4 Superspace and superfields

The usual space-time Lagrangian formulation is not the most convenient one for describing supersymmetric field theories. This is because in ordinary space-time supersymmetry is not manifest. In fact, an extension of ordinary space-time, known as superspace, happens to be the best and most natural framework in which to formulate supersymmetric theories. The basic idea of \( (N = 1) \) superspace is to enlarge the space-time labelled with coordinates \( x^\mu \), associated to the generators \( P_\mu \), by adding \( 2 + 2 \) anti-commuting Grassmann coordinates \( \theta_\alpha, \bar{\theta}_\dot{\alpha} \), associated to the supersymmetry generators \( Q_\alpha, \bar{Q}_{\dot{\alpha}} \), and obtain a eight coordinate superspace labelled by \( (x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \). In such an apparently exotic space many mysterious (or hidden) properties of supersymmetric field theories become manifest. As we will see, at the price of learning a few mathematical new ingredients, the goal of constructing supersymmetric field theories will be achieved much more easily, and within a framework in which many classical and quantum properties of supersymmetry will be more transparent.

In this lecture we will introduce superspace and superfields. In subsequent lectures we will use this formalism to construct supersymmetric field theories and study their dynamics.

4.1 Superspace as a coset

Let us start recalling the relation between ordinary Minkowski space and the Poincaré group. Minkowski space is a four-dimensional coset space defined as

\[
\mathcal{M}_{1,3} = \frac{ISO(1,3)}{SO(1,3)},
\]

where \( ISO(1,3) \) is the Poincaré group and \( SO(1,3) \) the Lorentz group. The Poincaré group \( ISO(1,3) \) is nothing but the isometry group of this coset space, which means that any point of \( \mathcal{M}_{1,3} \) can be reached from the origin with a Poincaré transformation. This transformation, however, is defined up to Lorentz transformations. Therefore, each coset class (≡ a point in space-time) has a unique representative which is just a translation and can be parametrized by a coordinate \( x^\mu \)

\[
x^\mu \leftrightarrow e^{(x^\mu P_\mu)}.
\]

Superspace can be defined along similar lines. The first thing we need to do is to extend the Poincaré group to the so-called superPoincaré group. In order to do this,
given that a group is the exponent of the algebra, we have to rewrite the whole supersymmetry algebra in terms of commutators, namely as a Lie algebra. This is achieved by introducing a set of constant Grassmann numbers \( \theta_\alpha, \bar{\theta}_\beta \), defined as to anti-commute with everything fermionic and commute with everything bosonic

\[
\{\theta^\alpha, \theta^\beta\} = 0 \quad , \quad \{\bar{\theta}_\alpha, \bar{\theta}_\beta\} = 0 \quad , \quad \{\theta^\alpha, \bar{\theta}_\beta\} = 0 .
\] (4.3)

This allows to transform anti-commutators of the supersymmetry algebra into commutators, and get

\[
[\theta Q, \bar{\theta} Q] = 2 \theta \sigma^\mu \bar{\theta} P_\mu \quad , \quad [\theta Q, \theta Q] = [\bar{\theta} Q, \bar{\theta} Q] = 0 ,
\] (4.4)

where as usual \( \theta Q \equiv \theta_\alpha Q_\alpha \), \( \bar{\theta} Q \equiv \bar{\theta}_\beta \bar{Q}_\beta \). This way, one can write the supersymmetry algebra solely in terms of commutators. Exponentiating this algebra one gets the superPoincaré group. A generic group element can then be written as

\[
G(x, \theta, \bar{\theta}, \omega) = \exp(ixP + i\theta Q + i\bar{\theta} Q + \frac{1}{2}i\omega M) ,
\] (4.5)

where \( xP \) is a shorthand notation for \( x^\mu P_\mu \) and \( \omega M \) a shorthand notation for \( \omega^{\mu\nu} M_{\mu\nu} \).

