2 The supersymmetry algebra

In this lecture we introduce the supersymmetry algebra, which is the algebra encoding the set of symmetries a supersymmetric theory should enjoy.

2.1 Lorentz and Poincaré groups

The Lorentz group $SO(1,3)$ is the subgroup of matrices Λ of $GL(4,R)$ with unit determinant, $\det\Lambda = 1$, and which satisfy the following relation

$$\Lambda^T \eta \, \Lambda = \eta \tag{2.1}$$

where η is the (mostly minus in our conventions) flat Minkowski metric

$$\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-) \ . \tag{2.2}$$

The Lorentz group has six generators (associated to space rotations and boosts) enjoying the following commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad , \quad [J_i, K_j] = i\epsilon_{ijk} K_k \quad , \quad [K_i, K_j] = -i\epsilon_{ijk} J_k \ . \tag{2.3}$$

Notice that while the J_i are hermitian, the boosts K_i are anti-hermitian, this being related to the fact that the Lorentz group is non-compact (topologically, the Lorentz group is $\mathbb{R}_3 \times S_3/\mathbb{Z}_2$, the non-compact factor corresponding to boosts and the doubly connected $S_3/\mathbb{Z}_2$ corresponding to rotations). In order to construct representations of this algebra it is useful to introduce the following complex linear combinations of the generators J_i and K_i

$$J_i^{\pm} = \frac{1}{2}\left(J_i \pm i K_i\right) \ , \tag{2.4}$$

where now the $J_i^{\pm}$ are hermitian. In terms of $J_i^{\pm}$ the algebra (2.3) becomes

$$[J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk} J_k^{\pm} \quad , \quad [J_i^{\pm}, J_j^{\mp}] = 0 \ . \tag{2.5}$$

This shows that the Lorentz algebra is equivalent to two $SU(2)$ algebras. As we will see later, this simplifies a lot the study of the representations of the Lorentz group, which can be organized into (couples of) $SU(2)$ representations. This isomorphism comes from the theory of Lie Algebra which says that at the level of complex algebras

$$SO(4) \simeq SU(2) \times SU(2) \ . \tag{2.6}$$

In fact, the Lorentz algebra is a specific real form of that of $SO(4)$. This difference can be seen from the defining commutation relations (2.3): for $SO(4)$ one would have had a plus sign on the right hand side of the third such commutation relations. This difference has some consequence when it comes to study representations. In particular, while in Euclidean space all representations are real or pseudoreal, in Minkowski space complex conjugation interchanges the two $SU(2)$'s. This can also be seen at the level of the generators $J_i^{\pm}$. In order for all rotation and boost parameters to be real, one must take all the J_i and K_i to be imaginary and hence from eq. (2.4) one sees that

$$(J_i^{\pm})^* = -J_i^{\mp} \ . \tag{2.7}$$

In terms of algebras, all this discussion can be summarized noticing that for the Lorentz algebra the isomorphism (2.6) changes into

$$SO(1,3) \simeq SU(2) \times SU(2)^* \ . \tag{2.8}$$

For later purpose let us introduce a four-vector notation for the Lorentz generators, in terms of an anti-symmetric tensor $M_{\mu\nu}$ defined as

$$M_{\mu\nu} = -M_{\nu\mu} \quad \text{with} \quad M_{0i} = K_i \quad \text{and} \quad M_{ij} = \epsilon_{ijk} J_k \ , \tag{2.9}$$

where $\mu = 0, 1, 2, 3$. In terms of such matrices, the Lorentz algebra reads

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho} M_{\nu\sigma} - i\eta_{\nu\sigma} M_{\mu\rho} + i\eta_{\mu\sigma} M_{\nu\rho} + i\eta_{\nu\rho} M_{\mu\sigma} \ . \tag{2.10}$$

Another useful relation one should bear in mind is the relation between the Lorentz group and $SL(2,\mathbb{C})$, the group of 2×2 complex matrices with unit determinant. More precisely, there exists a homomorphism between $SL(2,\mathbb{C})$ and $SO(1,3)$, which means that for any matrix $A \in SL(2,\mathbb{C})$ there exists an associated Lorentz matrix Λ, and that

$$\Lambda(A)\,\Lambda(B) = \Lambda(AB) \ , \tag{2.11}$$

where A and B are $SL(2,\mathbb{C})$ matrices. This can be proved as follows. Lorentz transformations act on four-vectors as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \ , \tag{2.12}$$

where the matrices Λ's are a representation of the generators $M_{\mu\nu}$ defined above. Let us introduce 2×2 matrices σ_μ where σ_0 is the identity matrix and σ_i are the Pauli matrices defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{2.13}$$

Let us also define the matrices with upper indices, σ^μ, as

$$\sigma^\mu = (\sigma^0, \sigma^i) = (\sigma_0, -\sigma_i) \; . \tag{2.14}$$

The matrices σ_μ are a complete set, in the sense that any 2×2 complex matrix can be written as a linear combination of them. For every four-dimensional vector x^μ let us construct the 2×2 complex matrix

$$\rho : x^\mu \to x^\mu \sigma_\mu = X \; . \tag{2.15}$$

The matrix X is hermitian, since the Pauli matrices are hermitian, and has determinant equal to $x^\mu x_\mu$, which is a Lorentz invariant quantity. Therefore, ρ is a map from Minkowski space to H, the space of 2×2 hermitian complex matrices

$$M_4 \underset{\rho}{\longrightarrow} H \; . \tag{2.16}$$

Let us now act on X with a $SL(2, \mathbb{C})$ transformation A

$$A : X \to A X A^\dagger = X' \; . \tag{2.17}$$

This transformation preserves the determinant since $\det A = 1$ and also preserves the hermicity of X since

$$X'^\dagger = (A X A^\dagger)^\dagger = A X^\dagger A^\dagger = A X A^\dagger = X' \; . \tag{2.18}$$

