of M-theory and is therefore difficult to pursue such a program.

Putting together all what we have been learning so far, we can extract a duality-web, which we summarize in figure 4.

All what we have been saying, both in the branes-suspended-between-branes context as well as in the present one, can be extended to cases with four supercharges (i.e. $\mathcal{N} = 1$ in four dimensions) and a similar duality web as the one of figure 4 can be drawn. As already noticed, by tilting or rotating the NS5-brane with respect to each other one can construct T-duals of fractional branes on $\mathcal{N} = 1$ conifold and orbifolds respectively, and the M-theory picture of that correspond to M5-branes wrapped on suitable supersymmetric two-cycles.

4.3 Fractional Branes on Conifolds

Another notable example of a supergravity dual of $\mathcal{N} = 1$ SYM theory was obtained by considering D-branes on the conifold, i.e. a CY three-fold whose topology is that of a cone on $T^{1,1}$ (recall that $T^{1,1} \sim S^2 \times S^3$). This is a singular manifold since at the apex both the two-sphere and the three-sphere shrink to zero size. By considering a bound state of N D3-branes and M D5-branes wrapped on the two-sphere (which are nothing but fractional D3-branes, recall lecture two), Klebanov and Strassler (KS) were able to find the corresponding smooth supergravity solution. This is expected to be dual, in the same sense of what we have been doing in the previous lectures, to the gauge theory living on the world volume of these bound state of D-branes. This is an $\mathcal{N} = 1$ SYM theory with gauge group $SU(N + M) \times SU(N)$, matter in the bi-fundamental and a

