3 Representations of the supersymmetry algebra

In this lecture we will discuss representations of the supersymmetry algebra. Let us first briefly recall how things go for the Poincaré algebra. The Poincaré algebra (2.25) has two Casimir (i.e. two operators which commute with all generators)

\[ P^2 = P_\mu P^\mu \quad \text{and} \quad W^2 = W_\mu W^\mu , \tag{3.1} \]

where \( W^\mu = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} P_\nu M_{\rho \sigma} \) is the so-called Pauli-Lubanski vector. Casimir operators are useful to classify irreducible representations of a group. In the case of the Poincaré group such representations are nothing but what we usually call particles. Let us see how this is realized for massive and massless particles, respectively.

Let us first consider a massive particle with mass \( m \) and go to the rest frame, \( P_\mu = (m, 0, 0, 0) \). In this frame it is easy to see that the two Casimir reduce to \( P^2 = m^2 \) and \( W^2 = -m^2 j(j + 1) \) where \( j \) is the spin. The second equality can be proven by noticing that \( W_\mu P^\mu = 0 \) which implies that in the rest frame \( W_0 = 0 \). Therefore in the rest frame \( W_\mu = (0, \frac{1}{2} \epsilon_{ijk} m M^{jk}) \) from which one immediately gets \( W^2 = -m^2 \vec{J}^2 \). So we see that massive particles are distinguished by their mass and their spin.

Let us now consider massless particles. In the rest frame \( P_\mu = (E, 0, 0, E) \). In this case we have that \( P^2 = 0 \) and \( W^2 = 0 \), and \( W^\mu = M_{12} P^\mu \). In other words, the two operators are proportional for a massless particle, the constant of proportionality being \( M_{12} = \pm j \), the helicity. For these representations the spin is then fixed and the different states are distinguished by their energy and by the sign of the helicity (e.g. the photon is a massless particle with two helicity states, \( \pm 1 \)).

Now, as a particle is an irreducible representation of the Poincaré algebra, we call superparticle an irreducible representation of the supersymmetry algebra. Since the Poincaré algebra is a subalgebra of the supersymmetry algebra, it follows that any irreducible representation of the supersymmetry algebra is a representation of the Poincaré algebra, which in general will be reducible. This means that a superparticle corresponds to a collection of particles, the latter being related by the action of the supersymmetry generators \( Q_\alpha^I \) and \( \bar{Q}_\dot{\alpha}^I \) and having spins differing by units of half. Being a multiplet of different particles, a superparticle is often called supermultiplet.

Before discussing in detail specific representations of the supersymmetry algebra, let us list three generic properties any such representation enjoys, all of them having very important physical implications.
1. As compared to the Poincaré algebra, in the supersymmetry algebra $P^2$ is still a Casimir, but $W^2$ is not anymore (this follows from the fact that $M_{\mu \nu}$ does not commute with the supersymmetry generators). Therefore, particles belonging to the same supermultiplet have the same mass and different spin, since the latter is not a conserved quantum number of the representation. The mass degeneracy between bosons and fermions is something we do not observe in known particle spectra; this implies that supersymmetry, if at all realized, must be broken in Nature.

2. In a supersymmetric theory the energy of any state is always $\geq 0$. Consider an arbitrary state $|\phi\rangle$. Using the supersymmetry algebra, we easily get

$$
\langle \phi | \{ Q^I_{\alpha}, \bar{Q}^I_{\dot{\alpha}} \} | \phi \rangle = 2\sigma^\mu_{\alpha \dot{\alpha}} \langle \phi | P_\mu | \phi \rangle \delta^{IJ}
$$

$$
(\bar{Q}^I_{\dot{\alpha}} = (Q^I_{\alpha})^\dagger) = \langle \phi | (Q^I_{\alpha}(Q^I_{\alpha})^\dagger + (Q^I_{\alpha})^\dagger Q^I_{\alpha}) | \phi \rangle
$$

$$
= ||(Q^I_{\alpha})^\dagger | \phi \rangle ||^2 + ||Q^I_{\alpha} | \phi \rangle ||^2 \geq 0 .
$$

The last inequality follows from positivity of the Hilbert space. Summing over $\alpha = \dot{\alpha} = 1, 2$ and recalling that $\text{Tr} \sigma^\mu = 2\delta^{\mu 0}$ we get

$$
4 \langle \phi | P_0 | \phi \rangle \geq 0 ,
$$

as anticipated.

3. A supermultiplet contains an equal number of bosonic and fermionic d.o.f., $n_B = n_F$. Define a fermion number operator

$$
(-1)^{N_F} = \begin{cases} 
-1 & \text{fermionic state} \\
+1 & \text{bosonic state}
\end{cases}
$$

$N_F$ can be taken to be twice the spin, $N_F = 2s$. Such an operator, when acting on a bosonic, respectively a fermionic state, gives indeed

$$
(-1)^{N_F} |B\rangle = |B\rangle , \quad (-1)^{N_F} |F\rangle = -|F\rangle .
$$

We want to show that $\text{Tr} (-1)^{N_F} = 0$ if the trace is taken over a finite dimensional representation of the supersymmetry algebra. First notice that

$$
\{ Q^I_{\alpha}, (-1)^{N_F} \} = 0 \rightarrow Q^I_{\alpha} (-1)^{N_F} = -(-1)^{N_F} Q^I_{\alpha} .
$$
Using this property and the cyclicity of the trace one easily sees that

\[
0 = \text{Tr} \left( -Q^I_\alpha (-1)^{N_F} \bar{Q}^J_\beta + (-1)^{N_F} \bar{Q}^J_\beta Q^I_\alpha \right)
\]

\[
= \text{Tr} \left( (-1)^{N_F} \left\{ Q^I_\alpha, \bar{Q}^J_\beta \right\} \right) = 2\sigma^\mu_{\alpha\beta} \text{Tr} \left[ (-1)^{N_F} \right] P_\mu \delta^{IJ}.
\]

Summing on \( I, J \) and choosing any \( P_\mu \neq 0 \) it follows that \( \text{Tr} (-1)^{N_F} = 0 \), which implies that \( n_B = n_F \).