Therefore A is a map between H and itself

$$H \underset{A}{\longrightarrow} H \; . \tag{2.19}$$

We finally apply the inverse map ρ^{-1} to X' and get a four-vector x'^μ. The inverse map is defined as

$$\rho^{-1} = \frac{1}{2}\, \mathrm{Tr}[\bullet\; \bar{\sigma}^\mu] \tag{2.20}$$

(where, as we will later see more rigorously, as a complex 2×2 matrix $\bar{\sigma}^\mu$ is the same as σ_μ). Indeed

$$\rho^{-1} X = \frac{1}{2}\, \mathrm{Tr}[X \bar{\sigma}^\mu] = \frac{1}{2}\, \mathrm{Tr}[x_\nu \sigma^\nu \bar{\sigma}^\mu] = \frac{1}{2}\, \mathrm{Tr}[\sigma^\nu \bar{\sigma}^\mu] x_\nu = \frac{1}{2}\, 2\, \eta^{\mu\nu} x_\nu = x^\mu \; . \tag{2.21}$$

Assembling everything together we then get a map from Minkowski space into itself via the following chain

$$\begin{array}{ccccccc} M_4 & \underset{\rho}{\longrightarrow} & H & \underset{A}{\longrightarrow} & H & \underset{\rho^{-1}}{\longrightarrow} & M_4 \\ x_\nu & \underset{\rho}{\longrightarrow} & x_\nu \sigma^\nu & \underset{A}{\longrightarrow} & A x_\nu \sigma^\nu A^\dagger & \underset{\rho^{-1}}{\longrightarrow} & \frac{1}{2}\, \mathrm{Tr}[A x_\nu \sigma^\nu A^\dagger \bar{\sigma}^\mu] = x'^\mu \end{array} \tag{2.22}$$

This is nothing but a Lorentz transformation obtained by the $SL(2,\mathbb{C})$ transformation A as

$$\Lambda^\mu{}_\nu(A) = \frac{1}{2}\mathrm{Tr}[\bar\sigma^\mu A\,\sigma_\nu A^\dagger]\ . \tag{2.23}$$

It is now a trivial exercise, provided eq. (2.23), to prove the homomorphism (2.11).

Notice that the relation (2.23) can in principle be inverted, in the sense that for a given Λ one can find a corresponding $A \in SL(2,\mathbb{C})$. However, the relation is not an isomorphism, since it is double valued. The isomorphism holds between the Lorentz group and $SL(2,\mathbb{C})/\mathbb{Z}_2$ (in other words $SL(2,\mathbb{C})$ is a double cover of the Lorentz group). This can be seen as follows. Consider the 2×2 matrix

$$M(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tag{2.24}$$

which corresponds to a Lorentz transformation producing a rotation by an angle θ about the z-axis. Taking $\theta = 2\pi$ which corresponds to the identity in the Lorentz group, one gets $M = -\mathbf{1}$ which is a non-trivial element of $SL(2,\mathbb{C})$. It then follows that the elements of $SL(2,\mathbb{C})$ are identified two-by-two under a $\mathbb{Z}_2$ transformation in the Lorentz group. Note that this $\mathbb{Z}_2$ identification holds also in Euclidean space: at the level of groups $SU(2)\times SU(2) = Spin(4)$, where $Spin(4)$ is a double cover of $SO(4)$ as a group (it has an extra $\mathbb{Z}_2$).

The Poincaré group is the Lorentz group augmented by the space-time translation generators P_μ. In terms of the generators $P_\mu, M_{\mu\nu}$ the Poincaré algebra reads

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} \\ [M_{\mu\nu}, P_\rho] &= -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu\ . \end{aligned} \tag{2.25}$$

2.2 Spinors and representations of the Lorentz group

We are now ready to discuss representations of the Lorentz group. Thanks to the isomorphism (2.8) they can be easily organized in terms of those of $SU(2)$ which can be labeled by the spins. In this respect, let us introduce two-component spinors as the objects carrying the basic representations of $SL(2,\mathbb{C})$. There exist two such representations. A spinor transforming in the self-representation $\mathcal{M}$ is a two complex component object

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.26}$$

where ψ_1 and ψ_2 are complex Grassmann numbers, which transform under a matrix $\mathcal{M} \in SL(2,\mathbb{C})$ as

$$\psi_\alpha \to \psi'_\alpha = \mathcal{M}_\alpha^{\ \beta}\,\psi_\beta \qquad \alpha,\beta = 1,2\ . \tag{2.27}$$

The complex conjugate representation is defined from $\mathcal{M}^*$, where $\mathcal{M}^*$ means complex conjugation, as

$$\bar\psi_{\dot\alpha} \to \bar\psi'_{\dot\alpha} = \mathcal{M}^{*\ \dot\beta}_{\ \dot\alpha}\,\bar\psi_{\dot\beta} \qquad \dot\alpha,\dot\beta = 1,2\ . \tag{2.28}$$

These two representations are *not* equivalent, that is it does not exist a matrix C such that $\mathcal{M} = C\mathcal{M}^*C^{-1}$.