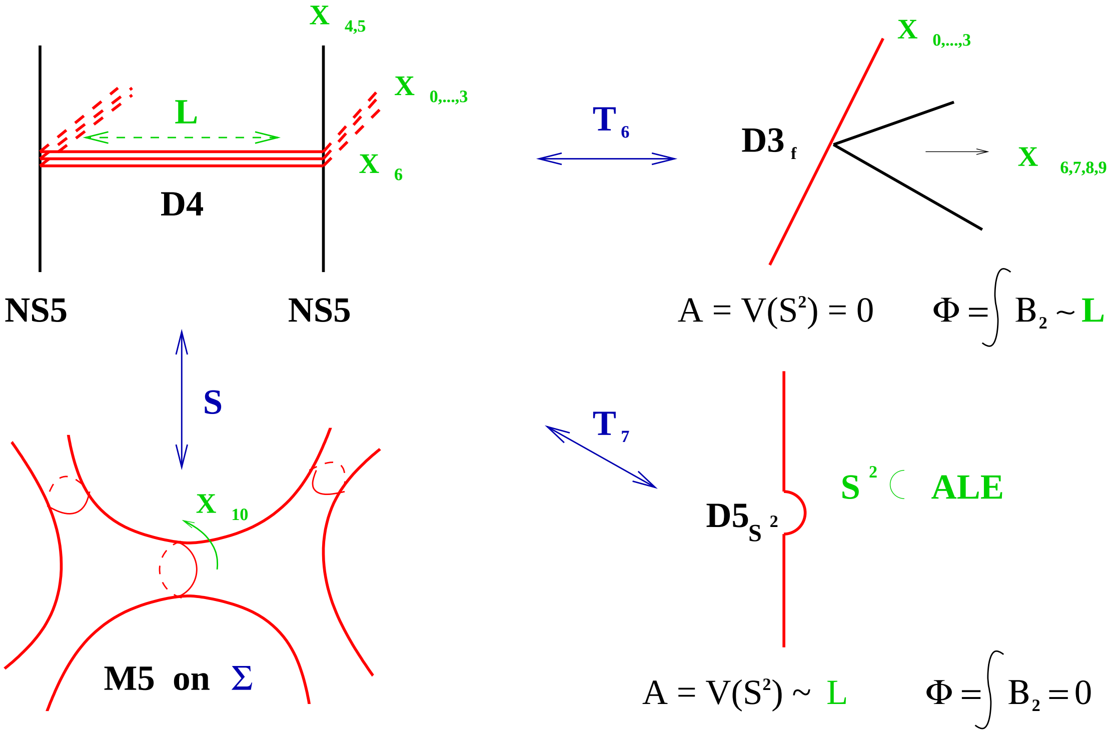

Figure 4: The $\mathcal{N} = 2$ duality web. On the left hand side up we have D4-branes stretched between two NS5-branes. By performing a T-duality along x_6 we get fractional D3-branes on the orbifold $\mathbb{C}_2/\mathbb{Z}_2$. The distance L between the NS5-branes translates into the (background) value of the $B_{(2)}$ flux. Performing a T-duality along x_7, instead, we get D5-branes wrapped on a two-sphere inside an A_1 ALE space. The distance L between the NS5-branes becomes now the (background) value of the volume of the two-sphere. Finally, performing a S-duality (left hand side down) we end up in M-theory with a M5-brane wrapped on a Riemann surface. All these different configurations describe the same physics at low energy: four-dimensional $\mathcal{N} = 2$ SYM.

quartic superpotential. There is a lot interesting physics in here, but at first sight we do not see how pure $\mathcal{N} = 1$ SYM could come about. In fact, it turns out that through a chain of Seiberg dualities dubbed "duality cascade" the above theory flows to pure $SU(M)$ SYM in the IR. Indeed, studying the KS solution in the corresponding dual region, i.e. near the tip of the cone, most of the results we obtained in the last lecture about $\mathcal{N} = 1$ SYM using the MN solution, can be similarly obtained here.

There are a couple of things that may be worth pointing out, though.

As in the IR both the MN and KS solution are expected to be dual to the same theory, they could not look so different, geometrically. Indeed, they do not. As it is the case for the MN solution, the KS solution realizes a geometric transition, too. The metric, in the deep IR, is nothing but that of a deformed conifold, similarly to what we found for the MN solution (in this case this statement is even more precise as the topology of the six-dimensional transverse space is that of a cone over $T^{1,1}$ for *any* value of ρ). Starting from a singular manifold, with a shrinking two-cycle at the apex, and putting fractional branes there, one ends up with a smooth space, where a three-cycle has blown-up. At $\rho = 0$ the metric is exactly equal to eq. (3.20). As it was the case for the MN solution, the deformation of the conifold is related to the IR physics of the gauge theory, more precisely to the phenomenon of gaugino condensation.

Another interesting feature of the KS model is that it can be obtained as a deformation of a $\mathcal{N} = 2$ model. Let us start by considering N regular D3 branes on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. The gauge theory is a superconformal $\mathcal{N} = 2$ theory with gauge group $SU(N) \times SU(N)$ and the correspoding supergravity dual has an AdS$_5$ $\times$ $S^5/\mathbb{Z}_2$ geometry. Adding a relevant perturbation to the superpotential of the $\mathcal{N} = 2$ theory and integrating out the adjoint scalars, one breaks $\mathcal{N} = 2$ to $\mathcal{N} = 1$ and a quartic superpotential is generated. From the geometric point of view this operation corresponds to blow-up the fixed circle of the S^5, ending up with the conifold on $T^{(1,1)}$. Adding M fractional D3 branes, one can repeat the same reasoning and ends up with the non-conformal $\mathcal{N} = 1$ SYM theory we described above.

4.4 Dualities As Geometric Transitions

The gauge/gravity dualities we have been discussed so far can be rephrased in a more geometrical language. This is sometime called the gauge/geometry correspondence and its formulation is mainly due to work of Vafa with different collaborators in the last few years.

The original duality was obtained embedding in string theory (in fact, type IIA string theory) a previously discovered large N duality between Chern-Simon theory and topological closed string theory. Upon mirror symmetry it was then possible to obtain a similar duality in type IIB string theory. These example were further generalized by subsequent works and it turns out a large class of $\mathcal{N} = 1$ SYM theory dualities can be casted in this geometric way.