In the following, we discuss (some) representations in detail. Since the mass is a conserved quantity in a supermultiplet, it is meaningful distinguishing between massless and massive representations. Let us start from the former.

### 3.1 Massless supermultiplets

Let us first assume that all central charges vanish, i.e. \( Z^{IJ} = 0 \) (we will see later that this is the only relevant case, for massless representations). Notice that in this case it follows from eqs. (2.62) and (2.63) that all \( Q \)'s and all \( \bar{Q} \)'s anticommute among themselves. The steps to construct the irreps are as follows:

1. Go to the rest frame where \( P_\mu = (E, 0, 0, E) \). In such frame we get

\[
\sigma^\mu P_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}
\]

Pluggin this into eq. (2.61) we get

\[
\left\{ Q^I_\alpha, \bar{Q}^J_\beta \right\} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix} \delta^{IJ}_{\alpha\beta} \quad \rightarrow \quad \left\{ Q^I_1, \bar{Q}^J_1 \right\} = 0.
\]

Due to the positiveness of the Hilbert space, this implies that both \( Q^I_1 \) and \( \bar{Q}^I_1 \) are trivially realized. Indeed, from the equation above we get

\[
0 = \langle \phi | \left\{ Q^I_1, \bar{Q}^I_1 \right\} | \phi \rangle = ||Q^I_1 | \phi ||^2 + ||\bar{Q}^I_1 | \phi ||^2,
\]

whose only solution is \( Q^I_1 = \bar{Q}^I_1 = 0 \). We are then left with just \( Q^I_2 \) and \( \bar{Q}^I_2 \), hence only half of the generators.

2. From the non-trivial generators we can define

\[
a_I \equiv \frac{1}{\sqrt{4E}} Q^I_2, \quad a_I^\dagger \equiv \frac{1}{\sqrt{4E}} \bar{Q}^I_2.
\]
These operators satisfy the anticommutation relations of a set of $N$ creation and $N$ annihilation operators

$$\{a_I, a_J^\dagger\} = \delta^{IJ}, \quad \{a_I, a_J\} = 0, \quad \{a_I^\dagger, a_J^\dagger\} = 0. \quad (3.10)$$

These are the basic tools we need in order to construct irreps of the supersymmetry algebra. Notice that when acting on some state, the operators $Q_I^2$ and $\bar{Q}_I^2$ (and hence $a_I$ and $a_I^\dagger$) lower respectively raise the helicity of half unit, since

$$[M_{12}, Q_I^2] = i(\sigma_{12})_2^2 Q_I^2 = -\frac{1}{2} Q_I^2, \quad [M_{12}, \bar{Q}_I^2] = \frac{1}{2} \bar{Q}_I^2, \quad (3.11)$$

and $J_3 = M_{12}$.

3. To construct a representation, one can start by choosing a state annihilated by all $a_I$'s (known as the Clifford vacuum): such state will carry some irrep of the Poincaré algebra. Besides having $m = 0$, it will carry some helicity $\lambda_0$, and we call it $|E, \lambda_0\rangle$ ($|\lambda_0\rangle$ for short). For this state

$$a_I |\lambda_0\rangle = 0. \quad (3.12)$$

Note that this state can be either bosonic or fermionic, and should not be confused with the actual vacuum of the theory, which is the state of minimal energy: the Clifford vacuum is a state with quantum numbers $(E, \lambda_0)$ and which satisfies eq. (3.12).

4. The full representation (aka supermultiplet) is obtained acting on $|\lambda_0\rangle$ with the creation operators $a_I^\dagger$ as follows

$$|\lambda_0\rangle, \quad a_I^\dagger |\lambda_0\rangle \equiv |\lambda_0 + \frac{1}{2}\rangle_I, \quad a_I^\dagger a_I^\dagger |\lambda_0\rangle \equiv |\lambda_0 + 1\rangle_{IJ}, \quad \ldots, \quad a_I^\dagger a_2^\dagger \ldots a_N^\dagger |\lambda_0\rangle \equiv |\lambda_0 + \frac{N}{2}\rangle.$$

Hence, starting from a Clifford vacuum with helicity $\lambda_0$, the state with highest helicity in the representation has helicity $\lambda = \lambda_0 + \frac{N}{2}$. Due to the antisymmetry in $I \leftrightarrow J$, at helicity level $\lambda = \lambda_0 + \frac{k}{2}$ we have

$$\# \text{ of states with helicity } \lambda_0 + \frac{k}{2} = \binom{N}{k}, \quad (3.13)$$
where \( k = 0, 1, \ldots, N \). The total number of states in the irrep will then be

\[
\sum_{k=0}^{N} \binom{N}{k} = 2^N = (2^{N-1})_B + (2^{N-1})_F ,
\]

(3.14)

half of them having integer helicity (bosons), half of them half-integer helicity (fermions).

5. CPT flips the sign of the helicity. Therefore, unless the helicity is distributed symmetrically around 0, which is not the case in general, a supermultiplet is not CPT-invariant. This means that in order to have a CPT-invariant theory one should in general double the supermultiplet we have just constructed adding its CPT conjugate. This is not needed if the supermultiplet is self-CPT conjugate, which can happen only if \( \lambda_0 = -\frac{N}{4} \) (in this case the helicity is indeed distributed symmetrically around 0).

Let us now apply the above procedure and construct several (physically interesting) irreps of the supersymmetry algebra.

