2 The supersymmetry algebra

In this lecture we will introduce the supersymmetry algebra, which is the algebra encoding the set of symmetries a supersymmetric theory should enjoy. The supersymmetry algebra is an extension of the Poincaré algebra so in the first two sections we will start recalling a few facts about Lorentz and Poincaré algebras, the corresponding groups and their representations. In particular, we will emphasize the relation of the Lorentz algebra with the \( SU(2) \) and \( SL(2, \mathbb{C}) \) algebras and define (two component) spinors as the basic representation of the Lorentz group. In the last section we will introduce the concept of graded Lie algebra and, finally, the supersymmetry algebra, which is a specific instance of a graded Lie algebra.

2.1 Lorentz and Poincaré groups

The Lorentz group \( SO(1,3) \) is the subgroup of matrices \( \Lambda \) of \( GL(4, \mathbb{R}) \) with unit determinant, \( \det \Lambda = 1 \), and which satisfy the following relation

\[
\Lambda^T \eta \Lambda = \eta ,
\]

where \( \eta \) is the (mostly minus in our conventions) flat Minkowski metric

\[
\eta_{\mu \nu} = \text{diag}(+, -, -, -) .
\]

The Lorentz group has six generators (associated to space rotations and boosts) enjoying the following commutation relations

\[
[J_i, J_j] = i \epsilon_{ijk} J_k \quad , \quad [J_i, K_j] = i \epsilon_{ijk} K_k \quad , \quad [K_i, K_j] = -i \epsilon_{ijk} J_k .
\]

Notice that while the \( J_i \) are hermitian, the boosts \( K_i \) are anti-hermitian, this being related to the fact that the Lorentz group is non-compact (topologically, the Lorentz group is \( \mathbb{R}^3 \times S^3/\mathbb{Z}_2 \), the non-compact factor corresponding to boosts and the doubly connected \( S^3/\mathbb{Z}_2 \) corresponding to rotations). In order to construct representations of this algebra it is useful to introduce the following complex linear combinations of the generators \( J_i \) and \( K_i \)

\[
J_i^\pm = \frac{1}{2} (J_i \pm i K_i) ,
\]

where now the \( J_i^\pm \) are hermitian. In terms of \( J_i^\pm \) the algebra (2.3) becomes

\[
[J_i^\pm , J_j^\pm] = i \epsilon_{ijk} J_k^\pm \quad , \quad [J_i^\pm , J_j^\mp] = 0 .
\]
This shows that the Lorentz algebra is equivalent to two $SU(2)$ algebras. As we will see later, this simplifies a lot the study of representations of the Lorentz group, which can be organized into (couples of) $SU(2)$ representations. This isomorphism comes from the theory of Lie Algebra which says that at the level of complex algebras

$$SO(4) \simeq SU(2) \times SU(2) .$$

(2.6)

The Lorentz algebra is a specific real form of that of $SO(4)$. This difference can be seen from the defining commutation relations (2.3): for $SO(4)$ one would have had a plus sign on the right hand side of the third such commutation relations. This difference has some consequence when it comes to study representations. In particular, while in Euclidean space all representations are real or pseudoreal, in Minkowski space complex conjugation interchanges the two $SU(2)$'s. This can also be seen at the level of the generators $J_i^\pm$. In order for all rotation and boost parameters to be real, one must take all the $J_i$ and $K_i$ to be imaginary and hence from eq. (2.4) one sees that

$$(J_i^\pm)^* = -J_i^\mp .$$

(2.7)

In terms of algebras, all this discussion can be summarized noticing that for the Lorentz algebra the isomorphism (2.6) changes into

$$SO(1, 3) \simeq SU(2) \times SU(2)^* .$$

(2.8)

For later purpose let us introduce a four-vector notation for the Lorentz generators, in terms of an anti-symmetric tensor $M_{\mu \nu}$ defined as

$$M_{\mu \nu} = -M_{\nu \mu} \quad \text{with} \quad M_{0i} = K_i \quad \text{and} \quad M_{ij} = \epsilon_{ijk} J_k ,$$

(2.9)

where $\mu = 0, 1, 2, 3$. In terms of such matrices, the Lorentz algebra reads

$$[M_{\mu \nu}, M_{\rho \sigma}] = -i\eta_{\mu \rho} M_{\nu \sigma} - i\eta_{\nu \sigma} M_{\rho \mu} + i\eta_{\mu \sigma} M_{\nu \rho} + i\eta_{\nu \rho} M_{\mu \sigma} .$$

(2.10)

Another useful relation one should bear in mind is the relation between the Lorentz group and $SL(2, \mathbb{C})$, the group of $2 \times 2$ complex matrices with unit determinant. More precisely, there exists a homomorphism between $SL(2, \mathbb{C})$ and $SO(1, 3)$, which means that for any matrix $A \in SL(2, \mathbb{C})$ there exists an associated Lorentz matrix $\Lambda$, and that

$$\Lambda(A) \Lambda(B) = \Lambda(AB) ,$$

(2.11)
where $A$ and $B$ are $SL(2, \mathbb{C})$ matrices. This can be proved as follows. Lorentz transformations act on four-vectors as

$$x'^\mu = \Lambda^\mu_\nu x^\nu ,$$

(2.12)

where the matrices $\Lambda$’s are a representation of the generators $M_{\mu\nu}$ defined above. Let us introduce $2 \times 2$ matrices $\sigma_\mu$ where $\sigma_0$ is the identity matrix and $\sigma_i$ are the Pauli matrices defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(2.13)

Let us also define the matrices with upper indexes, $\sigma^\mu$, as

$$\sigma^\mu = (\sigma^0, \sigma^i) = (\sigma_0, -\sigma_i) .$$

(2.14)