In what follows we refer to the type IIB version of the duality since this is where comparison with our findings are straightforward. The general idea is to engineer an $\mathcal{N} = 1$

supersymmetric gauge theory by means of D5-branes wrapped on a topologically non-trivial two-cycle of some CY space. In the corresponding dual closed string realization, the geometry goes through a geometric transition: the two-cycle shrinks to zero size and a three-cycle blows-up. The geometry is completely smooth, no singularities are there, i.e. the D-branes have disappeared. They get replaced by NSNS and RR fluxes. Most of the gauge theory quantities, as most notably the effective superpotential, can be obtained by integrating these fluxes, as well as other geometric quantities, on suitable cycles inside the CY.

Another feature of the $\mathcal{N}=1$ theories considered in this construction is that they can all be seen as deformations of $\mathcal{N}=2$ SYM. One starts with N D5-branes wrapped on a supersymmetric two-cycle in a CY two-fold, say a (non-compact) $K3$ manifold. The geometry of the ten-dimensional space-time is then $\mathbb{R}^{1,3}\times K3\times\mathbb{C}$ and the gauge theory living on the D5-branes is four-dimensional $\mathcal{N}=2$ gauge theory (at energies where we can forget KK excitations on the two-cycle). Suppose now to add a scalar superpotential $W(\phi)$ (ϕ being the complex scalar field of the $\mathcal{N}=2$ vector multiplet) which breaks $\mathcal{N}=2$ to $\mathcal{N}=1$. From the target space geometry point of view, it turns out this corresponds to make the direct product $K3\times\mathbb{C}$ a non-trivial fibration, and the space a proper Calabi-Yau three-fold with $SU(3)$ holonomy, ending up with the configuration described above. This means all these $\mathcal{N}=1$ theories are strict relatives of the $\mathcal{N}=2$ theory, and this is something which makes the study of their properties more manageable.

All what we have been saying should alert the reader: there is more than one thing pointing to the configurations we have discussed, the MN and the KS models. Strictly speaking none of them is really a Vafa model (although the KS solution is pretty near to it), but both of them do realize a geometric transition, as we have already discussed: both the MN and the KS brane configurations modify the original CY space into a deformed one, and branes get replaced by fluxes (in fact, it is quite a general feature of all these gauge/gravity duals to realize a geometric transition). Moreover, the gaugino condensate is related to the deformation parameter, this also being the case for Vafa models. There are also some differences, especially in the MN solution. For one thing, we have a dilaton there, which is absent in the Vafa model, and we do not know the exact expression for the equations giving the superpotential for cases with running dilaton. Moreover, in the MN solution we do not have NSNS three-form flux but only RR one. On the other hand, the KS solution has much in common with the general picture just discussed. It realizes a geometric transition. It has non-trivial RR and NSNS three-form fluxes. It has trivial dilaton. It has a superpotential, which can be seen as generated deforming a $\mathcal{N} = 2$ SYM theory, this corresponding, from the geometry point of view, to start from $\mathbb{C}_2/\mathbb{Z}_2$ and partially resolve the orbifold singularity into a CY three-fold (a conifold in this case). On the other hand, the starting point of the KS configuration is different, i.e. a singular conifold and not a resolved one. Moreover, the UV completion of the KS solution is a chain of Seiberg dualities and the gauge theory remains four-dimensional all way long (this is typical for any kind of fractional brane as the cycles they are wrapped on are geometrically vanishing). In Vafa models, instead, at very high energy one start seeing the KK states on the two-cycle the brane are wrapped on, therefore far in the UV the theory becomes six-dimensional (this same thing happens for the MN solution). In fact, all these theories have in general different UV completions but are equivalent in the IR.