The superPoincaré group, mathematically, is \( Osp(4|1) \). Let us open a brief parenthesis and explain such a notation. Let us define the graded Lie algebra \( Osp(2p|N) \) as the grade one Lie algebra \( \mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1 \) whose generic element can be written as a matrix of complex dimension \( (2p + N) \times (2p + N) \)

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\] (4.6)

where \( A \) is a \( (2p \times 2p) \) matrix, \( B \) a \( (2p \times N) \) matrix, \( C \) a \( (N \times 2p) \) matrix and \( D \) a \( (N \times N) \) matrix. An element of \( \mathbb{L}_0 \) respectively \( \mathbb{L}_1 \) has entries

\[
\begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix}
\] respectively

\[
\begin{pmatrix}
0 & B \\
C & 0
\end{pmatrix}
\] (4.7)

where

\[
A^T \Omega_{(2p)} + \Omega_{(2p)} A = 0 \\
D^T \Omega_{(N)} + \Omega_{(N)} D = 0 \\
C = \Omega_{(N)} B^T \Omega_{(2p)}
\]
and
\[ \Omega^2_{(2p)} = -\mathbb{1}, \quad \Omega^T_{(N)} = \Omega_{(N)}, \quad \Omega^R_{(2p)} = -\Omega_{(2p)}. \tag{4.8} \]

This implies that the matrices \( A \) span a \( Sp(2p, \mathbb{C}) \) algebra and the matrices \( D \) a \( O(N, \mathbb{C}) \) algebra. Therefore we have that
\[ \mathbb{L}_0 = Sp(2p) \otimes O(N), \tag{4.9} \]
hence the name \( Osp(2p|N) \) for the whole superalgebra. A generic element of the superalgebra has the form
\[ Q = q^a t_a + q^I t_I, \tag{4.10} \]
where \( t_a \in \mathbb{L}_0 \) and \( t_I \in \mathbb{L}_1 \) are a basis of the corresponding vector spaces, and we have introduced complex numbers \( q^a \) for \( \mathbb{L}_0 \) and Grassman numbers \( q^I \) for \( \mathbb{L}_1 \) (recall why and how we introduced the fermionic parameters \( \theta_\alpha, \bar{\theta}_{\dot{\alpha}} \) before).

Taking now \( p = 2 \) we have the algebra \( Osp(4|N) \). This is not yet what we are after. The last step, which we do not describe in detail here, amounts to take the so-called Inonu-Wigner contraction. Essentially, one has to rescale (almost) all generators by a constant \( 1/\bar{e} \), rewrite the algebra in terms of the rescaled generators and take the limit \( \bar{e} \to 0 \). What one ends up with is the \( N \)-extended supersymmetry algebra in Minkowski space we all know, dubbed \( Osp(4|N) \), where in the aforementioned limit one gets the identification
\[ A \to P_\mu, M_{\mu\nu} \quad D \to Z^{IJ}, \quad B,C \to Q_I, \bar{Q}_\dot{I}. \tag{4.11} \]
Taking \( N = 1 \) one finally gets the unextended supersymmetry algebra \( Osp(4|1) \).

Given the generic group element of the superPoincaré group (4.5), \( N = 1 \) superspace is defined as the \((4+4 \text{ dimensional})\) group coset
\[ \mathcal{M}_{4|1} = \frac{Osp(4|1)}{SO(1,3)}. \tag{4.12} \]
A point in superspace (point in a loose sense, of course, given the non-commutative nature of the Grassman parameters \( \theta_\alpha, \bar{\theta}_{\dot{\alpha}} \) gets identified with the coset representative corresponding to a so-called super-translation through the one-to-one map
\[ (x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \longleftrightarrow e^{(x^\mu P_\mu)} e^{(\theta Q + \bar{\theta} \bar{Q})}. \tag{4.13} \]
The \( 2 + 2 \) anti-commuting Grassmann numbers \( \theta_\alpha, \bar{\theta}_{\dot{\alpha}} \) can then be thought of as coordinates in superspace (in four-component notation they correspond to a Marojana spinor \( \theta \)). For these Grassmann numbers all usual spinor identities hold.
Thus far we have introduced what is known as $N = 1$ superspace. If discussing extended supersymmetry one should introduce, in principle, a larger superspace. There exist (two, at least) formulations of $N = 2$ superspace. However, these formulations present some subtleties and problems whose discussion is beyond the scope of this course. And no formulation is known of $N = 4$ superspace. In this course we will use $N = 1$ superspace even when discussing extended supersymmetry, as it is typically done in most of the literature.