The matrices $\sigma_\mu$ are a complete set, in the sense that any $2 \times 2$ complex matrix can be written as a linear combination of them. For every four-dimensional vector $x^\mu$ let us construct the $2 \times 2$ complex matrix

$$\rho : x^\mu \rightarrow x^\mu \sigma_\mu = X .$$

(2.15)

The matrix $X$ is hermitian, since the Pauli matrices are hermitian, and has determinant equal to $x^\mu x_\mu$, which is a Lorentz invariant quantity. Therefore, $\rho$ is a map from Minkowski space to $H$, the space of $2 \times 2$ hermitian complex matrices

$$M_4 \xrightarrow{\rho} H .$$

(2.16)

Let us now act on $X$ with a $SL(2, \mathbb{C})$ transformation $A$

$$A : X \rightarrow AXA^\dagger = X' .$$

(2.17)

This transformation preserves the determinant since $\det A = 1$ and also preserves the hermicity of $X$ since

$$X'^\dagger = (AXA^\dagger)^\dagger = AXA^\dagger = AXA^\dagger = X' .$$

(2.18)

Therefore $A$ is a map between $H$ and itself

$$H \xrightarrow{A} H .$$

(2.19)
We finally apply the inverse map \( \rho^{-1} \) to \( X' \) and get a four-vector \( x'^\mu \). The inverse map is defined as

\[
\rho^{-1} = \frac{1}{2} \text{Tr}[\bullet \bar{\sigma}^\mu] \tag{2.20}
\]

(where, as we will later see more rigorously, as a complex \( 2 \times 2 \) matrix \( \bar{\sigma}^\mu \) is the same as \( \sigma_\mu \)). Indeed

\[
\begin{align*}
\rho^{-1} X &= \frac{1}{2} \text{Tr}[X \bar{\sigma}^\mu] = \frac{1}{2} \text{Tr}[x_\nu \sigma^\nu \bar{\sigma}^\mu] = \frac{1}{2} \text{Tr}[\sigma^\nu \bar{\sigma}^\mu] x_\nu = \frac{1}{2} 2 \eta^{\nu\mu} x_\nu = x^\mu . \tag{2.21}
\end{align*}
\]

Assembling everything together we then get a map from Minkowski space into itself via the following chain

\[
\begin{align*}
M_4 & \xrightarrow[\rho]{} H & \xrightarrow[A]{} H & \xrightarrow[\rho^{-1}]{} M_4 \\
x_\nu & \xrightarrow[\rho]{} x_\nu \sigma^\nu & \xrightarrow[A]{} A x_\nu \sigma^\nu A^\dagger & \xrightarrow[\rho^{-1}]{} \frac{1}{2} \text{Tr}[A x_\nu \sigma^\nu A^\dagger \bar{\sigma}^\mu] = x'^\mu \tag{2.22}
\end{align*}
\]

This is nothing but a Lorentz transformation obtained by the \( SL(2, \mathbb{C}) \) transformation \( A \) as

\[
A^\mu_\nu(A) = \frac{1}{2} \text{Tr}[\bar{\sigma}^\mu A \sigma_\nu A^\dagger] . \tag{2.23}
\]

It is now a trivial exercise, provided eq. (2.23), to prove the homomorphism (2.11).

Notice that the relation (2.23) can in principle be inverted, in the sense that for a given \( A \) one can find a corresponding \( A \in SL(2, \mathbb{C}) \). However, the relation is not an isomorphism, since it is double valued. The isomorphism holds between the Lorentz group and \( SL(2, \mathbb{C})/\mathbb{Z}_2 \) (in other words \( SL(2, \mathbb{C}) \) is a double cover of the Lorentz group). This can be seen as follows. Consider the \( 2 \times 2 \) matrix

\[
M(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tag{2.24}
\]

which corresponds to a Lorentz transformation producing a rotation by an angle \( \theta \) about the \( z \)-axis. Taking \( \theta = 2\pi \) which corresponds to the identity in the Lorentz group, one gets \( M = -1 \) which is a non-trivial element of \( SL(2, \mathbb{C}) \). It then follows that the elements of \( SL(2, \mathbb{C}) \) are identified two-by-two under a \( \mathbb{Z}_2 \) transformation in the Lorentz group. Note that this \( \mathbb{Z}_2 \) identification holds also in Euclidean space: at the level of groups \( SU(2) \times SU(2) = Spin(4) \), where \( Spin(4) \) is a double cover of \( SO(4) \) as a group (it has an extra \( \mathbb{Z}_2 \)).

The Poincaré group, \( ISO(1, 3) \), is the Lorentz group augmented by the space-time translation generators \( P_\mu \). In terms of the generators \( P_\mu \) and \( M_{\mu\nu} \), the Poincaré
algebra reads

\[ [P_\mu, P_\nu] = 0 \]
\[ [M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} \quad (2.25) \]
\[ [M_{\mu\nu}, P_\rho] = -i\eta_{\rho\nu}P_\mu + i\eta_{\mu\rho}P_\nu . \]

### 2.2 Spinors and representations of the Lorentz group

We are now ready to discuss representations of the Lorentz group. Thanks to the isomorphism (2.8) they can be easily organized in terms of those of \( SU(2) \) which can be labeled by the spins. In this respect, let us introduce two-component spinors as the objects carrying the basic representations of \( SL(2, \mathbb{C}) \). There exist two such representations. A spinor transforming in the self-representation \( \mathcal{M} \) is a two complex component object

\[ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (2.26) \]

where \( \psi_1 \) and \( \psi_2 \) are complex Grassmann numbers, which transform under a matrix \( \mathcal{M} \in SL(2, \mathbb{C}) \) as

\[ \psi_\alpha \rightarrow \psi'_\alpha = \mathcal{M}_\alpha^\beta \psi_\beta \quad \alpha, \beta = 1, 2 . \quad (2.27) \]

The complex conjugate representation is defined from \( \mathcal{M}^* \), where \( \mathcal{M}^* \) means complex conjugation, as

\[ \bar{\psi}_\dot{\alpha} \rightarrow \bar{\psi}'_{\dot{\alpha}} = \mathcal{M}^*_{\dot{\alpha}}^\dot{\beta} \bar{\psi}_{\dot{\beta}} \quad \dot{\alpha}, \dot{\beta} = 1, 2 . \quad (2.28) \]

These two representations are *not* equivalent, that is it does not exist a matrix \( C \) such that \( \mathcal{M} = C \mathcal{M}^* C^{-1} \).