There are, however, other representations which are equivalent to the former. Let us first introduce the invariant tensor of $SU(2)$, $\epsilon_{\alpha\beta}$, and similarly for the other $SU(2)$, $\epsilon_{\dot\alpha\dot\beta}$, which one uses to raise and lower spinorial indices as well as to construct scalars and higher spin representations by spinor contractions

$$\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ . \tag{2.29}$$

We can then define

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \quad , \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \quad , \quad \bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta} \quad , \quad \bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}\ . \tag{2.30}$$

The convention here is that adjacent indices are always contracted putting the epsilon tensor on the left.

Using above conventions one can easily prove that $\psi'^\alpha = (\mathcal{M}^{-1T})^\alpha_{\ \beta}\psi^\beta$. Since $\mathcal{M}^{-1T} \simeq \mathcal{M}$ (the matrix C being in fact the epsilon tensor $\epsilon_{\alpha\beta}$), it follows that the fundamental (ψ_α) and anti-fundamental (ψ^α) representations of $SU(2)$ are equivalent (note that this does not hold for $SU(N)$ with $N > 2$, for which the fundamental and anti-fundamental representations are not equivalent). A similar story holds for $\bar\psi^{\dot\alpha}$ which transforms in the representation $\mathcal{M}^{*-1T}$, that is $\bar\psi'^{\dot\alpha} = (\mathcal{M}^{*-1T})^{\dot\alpha}_{\ \dot\beta}\bar\psi^{\dot\beta}$, which is equivalent to the complex conjugate representation $\bar\psi_{\dot\alpha}$ (the matrix C connecting $\mathcal{M}^{*-1T}$ and $\mathcal{M}^*$ is now the epsilon tensor $\epsilon_{\dot\alpha\dot\beta}$). From our conventions one can easily see that the complex conjugate matrix $(\mathcal{M}_\alpha^{\ \beta})^*$ (that is, the matrix obtained from $\mathcal{M}_\alpha^{\ \beta}$ by taking the complex conjugate of each entry), once expressed in terms of dotted indices, is not $\mathcal{M}^{*\ \dot\beta}_{\ \dot\alpha}$, but rather $(\mathcal{M}_\alpha^{\ \beta})^* = (\mathcal{M}^{*-1T})^{\dot\alpha}_{\ \dot\beta}$. Finally, lower undotted indices are row indices, while upper ones are column indices. Dotted indices follow instead the opposite convention. This implies that $(\psi_\alpha)^* = \bar\psi^{\dot\alpha}$, while under hermitian conjugation (which also includes transposition), we have, e.g. $\bar{\psi}_{\dot{\alpha}} = (\psi_\alpha)^\dagger$, as operator identity.

Due to the homomorphism between $SL(2,\mathbb{C})$ and $SO(1,3)$, it turns out that the two spinor representations ψ_α and $\bar{\psi}^{\dot{\alpha}}$ are representations of the Lorentz group, and, because of the isomorphism (2.8), they can be labeled in terms of $SU(2)$ representations as

$$\psi_\alpha \equiv \left(\frac{1}{2}, 0\right) \tag{2.31}$$

$$\bar{\psi}^{\dot{\alpha}} \equiv \left(0, \frac{1}{2}\right) . \tag{2.32}$$

To understand the identifications above just note that $\sum_i (J_i^+)^2$ and $\sum_i (J_i^-)^2$ are Casimir of the two $SU(2)$ algebras (2.5) with eigenvalues $n(n+1)$ and $m(m+1)$ with $n, m = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ being the eigenvalues of J_3^+ and J_3^-, respectively. Hence we can indeed label the representations of the Lorentz group by pairs (n,m) and since $J_3 = J_3^+ + J_3^-$ we can identify the spin of the representation as $n+m$, its dimension being $(2n+1)(2m+1)$. The two spinor representations (2.31) and (2.32) are just the basic such representations.

Recalling that Grassmann variables anticommute (that is $\psi_1\chi_2 = -\chi_2\psi_1$, $\psi_1\bar{\chi}_{\dot{2}} = -\bar{\chi}_{\dot{2}}\psi_1$, etc...) we can now define a scalar product for spinors as

$$\psi\chi \equiv \psi^\alpha\chi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \tag{2.33}$$

$$\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}\bar{\chi}^{\dot{\alpha}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\alpha}}\bar{\chi}^{\dot{\beta}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} . \tag{2.34}$$

Under hermitian conjugation we have

$$(\psi\chi)^\dagger = (\psi^\alpha\chi_\alpha)^\dagger = \chi_\alpha{}^\dagger\psi^{\alpha\dagger} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} . \tag{2.35}$$

In our conventions, undotted indices are contracted from upper left to lower right while dotted indices from lower left to upper right (this rule does not apply when raising or lowering indices with the epsilon tensor). Recalling eq. (2.17), namely that under $SL(2,\mathbb{C})$ the matrix $X = x^\mu\sigma_\mu$ transforms as $AXA^\dagger$ and that the index structure of A and $A^\dagger$ is $A_\alpha{}^\beta$ and $A^{*\dot{\beta}}{}_{\dot{\alpha}}$, respectively, we see that σ_μ naturally has a dotted and an undotted index and can be contracted with an undotted and a dotted spinor as

$$\psi\sigma^\mu\bar{\chi} \equiv \psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} . \tag{2.36}$$