4.2 Superfields as fields in superspace

Superfields are nothing but fields in superspace: functions of the superspace coordinates $(x^\mu, \theta, \bar{\theta})$. Since $\theta$ and $\bar{\theta}$ anticommute, any product involving more than two $\theta$’s or two $\bar{\theta}$’s vanishes: given that $\theta_\alpha \theta_\beta = -\theta_\beta \theta_\alpha$, we have that $\theta_\alpha \theta_\beta = 0$ for $\alpha = \beta$ and therefore $\theta_\alpha \theta_\beta \theta_\gamma = 0$, since at least two indexes in this product are the same. Hence, the most general superfield $Y = Y(x, \theta, \bar{\theta})$ has the following simple Taylor-like expansion

$$Y(x, \theta, \bar{\theta}) = f(x) + \theta \psi(x) + \bar{\theta} \bar{\psi}(x) + \theta \theta m(x) + \bar{\theta} \bar{\theta} n(x) + \theta \sigma^\mu \bar{\theta} v_\mu(x) + \theta \theta \bar{\theta} \bar{\theta} d(x) + \bar{\theta} \bar{\theta} \theta \rho(x) + \theta \theta \bar{\theta} \bar{\theta} d(x).$$

Each entry above is a field (possibly with some non-trivial tensor structure). In this sense, a superfield is nothing but a finite collection (a multiplet) of ordinary fields.

We aim at constructing supersymmetric Lagrangians out of superfields. In such Lagrangians superfields get multiplied one another, sometime we should act on them with derivatives, etc... Moreover, integration in superspace will be needed, eventually. Therefore, it is necessary to pause a bit and recall how operations of this kind work for Grassman variables.

Derivation in superspace is defined as follows

$$\partial_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} \quad \text{and} \quad \partial^\alpha = \epsilon^{\alpha \beta} \partial_\beta \quad \text{and} \quad \bar{\partial}_\dot{\alpha} \equiv \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \quad \text{and} \quad \bar{\partial}^{\dot{\alpha}} = \epsilon^{\dot{\alpha} \dot{\beta}} \bar{\partial}_{\dot{\beta}},$$

where

$$\partial_\alpha \theta_\beta = \delta_\alpha^\beta, \quad \bar{\partial}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}}, \quad \partial_\alpha \bar{\theta}_\beta = 0, \quad \bar{\partial}^{\dot{\alpha}} \theta^\beta = 0.$$

Note that this implies that $\partial^\alpha \theta_\beta = -\delta^\alpha_\beta$ and $\bar{\partial}^{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = -\delta^{\dot{\alpha}}_{\dot{\beta}}$ (that’s life!).

Let us consider a single Grassmann variable $\theta$ (either $\theta_1, \theta_2, \bar{\theta}_1$ or $\bar{\theta}_2$ in our case). Integration in $\theta$ is defined as follows

$$\int d\theta = 0 \quad \int d\theta \theta = 1.$$
This implies that for a generic function \( f(\theta) = f_0 + \theta f_1 \), the following results hold
\[
\int d\theta \; f(\theta) = f_1 , \quad \int d\theta \; \delta(\theta) f(\theta) = f_0 \quad \longrightarrow \quad \int = \partial , \quad \theta = \delta(\theta) . \quad (4.18)
\]
These relations can be easily generalized to \( N = 1 \) superspace, provided
\[
d^2 \theta \equiv \frac{1}{2} d\theta^1 d\theta^2 , \quad d^2 \bar{\theta} \equiv \frac{1}{2} d\bar{\theta}^2 d\bar{\theta}^1 . \quad (4.19)
\]
With these definitions one can prove the following useful identities
\[
\int d^2 \theta \; \theta \theta = \int d^2 \bar{\theta} \; \bar{\theta} \bar{\theta} = 1 , \quad \int d^2 \theta d^2 \bar{\theta} \; \theta \theta = 1
\]
\[
\int d^2 \theta = \frac{1}{4} \epsilon^{\alpha\beta} \partial_{\alpha} \partial_{\beta} , \quad \int d^2 \bar{\theta} = -\frac{1}{4} \bar{\epsilon}^{\dot{\alpha}\dot{\beta}} \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} . \quad (4.20)
\]
Another crucial question we need to answer is: how does a superfield transform under supersymmetry transformations? In order to answer this question we first need to realize the supersymmetry generators \( Q_{\alpha}, Q_{\dot{\alpha}} \) as differential operators, in the same way we do for the generators of Poincaré algebra (translations, rotations and boosts).