**N = 1 supersymmetry**

- Matter multiplet (aka chiral multiplet):
  
  \[ \lambda_0 = 0 \rightarrow \left( 0, \frac{1}{2} \right) \oplus_{CPT} \left( -\frac{1}{2}, 0 \right) . \]

  (3.15)

  The degrees of freedom of this representation are those of one Weyl fermion and one complex scalar (on shell; recall we are constructing supersymmetry representations on states!). In a \( N = 1 \) supersymmetric theory this is the representation where matter sits; this is why such multiplets are called matter multiplets. For historical reasons, these are also known as Wess-Zumino multiplets.

- Gauge (or vector) multiplet:
  
  \[ \lambda_0 = \frac{1}{2} \rightarrow \left( \frac{1}{2}, +1 \right) \oplus_{CPT} \left( -1, -\frac{1}{2} \right) . \]

  (3.16)

  The degrees of freedom are those of one vector and one Weyl fermion. This is the representation one needs to describe gauge fields in a supersymmetric theory. Notice that since internal symmetries (but the \( R \)-symmetry) commute
with the supersymmetry algebra, the representation the Weyl fermion should transform under gauge transformations should be the same as the vector field, i.e. the adjoint. Hence, usual SM matter (quarks and leptons) cannot be accommodated in these multiplets.

Although in this course we will focus on rigid supersymmetry and hence not consider supersymmetric theories with gravity, let us list for completeness (and future reference) also representations containing states with higher helicity.

- Spin 3/2 multiplet:

\[
\lambda_0 = 1 \rightarrow (1, +\frac{3}{2}) \bigoplus_{CPT} \left( -\frac{3}{2}, -1 \right) .
\]

(3.17)

The degrees of freedom are those of a spin 3/2 particle and one vector.

- Graviton multiplet:

\[
\lambda_0 = \frac{3}{2} \rightarrow \left( +\frac{3}{2}, +2 \right) \bigoplus_{CPT} \left( -2, -\frac{3}{2} \right) .
\]

(3.18)

The degrees of freedom are those of a graviton, which has helicity 2, and a particle of helicity 3/2, known as the gravitino (which is indeed the supersymmetric partner of the graviton).

Representations constructed from a Clifford vacuum with higher helicity, will inevitably include states with helicity higher than 2. Hence, if one is interested in interacting local field theories, the story stops here. Recall that massless particles with helicity higher than \( \frac{1}{2} \) should couple to conserved quantities at low momentum. The latter are: conserved internal symmetry generators for (soft) massless vectors, supersymmetry generators for (soft) gravitinos and four-vector \( P_\mu \) for (soft) gravitons. The supersymmetry algebra does not allow for generators other than these ones; hence, supermultiplets with helicity \( \lambda \geq \frac{5}{2} \) are ruled out: they may exist, but they cannot have couplings that survive in the low energy limit.

The above discussion also implies that in a local interacting field theory a spin 3/2 particle is inevitably associated to local supersymmetry and hence, in turn, with gravity. Hence there is no much meaning for a theory without the graviton multiplet an a spin 3/2 multiplet, which would be a non-intercating one in fact. In other words, the physical gravitino is the one in the graviton multiplet.

\[ \mathbf{N = 2} \text{ supersymmetry} \]
• Matter multiplet (aka hypermultiplet):

\[ \lambda_0 = -\frac{1}{2} \rightarrow \left( -\frac{1}{2}, 0, 0, +\frac{1}{2} \right) \oplus_{\text{CPT}} \left( -\frac{1}{2}, 0, 0, +\frac{1}{2} \right). \]  

The degrees of freedom are those of two Weyl fermions and two complex scalars. This is where matter sits in a \( N = 2 \) supersymmetric theory. In \( N = 1 \) language this representation corresponds to two Wess-Zumino multiplets with opposite chirality (CPT flips the chirality). Notice that in principle this representation enjoys the CPT self-conjugate condition \( \lambda_0 = -\frac{N}{4} \). However, a closer look shows that an hypermultiplet cannot be self-conjugate (that’s why we added the CPT conjugate representation). The way the various states are constructed out of the Clifford vacuum, shows that under \( SU(2) \) \( R \)-symmetry the helicity 0 states behave as a doublet while the fermionic states are singlets. If the representation were CPT self-conjugate the two scalar degrees of freedom would have been both real. Such states cannot form a \( SU(2) \) doublet since a two-dimensional representation of \( SU(2) \) is pseudoreal, and hence the doublet should be complex.

• Gauge (or vector) multiplet:

\[ \lambda_0 = 0 \rightarrow \left( 0, +\frac{1}{2}, +\frac{1}{2}, +1 \right) \oplus_{\text{CPT}} \left( -1, -\frac{1}{2}, -\frac{1}{2}, 0 \right). \]  

The degrees of freedom are those of one vector, two Weyl fermions and one complex scalar. In \( N = 1 \) language this is just a vector and a matter multiplet (beware: both transforming in the same, i.e. adjoint representation of the gauge group).

• Spin 3/2 multiplet:

\[ \lambda_0 = -\frac{3}{2} \rightarrow \left( -\frac{3}{2}, -1, -1, -\frac{1}{2} \right) \oplus_{\text{CPT}} \left( +\frac{1}{2}, +1, +1, +\frac{3}{2} \right). \]  

The degrees of freedom are those of a spin 3/2 particle, two vectors and one Weyl fermion.

• Graviton multiplet:

\[ \lambda_0 = -2 \rightarrow \left( -2, -\frac{3}{2}, -\frac{3}{2}, -1 \right) \oplus_{\text{CPT}} \left( +1, +\frac{3}{2}, +\frac{3}{2}, +2 \right). \]  

The degrees of freedom are those of a graviton, two gravitini and a vector, which is usually called graviphoton in the supergravity literature.
$N = 4$ supersymmetry

- Gauge (or vector) multiplet:

$$
\lambda_0 = -1 \rightarrow \left( -1, 4 \times \frac{1}{2}, 6 \times 0, 4 \times +\frac{1}{2}, +1 \right). \quad (3.23)
$$

The degrees of freedom are those of one vector, four Weyl fermions and three complex scalars. In $N = 1$ language this corresponds to one vector multiplet and three matter multiplets (all transforming in the adjoint). Notice that this multiplet is CPT self-conjugate. This time there are no problems with $R$-symmetry transformations. The vector is a singlet under $SU(4)$, fermions transform under the fundamental representation, and scalars under the two times anti-symmetric representation, which is the fundamental of $SO(6)$, and is real. The fact that the representation under which the scalars transform is real also explains why for $N = 4$ supersymmetry, the $R$-symmetry group is not $U(4)$ but actually $SU(4)$.