There are, however, other representations which are equivalent to the former. Let us first introduce the invariant tensor of \( SU(2) \), \( \epsilon_{\alpha\beta} \), and similarly for the other \( SU(2) \), \( \epsilon_{\dot{\alpha}\dot{\beta}} \), which one uses to raise and lower spinorial indexes as well as to construct scalars and higher spin representations by spinor contractions

\[ \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon^\alpha_{\beta} = \epsilon^\dot{\alpha}_{\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \quad (2.29) \]

We can then define

\[ \psi^\alpha = \epsilon^\alpha_{\beta} \psi_\beta , \quad \psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta , \quad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}} , \quad \bar{\psi}^\dot{\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}} . \quad (2.30) \]
The convention here is that adjacent indexes are always contracted putting the epsilon tensor on the left.

Using above conventions one can easily prove that \( \psi'^\alpha = (\mathcal{M}^{-1T})^\alpha_\beta \psi^\beta \). Since \( \mathcal{M}^{-1T} \simeq \mathcal{M} \) (the matrix \( C \) being in fact the epsilon tensor \( \epsilon_{\alpha\beta} \)), it follows that the self-representation \( \bar{\psi}_\alpha \) and the representation \( \psi^\alpha \) are equivalent. A similar story holds for \( \bar{\psi}^\alpha \) which transforms in the representation \( \mathcal{M}^{*^{-1T}} \), that is \( \bar{\psi}'^\alpha = (\mathcal{M}^{*^{-1T}})^{\alpha}_{\beta} \bar{\psi}^\beta \), which is equivalent to the complex conjugate representation \( \bar{\psi}_\alpha \) (the matrix \( C \) connecting \( \mathcal{M}^{*^{-1T}} \) and \( \mathcal{M} \) is now the epsilon tensor \( \epsilon_{\alpha\beta} \)). From our conventions one can easily see that the complex conjugate matrix \((\mathcal{M}^{-1})^*\) (that is, the matrix obtained from \( \mathcal{M}^{-1} \) by taking the complex conjugate of each entry), once expressed in terms of dotted indexes, is not \( \mathcal{M}^{*}_{\alpha\beta} \), but rather \((\mathcal{M}^{*^{-1T}})^{\alpha}_{\beta} \). Finally, lower undotted indexes are row indexes, while upper ones are column indexes. Dotted indexes follow instead the opposite convention. This implies that \((\psi_\alpha)^* = \bar{\psi}^\alpha\), while under hermitian conjugation (which also includes transposition), we have, e.g. \( \bar{\psi}_\alpha = (\psi_\alpha)^1 \), as operator identity.

Due to the homomorphism between \( SL(2, \mathbb{C}) \) and \( SO(1, 3) \), it turns out that the two spinor representations \( \psi_\alpha \) and \( \psi^\alpha \) are representations of the Lorentz group, and, because of the isomorphism (2.8), they can be labeled in terms of \( SU(2) \) representations as

\[
\psi_\alpha \equiv \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad \bar{\psi}^\alpha \equiv \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}.
\]

To understand the identifications above just note that \( \sum_i (J_i^+)^2 \) and \( \sum_i (J_i^-)^2 \) are Casimir of the two \( SU(2) \) algebras (2.5) with eigenvalues \( n(n+1) \) and \( m(m+1) \) with \( n, m = 0, 1, 2, 3, \ldots \) being the eigenvalues of \( J_3^+ \) and \( J_3^- \), respectively. Hence we can indeed label the representations of the Lorentz group by pairs \((n, m)\) and since \( J_3 = J_3^+ + J_3^- \) we can identify the spin of the representation as \( n + m \), its dimension being \((2n+1)(2m+1) \). The two spinor representations (2.31) and (2.32) are just the basic such representations. Note, in passing, that the representations \( \psi_\alpha \) and \( \psi^\alpha \) are nothing but the fundamental and anti-fundamental representations of \( SU(2) \). That they are equivalent is specific to \( SU(2) \) and does not hold for \( SU(N) \) with \( N > 2 \).