Similarly one can define $\bar{\sigma}^{\mu}$ as

$$\bar{\sigma}^{\mu\ \dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^{\mu}_{\beta\dot{\beta}} = (\sigma_0, \sigma_i)\ , \tag{2.37}$$

and define the product of $\bar{\sigma}^{\mu}$ with a dotted and an undotted spinor as

$$\bar{\psi}\bar{\sigma}^{\mu}\chi \equiv \bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{\mu\ \dot{\alpha}\beta}\chi_{\beta}\ . \tag{2.38}$$

A number of useful identities one can prove are

$$\begin{aligned}
\psi^{\alpha}\psi^{\beta} &= -\frac{1}{2}\epsilon^{\alpha\beta}\psi\psi \quad , \quad (\theta\phi)(\theta\psi) = -\frac{1}{2}(\phi\psi)(\theta\theta) \\
\chi\sigma^{\mu}\bar{\psi} &= -\bar{\psi}\bar{\sigma}^{\mu}\chi \quad , \quad \chi\sigma^{\mu}\bar{\sigma}^{\nu}\psi = \psi\sigma^{\nu}\bar{\sigma}^{\mu}\chi \\
(\chi\sigma^{\mu}\bar{\psi})^{\dagger} &= \psi\sigma^{\mu}\bar{\chi} \quad , \quad (\chi\sigma^{\mu}\bar{\sigma}^{\nu}\psi)^{\dagger} = \bar{\psi}\bar{\sigma}^{\nu}\sigma^{\mu}\bar{\chi} \\
(\theta\psi)\left(\theta\sigma^{\mu}\bar{\phi}\right) &= -\frac{1}{2}(\theta\theta)\left(\psi\sigma^{\mu}\bar{\phi}\right) \quad , \quad \left(\bar{\theta}\bar{\psi}\right)\left(\bar{\theta}\bar{\sigma}^{\mu}\phi\right) = -\frac{1}{2}\left(\bar{\theta}\bar{\theta}\right)\left(\bar{\psi}\bar{\sigma}^{\mu}\phi\right) \\
(\phi\psi)\cdot\bar{\chi}_{\dot{\alpha}} &= \frac{1}{2}(\phi\sigma^{\mu}\bar{\chi})(\psi\sigma_{\mu})_{\dot{\alpha}}\ .
\end{aligned} \tag{2.39}$$

As some people might be more familiar with four component spinor notation, let us close this section by briefly mentioning the connection with Dirac spinors. In the Weyl representation Dirac matrices read

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix} \quad , \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{2.40}$$

and a Dirac spinor is

$$\psi = \begin{pmatrix} \psi_{\alpha} \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \ \text{ implying }\ r(\psi) = \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\ . \tag{2.41}$$

This shows that a Dirac spinor carries a reducible representation of the Lorentz algebra. Using this four component spinor notation one sees that

$$\begin{pmatrix} \psi_{\alpha} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{2.42}$$

are Weyl (chiral) spinors, with chirality $+1$ and -1, respectively. One can easily show that a Majorana spinor ($\psi^{\mathcal{C}} = \psi$) is a Dirac spinor such that $\chi_{\alpha} = \psi_{\alpha}$. To prove this, just recall that in four component notation the conjugate Dirac spinor is defined as $\bar{\psi} = \psi^\dagger \gamma_0$ and the charge conjugate is $\psi^{\mathcal{C}} = \mathcal{C}\bar{\psi}^T$ with the charge conjugate matrix in the Weyl representation being

$$\mathcal{C} = \begin{pmatrix} -\epsilon_{\alpha\beta} & 0 \\ 0 & -\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \ . \tag{2.43}$$

Finally, Lorentz generators are

$$\Sigma^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu} \quad , \quad \gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{1}{2}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} \tag{2.44}$$

while the 2-index Pauli matrices are defined as

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\gamma}}(\bar{\sigma}^\nu)^{\dot{\gamma}\beta} - (\mu \leftrightarrow \nu)\right) \quad , \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} = \frac{1}{4}\left((\bar{\sigma}^\mu)^{\dot{\alpha}\gamma}\sigma^\nu_{\gamma\dot{\beta}} - (\mu \leftrightarrow \nu)\right) \tag{2.45}$$

From the last equations one then sees that $i\sigma^{\mu\nu}$ acts as a Lorentz generator on ψ_α while $i\bar{\sigma}^{\mu\nu}$ acts as a Lorentz generator on $\bar{\psi}^{\dot{\alpha}}$.

2.3 The supersymmetry algebra

As we have already mentioned, a no-go theorem provided by Coleman and Mandula implies that, under certain assumptions (locality, causality, positivity of energy, finiteness of number of particles, etc...), the only possible symmetries of the S-matrix are, besides C, P, T,

- Poincaré symmetries, with generators $P_\mu, M_{\mu\nu}$
- Some *internal* symmetry group with generators B_l which are Lorentz scalars, and which are typically related to some conserved quantum number like electric charge, isopin, etc...