We cannot discuss these interesting issues further, here. Still, it should be already surprising what we have learnt so far. The lesson is that all these apparently different approaches are related to each other: fractional branes on orbifolds, fractional branes on conifolds, M5-branes wrapped on Riemann surfaces, D-branes wrapped on supersymmetric cycles of CY spaces, geometric transitions, etc... seem to be part of a unique picture. Each approach has of course its advantages and disadvantages, but having an understanding and a control on all of them could considerably increase the chances of being able to solve, one day, the still open problems lying on the carpet. This would not only mean to eventually distil an exact duality for non-conformal supersymmetric (and may be non-supersymmetric) Yang-Mills theories, but also to have a chance to understand more deeply the ultimate structure of the underlying microscopic theory.

Actually, before we end, there is one more topic we would like to discuss. And this hides some more (and conceptually remarkable) surprises.

4.5 M-theory On G_2-holonomy Manifolds

The study of compactifications of Heterotic string on CY three-folds turned out to be a very useful way to study $\mathcal{N} = 1$ supersymmetric theories in four dimensions by means of string theory. There is an M-theory analogue of that and amounts to consider M-theory on G_2-holonomy manifolds. These are seven-dimensional manifolds with G_2 holonomy. They preserve 1/8 supercharges and therefore, when considering M-theory propagating on these spaces, one obtains four-dimensional $\mathcal{N} = 1$ supersymmetric theories at low energy.

An obstruction to find phenomenologically interesting four-dimensional models out of M-theory on G_2-holonomy manifolds is that one cannot get chiral fermions in four dimensions if the manifold is smooth. This problem can be overcome taking a singular G_2-manifold where some three-cycle has shrunk to zero size. M-theory behaves smoothly on these spaces since the effective volume gets complexified as we have a three-form potential in the spectrum (this has the same rôle played by the NSNS two-form potential when studying ten-dimensional string theory on singular spaces). In this case, at very low energy, one can show the theory is described solely by gauge degrees of freedom near the singularity (i.e. gravitational modes are decoupled) and one can indeed get chiral matter. The nature of the singularity dictates the structure of the four-dimensional gauge theory (gauge group, matter content, etc...) so in principle one can study a plethora of $\mathcal{N} = 1$ gauge theories by means of different G_2-holonomy spaces. Note that branes are not present in this construction.

So far, so good. But where is the connection with what we have been discussing before? It turns out the connection is there, and is rather profound. It has been shown that the gauge/geometry duality that Vafa originally studied in type IIA theory, given in terms of D6-branes wrapped on a three-cycle of the deformed conifold which are dual to the resolved conifold with fluxes, can be up-lifted in M-theory. What one ends up with is just M-theory on two different points in the moduli space of a G_2-manifold! Moreover no branes are present on either sides of the correspondence (this is not too surprising though, as D6-branes are purely gravitational objects in M-theory, so, once the embedding is performed, it is natural ending up with something where branes do not appear). The D6-brane/deformed conifold (gauge theory) side of the duality corresponds to a singular G_2-manifold while the fluxes/resolved conifold (gravity) side corresponds to a smooth G_2-manifold. These two manifolds are related by a so-called flop transition. So, the duality between the $\mathcal{N} = 1$ SYM theory and supergravity gets reformulated as a purely geometric transition when studying M-theory propagating on a G_2-holonomy manifold.

The remarkable thing is that it turns out the moduli space is smoothly connected, i.e. there is no phase transition at the quantum level in going from the smooth to the singular points mentioned above. This means that the equivalence between the two descriptions (the gauge theory description and the supergravity description) is guaranteed as this is the "same" physics from the M-theory point of view. This may suggest the idea that the gauge/gravity duality could be really proved in M-theory. Once again, M-theory seems to be the right place where to look for answers to fundamental questions. However, in order to make this more concrete, we need a deeper understanding of M-theory of what we presently have.