Recalling that Grassmann variables anticommute (that is \( \psi_1 \chi_2 = -\chi_2 \psi_1, \psi_1 \bar{\chi}_2 = -\bar{\chi}_2 \psi_1 \)),
we can now define a scalar product for spinors as
\[ \psi \chi = \psi^\alpha \chi_\alpha = \psi^\beta \chi_\beta = -\psi_\alpha \chi^\alpha = -\psi_\beta \chi^\beta = \chi \psi \] (2.33)
\[ \bar{\psi} \bar{\chi} = \bar{\psi}_\alpha \bar{\chi}^\alpha = \epsilon_{\alpha \beta} \bar{\psi}_\beta \bar{\chi}^\alpha = -\epsilon_{\alpha \beta} \bar{\psi}_\beta \bar{\chi}^\alpha = -\bar{\psi}^\alpha \bar{\chi}_\alpha = \bar{\chi} \bar{\psi} . \] (2.34)

Under hermitian conjugation we have
\[ (\psi \chi)^\dagger = (\psi^\alpha \chi_\alpha)^\dagger = \chi^\alpha \psi^\alpha = \bar{\chi} \bar{\psi} . \] (2.35)

In our conventions, undotted indexes are contracted from upper left to lower right while dotted indexes from lower left to upper right (this rule does not apply when raising or lowering indexes with the epsilon tensor). Recalling eq. (2.17), namely that under $SL(2, \mathbb{C})$ the matrix $X = x^\mu \sigma_\mu$ transforms as $AXA^\dagger$ and that the index structure of $A$ and $A^\dagger$ is $A_\alpha^\beta$ and $A^\beta_\alpha$, respectively, we see that $\sigma_\mu$ naturally has a dotted and an undotted index and can be contracted with an undotted and a dotted spinor as
\[ \psi \sigma^\mu \bar{\chi} \equiv \psi^\alpha \sigma^\mu_\alpha \chi^\alpha . \] (2.36)

Similarly one can define $\bar{\sigma}^\mu$ as
\[ \bar{\sigma}^\mu_\alpha^\alpha = \epsilon_{\alpha \beta} \epsilon^{\alpha \beta} \sigma^\mu_\beta = (\sigma_0, \sigma_1) , \] (2.37)
and define the product of $\bar{\sigma}^\mu$ with a dotted and an undotted spinor as
\[ \bar{\psi} \bar{\sigma}^\mu \chi \equiv \bar{\psi}_\alpha \bar{\sigma}^\mu_\alpha \chi_\beta . \] (2.38)

A number of useful identities one can prove are
\[ \psi^\alpha \psi^\beta = \frac{1}{2} \epsilon^{\alpha \beta} \psi \psi , \quad (\theta \phi) (\theta \psi) = -\frac{1}{2} (\phi \psi) (\theta \theta) \]
\[ \chi \sigma^\mu \bar{\psi} = -\bar{\psi} \sigma^\mu \chi , \quad \chi \sigma^\mu \bar{\sigma}^\nu \psi = \psi \sigma^\nu \bar{\sigma}^\mu \chi \]
\[ (\chi \sigma^\mu \bar{\psi})^\dagger = \psi \sigma^\mu \bar{\chi}^\dagger , \quad (\chi \sigma^\mu \bar{\sigma}^\nu \psi)^\dagger = \bar{\psi} \sigma^\nu \sigma^\mu \bar{\chi}^\dagger \]
\[ (\theta \psi) (\theta \sigma^\mu \phi) = -\frac{1}{2} (\theta \theta) (\psi \sigma^\mu \phi) , \quad (\bar{\theta} \psi) (\bar{\theta} \sigma^\mu \phi) = -\frac{1}{2} (\bar{\theta} \bar{\theta}) (\bar{\psi} \sigma^\mu \phi) \]
\[ (\phi \psi) \cdot \bar{\sigma}^\alpha = \frac{1}{2} (\phi \sigma^\mu \bar{\chi}) (\psi \sigma_\mu)_\alpha . \] (2.39)

As some people might be more familiar with four component spinor notation, let us close this section by briefly mentioning the connection with Dirac spinors. In the Weyl representation Dirac matrices read
\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \quad \gamma_5 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (2.40)
and a Dirac spinor is

\[ \psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^\dot{\alpha} \end{pmatrix} \]  

implying \( r(\psi) = \begin{pmatrix} 1/2, 0 \\ 0, 1/2 \end{pmatrix} \). \hspace{1cm} (2.41)

This shows that a Dirac spinor carries a reducible representation of the Lorentz algebra. Using this four component spinor notation one sees that

\[ \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} \]  

and \( \begin{pmatrix} 0 \\ \bar{\chi}^\dot{\alpha} \end{pmatrix} \) are Weyl (chiral) spinors, with chirality +1 and −1, respectively. One can easily show that a Majorana spinor (\( \psi^C = \psi \)) is a Dirac spinor such that \( \chi_\alpha = \psi_\alpha \). To prove this, just recall that in four component notation the conjugate Dirac spinor is defined as \( \overline{\psi} = \psi\gamma_0 \) and the charge conjugate is \( \psi^C = C\overline{\psi}^T \) with the charge conjugate matrix in the Weyl representation being

\[ C = \begin{pmatrix} -\epsilon_{\alpha\beta} & 0 \\ 0 & -\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \] \hspace{1cm} (2.43)

Finally, Lorentz generators are

\[ \Sigma^{\mu\nu} = \frac{i}{2} \gamma^{\mu\nu}, \quad \gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{1}{2} \begin{pmatrix} \sigma^\mu\sigma^\nu - \sigma^\nu\sigma^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} \] \hspace{1cm} (2.44)

while the 2-index Pauli matrices are defined as

\[ (\sigma^{\mu\nu})_\alpha^\beta = \frac{1}{4} \left( \sigma^\mu_{\alpha\gamma}(\bar{\sigma}^\nu)^{\gamma\beta} - (\mu \leftrightarrow \nu) \right) \quad , \quad (\bar{\sigma}^{\mu\nu})^\dot{\alpha}\dot{\beta} = \frac{1}{4} \left( (\bar{\sigma}^\mu)^{\dot{\alpha}\gamma}(\bar{\sigma}^\nu)^{\gamma\beta} - (\mu \leftrightarrow \nu) \right) \] \hspace{1cm} (2.45)

From the last equations one then sees that \( i\sigma^{\mu\nu} \) acts as a Lorentz generator on \( \psi_\alpha \)

while \( i\bar{\sigma}^{\mu\nu} \) acts as a Lorentz generator on \( \overline{\psi}^{\dot{\alpha}} \).