Let us recall how the story goes in ordinary space-time and consider a translation generated by \( P_\mu \) with infinitesimal parameter \( a^\mu \), on a field \( \phi(x) \) (for notational convenience, we will use momentarily calligraphic letters for the abstract operator and latin ones for the representation of the same operator as a differential operator in field space). This is defined as
\[
\phi(x + a) = e^{-ia^\mu P_\mu} \phi(x) e^{ia^\mu P_\mu} = \phi(x) - ia^\mu [P_\mu, \phi(x)] + \ldots . \quad (4.21)
\]
On the other hand, Taylor expanding the left hand side we get
\[
\phi(x + a) = \phi(x) + a^\mu \partial_\mu \phi(x) + \ldots \quad (4.22)
\]
Equating the right hand sides of the two equations above we then get
\[
[\phi(x), P_\mu] = -i \partial_\mu \phi(x) \equiv P_\mu \phi(x) , \quad (4.23)
\]
where \( P_\mu \) is the generator of translations and \( P_\mu \) is its representation as a differential operator in field space (recall that \( \partial_\mu \) is an operator and from \( (\partial_\mu)^* = \partial_\mu \) one gets that \( (\partial_\mu)^\dagger = -\partial_\mu \); hence \( P_\mu \) is hermitian, as it should). So, a translation of a field by parameter \( a^\mu \) induces a change on the field itself as
\[
\delta_a \phi \equiv \phi(x + a) - \phi(x) = ia^\mu P_\mu \phi . \quad (4.24)
\]
Notice that here and below we are using right multiplication, when acting on fields.

We now want to apply the same procedure to a superfield. A translation in superspace (i.e. a supersymmetry transformation) on a superfield $Y(x, \theta, \bar{\theta})$ by a quantity $(\epsilon_\alpha, \bar{\epsilon}_\dot{\alpha})$, where $\epsilon_\alpha, \bar{\epsilon}_\dot{\alpha}$ are spinorial parameters, is defined as

$$Y(x + \delta x, \theta + \delta \theta, \bar{\theta} + \delta \bar{\theta}) = e^{-i(\epsilon \partial + \bar{\epsilon} \bar{\partial})} Y(x, \theta, \bar{\theta}) e^{i(\epsilon \partial + \bar{\epsilon} \bar{\partial})},$$

(4.25)

with

$$\delta_{\epsilon_\alpha} Y(x, \theta, \bar{\theta}) \equiv Y(x + \delta x, \theta + \delta \theta, \bar{\theta} + \delta \bar{\theta}) - Y(x, \theta, \bar{\theta})$$

(4.26)

the variation of the superfield under the supersymmetry transformation.

To find the representation of $Q$ and $\bar{Q}$ as differential operators, there are two questions we need to answer, first. What is the explicit expression for $\delta x, \delta \theta, \delta \bar{\theta}$? Why are we supposing here $\delta x \neq 0$, given we are not acting with the generator of space-time translations $P_\mu$, but just with supersymmetry generators?

First notice that eq. (4.25) can be written as

$$Y(x + \delta x, \theta + \delta \theta, \bar{\theta} + \delta \bar{\theta}) = e^{-i(\epsilon \partial + \bar{\epsilon} \bar{\partial})} e^{-i(x \partial + \theta \bar{\partial} + \bar{\theta} \partial)} Y(0; 0, 0) e^{i(x \partial + \theta \bar{\partial} + \bar{\theta} \partial)} e^{i(\epsilon \partial + \bar{\epsilon} \bar{\partial})}$$

(4.27)

Let us now evaluate the last two exponentials, for which we need to recall the Baker-Campbell-Hausdorff formula for non-commuting objects which says that

$$e^{A} e^{B} = e^{C} \quad \text{where} \quad C = \sum_{n=1}^{\infty} \frac{1}{n!} C_n(A, B)$$