For $N = 4$ it is not possible to have matter in the usual sense, since the number of supersymmetry generators is too high to avoid helicity one states. Therefore, $N = 4$ supersymmetry cannot accommodate fermions transforming in fundamental representations. Besides the vector multiplet there are of course also representations with higher helicity, but we refrain to report them here.

One might wonder why we did not discuss $N = 3$ representations. This is just because as far as non-gravitational theories are concerned, $N = 3$ and $N = 4$ are equivalent: when constructing $N = 3$ representations, once the CPT conjugate representation is added (in this case we cannot satisfy the condition $\lambda_0 = -\frac{N}{4}$) one ends up with a multiplet which is the same as the $N = 4$ vector multiplet.

$N > 4$ supersymmetry

In this case one can easily get convinced that it is not possible to avoid gravity since there do not exist representations with helicity smaller than $\frac{3}{2}$ when $N > 4$. Hence, theories with $N > 4$ are all supergravity theories. It is interesting to note that $N = 8$ supergravity allows only one possible representation with highest helicity smaller than $\frac{5}{2}$ and that for higher $N$ one cannot avoid states with helicity $\frac{5}{2}$ or higher. Therefore, $N = 8$ is an upper bound on the number of supersymmetry generators, as far as interacting local field theories are concerned. Beware: as stated, all
these statements are valid in four space-time dimensions. The way to count super-
symmetries depends on the dimension of space-time, since spinorial representations
get bigger, the more the dimensions. Obviously, completely analogous statements
can be made in higher dimensions. For instance, in ten space-time dimensions the
maximum allowed supersymmetry to avoid states with helicity $\frac{5}{2}$ or higher is $N = 2$.
A dimension-independent statement can be made counting the number of single
component supersymmetry generators. Using this language, the maximum allowed
number of supersymmetry generators for non-gravitational theories is 16 (which is
indeed $N = 4$ in four dimensions) and 32 for theories with gravity (which is $N = 8$
in four dimensions).

The table below summarizes all results we have discussed.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\lambda_{\text{max}} = 1$</th>
<th>$\lambda_{\text{max}} = \frac{1}{2}$</th>
<th>$\lambda_{\text{max}} = 2$</th>
<th>$\lambda_{\text{max}} = \frac{3}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>none</td>
<td>$[2(\frac{1}{2}), 8(\frac{3}{2}), 28(1), 56(\frac{1}{2}), 70(0)]$</td>
<td>$[\frac{5}{2}, 6(1), 15(\frac{1}{2}), 20(0)]$</td>
<td>none</td>
</tr>
<tr>
<td>6</td>
<td>none</td>
<td>$[2(\frac{1}{2}), 6(\frac{3}{2}), 16(1), 26(\frac{1}{2}), 30(0)]$</td>
<td>$[\frac{5}{2}, 6(1), 15(\frac{1}{2}), 20(0)]$</td>
<td>none</td>
</tr>
<tr>
<td>5</td>
<td>none</td>
<td>$[2(\frac{1}{2}), 5(\frac{3}{2}), 10(1), 11(\frac{1}{2}), 10(0)]$</td>
<td>$[\frac{5}{2}, 4(1), 7(\frac{1}{2}), 8(0)]$</td>
<td>none</td>
</tr>
<tr>
<td>4</td>
<td>$[(1), 4(\frac{1}{2}), 6(0)]$</td>
<td>$[2(\frac{1}{2}), 4(\frac{3}{2}), 6(1), 4(\frac{1}{2}), 2(0)]$</td>
<td>$[\frac{5}{2}, 3(1), 3(\frac{1}{2}), 2(0)]$</td>
<td>none</td>
</tr>
<tr>
<td>3</td>
<td>$[(1), 4(\frac{1}{2}), 6(0)]$</td>
<td>$[2(\frac{1}{2}), 3(\frac{3}{2}), 3(1), (\frac{1}{2})]$</td>
<td>$[\frac{5}{2}, 2(1), (\frac{1}{2})]$</td>
<td>none</td>
</tr>
<tr>
<td>2</td>
<td>$[(1), 2(\frac{1}{2}), 2(0)]$</td>
<td>$[2(\frac{1}{2}), 4(0)]$</td>
<td>$[2(\frac{1}{2}), 2(\frac{3}{2}), (1)]$</td>
<td>$[\frac{5}{2}, 2(1), (\frac{1}{2})]$</td>
</tr>
<tr>
<td>1</td>
<td>$[(1), (\frac{1}{2})]$</td>
<td>$[(\frac{1}{2})2(0)]$</td>
<td>$[2(\frac{1}{2})]$</td>
<td>$[\frac{5}{2}, (\frac{1}{2})]$</td>
</tr>
</tbody>
</table>

The numbers in parenthesis represent the helicity, while other numbers represent
the multiplicity of states with given helicity. Notice that, as anticipated, any super-
multiplet contains particles with spin at least as large as $\frac{1}{2}N$.