The full symmetry algebra hence reads

$$[P_\mu, P_\nu] = 0 \tag{2.46}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} \tag{2.47}$$

$$[M_{\mu\nu}, P_\rho] = -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu \tag{2.48}$$

$$[B_l, B_m] = i f_{lm}^{\ \ n} B_n \tag{2.49}$$

$$[P_\mu, B_l] = 0 \tag{2.50}$$

$$[M_{\mu\nu}, B_l] = 0 \ , \tag{2.51}$$

where ${f_{lm}}^n$ are structure constants and the last two commutation relations simply say that the full algebra is the direct sum of the Poincaré algebra and the algebra G spanned by the scalar bosonic generators B_l, that is

$$ISO(1,3) \times G \; . \tag{2.52}$$

at the level of groups (a nice proof of the Coleman-Mandula theorem can be found in Weinberg's book, Vol. III, chapter 24.B).

The Coleman-Mandula theorem can be evaded by weakening one (or more) of its assumptions. For instance, the theorem assumes that the symmetry algebra involves only commutators. Haag, Lopuszanski and Sohnius generalized the notion of Lie algebra to include algebraic systems involving, in addition to commutators, also *anticommutators*. This extended Lie algebra goes under the name of Graded Lie algebra. Allowing for a graded Lie algebra weakens the Coleman-Mandula theorem enough to allow for supersymmetry, which is nothing but a specific graded Lie algebra.

Let us first define what a graded Lie algebra is. Recall that a Lie algebra is a vector space (over some field, say $\mathbb{R}$ or $\mathbb{C}$) which enjoys a composition rule called product

$$[\quad , \quad] : L \times L \to L \tag{2.53}$$

with the following properties

$$\begin{aligned}
&[v_1, v_2] \in L \\
&[v_1, (v_2 + v_3)] = [v_1, v_2] + [v_1, v_3] \\
&[v_1, v_2] = -[v_2, v_1] \\
&[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0 \; ,
\end{aligned}$$

where v_i are elements of the algebra. A *graded* Lie algebra of grade n is a vector space

$$L = \oplus_{i=0}^{i=n} \; L_i \tag{2.54}$$

where L_i are all vector spaces, and the product

$$[\quad , \quad \} : L \times L \to L \tag{2.55}$$

has the following properties

$$[L_i, L_j\} \in L_{i+j} \quad \text{mod } n+1$$
$$[L_i, L_j\} = -(-1)^{ij}[L_j, L_i\}$$
$$[L_i, [L_j, L_k\}\}(-1)^{ik} + [L_j, [L_k, L_i\}\}(-1)^{ij} + [L_k, [L_i, L_j\}\}(-1)^{jk} = 0 \ .$$

First notice that from the first such properties it follows that L_0 is a Lie algebra while all other L_i's with $i \neq 0$ are not. The second property is called supersymmetrization while the third one is nothing but the generalization to a graded algebra of the well known Jacobi identity any algebra satisfies.

The supersymmetry algebra is a graded Lie algebra of grade one, namely

$$L = L_0 \oplus L_1 \ , \tag{2.56}$$

where L_0 is the Poincaré algebra and $L_1 = (Q^I_\alpha \ , \ \bar{Q}^I_{\dot{\alpha}})$ with $I = 1, \ldots, N$ where $Q^I_\alpha \ , \ \bar{Q}^I_{\dot{\alpha}}$ is a set of $N + N = 2N$ anticommuting fermionic generators transforming in the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group, respectively. Haag, Lopuszanski and Sohnius proved that this is the only possible consistent extension of the Poincaré algebra, given the other (very physical) assumptions one would not like to relax of the Coleman-Mandula theorem. For instance, generators with spin higher than one, like those transforming in the $(\frac{1}{2}, 1)$ representation of the Lorentz group, cannot be there.

The generators of L_1 are spinors and hence they transform non-trivially under the Lorentz group. Therefore, supersymmetry is not an internal symmetry. Rather it is an extension of Poincaré space-time symmetries. Moreover, acting on bosons, the supersymmetry generators transform them into fermions (and viceversa). Hence, this symmetry naturally mixes radiation with matter.

The supersymmetry algebra, besides commutators (2.46)-(2.51), contains the following (anti)commutators

$$\begin{aligned}
\left[P_\mu, Q^I_\alpha\right] &= 0 && (2.57)\\
\left[P_\mu, \bar{Q}^I_{\dot\alpha}\right] &= 0 && (2.58)\\
\left[M_{\mu\nu}, Q^I_\alpha\right] &= i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta && (2.59)\\
\left[M_{\mu\nu}, \bar{Q}^{I\dot\alpha}\right] &= i(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{I\dot\beta} && (2.60)\\
\left\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\right\} &= 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^{IJ} && (2.61)\\
\left\{Q^I_\alpha, Q^J_\beta\right\} &= \epsilon_{\alpha\beta}Z^{IJ} \quad , \quad Z^{IJ} = -Z^{JI} && (2.62)\\
\left\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\right\} &= \epsilon_{\dot\alpha\dot\beta}(Z^{IJ})^* && (2.63)
\end{aligned}$$

Several comments are in order at this point.