(4.28)

with

$$C_1 = A + B \quad , \quad C_2 = [A, B] \quad , \quad C_3 = \frac{1}{2} [A, [A, B]] - \frac{1}{2} [B, [B, A]] \quad \ldots \quad (4.29)$$

We then have

$$\exp\{i(x \partial + \theta \bar{\partial} + \bar{\theta} \partial)\} \exp\{i(\epsilon \partial + \bar{\epsilon} \bar{\partial})\} =$$

$$= \exp\{ix^\mu P_\mu + i(\epsilon + \theta) \bar{Q} + i(\bar{\epsilon} + \bar{\theta}) Q - \frac{1}{2} [\theta \bar{Q}, \epsilon Q] - \frac{1}{2} [\bar{\theta} Q, \bar{\epsilon} \bar{Q}]\}$$

$$= \exp\{ix^\mu P_\mu + i(\epsilon + \theta) \bar{Q} + i(\bar{\epsilon} + \bar{\theta}) Q + \epsilon \sigma^\mu \bar{\theta} P_\mu - \theta \sigma^\mu \epsilon \bar{P}_\mu\}$$

$$= \exp\{i(x^\mu + i \theta \sigma^\mu \bar{\epsilon} - i \epsilon \sigma^\mu \bar{\theta}) P_\mu + i(\epsilon + \theta) \bar{Q} + i(\bar{\epsilon} + \bar{\theta}) Q\}$$

(4.30)

which means that

$$\begin{cases}
\delta x^\mu = i \theta \sigma^\mu \bar{\epsilon} - i \epsilon \sigma^\mu \bar{\theta} \\
\delta \theta^\alpha = \epsilon^\alpha \\
\delta \bar{\theta}^{\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha}}
\end{cases}$$

(4.31)
This answers the first question. Notice, further, the expression for $\delta x^\mu$, which is non-vanishing. This is needed, in order to be consistent with the supersymmetry algebra, $\{ Q_\alpha, \bar{Q}_\dot{\alpha} \} \sim P_\mu$: two subsequent supersymmetry transformations generate a space-time translation. This answers the second question.

We can now give a more precise definition for what a superfield actually is: a superfield is a space-time translation. This answers the second question.

We can now find the representation of the supersymmetry generators $Q_\alpha$ and $\bar{Q}_\dot{\alpha}$ as differential operators. Let us take eq. (4.26) and, recalling eqs. (4.31), let us Taylor expand the right hand side which becomes

$$
\delta_\varepsilon Y(x, \theta, \bar{\theta}) = Y(x, \theta, \bar{\theta}) + i (\theta \sigma^\mu \bar{\epsilon} - \epsilon \sigma^\mu \theta) \partial_\mu Y(x, \theta, \bar{\theta}) + \epsilon^a \partial_a Y(x, \theta, \bar{\theta}) + \cdots - Y(x, \theta, \bar{\theta})
$$

On the other hand, from eq. (4.25) we get

$$
\delta_\varepsilon Y(x, \theta, \bar{\theta}) = (1 - i \varepsilon \mathcal{Q} - i \varepsilon \bar{\mathcal{Q}} + \ldots) Y(x, \theta, \bar{\theta}) (1 + i \varepsilon \mathcal{Q} + i \varepsilon \bar{\mathcal{Q}} + \ldots) - Y(x, \theta, \bar{\theta})
$$

(4.33)

On the other hand, from eq. (4.25) we get

$$
\delta_\varepsilon Y(x, \theta, \bar{\theta}) = 1 + i \varepsilon \mathcal{Q} + i \varepsilon \bar{\mathcal{Q}} + \ldots
$$

(4.34)

the previous result implies that the supersymmetry variation of a superfield by parameters $\varepsilon, \bar{\epsilon}$ is represented as

$$
\delta_\varepsilon Y = (i \varepsilon \mathcal{Q} + i \varepsilon \bar{\mathcal{Q}}) Y.
$$

(4.35)

Comparing with eq. (4.32) we get the following expression for the differential operators $Q_\alpha, \bar{Q}_\dot{\alpha}$

$$
\left\{ \begin{array}{l}
Q_\alpha = -i \partial_\alpha - \sigma^\mu_{\alpha \dot{\beta}} \bar{\partial}_\mu \bar{\partial}_{\dot{\beta}} \\
\bar{Q}_{\dot{\alpha}} = +i \partial_{\dot{\alpha}} + \theta^\beta \sigma^\mu_{\beta \dot{\alpha}} \partial_\mu
\end{array} \right.
$$

(4.36)

Notice that, consistently, $Q_\alpha = \bar{Q}_{\dot{\alpha}}$ (to prove this, recall that $(\sigma^\mu_{\alpha \dot{\beta}})^\dagger = \sigma^\mu_{\dot{\beta} \alpha}$).