2.3 The supersymmetry algebra

As we have already mentioned, a no-go theorem provided by Coleman and Mandula implies that, under certain assumptions (locality, causality, positivity of energy, finiteness of number of particles, etc...), the only possible symmetries of the S-matrix are, besides \( C, P, T \)

- Poincaré symmetries, with generators \( P_\mu, M_{\mu\nu} \)
• An internal symmetry group $G$ with generators $B_l$ being Lorentz scalars and with the structure of a compact semi-simple group times $U(1)$ factors.

The full symmetry algebra hence reads

$$[P_\mu, P_\nu] = 0$$ (2.46)

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\nu}M_{\rho\sigma} - i\eta_{\rho\sigma}M_{\mu\nu} + i\eta_{\mu\rho}M_{\nu\sigma} + i\eta_{\nu\sigma}M_{\mu\rho}$$ (2.47)

$$[M_{\mu\nu}, P_\rho] = -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu$$ (2.48)

$$[B_l, B_m] = if_{lm}^n B_n$$ (2.49)

$$[P_\mu, B_l] = 0$$ (2.50)

$$[M_{\mu\nu}, B_l] = 0,$$ (2.51)

where $f_{lm}^n$ are structure constants and the last two commutation relations simply say that the algebra is the direct sum of the Poincaré algebra and the algebra $G$ spanned by the generators $B_l$, that is

$$ISO(1,3) \times G,$$ (2.52)

at the level of groups.

The Coleman-Mandula theorem can be evaded by weakening one (or more) of its assumptions. The theorem assumes, in particular, that the symmetry algebra involves only commutators but there are not any specific physical requirements for this to be needed. Haag, Lopuszanski and Sohnius generalized the notion of Lie algebra to include algebraic systems involving, in addition to commutators, also anticommutators. This extended Lie algebra goes under the name of graded Lie algebra. Allowing for a graded Lie algebra weakens the Coleman-Mandula theorem enough to allow for supersymmetry, which is nothing but a specific graded Lie algebra.

Let us first define what a graded Lie algebra is. Recall that a Lie algebra is a vector space (over some field, say $\mathbb{R}$ or $\mathbb{C}$) which enjoys an additional composition rule, called product

$$[\ ,
\ ] : L \times L \to L,$$ (2.53)
with the following properties

\[
\begin{align*}
[v_1, v_2] &\in L \\
[v_1, (v_2 + v_3)] &\equiv [v_1, v_2] + [v_1, v_3] \\
[v_1, v_2] &\equiv -[v_2, v_1] \\
[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] &\equiv 0 ,
\end{align*}
\]

where \( v_i \) are elements of the algebra. A graded Lie algebra of grade \( n \) is a vector space

\[
L = \bigoplus_{i=0}^{n} L_i
\]

where \( L_i \) are all vector spaces, and the product

\[
\{ \ , \ \} : L \times L \to L
\]

has the following properties

\[
\begin{align*}
[L_i, L_j] &\in L_{i+j} \mod n + 1 \\
[L_i, L_j] &\equiv -(-1)^{ij}[L_j, L_i] \\
[L_i, [L_j, L_k]] &\equiv (-1)^{ik} + [L_j, [L_k, L_i]](-1)^{ij} + [L_k, [L_i, L_j]](-1)^{jk} = 0 .
\end{align*}
\]

From the first such properties it follows that \( L_0 \) is a Lie algebra while all other \( L_i \)'s with \( i \neq 0 \) are not. The second property is called supersymmetrization while the third one is nothing but the generalization to a graded algebra of the well known Jacobi identity any algebra satisfies.

The supersymmetry algebra is a graded Lie algebra of grade one, namely

\[
L = L_0 \oplus L_1 ,
\]

where \( L_0 \) is the Poincaré algebra and \( L_1 = (Q^I_\alpha , \overline{Q}^I_\dot{\alpha}) \) with \( I = 1, \ldots, N \), where \( Q^I_\alpha , \overline{Q}^I_\dot{\alpha} \) is a set of \( N + N = 2N \) anticommuting fermionic generators transforming in the representations \( (\frac{1}{2}, 0) \) and \( (0, \frac{1}{2}) \) of the Lorentz group, respectively. Haag, Lopuszanski and Sohnius proved that this is the only possible consistent extension of the Poincaré algebra, given the other (very physical) assumptions one would not like to relax of the Coleman-Mandula theorem. For instance, generators with spin higher than one, like those transforming in representation \( (\frac{1}{2}, 1) \), cannot be there.

The generators of \( L_1 \) are spinors and hence they transform non-trivially under the Lorentz group. Therefore, supersymmetry is not an internal symmetry. Rather
it is an extension of Poincaré space-time symmetries. Moreover, acting on bosons, the supersymmetry generators transform them into fermions (and vice versa). Hence, this symmetry naturally mixes radiation with matter.

The supersymmetry algebra, besides the commutators (2.46)-(2.51), contains the following (anti)commutators

\[ [P_\mu, Q^I_{\dot{\alpha}}] = 0 \]  
\[ [P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0 \]  
\[ [M_{\mu\nu}, Q^I_{\alpha}] = i(\sigma_{\mu\nu})_{\beta}^\alpha Q^I_{\beta} \]  
\[ [M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}} \]  
\[ \{Q^I_{\alpha}, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma_{\alpha\beta}^\mu P_\mu \delta^{IJ} \]  
\[ \{Q^I_{\alpha}, Q^J_{\beta}\} = \epsilon_{\alpha\beta} Z^{IJ} \text{ , } Z^{IJ} = -Z^{JI} \]  
\[ \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^* \]  

Let us discuss a bit the above structure.