A final, very important comment regards chirality. The SM is a chiral the-
ory, in the sense that there exist particles in the spectrum whose chiral and anti-
chiral components transform differently under the gauge group (weak interactions
are chiral). When it comes to supersymmetric extensions, it is easy to see that
only $N = 1$ theories allow for chiral matter. That $N = 1$ irreps can be chiral is
obvious: Wess-Zumino multiplets contain one single Weyl fermion. Therefore, in
$N = 1$ supersymmetric extensions of the Standard Model (SM) one can accomo-
date left and right components of leptons and quarks in different multiplets, which
then transform differently under the $SU(2)$ gauge group. Let us now consider
extended supersymmetry. First notice that all helicity $\frac{1}{2}$ states belonging to multi-
plets containing vector fields should transform in the adjoint representation of the
gauge group, which is real. Therefore, the only other representation which might
allow for helicity $\frac{1}{2}$ states transforming in fundamental representations is the $N = 2$
hypermultiplet. However, as already noticed, a hypermultiplet contains two Wess-
Zumino multiplets with opposite chirality. Since for any internal symmetry group $G$,
we have that $[G, \text{SuperPoincaré}] = 0$, these two Wess-Zumino multiplets transform
in the same representation under $G$. Therefore, $N = 2$ is non-chiral: left and right
components of leptons and quarks would belong to the same matter multiplet and
could not transform differently under the $SU(2)$ SM gauge group. Summarizing,
if extended supersymmetry is realized in Nature, it should be broken at some high
enough energy scale to an effective $N = 1$ model. This is why at low energy people
typically focus just on $N = 1$ extensions of the SM.

3.2 Massive supermultiplets

The logical steps one should follow for massive representations are similar to previous
ones. There is however one important difference. Let us consider a state with mass
$m$ in its rest frame $P_\mu = (m, 0, 0, 0)$. One can easily see that, differently from
the massless case, the number of non-trivial generators gets not diminished: there
remain the full set of $2N$ creation and $2N$ annihilation operators. Indeed, eq. (2.61)
is now

$$\left\{ Q^I_\alpha, \bar{Q}^J_\dot{\beta} \right\} = 2m \delta_{\alpha\dot{\beta}} \delta^{IJ}$$

and no supersymmetric generators are trivially realized. This means that, gener-
ically, massive representations are longer than massless ones. Another important
difference is that we better speak of spin rather than helicity, now. A given Clifford
vacuum will be defined by mass $m$ and spin $j$, with $j(j + 1)$ being the eigenvalue
of $J^2$. Hence, the Clifford vacuum will have itself degeneracy $2j + 1$ since $j_3$ takes
values from $-j$ to $+j$.

$N = 1$ supersymmetry

The annihilation and creation operators, satisfying the usual oscillator algebra,
now read

$$a_{1,2} = \frac{1}{\sqrt{2m}} Q_{1,2} \quad , \quad a_{1,2}^\dagger = \frac{1}{\sqrt{2m}} \bar{Q}_{1,2}.$$  

As anticipated these are twice those for the massless case. Notice that $a_{1,2}^\dagger$ lowers the
spin by half unit while $a_{1,2}^\dagger$ raises it. We can now define a Clifford vacuum as a state
with mass $m$ and spin $j_0$ which is annihilated by both $a_1$ and $a_2$ and act with the creation operators to construct the corresponding massive representations.

- **Matter multiplet:**
  \[ j = 0 \rightarrow \left(-\frac{1}{2}, 0, 0', +\frac{1}{2}\right). \]  
  (3.26)
  The number of degrees of freedom is the same as the massless case (but with no need to add any CPT conjugate, of course). It is worth noticing that the second scalar state, dubbed $0'$, has opposite parity with respect to the state 0, that is, it is a pseudoscalar (the proof is left to the reader as a trivial exercise; hint: use anticommutation properties of the creation operators). So the scalar is indeed a complex scalar. Summarizing, the multiplet is made of a massive complex scalar and a massive Majorana fermion.

- **Gauge (or vector) multiplet:**
  \[ j = \frac{1}{2} \rightarrow \left(-1, 2 \times -\frac{1}{2}, 2 \times 0, 2 \times +\frac{1}{2}, 1\right). \]  
  (3.27)
  The degrees of freedom one ends-up with are those of one massive vector, one massive Dirac fermion and one massive real scalar (recall the comment after eq. (3.24)!)! The representation is longer than that of a massless vector supermultiplet, as expected. Notice that these degrees of freedom are the same as those of a massless vector multiplet plus one massless matter multiplet. This is self-consistent, since we do not like massive vectors to start with, and only allow Higgs-like mechanisms to generate masses for vector fields. In a renormalizable supersymmetric theory, one can generate massive vector multiplets by the supersymmetric generalization of the Higgs mechanism, in which a massless vector multiplet eats-up a chiral multiplet.

Since we cannot really make sense of massive particles with spin higher than one (and we are not much interested in supergravity theories in this course, anyway), we stop here and move to extended supersymmetry representations.

**Extended supersymmetry**

Let us then consider $N > 1$ and allow also for non-trivial central charges. A change of basis in the space of supersymmetry generators turns out to be useful for the following analysis. Since the central charge matrix $Z^{IJ}$ is antisymmetric, with a $U(N)$ rotation one can put it in the standard block-diagonal form.
(we have assumed for simplicity that $N$ is even). One can now define

$$a_r^{\alpha} = \frac{1}{\sqrt{2}} \left( Q_\alpha^1 + \epsilon_{\alpha\beta}(Q_\beta^2)^\dagger \right)$$

$$b_r^{\alpha} = \frac{1}{\sqrt{2}} \left( Q_\alpha^1 - \epsilon_{\alpha\beta}(Q_\beta^2)^\dagger \right)$$

$$a_s^{\alpha} = \frac{1}{\sqrt{2}} \left( Q_\alpha^2 + \epsilon_{\alpha\beta}(Q_\beta^4)^\dagger \right)$$

$$b_s^{\alpha} = \frac{1}{\sqrt{2}} \left( Q_\alpha^2 - \epsilon_{\alpha\beta}(Q_\beta^4)^\dagger \right)$$