One can check the validity of the expressions (4.36) by showing that the two differential operators close the supersymmetry algebra, namely that

$$
\{ Q_\alpha, Q_\beta \} = \{ \bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}} \} = 0, \quad \{ Q_\alpha, \bar{Q}_{\dot{\beta}} \} = 2 \sigma^\mu_{\alpha \dot{\beta}} P_\mu.
$$

(4.37)

We can now give a more precise definition for what a superfield actually is: a superfield is a field in superspace which transforms under a super-translation according to eq. (4.25). This implies, in particular, that a product of superfields is still a superfield.
4.3 Supersymmetric invariant actions - general philosophy

Having seen that a supersymmetry transformation is simply a translation in super-

space, it is now easy to construct supersymmetric invariant actions. In order for

an action to be invariant under superPoincaré transformations it is enough that the

Lagrangian is Poincaré invariant (actually, it should transform as a scalar density)

and that its supersymmetry variation is a total space-time derivative.

Here is where the formalism we have introduced starts to manifest its powerful-

ness. The basic point is that the integral in superspace of any arbitrary superfield

is a supersymmetric invariant quantity. In other words, the following integral

\[ \int d^4x \, d^2\theta \, d^2\bar{\theta} Y(x, \theta, \bar{\theta}) \] (4.38)

is manifestly supersymmetric invariant, if \( Y \) is a superfield. This can be proven as

follows. The integration measure is translationally invariant by construction since

\[ \int d\theta \theta = \int d(\theta + \xi)(\theta + \xi) = 1 \] (4.39)

This implies that

\[ \delta_{\epsilon, \bar{\epsilon}} \int d^4x \, d^2\theta \, d^2\bar{\theta} Y(x, \theta, \bar{\theta}) = \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \delta_{\epsilon, \bar{\epsilon}} Y(x, \theta, \bar{\theta}) . \] (4.40)

Now, using eqs. (4.35) and (4.36) we get

\[ \delta_{\epsilon, \bar{\epsilon}} Y = \epsilon^\alpha \partial_\alpha Y + \bar{\epsilon}_\dot{\alpha} \bar{\partial}^\dot{\alpha} Y + \partial_\mu \left[ -i (\epsilon \sigma \bar{\theta} - \theta \sigma \bar{\epsilon}) Y \right] . \] (4.41)

Integration in \( d^2\theta d^2\bar{\theta} \) kills the first two terms since they do not have enough \( \theta \)'s or \( \bar{\theta} \)'s to compensate for the measure, and leaves only the last term, which is a total
derivative. In other words, under supersymmetry transformations the integrand in
eq (4.40) transforms as a total space-time derivative plus terms which get killed by
integration in superspace. Hence the full integral is supersymmetric invariant

\[ \delta_{\epsilon, \bar{\epsilon}} \int d^4x \, d^2\theta \, d^2\bar{\theta} Y(x, \theta, \bar{\theta}) = 0 . \] (4.42)

Supersymmetric invariant actions are constructed this way, \( i.e. \) by integrating in
superspace a suitably defined superfield. Such superfield, call it \( \mathcal{A} \), should not be
generic, of course. It should have the right structure to give rise, upon integration
on Grassman coordinates, to a Lagrangian density, which is a real, dimension-four
operator, transforming as a scalar density under Poincaré transformations. The end result will be a supersymmetric invariant action $S$

$$S = \int d^4x \, d^2\theta \, d^2\bar{\theta} \, A(x; \theta, \bar{\theta}) = \int d^4x \, \mathcal{L}(\phi(x), \psi(x), A_\mu(x), \ldots) \ .$$

(4.43)

Let us emphasize again: one does not need to prove $S$ to be invariant under supersymmetry transformations. If it comes from an integral of a superfield in superspace, this is just automatic: by construction, the Lagrangian $\mathcal{L}$ on the r.h.s. of eq. (4.43), an apparently innocent-looking function of ordinary fields, is guaranteed to be supersymmetric invariant, up to total space-time derivatives.