5 A (Biased And Incomplete) Guided Tour Through The Literature

Let me finally give some references on the material presented in these lectures. The literature on the subject is endless and I cannot provide an exhaustive list here. I just mention some of the papers I think could help the reader in deepening the issues I discussed. Essentially, this amounts to list those papers I found useful myself in preparing the material presented in this course. Often, these papers do not even coincide with the original works. This is why what follows is biased and incomplete.

A comprehensive review on the AdS/CFT correspondence is [1] while an updated discussion of the conceptual problems in finding exact dualities for non-conformal theories can be found in [2], which is a recent review on similar topics as those presented in these lectures.

- *Fractional Branes*

The first work where the supergravity solution for fractional branes was presented and the corresponding gauge dual discussed is [3] (see also [4]). This dealt with the case of fractional D3-branes on the most simple orbifold, $\mathbb{C}_2/\mathbb{Z}_2$. The corresponding compact case was discussed in [5] while in [6] the analysis was extended to orbifolds of the full ADE series. Addition of D7-branes was considered in [7] and [8], this giving $\mathcal{N} = 2$ SYM coupled to fundamental matter. Cases with $\mathcal{N} = 1$ supersymmetry were discussed in [9] and extended to include fundamental matter in [10]. Finally, in [11] a complete treatment on the correspondence between the $U(1)_R$ anomaly and the supergravity duals was given (see also [12] for the related case of fractional branes on conifolds). The work [13] is a (partial) review which has some overlap with our second lecture. I say partial because it includes theories with 8 supercharges, only, and misses some findings which were achieved later. Still, it can be rather useful as various computational aspects (the derivation of the string spectrum, the solution of the equations of motion, the condition of supersymmetry preservation, etc...) are given in great detail.

Fractional branes were introduced long ago in [14] and [15] (see also the more recent and very useful [16]). Looking to the citation list of these papers one can find much more fractional branes literature. Let me just mention the two original papers dealing with strings and D-branes on orbifolds, [17] and [18], discussing orbifolds of the A and of the DE series, respectively, and [19], where first investigations on the rôle played by fractional branes in the gauge/gravity correspondence were done.

The enhançon phenomenon was discovered in [20] and discussed much further in many contexts. To our purpose, let us just mention three papers. In [21] and [22] (for the case of wrapped and fractional branes, respectively) the excision criteria, outlined in [20], was systematically studied and proved to be consistent at the pure supergravity level, while [23] is the only paper, to our acknowledge, where an honest supergravity attempt to solve the enhançon, by including the new light degrees of freedom, was pursued. This was done for the case of three-dimensional supersymmetric gauge theory, though. Finally, in the nice work [24] the relation between the enhançon, the running of the five-form flux and the Seiberg-Witten curve as predicted by the dual SYM theory was discussed, showing agreement between supergravity, the enhançon picture and the field theory expectations (this was done exploiting the $T \times S$-dual picture of M5-branes wrapped on Riemann surfaces, where Seiberg-Witten curves naturally arise).

Fractional branes on conifolds is a subject that was pioneered by I. Klebanov with a number of collaborators during the last few years. The original paper, containing the smooth supergravity solution, is [25] , which elaborated on some previous works, [26], [27] and [28] (another interesting paper discussing the basis of D-branes on conifolds is [29]). A recent, complete and very nice review on this subject is [30], where much more references can be found. The complete $\mathcal{N} = 1$ β-function for the case of fractional branes on the conifold was found in [31], after the above mentioned review appeared on the archive.

- *Wrapped Branes*

Let us start mentioning those papers discussing (and refining) the proposal originally made by Maldacena and Nuñez which is relevant to study pure $\mathcal{N} = 1$ SYM in four dimensions. The first paper is of course [32], where the supergravity solution was found (see also the preceding work [33] for a discussion of the gauged supergravity approach to curved branes supergravity solutions). This solution was obtained up-lifting in ten dimensions a previously found non-abelian monopole solution in four dimensions [34].