- Eqs. (2.59) and (2.60) follow from the fact that \( Q^I_{\alpha} \) and \( \bar{Q}^I_{\dot{\alpha}} \) are spinors of the Lorentz group, recall eq.(2.45). From these same equations, recalling that \( M_{12} = J_3 \), one also sees that

\[ [J_3, Q^I_{\alpha}] = \frac{1}{2} Q^I_{\alpha} \text{ , } [J_3, Q^I_{\dot{\beta}}] = -\frac{1}{2} Q^I_{\dot{\beta}} . \]  

Taking the hermitian conjugate of the above relations we get

\[ [J_3, \bar{Q}^I_{\dot{\alpha}}] = -\frac{1}{2} \bar{Q}^I_{\dot{\alpha}} \text{ , } [J_3, \bar{Q}^I_{\dot{\beta}}] = \frac{1}{2} \bar{Q}^I_{\dot{\beta}} \]  

and so we see that \( Q^I_{\alpha} \) and \( \bar{Q}^I_{\dot{\alpha}} \) rise the z-component of the spin by half unit while \( Q^I_{\dot{\beta}} \) and \( \bar{Q}^I_{\dot{\beta}} \) lower it by half unit.

- Eq.(2.61) has a very important implication. First notice that given the transformation properties of \( Q^I_{\alpha} \) and \( \bar{Q}^I_{\dot{\alpha}} \) under Lorentz transformations, their anticommutator should be symmetric under \( I \leftrightarrow J \) and should transform as

\[ \left( \frac{1}{2}, 0 \right) \otimes \left( 0, \frac{1}{2} \right) = \left( \frac{1}{2}, \frac{1}{2} \right) . \]
The obvious such candidate is $P_\mu$ which is the only generator in the algebra with such transformation properties (the $\delta^{IJ}$ in eq. (2.61) is achieved by diagonalizing an arbitrary symmetric matrix and rescaling the $Q$'s and the $\overline{Q}$'s). Hence, the commutator of two supersymmetry transformations is a translation. In theories with local supersymmetry (i.e. where the spinorial infinitesimal parameter of the supersymmetry transformation depends on $x^\mu$), the commutator is an infinitesimal translation whose parameter depends on $x^\mu$. This is nothing but a theory invariant under general coordinate transformation, namely a theory of gravity! The upshot is that theories with local supersymmetry automatically incorporate gravity, the two things are tight together. Such theories are called supergravity theories, SUGRA for short.

- Eqs.(2.57) and (2.58) are not at all obvious. Compatibility with Lorentz symmetry would imply the right hand side of eq. (2.57) to transform as

$$
\left(\frac{1}{2},\frac{1}{2}\right) \otimes \left(\frac{1}{2},0\right) = \left(0,\frac{1}{2}\right) \oplus \left(1,\frac{1}{2}\right),
$$

and similarly for eq. (2.58). The second term on the right hand side cannot be there, due to the theorem of Haag, Lopuszanski and Sohnius which says that the only allowed fermionic generators in the algebra are supersymmetry generators, which are spin $\frac{1}{2}$. In other words, there cannot be a consistent extension of the Poincaré algebra including generators transforming in the $(1, \frac{1}{2})$ under the Lorentz group. Still, group theory arguments by themselves do not justify eqs.(2.57) and (2.58) but rather something like

$$
[P_\mu, Q^I] = C^I_J \sigma^\mu_{\alpha\beta} \overline{Q}^{J\beta}
$$

and

$$
[P_\mu, \overline{Q}^I] = (C^I_J)^* \overline{\sigma}^\mu_{\alpha\beta} Q^{J\beta}.
$$

where $C^I_J$ is an undetermined matrix. We want to prove that this matrix vanishes. Let us first consider the generalized Jacobi identity which the supersymmetry algebra should satisfy and let us apply it to the $(Q, P, P)$ system. We get

$$
[[Q^I_\mu, P_\mu], P_\nu] + [[P_\mu, P_\nu], Q^I_\alpha] + [[P_\nu, Q^I_\alpha], P_\mu] =
$$

$$
- C^I_J \sigma^\mu_{\alpha\beta} \overline{Q}^{J\beta}, P_\nu] + C^I_J \sigma^\nu_{\alpha\beta} \overline{Q}^{J\beta}, P_\mu] =
$$

$$
C^I_J C^J_K \sigma^\mu_{\alpha\beta} \overline{\sigma}^\beta_{\gamma\delta} Q^{K\gamma} - C^I_J C^J_K \sigma^\mu_{\nu\alpha\beta} \overline{\sigma}^\beta_{\gamma\delta} Q^{K\gamma} =
$$

$$
4 (C^I K^J) (\sigma_{\mu\nu})_{\alpha\gamma} Q^{K\gamma} = 0.
$$

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This implies that
\[ C C^* = 0 \]  
(2.70)
as a matrix equation. Note that this is not enough to conclude, as we would, that \( C = 0 \). For that, we also need to show, in addition, that \( C \) is symmetric. To this aim we have to consider other equations, as discussed below.

Let us now consider eqs. (2.62) and (2.63). As for the first, from Lorentz representation theory we would expect
\[
\left( \frac{1}{2}, 0 \right) \otimes \left( \frac{1}{2}, 0 \right) = (0, 0) \oplus (1, 0)
\]  
(2.71)
which explicitly means
\[
\left\{ Q^I_a, Q^J_b \right\} = \epsilon_{ab} Z^{IJ} + \epsilon_{bc} (\sigma^{\mu\nu})_a^c M_{\mu\nu} Y^{IJ}.
\]  
(2.72)
The \( Z^{IJ} \), being Lorentz scalars, should be some linear combination of the internal symmetry generators \( B_i \) and, given the antisymmetric properties of the epsilon tensor under \( \alpha \leftrightarrow \beta \), they should be anti-symmetric under \( I \leftrightarrow J \). On the contrary, given that \( \epsilon_{bc} (\sigma^{\mu\nu})_a^c \) is symmetric in \( \alpha \leftrightarrow \beta \), the matrix \( Y^{IJ} \) should be symmetric under \( I \leftrightarrow J \).