$$\ldots = \ldots$$

$$\ldots = \ldots$$

$$a_r^{N/2} = \frac{1}{\sqrt{2}} \left( Q_\alpha^{N-1} + \epsilon_{\alpha\beta}(Q_\beta^N)^\dagger \right)$$

$$b_r^{N/2} = \frac{1}{\sqrt{2}} \left( Q_\alpha^{N-1} - \epsilon_{\alpha\beta}(Q_\beta^N)^\dagger \right)$$

which satisfy the oscillator algebra

$$\{ a_r^{\alpha}, (a_s^\beta)^\dagger \} = (2m + Z_r) \delta_{rs} \delta_{\alpha\beta}$$

$$\{ b_r^{\alpha}, (b_s^\beta)^\dagger \} = (2m - Z_r) \delta_{rs} \delta_{\alpha\beta}$$

$$\{ a_r^{\alpha}, (b_s^\beta)^\dagger \} = \{ a_r^{\alpha}, a_s^\beta \} = \ldots = 0$$

where $r, s = 1, \ldots, N/2$. As anticipated, we have now $2N$ creation operators

$$(a_r^{\alpha})^\dagger, \quad (b_r^{\alpha})^\dagger \quad r = 1, \ldots, N/2 \quad \alpha = 1, 2$$

which we can use to construct massive representations starting from some given Clifford vacuum. Notice that, from their very definition, it follows that creation
operators with spinorial index $\alpha = 1$ lower the spin by half unit, while those with spinorial index $\alpha = 2$ raise it.

Several important comments are in order, at this point. Due to the positiveness of the Hilbert space, from the oscillator algebra above we get

$$2m \geq |Z_r|, \quad r = 1, \ldots, \frac{N}{2}.$$ \hspace{1cm} (3.30)

This means that the mass of a given irrep is always larger (or equal) than (half) the modulus of any central charge eigenvalue. The first important consequence of the bound (3.30) is that for massless representations (for which the left hand side of the above equation is identically 0) the central charges are always trivially realized, i.e. $Z^{IJ} = 0$. That’s why we did not discuss massless multiplets with non vanishing central charges in the previous section.

Suppose none of the central charge eigenvalues saturates the bound (3.30), namely $2m > |Z_r|$, $\forall r$. Proceeding as before, starting from a Clifford vacuum $\lambda_0$ annihilated by all (undaggered version of) operators (3.29), one creates $2^{2N}$ states, $2^{2N-1}$ bosonic and $2^{2N-1}$ fermionic, with spin going from $\lambda_0 - N/2$ to $\lambda_0 + N/2$. Therefore, the representation has states with spins spanning $2N + 1$ half-integer values.

Suppose instead that some $Z_r$ saturate the bound (3.30), say $k \leq N/2$ of them do so. Looking at the oscillator algebra one immediately sees that $k b$-type operators become trivial (we are supposing, without loss of generality, that all $Z_r$ are positive), and the dimension of the representation diminishes accordingly. The multiplet contains only $2^{2(N-k)}$ states now. These are called short multiplets. The extreme case is when all $Z_r$ saturate the bound ($k = N/2$). In this case half the creation operators trivialize and we get a multiplet, known as ultrashtort, whose dimension is identical to that of a massless one: the number of states is indeed $2^N$, $2^{N-1}$ bosonic and $2^{N-1}$ fermionic.

The upshot of the discussion above is that in theories with extended supersymmetry one can have massive multiplets with different lengths: degrees of freedom):

\begin{align*}
\text{long multiplets} & \quad 2^{2N} = (2^{2N-1})_B + (2^{2N-1})_F \\
\text{short multiplets} & \quad 2^{2N-2k} = (2^{2(N-k)-1})_B + (2^{2(N-k)-1})_F \\
\text{ultra-short multiplets} & \quad 2^{2N-2N/2} = 2^N = (2^{N-1})_B + (2^{N-1})_F.
\end{align*}

One final comment is that all states belonging to some representation of supersymmetry also transform into given representations of the R-symmetry group,
since the supercharges do so. This can be checked case by case, knowing how states
are constructed out of the Clifford vacuum. One should just remember that the
R-symmetry group is $U(N)$ in absence of central charges and $USp(N)$ if central
charges are present.

$N = 2$ supersymmetry

In this case we have one only central charge eigenvalue, $Z$, and we have four
oscillators $a_\alpha$ and $b_\alpha$ (we have drop the now inessential upper index $r$).

In the following, the Clifford vacuum will be defined as a state annihilated by all
undaggered operators, unless otherwise stated. Let us first consider the case of long
multiplets, namely a situation in which the bound (3.30) is not saturated. In this
case we cannot have massive matter since we have too many creation operators to
avoid spins higher than $\frac{1}{2}$. So the only possibility are (massive) vector multiplets.

- **Gauge (or vector) multiplet:**

  \[
  j = 0 \rightarrow \left( -1, 4 \times -\frac{1}{2}, 6 \times 0, 4 \times +\frac{1}{2}, 1 \right). \tag{3.31}
  \]

  The degrees of freedom correspond to a massive vector, two Dirac fermions,
  and five real scalars. Their number equals that of a massless $N = 2$ vector
  multiplet and a massless $N = 2$ hypermultiplet. As before, such massive
  vector multiplet should be thought of as obtained via some supersymmetric
  Higgs-like mechanism.

Let us now consider shorter representations. Since the central charge matrix is
$2 \times 2$, we have only one eigenvalue, $Z$, and the only possible short representation is
in fact the ultrashort, whose length should equal that of the corresponding massless
representations. The only non-trivial oscillators are now $a_\alpha$, since the $b_\alpha$ are trivially
realized.