- Eqs. (2.59) and (2.60) follow from the fact that Q^I_α and $\bar{Q}^I_{\dot\alpha}$ are spinors of the Lorentz group, recall eq.(2.45). From these same equations one also sees that, since $M_{12} = J_3$, we have

$$\left[J_3, Q^I_1\right] = \frac{1}{2}Q^I_1 \quad , \quad \left[J_3, Q^I_2\right] = -\frac{1}{2}Q^I_2 \ . \qquad (2.64)$$

  Taking the hermitian conjugate of the above relations we get

$$\left[J_3, \bar{Q}^I_{\dot 1}\right] = -\frac{1}{2}\bar{Q}^I_{\dot 1} \quad , \quad \left[J_3, \bar{Q}^I_{\dot 2}\right] = \frac{1}{2}\bar{Q}^I_{\dot 2} \qquad (2.65)$$

  and so we see that Q^I_1 and $\bar{Q}^I_{\dot 2}$ rise the z-component of the spin by half unit while Q^I_2 and $\bar{Q}^I_{\dot 1}$ lower it by half unit.

- Eq.(2.61) has a very important implication. First notice that given the transformation properties of Q^I_α and $\bar{Q}^J_{\dot\beta}$ under Lorentz transformations, their anticommutator should be symmetric under $I \leftrightarrow J$ and should transform as

$$\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right) \ . \qquad (2.66)$$

  The obvious such candidate is P_μ which is the only generator in the algebra with such transformation properties (the δ^{IJ} in eq. (2.61) is achieved by diagonalizing an arbitrary symmetric matrix and rescaling the Q's and the $\bar{Q}$'s). Hence, the commutator of two supersymmetry transformations is a translation. In theories with *local* supersymmetry (*i.e.* where the spinorial infinitesimal parameter of the supersymmetry transformation depends on x^μ), the commutator is an infinitesimal translation whose parameter depends on x^μ. This is nothing but a theory invariant under general coordinate transformation, namely a theory of gravity! The upshot is that theories with local supersymmetry automatically incorporate gravity. Such theories are called *supergravity* theories, SUGRA for short.

- Eqs.(2.57) and (2.58) are not at all obvious. Compatibility with Lorentz symmetry would imply the right hand side of eq. (2.57) to transform as
$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},0\right)=\left(0,\frac{1}{2}\right)\oplus\left(1,\frac{1}{2}\right)\ , \tag{2.67}$$
and similarly for eq. (2.58). The second term on the right hand side cannot be there, due to the theorem of Haag, Lopuszanski and Sohnius which says that the only allowed fermionic generators in the algebra are supersymmetry generators, which are spin $\frac{1}{2}$. In other words, there cannot be a consistent extension of the Poincaré algebra including generators transforming in the $(1,\frac{1}{2})$ under the Lorentz group. Still, group theory arguments by themselves do not justify eqs.(2.57) and (2.58) but rather something like
$$\left[P_\mu, Q^I_\alpha\right] \;=\; C^I{}_J \sigma_{\mu\,\alpha\dot\beta}\,\bar Q^{J\dot\beta} \tag{2.68}$$
$$\left[P_\mu, \bar Q^I_{\dot\alpha}\right] \;=\; (C^I{}_J)^* \bar\sigma_{\mu\,\dot\alpha\beta}\,Q^{J\beta}\ . \tag{2.69}$$
where $C^I{}_J$ is an undetermined matrix. We want to prove that this matrix vanishes. Let us first consider the generalized Jacobi identity which the supersymmetry algebra should satisfy and let us apply it to the (Q,P,P) system. We get
$$\begin{aligned}\left[\left[Q^I_\alpha,P_\mu\right],P_\nu\right]+\left[\left[P_\mu,P_\nu\right],Q^I_\alpha\right]+\left[\left[P_\nu,Q^I_\alpha\right],P_\mu\right] &= \\ -C^I{}_J\sigma_{\mu\,\alpha\dot\beta}\left[\bar Q^{J\dot\beta},P_\nu\right]+C^I{}_J\sigma_{\nu\,\alpha\dot\beta}\left[\bar Q^{J\dot\beta},P_\mu\right] &= \\ C^I{}_J C^J{}_K{}^*\sigma_{\mu\,\alpha\dot\beta}\,\bar\sigma_\nu{}^{\dot\beta}{}_\gamma Q^{K\gamma}-C^I{}_J C^J{}_K{}^*\sigma_{\nu\,\alpha\dot\beta}\,\bar\sigma_\mu{}^{\dot\beta}{}_\gamma Q^{K\gamma} &= \\ 4\,(C\ C^*)^I{}_K\ (\sigma_{\mu\nu})_{\alpha\gamma}Q^{K\gamma} &= 0\ .\end{aligned}$$
This implies that
$$C\ C^*=0\ , \tag{2.70}$$
as a matrix equation. Note that this is not enough to conclude, as we would, that $C=0$. For that, we also need to show, in addition, that C is symmetric. To this aim we have to consider other equations, as detailed below.