The superfield $A$ will be in general a product of superfields (recall that a product of superfields is still a superfield). However, the general superfield (4.14) cannot be the basic object of this construction: it contains too many field components to correspond to an irreducible representation of the supersymmetry algebra. We have to put (supersymmetric invariant) constraints on $Y$ and restrict its form to contain only a subset of fields. Being the constraint supersymmetric invariant, this reduced set of fields will still be a superfield, and hence it will carry a representation of the supersymmetry algebra. In what follows, we will start discussing a few such constraints, the so-called chiral and real constraints. These will be the relevant ones for our purposes, as they will lead to chiral and vector superfields, the right superfields to accommodate matter and radiation, respectively, and to linear superfields, where conserved currents sit.

### 4.4 Chiral superfields

One can construct covariant derivatives $D_\alpha, \overline{D}_\dot{\alpha}$ defined as

$$\left\{ \begin{array}{l}
D_\alpha = \partial_\alpha + i \sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^\dot{\beta} \partial_\mu \\
\overline{D}_\dot{\alpha} = \partial_\dot{\alpha} + i \theta^{\dot{\beta}} \sigma^\mu_{\mu\dot{\alpha}} \partial_\mu
\end{array} \right.$$  

(4.44)

and which anticommute with the supersymmetry generators $Q_\alpha, \overline{Q}_{\dot{\alpha}}$. More precisely we have

$$\{D_\alpha, \overline{D}_{\dot{\beta}}\} = 2i \sigma^\mu_{\alpha\dot{\beta}} \partial_\mu = -2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \ ,$$

$$\{D_\alpha, D_\beta \text{ or } \overline{Q}_{\dot{\beta}} \text{ or } \overline{Q}_{\dot{\beta}}\} = 0 \ (\text{similarly for } \overline{D}_{\dot{\alpha}}) \ .$$  

(4.45, 4.46)

This implies that

$$\delta_{\epsilon, \dot{\epsilon}} (D_\alpha Y) = D_\alpha (\delta_{\epsilon, \dot{\epsilon}} Y) \ ,$$

(4.47)
since $D_\alpha$ commutes both with $\epsilon Q$ and $\bar{\epsilon} \bar{Q}$. Therefore, if $Y$ is a superfield, that is a field in superspace transforming as dictated by eq. (4.25) under a supersymmetry transformation, so is $D_\alpha Y$. This means that $D_\alpha Y = 0$ is a supersymmetric invariant constraint we can impose on a superfield $Y$ to reduce the number of its components, while still having the field carrying a representation of the supersymmetry algebra (the same holds for the constraint $\overline{D}_\dot{\alpha} Y = 0$).

Recall the generic expression (4.14) for $Y$ and consider $\partial_\dot{\alpha} Y$: this has fewer components with respect to $Y$ itself, since, for instance, there is no $\theta \theta \theta \theta$ term. However

$$\left[ \partial_\dot{\alpha}, \epsilon Q \right] = \epsilon^\beta \sigma_\mu^{\dot{\beta}} \partial_\mu . \quad (4.48)$$

This implies that a supersymmetry transformation on $\overline{X}_\dot{\alpha} = \partial_\dot{\alpha} Y$ would generate a $\theta \theta \theta \theta$ term which $\overline{X}_\dot{\alpha}$ did not contain. Hence $\partial_\dot{\alpha} Y$ is not a superfield, in the sense of providing a basis for a representation of supersymmetry. On the other hand, the covariant derivatives defined in (4.44) anticommute with $Q_\alpha$ and $\overline{Q}_{\dot{\alpha}}$. Hence, if $Y$ is a superfield, $D_\alpha Y, \overline{D}_{\dot{\alpha}} Y$ are also superfields (and so is $\partial_\mu Y$, since $\Gamma_\mu^{\dot{\alpha}}$ commutes with $Q_\alpha$ and $\overline{Q}_{\dot{\alpha}}$).