- **Matter multiplet (short rep.):**

  \[
  j = 0 \rightarrow \left( 2 \times -\frac{1}{2}, 4 \times 0, 2 \times +\frac{1}{2} \right), \tag{3.32}
  \]

  (where the doubling of states arises for similar reasons as for the massless hyper-
  multiplet). The degrees of freedom are those of one massive Dirac fermion
  and two massive complex scalars. As expected the number of degrees of free-
  dom equals those of a massless hypermultiplet.
Vector multiplet (short rep.):

In this case it is more convenient to define the Clifford vacuum as a spin -1 state annihilated by $a_2$ and $a_1^\dagger$ so both creation operators rise the spin by $1/2$ unit. We get

$$j = -1 \rightarrow \left( -1, 2 \times \frac{1}{2}, 2 \times 0, 2 \times \frac{1}{2}, +1 \right).$$

The degrees of freedom are those of one massive vector, one Dirac fermion and one real scalar. While rearranged differently in terms of fields, the number of bosonic and fermionic degrees of freedom equals that of a massless vector multiplet. What’s interesting here is that a massive ultrashort vector multiplet can arise dynamically, via some Higgs-like mechanism involving only a massless vector multiplet, something peculiar to $N = 2$ supersymmetry and related to the fact that massless vector multiplets contain scalars, and can then self-Higgs.

$N = 4$ supersymmetry

For $N = 4$ supersymmetry long multiplets are not allowed since the number of states (actually $256!$) would include at least spin 2 states; such a theory would then include a massive spin 2 particle which is not believed to be possible in a local quantum field theory. However, in $N = 4$ supersymmetry one can allow for ultrashort vector multiplets, whose construction is left to the reader and whose field content simply amounts to rearrange the fields characterizing a massless vector multiplet into massive states: one would get a massive vector, two Dirac fermions and five real scalars.

Let us end this section with an important remark. All short multiplets are supersymmetry preserving. Indeed, they are annihilated by some supersymmetry generators (those whose corresponding central charge eigenvalue saturates the bound). In general one can then have $\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N/2}{N}$ supersymmetry preserving multiplets (the numerator is nothing but the integer $k$ previously defined). For instance, ultrashort multiplets, for which $k = N/2$, are $\frac{1}{2}$ supersymmetry preserving states. Such multiplets have very important properties at the quantum level; most notably, it turns out that they are more protected against quantum corrections with respect to long multiplets. Short multiplets are also called BPS, since the bound (3.30) is very much reminiscent of the famous Bogomolnyi-Prasad-Sommerfeld bound which is saturated by solitons, typically. This is not just a mere analogy, since the bound
(3.30) is in fact not just an algebraic relation but it has a very concrete physical meaning: it is nothing but a specific BPS-like bound. Indeed, short multiplets often arise as solitons in supersymmetric field theories, and central charges correspond to physical (topological) charges. We will see examples of BPS states later in this course.

3.3 Representation on fields: a first try

So far we have discussed supersymmetry representations on states. However, we would like to discuss supersymmetric field theories, eventually. Therefore, we need to construct supersymmetric representations in terms of multiplets of fields rather than multiplets of states. In principle, following our previous strategy this can be done quite easily.

Let us focus on $N = 1$ supersymmetry, for simplicity. To build a representation of the supersymmetry algebra on fields, we start from some field $\phi(x)$ for which

$$[\tilde{Q}_\alpha, \phi(x)] = 0 \quad (3.34)$$

The field $\phi$ is the analogous of the Clifford vacuum $|\lambda_0\rangle$ we used previously, the ground state of the representation. Similarly as before, acting on this ground state $\phi(x)$ with the supersymmetry generator $Q_\alpha$, we can generate new fields out of it, all belonging to the same representation.

For definiteness, we choose $\phi(x)$ to be a scalar field, but one can also have ground states which are fields with some non-trivial tensor structure, as we will later see. Not much of what we want to say here depends on this choice. The first thing to notice is that the scalar field $\phi(x)$ is actually complex. Suppose it were real. Then, taking the hermitian conjugate of eq. (3.34) one would have obtained

$$[Q_\alpha, \phi(x)] = 0 \quad (3.35)$$

One can now use the generalized Jacobi identity for $(\phi, Q, \tilde{Q})$ and get

$$[\phi(x), \{Q_\alpha, \tilde{Q}_\alpha\}] + \{Q_\alpha, [\tilde{Q}_\alpha, \phi(x)]\} - \{\tilde{Q}_\alpha, [\phi(x), Q_\alpha]\} = 0$$

$$\rightarrow 2\sigma^\mu [\phi(x), P_\mu] = 0 \quad \rightarrow [P_\mu, \phi(x)] \sim \partial_\mu \phi(x) = 0 \quad (3.36)$$

which should then imply that the field is actually a constant (not a field, really!). So better $\phi(x)$ to be complex. In this case eq. (3.35) does not hold, but rather

$$[Q_\alpha, \phi(x)] \equiv \psi_\alpha(x) \quad (3.37)$$
This automatically defines a new field \( \psi_\alpha \) belonging to the same representation (since \( \phi \) is a scalar, \( \psi \) is a Weyl spinor). The next step is to see whether acting with the supersymmetry generators on \( \psi_\alpha \) gives new fields or just derivatives (or combinations) of fields already present in the representation. In principle we have

\[
\{ Q_\alpha, \psi_\beta(x) \} = F_{\alpha\beta}(x) \tag{3.38}
\]

\[
\{ \bar{Q}_\dot{\alpha}, \psi_\beta(x) \} = X_{\dot{\alpha}\beta}(x) . \tag{3.39}
\]