- Let us now consider eqs. (2.62) and (2.63). As for the first, from Lorentz representation theory we would expect

$$\left(\frac{1}{2},0\right)\otimes\left(\frac{1}{2},0\right)=(0,0)\oplus(1,0)\ , \tag{2.71}$$

which explicitly means

$$\{Q^I_\alpha,Q^J_\beta\}=\epsilon_{\alpha\beta}Z^{IJ}+\epsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma M_{\mu\nu}Y^{IJ}\ . \tag{2.72}$$

The Z^{IJ}, being Lorentz scalars, should be some linear combination of the internal symmetry generators B_l and, given the antisymmetric properties of the epsilon tensor under $\alpha\leftrightarrow\beta$, should be anti-symmetric under $I\leftrightarrow J$. On the contrary, given that $\epsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma$ is symmetric in $\alpha\leftrightarrow\beta$, the matrix Y^{IJ} should be symmetric under $I\leftrightarrow J$. Let us now consider the generalized Jacobi identity between (Q,Q,P), which can be written as

$$[\{Q_{I\alpha},Q_{J\beta}\},P_\mu]=\{Q_{I\alpha},[Q_{J\beta},P_\mu]\}+\{Q_{J\beta},[Q_{I\alpha},P_\mu]\}\ . \tag{2.73}$$

If one multiplies it by $\epsilon^{\alpha\beta}$, only the anti-symmetric part under $\alpha\leftrightarrow\beta$ of the left hand side survives, which is 0, since the matrix Z_{IJ}, see eq. (2.72), commutes with P_μ. So we get

$$\begin{aligned}0&=\epsilon^{\alpha\beta}\{Q_{I\alpha},[Q_{J\beta},P_\mu]\}+\epsilon^{\alpha\beta}\{Q_{J\beta},[Q_{I\alpha},P_\mu]\}\\&=\epsilon^{\alpha\beta}C_I{}^K\sigma_{\mu\,\beta\dot\beta}\{Q_{J\alpha},\bar Q^{\dot\beta}_K\}-\epsilon^{\alpha\beta}C_J{}^K\sigma_{\mu\,\beta\dot\beta}\{Q_{I\alpha},\bar Q^{\dot\beta}_K\}\sim(C_{IJ}-C_{JI})\,\bar\sigma_\mu^{\dot\gamma\alpha}\sigma^\nu_{\alpha\dot\gamma}P_\nu\\&=2\,(C_{IJ}-C_{JI})P_\mu\ ,\end{aligned}$$

which implies that the matrix C is symmetric. So the previously found equation $C\ C^*=0$ can be promoted to $CC^\dagger=0$, which implies $C=0$ and hence eq. (2.57). A similar rationale leads to eq. (2.58).

Let us now come back to eq. (2.62), which we have not yet proven. To do so, we should plug eq. (2.57) into the Jacobi identity (2.73), getting

$$[\{Q_{I\alpha},Q_{J\beta}\},P_\mu]=0\ , \tag{2.74}$$

which implies, by (2.72), that the matrix Y^{IJ} vanishes because P_μ does not commute with $M_{\mu\nu}$. This finally proves eq. (2.62). Seemingly, one can prove eq. (2.63), which is just the hermitian conjugate of (2.62).

What about the commutation relations between supersymmetry generators and internal symmetry generators, if any? In general, the Q's will carry a representation of the internal symmetry group G. So one expects something like

$$\left[Q^I_\alpha, B_l\right] = (b_l)^I{}_J Q^J_\alpha \tag{2.75}$$

$$\left[\bar{Q}_{I\dot{\alpha}}, B_l\right] = -\bar{Q}_{J\dot{\alpha}} (b_l)^J{}_I \ . \tag{2.76}$$

The second commutation relation comes from the first under hermitian conjugation, recalling that the b_l are hermitian, because so are the generators B_l. The largest possible internal symmetry group which can act non trivially on the Q's is thus $U(N)$, and this is called the R-symmetry group (recall that the relation between a Lie algebra with generators S and the corresponding Lie group with elements U is $U = e^{iS}$; hence hermitian generators, $S^\dagger = S$, correspond to unitary groups, $U^\dagger = U^{-1}$). In fact, in presence of non-vanishing central charges one can prove that the R-symmetry group reduces to $USp(N)$, the compact version of the symplectic group $Sp(N)$, $USp(N) \cong U(N) \cap Sp(N)$.

As already noticed, the operators Z_{IJ}, being Lorentz scalars, should be some linear combination of the internal symmetry group generators B_l of the compact Lie algebra G, say

$$Z^{IJ} = a^{l|IJ} B_l \ . \tag{2.77}$$

Using the above equation together with eqs.(2.62), (2.75) and (2.76) we get

$$\begin{aligned} \left[Z^{IJ}, B_l\right] &= (b_l)^I_K Z^{KJ} + (b_l)^J_K Z^{IK} \\ \left[Z^{IJ}, Z^{KL}\right] &= a^{l|KL} (b_l)^I_M Z^{MJ} + a^{l|KL} (b_l)^J_M Z^{IM} \end{aligned}$$

This implies that the Z's span an invariant subalgebra of G. Playing with Jacobi identities one can see that the Z are in fact *central charges*, that is they commute with the whole supersymmetry algebra, and within themselves. In other words, they span an invariant *abelian* subalgebra of G and commute with all other generators

$$\begin{aligned} &\left[Z^{IJ}, B_l\right] = 0 \quad \left[Z^{IJ}, Z^{KL}\right] = 0 \\ &\left[Z^{IJ}, P_\mu\right] = 0 \quad \left[Z^{IJ}, M_{\mu\nu}\right] = 0 \\ &\left[Z^{IJ}, Q^K_\alpha\right] = 0 \quad \left[Z^{IJ}, \bar{Q}^K_{\dot{\alpha}}\right] = 0 \ . \end{aligned}$$

Notice that this does not at all imply they are uneffective. Indeed, central charges are not numbers but quantum operators and their value may vary from state to state. For a supersymmetric vacuum state, which is annihilated by all supersymmetry generators, they are of course trivially realized, recall eqs. (2.62) and (2.63). However, they do not need to vanish in general. For instance, as we will see in a subsequent lecture, massive representations are very different if Z^{IJ} vanishes or if it is non-trivially realized on the representation.