In [35] it was first proposed the identification between the supergravity field $a(\rho)$ and the gaugino condensate and some AdS/CFT arguments were given in favor of this identification. In [36] this identification was used to extract the radius/energy relation and the exact β-function was found (see however [37]). Finally, in [38], a refinement on the gauge/gravity dictionary was provided, pint-pointing the correct two-cycle to be used in the correspondence and solving a number of problems present in previous works. The third lecture of this course (which also includes some unpublished material) aims to be a self-contained and updated treatment of the MN solution and its gauge dual.

A useful reference making some interesting qualitative remarks on $\mathcal{N}=1$ duals is [39]. The $\mathcal{N}=2$ analogue of the MN solution, discussing D5-branes wrapped on a two-cycle inside a K3 manifold (this is the first twist we discussed) was provided in [40] and [41]. Some recent remarks on radius/energy relations for $\mathcal{N}=2$ and $\mathcal{N}=1$ can be found in [42] and [43], respectively.

The case of three-dimensional supersymmetric gauge theory with 8 supercharges was discussed in [44] and that of 4 supercharges in [45] (see also [46]). Other examples of wrapped brane solutions and their gauge duals can be found for instance in [47], [48] and many others.

Some useful technical details about the geometry of $T^{1,1}$ manifolds (coordinate transformation, identification of cycles, etc...) can be found, for instance, in [49], [50] and [27], this being relevant for studying both the MN as well as the KS solutions. Finally, in [51] the up-lift formulæ we used have been derived and their consistency checked. For works discussing general methods for obtaining smooth supergravity duals of non-maximally supersymmetric gauge theories, see for instance [52].

- *Other Approaches*

Let me end this brief tour giving some references on the other approaches we discussed in the last lecture (this will be even more incomplete and aims just to open a little crack on a big world).

The study of four-dimensional supersymmetric gauge theories using configuration of branes suspended between branes was initiated by Witten [53] and [54], elaborating on some previous work [55]. Some nice works where extensions are considered and also relations to fractional branes both on orbifolds and on conifolds are investigated are for instance [56], [57] and [58]. They can be a good starting point for studying these things. Looking to the citation list of these works, much more papers can be found.

The same paper of Witten [53] is where these NS5/D-branes configurations was lifted up in M-theory and the study of M5 wrapped on Riemann surfaces was initiated. Let me also mention the works [59] and [60] where concrete attempts to work out the corresponding (eleven-dimensional) supergravity/gauge theory duality, for $\mathcal{N} = 2$ and $\mathcal{N} = 1$ cases respectively, were done.

The use G_2-holonomy manifolds to study four-dimensional SYM theories was somehow initiated in [61] and received a boost in [62] and [63], where it was shown the connection with the geometric transition picture (or more generally with the D-brane engineer/geometry-with-fluxes duality of ten-dimensional string theory). In the work [64] it was shown the smooth interpolation between the different vacua (where SYM and supergravity live at low energy, respectively) of M-theory on G_2-holonomy manifolds. This is a crucial property in order to re-interpret the gauge/gravity (geometry) duality of ten-dimensional superstring theory in purely geometric terms in M-theory. There have been many papers discussing various aspects of M-theory on G_2-holonomy manifolds in the last few years. Two reviews, straightening complementary aspects of this subject, are [65] and [66], and much more references can be found there.

Finally, let us come to geometric transitions. This idea and in particular how gauge/gravity dualities can be thought of from a more geometrical point of view is due to work done by Vafa with different collaborators during the last few years. The first paper, where the type IIA superstring version of the conjecture appeared (this is where the up-lift from the pure topological Chern-Simon/topological string duality was done) is [67], which also describes the mirror type IIB version of it. There have been many papers after this first. Let me just mention a couple of them, [68] and [69], which generalize the duality discussed in [67] to a large class of matter coupled $\mathcal{N} = 1$ SYM theories.

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