Enforcing the generalized Jacobi identity on \((\phi, Q, \bar{Q})\), and using eq. (3.37), after some trivial algebra one gets

\[
X_{\dot{\alpha}\dot{\beta}} = \{ \psi_\beta(x), \bar{Q}_\dot{\alpha} \} = 2\sigma^\mu_{\dot{\beta}\dot{\alpha}} [P_\mu, \phi] \sim \partial_\mu \phi , \tag{3.40}
\]

which implies that \( X_{\dot{\alpha}\dot{\beta}} \) is not a new field but just the space-time derivative of the scalar field \( \phi \). Let us now enforce the generalized Jacobi identity on \((\phi, Q, Q)\). Since the \( Q \)’s anticommute (recall we are considering \( N = 1 \) supersymmetry and hence there are no central charges) one simply gets

\[
\{ Q_\alpha, [Q_\beta, \phi] \} - \{ Q_\beta, [\phi, Q_\alpha] \} = 0 \quad \rightarrow \quad F_{\alpha\beta} + F_{\beta\alpha} = 0 . \tag{3.41}
\]

This says that the field \( F_{\alpha\beta} \) is antisymmetric under \( \alpha \leftrightarrow \beta \), which implies that

\[
F_{\alpha\beta}(x) = \epsilon_{\alpha\beta} F(x) . \tag{3.42}
\]

So we find here a new scalar field \( F \). Acting on it with the supersymmetry generators produces new (?) fields as

\[
[ Q_\alpha, F ] = \lambda_\alpha \tag{3.43}
\]

\[
[ \bar{Q}_\dot{\alpha}, F ] = \bar{\chi}_{\dot{\alpha}} . \tag{3.44}
\]

Using the generalized Jacobi identities for \((\psi, Q, Q)\) and \((\psi, Q, \bar{Q})\), respectively, one can easily prove that \( \lambda_\alpha \) is actually vanishing and that \( \bar{\chi}_{\dot{\alpha}} \) is proportional to the space-time derivative of the field \( \psi \). So no new fields here: after a certain number of steps the representation closes. The multiplet of fields we have found is then

\[
(\phi, \psi, F) . \tag{3.45}
\]

If \( \phi \) is a scalar field, as we have supposed here, this multiplet is a matter multiplet since it contains particles with spin 0 and 1/2 only. It is called chiral or Wess-Zumino
multiplet and it is indeed the field theory counterpart of the chiral multiplet of states we have constructed before. Notice that the equality of the number of fermionic and bosonic states for a given representation still holds: we are now off-shell, and the spinor $\psi_a$ has four degrees of freedom; this is the same number of bosonic degrees of freedom, two coming from the scalar field $\phi$ and two from the scalar field $F$

\[(\text{Re}\phi, \text{Im}\phi, \text{Re}F, \text{Im}F)_B , \quad (\text{Re}\psi_1, \text{Im}\psi_1, \text{Re}\psi_2, \text{Im}\psi_2)_F . \quad (3.46)\]

While we see the expected degeneracy between bosonic and fermionic degrees of freedom, they do not match those of the chiral multiplet of states we have constructed before, which are just $2_B + 2_F$. This is because we are off-shell, now. In fact, going on-shell, the 4 fermionic degrees of freedom reduce to just 2 propagating degrees of freedom, due to Dirac equation. The same sort of reduction should occur for the bosonic degrees of freedom, so to match the $2_B + 2_F$ on-shell condition. But Klein-Gordon equation does not diminish the number of independent degrees of freedom! What happens is that $F$ turns out to be a non-dynamical auxiliary field: as we will see when constructing Lagrangians, the equation of motions for $F$ simply tells that this scalar field is not an independent field but rather a (specific) function of other fields, $F = F(\phi, \psi)$. This is not specific to the chiral multiplet we have constructed, but it is in fact a completely general phenomenon. We will come back to this point in next lectures.

The procedure we have followed to construct the multiplet $(3.45)$ can be easily generalized. Modifying the condition $(3.34)$ one can construct other kind of multiplets, like linear multiplets, vector multiplets, etc... And/or, construct chiral multiplets with different field content, simply by defining a ground state carrying some space-time index, letting $\phi$ being a spinor, a vector, etc...

Out of a set of multiplets with the desired field content, one can construct supersymmetric field theories via a suitable Lagrangian made out of these fields. In order for the theory to be supersymmetric, this Lagrangian should (at most) transform as a total space-time derivative under supersymmetry transformations. Indeed, in this case, the action constructed out of it

\[S = \int d^4x \mathcal{L} , \quad (3.47)\]

will be supersymmetric invariant.

In practice, to see whether a given action is invariant under supersymmetry is rather cumbersome: one should take any single term in the Lagrangian, act on it
with supersymmetry transformations and prove that the variations of all (possibly very many) terms sum-up to a total space-time derivative. This turns out to be very involved, in general. This difficulty is due to the fact that the formulation above is a formulation in which supersymmetry is not manifest.

Theoretical physicists came up with a brilliant idea to circumvent this problem. Ordinary field theories are naturally defined in Minkowski space and in such formulation it is easy to construct Lagrangians respecting Poincaré symmetry. It turns out that supersymmetric field theories are naturally defined on an extension of Minkowski space, known as *superspace*, which, essentially, takes into account the extra space-time symmetries associated to the supersymmetry generators. In such extended space it is much easier to construct supersymmetric Lagrangians, and indeed the superspace formalism is what is most commonly used nowadays to discuss supersymmetric field theories. This is the formalism we will use along this course, and next lecture will be devoted to a throughout introduction of superspace.

### 3.4 Exercises

1. Prove that \( P^2 \) and \( W^2 \) are Casimir of the Poincaré algebra.

2. Prove that CPT flips the sign of the helicity.

3. Construct explicitly the \( N=4,1/2 \) BPS vector multiplet. Discuss its (massive) field content and its relation with the massless vector multiplet. Can one construct a \( 1/4 \) BPS \( N=4 \) vector multiplet?

4. Enforcing the generalized Jacobi identity on \((\psi,Q,Q)\) and \( (\psi,Q,\bar{Q}) \), using eqs. (3.38), (3.39), (3.43) and (3.44), prove that

\[
\lambda_\alpha = 0 \quad , \quad \bar{\chi}_\beta \sim \partial_\mu (\sigma^\mu \psi)_{\bar{\beta}}.
\]

### References