Let us end this section with a few more comments. First, if $N = 1$ we have two supersymmetry generators, which correspond to one Majorana spinor, in four component notation. In this case we speak of unextended supersymmetry (and we do not have central charges whatsoever). For $N > 1$ we have extended supersymmetry (and we can have a central extension of the supersymmetry algebra, too). From an algebraic point of view there is no limit to N; but, as we will later see, increasing N the theory must contain particles of increasing spin. In particular we have

- $N \leq 4$ for theories without gravity (spin ≤ 1)
- $N \leq 8$ for theories with gravity (spin ≤ 2)

For $N = 1$ the R-symmetry group is just $U(1)$ (one can see it from the Jacobi identity between (Q, B, B) which implies that the ${f_{lm}}^n$ are trivially realized on the supersymmetry generators). In this case the hermitian matrices b_l are just real numbers and by rescaling the generators B_l one gets

$$[R, Q_\alpha] = -Q_\alpha \quad , \quad [R, \bar{Q}_{\dot{\alpha}}] = +\bar{Q}_{\dot{\alpha}} \ . \tag{2.78}$$

This implies that supersymmetric partners (which are related by the action of the Q's) have different R-charge. In particular, given eqs. (2.78), if a particle has $R = 0$ then its superpartner has $R = \pm 1$. An important physical consequence of this property is that in a theory where R-symmetry is preserved, the lightest supersymmetric particle (LSP) is stable.

Let us finally comment on the relation between two and four component spinor notation, when it comes to supersymmetry. In four component notation the $2N$ supersymmetry generators Q^I_α , $\bar{Q}^I_{\dot{\alpha}}$ constitute a set of N Majorana spinors

$$Q^I = \begin{pmatrix} Q^I_\alpha \\ \bar{Q}^{I\dot{\alpha}} \end{pmatrix} \qquad \bar{Q}^I = \begin{pmatrix} Q^{I\alpha} & \bar{Q}^I_{\dot{\alpha}} \end{pmatrix} \tag{2.79}$$

and the supersymmetry algebra reads

$$\begin{aligned} \{Q^I, \bar{Q}^J\} &= 2\delta^{IJ}\gamma^\mu P_\mu - i\,\mathbb{I}\,\mathrm{Im}Z^{IJ} - \gamma_5\,\mathrm{Re}Z^{IJ} \\ [Q^I, P_\mu] = 0 \quad [Q^I, M_{\mu\nu}] &= \frac{i}{2}\gamma_{\mu\nu}Q^I \quad [Q^I, R] = i\gamma_5 Q^I \ . \end{aligned} \tag{2.80}$$

Depending on what one needs to do, one notation can be more useful than the other. In the following we will mainly stick to the two component notation, though.

2.4 Exercises

1. Prove the following spinor identities

$$\begin{aligned}
\psi^\alpha \psi^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\psi\psi \quad , \quad (\theta\phi)(\theta\psi) = -\frac{1}{2}(\phi\psi)(\theta\theta) \\
\chi\sigma^\mu\bar{\psi} &= -\bar{\psi}\bar{\sigma}^\mu\chi \quad , \quad \chi\sigma^\mu\bar{\sigma}^\nu\psi = \psi\sigma^\nu\bar{\sigma}^\mu\chi \\
(\chi\sigma^\mu\bar{\psi})^\dagger &= \psi\sigma^\mu\bar{\chi} \quad , \quad (\chi\sigma^\mu\bar{\sigma}^\nu\psi)^\dagger = \bar{\psi}\bar{\sigma}^\nu\sigma^\mu\bar{\chi} \\
(\theta\psi)\left(\theta\sigma^\mu\bar{\phi}\right) &= -\frac{1}{2}(\theta\theta)\left(\psi\sigma^\mu\bar{\phi}\right) \quad , \quad \left(\bar{\theta}\bar{\psi}\right)\left(\bar{\theta}\bar{\sigma}^\mu\phi\right) = -\frac{1}{2}\left(\bar{\theta}\bar{\theta}\right)\left(\bar{\psi}\bar{\sigma}^\mu\phi\right) \\
(\phi\psi)\cdot\bar{\chi}_{\dot{\alpha}} &= \frac{1}{2}(\phi\sigma^\mu\bar{\chi})(\psi\sigma_\mu)_{\dot{\alpha}} \ .
\end{aligned}$$

2. The operators Z^{IJ} are linear combinations of the internal symmetries generators B_l. Hence, they commute with P_μ and $M_{\mu\nu}$. Prove that Z_{IJ} are in fact central charges of the supersymmetry algebra, namely that it also follows that

$$\left[Z^{IJ}, B_l\right] = 0 \ , \ \left[Z^{IJ}, Z^{KL}\right] = 0 \ , \ \left[Z^{IJ}, Q^K_\alpha\right] = 0 \ , \ \left[Z^{IJ}, \bar{Q}^K_{\dot{\alpha}}\right] = 0 \ .$$

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