Let us now consider the generalized Jacobi identity between \((Q, Q, P)\), which can be written as
\[
\left\{ \{Q_{Ia}, Q_{Jb}\}, P_\mu \right\} = \{Q_{Ia}, \{Q_{Jb}, P_\mu\}\} + \{Q_{Jb}, \{Q_{Ia}, P_\mu\}\}.
\]  
(2.73)
If one multiplies it by \( \epsilon^{\alpha\beta} \), only the anti-symmetric part under \( \alpha \leftrightarrow \beta \) of the left hand side survives, which, from eq. (2.72), can be seen to vanish since the matrix \( Z_{IJ} \) commutes with \( P_\mu \). So we get
\[
0 = \epsilon^{\alpha\beta} \{Q_{Ia}, \{Q_{Jb}, P_\mu\}\} + \epsilon^{\alpha\beta} \{Q_{Jb}, \{Q_{Ia}, P_\mu\}\}
\]
\[
= \epsilon^{\alpha\beta} C^K_I \sigma_{b\beta} \{Q_{Ja}, \overline{Q^b_K} \} - \epsilon^{\alpha\beta} C^K_J \sigma_{a\beta} \{Q_{Ja}, \overline{Q^b_K} \} \sim (C_{IJJ} - C_{IJI}) \sigma^{\nu\alpha} \sigma_{a\gamma} P_\nu
\]
\[
= 2 (C_{IJJ} - C_{IJI}) P_\mu,
\]
which implies that the matrix \( C \) is symmetric. So the previously found equation \( C C^* = 0 \) can be promoted to \( CC^\dagger = 0 \), which in turn implies \( C = 0 \) and hence eq. (2.57). A similar rationale leads to eq. (2.58).
Let us now come back to eq. (2.62), which we have not yet proven. To do so, we should plug eq. (2.57) into the Jacobi identity (2.73), getting

\[
\{\{Q_\alpha, Q_J\}, P_\mu\} = 0 ,
\]

which implies, by (2.72), that the matrix \(Y^{IJ}\) vanishes because \(P_\mu\) does not commute with \(M_{\mu\nu}\). This finally proves eq. (2.62). Similarly, one can prove eq. (2.63), which is just the hermitian conjugate of (2.62).

What about the commutation relations between supersymmetry generators and internal symmetry generators, if any? In general, the \(Q\)'s will carry a representation of the internal symmetry group \(G\). So one expects something like

\[
\begin{align*}
[Q^I_\alpha, B_l] & = (b_l)^I J Q^J_\alpha \\
[\bar{Q}_{I\dot{\alpha}}, B_l] & = -\bar{Q}_{J\dot{\alpha}} (b_l)^I J .
\end{align*}
\]

The second commutation relation comes from the first under hermitian conjugation, recalling that the \(b_l\) are hermitian, because so are the generators \(B_l\). The largest possible internal symmetry group which can act non trivially on the \(Q\)'s is thus \(U(N)\), and this is called the R-symmetry group (the relation between a Lie algebra with generators \(S\) and the corresponding Lie group with elements \(U = e^{iS}\); hence hermitian generators, \(S^\dagger = S\), correspond to unitary groups, \(U^\dagger = U^{-1}\)).

In presence of non-vanishing central charges one can prove that the R-symmetry group reduces to \(USp(N)\), the compact version of the symplectic group \(Sp(N)\), \(USp(N) \cong U(N) \cap Sp(N)\).

As already noticed, the operators \(Z_{IJ}\) are Lorentz scalars and should then correspond to some linear combination of the internal symmetry group generators \(B_l\) of the compact Lie algebra \(G\), say

\[
Z^{IJ} = a^{IJ} B_l .
\]

Using the above equation, the supersymmetry algebra (2.57)-(2.63) and eqs. (2.75) and (2.76) one can actually prove that the \(Z\)'s are central charges, that is they commute with the whole supersymmetry algebra, and within themselves. Contrary to what one could naively think, this does not imply they are ineffective. Indeed, central charges are not numbers but quantum operators and their value may vary from state to state. For a supersymmetric vacuum state, which is annihilated by all supersymmetry generators, they are trivially realized, recall eqs. (2.62) and (2.63).
However, they do not need to vanish in general. For instance, as we will see in the subsequent chapter, massive representations are very different if $Z^{IJ}$ vanishes or if it is non-trivially realized on the representation.

We have been using two-component spinor notation in writing the supersymmetry algebra, but one can equivalently use four-component notation. In four component notation the $2N$ supersymmetry generators $Q^I \alpha, \overline{Q}^I \dot{\alpha}$ constitute a set of $N$ Majorana spinors

$$Q^I = \begin{pmatrix} Q^I \alpha \\ \overline{Q}^I \dot{\alpha} \end{pmatrix} \quad \overline{Q}^I = \begin{pmatrix} Q^I \alpha \\ \overline{Q}^I \dot{\alpha} \end{pmatrix}$$

and the supersymmetry algebra reads

$$\{Q^I, Q^J\} = 2\delta^{IJ}\gamma^\mu P_\mu - i \Im Z^{IJ} - \gamma_5 \Re Z^{IJ}$$

$$[Q^I, P_\mu] = 0 \quad [Q^I, M_{\mu\nu}] = \frac{i}{2} \gamma_{\mu\nu} Q^I \quad [Q^I, R] = i\gamma_5 Q^I .$$

Depending on what one needs to do, one notation can be more useful than the other. In the following, we will stick to two component spinor notation, unless otherwise stated.

Let us close this section with a few more comments. First, if $N = 1$ there are only two supersymmetry generators, which correspond to one Majorana spinor in four component notation. In this case we then speak of unextended or minimal supersymmetry (and there are no non-trivial central charges). For $N > 1$ we have extended supersymmetry (and there can now be a central extension of the supersymmetry algebra). From an algebraic point of view there is no limit to $N$. However, as we will see later, increasing $N$ the theory must contain particles of increasing spin. In particular we have

- $N \leq 4$ for theories without gravity (spin $\leq 1$)
- $N \leq 8$ for theories with gravity (spin $\leq 2$)

Therefore, to avoid theories with spin higher than two (that is focusing on local, interacting theories) $N = 8$ is an upper bound. As discussed in the previous chapter, thus stated this statement is true in four space-time dimensions. Equivalent statements can be made in higher/lower dimensions, where the dimension of the spinor representation of the Lorentz group is bigger/smaller. What matters is the number of single state supersymmetry generators, which is a dimension-independent statement (e.g., $N = 8$ corresponds to 32 supercharges).
For $N = 1$ the R-symmetry group is just $U(1)$ (one can see it from the Jacobi identity between $(Q, B, B)$ which implies that the $f_{lm}^a$ are trivially realized on the supersymmetry generators). In this case the hermitian matrices $b_l$ are just real numbers and by rescaling the generators $B_l$ one gets

$$[R, Q_a] = -Q_a, \quad [R, \bar{Q}_a] = +\bar{Q}_a.$$  \hfill (2.80)

This implies that in minimal supersymmetric models, supersymmetric partners (which are related by the action of the $Q$’s) have different $U(1)$ R-charge. In particular, given eqs. (2.80), if a particle has $R = 0$ then its superpartner has $R = \pm 1$. An important physical consequence of this property is that in a theory where the R-symmetry is preserved, the lightest supersymmetric particle (LSP) is stable.

Let us finally comment on the relation between two and four component spinor notations, when it comes to supersymmetry. In four component notation the $2N$ supersymmetry generators $Q^I_a, \bar{Q}^I_{\dot{a}}$ constitute a set of $N$ Majorana spinors

$$Q^I = \left( \frac{Q^I_a}{\bar{Q}^I_{\dot{a}}} \right), \quad \bar{Q}^I = \left( Q^{\dot{I}}_a \right) \left( \bar{Q}^{\dot{I}}_{\dot{a}} \right)$$  \hfill (2.81)

and the supersymmetry algebra reads

$$\{Q^I, \bar{Q}^J\} = 2\delta^{IJ}\gamma^\mu P_\mu - i \text{Im}Z^{IJ} - \gamma_5 \text{Re}Z^{IJ}$$

$$[Q^I, P_\mu] = 0 \quad [Q^I, M_{\mu\nu}] = \frac{i}{2} \gamma_{\mu\nu}Q^I \quad [Q^I, R] = i\gamma_5 Q^I.$$  \hfill (2.82)

Depending on what one needs to do, one notation can be more useful than the other. In the following, we will stick to two component spinor notation, unless otherwise stated.

2.4 Exercises

1. Prove the following spinor identities

$$\psi^\alpha \psi^\beta = -\frac{1}{2} \epsilon^{\alpha\beta} \psi \psi, \quad (\theta \phi) (\bar{\theta} \bar{\phi}) = -\frac{1}{2} (\phi \bar{\psi}) (\theta \bar{\psi})$$

$$\chi \sigma^\mu \bar{\psi} = -\bar{\psi} \bar{\sigma}^\mu \chi, \quad \chi \sigma^\mu \bar{\sigma}^\nu \psi = \psi \sigma^\nu \bar{\sigma}^\mu \chi$$

$$\left( \chi \sigma^\mu \bar{\psi} \right)^\dagger = \bar{\psi} \sigma^\mu \bar{\chi}, \quad \left( \chi \sigma^\mu \bar{\sigma}^\nu \psi \right)^\dagger = \bar{\psi} \sigma^\nu \sigma^\mu \bar{\chi}$$

$$\left( \theta \phi \right) \left( \theta \sigma^\mu \bar{\phi} \right) = -\frac{1}{2} (\theta \theta) \left( \psi \sigma^\mu \bar{\phi} \right), \quad \left( \bar{\theta} \bar{\psi} \right) \left( \bar{\theta} \bar{\sigma}^\mu \phi \right) = -\frac{1}{2} (\bar{\theta} \bar{\theta}) \left( \bar{\psi} \sigma^\mu \bar{\phi} \right)$$

$$\left( \psi \sigma^\mu \bar{\chi} \right) \cdot \bar{\chi}_{\dot{a}} = \frac{1}{2} (\phi \sigma^\mu \chi) \left( \psi \sigma^\mu \right)_{\dot{a}}.$$
2. The operators $Z^{IJ}$ are linear combinations of the internal symmetries generators $B_l$, eq. (2.77). Hence, they commute with $P_\mu$ and $M_\mu$. Prove that $Z^{IJ}$ are in fact central charges of the supersymmetry algebra, namely that it also holds that

$$[Z^{IJ}, B_l] = 0, \quad [Z^{IJ}, Z^{KL}] = 0, \quad [Z^{IJ}, Q^K_\alpha] = 0, \quad [Z^{IJ}, \overline{Q}^K_\alpha] = 0.$$ 

References