A chiral superfield $\Phi$ is a superfield such that

$$\overline{D}_{\dot{\alpha}} \Phi = 0 . \quad (4.49)$$

Seemingly, an anti-chiral superfield $\Psi$ is a superfield such that

$$D_\alpha \Psi = 0 . \quad (4.50)$$

Notice that if $\Phi$ is chiral, its hermitian conjugate, $\overline{\Phi}$, is anti-chiral (unless otherwise stated, here and in the following we will use the symbol $\dagger$ instead of $\dagger$ to mean hermitian conjugation, to adapt to the two-component spinor notation for which $\psi_\dagger = \overline{\psi}_\dot{\alpha}$). This implies that a chiral superfield cannot be real (i.e. $\overline{\Phi} = \Phi$). Indeed, in this case it is easy to show that it should be a constant. Taking the hermitian conjugate of eq. (4.49) one would conclude that the field would also be anti-chiral. Acting now on it with the anticommutator in eq.(4.45) one would get $\partial_\mu \Phi = 0$. This is the superfield analogue of what we have seen in the previous lecture, when we constructed the chiral multiplet and we learned that the scalar field $\phi$ had to be complex.

We would like to find the most general expression for a chiral superfield in terms of ordinary fields, as we did for the general superfield (4.14). In other words, we have
to integrate the constraint (4.49). To this aim, it is useful to define new coordinates
\[ y^\mu = x^\mu + i \theta \sigma^\mu \bar{\theta} \quad , \quad \bar{y}^\mu = x^\mu - i \theta \sigma^\mu \bar{\theta} . \quad (4.51) \]

It easily follows that
\[ D_{\dot{a}} \dot{\theta}_\beta = \overline{D}_\dot{a} y^\mu = 0 \quad , \quad D_{\dot{a}} \bar{\theta}_\beta = \overline{D}_\dot{a} \bar{y}^\mu = 0 . \quad (4.52) \]

Recalling the definition (4.49) this implies that \( \Phi \) depends only on \( (y^\mu, \theta_\alpha) \) explicitly, but not on \( \bar{\theta}_\dot{a} \) (the \( \bar{\theta} \)-dependence is hidden inside \( y^\mu \)). In this (super)coordinate system the chiral constraint is easily solved by
\[ \Phi(y, \theta) = \phi(y) + \sqrt{2} \theta \psi(y) - \theta \theta F(y) . \quad (4.53) \]

Taylor-expanding the above expression around \( x \) we get for the actual \( \Phi(x, \theta, \bar{\theta}) \)
\[ \Phi(x, \theta, \bar{\theta}) = \phi(x) + \sqrt{2} \theta \psi(x) + i \theta \sigma^\mu \bar{\theta} \partial_\mu \phi(x) - \theta \theta F(x) - \frac{i}{\sqrt{2}} \theta \theta \theta \sigma^\mu \phi(x) - \frac{1}{4} \theta \theta \theta \theta \square \phi(x) , \quad (4.54) \]

which can also be conveniently recast as \( \Phi(x, \theta, \bar{\theta}) = e^{i \theta \sigma^\mu \partial_\mu \bar{\theta}} \Phi(x, \theta) \). We see that, as expected, this superfield has less components than the general superfield \( Y \), and some of them are related to each other.

The chiral superfield (4.54) is worth its name, since it is a superfield which encodes precisely the degrees of freedom of the chiral multiplet of fields we have previously constructed. On-shell, it corresponds to a \( N = 1 \) multiplet of states, hence carrying an irreducible representation of the \( N = 1 \) supersymmetry algebra.

A similar story holds for the anti-chiral superfield \( \Phi \) for which we would get
\[ \Phi(x, \theta, \bar{\theta}) = \phi(\bar{y}) + \sqrt{2} \theta \psi(\bar{y}) - \theta \theta F(\bar{y}) \quad (4.55) \]
\[ = \bar{\phi}(x) + \sqrt{2} \theta \psi(x) - i \theta \sigma^\mu \bar{\theta} \partial_\mu \bar{\phi}(x) - \theta \theta F(x) + \frac{i}{\sqrt{2}} \theta \theta \theta \sigma^\mu \partial_\mu \bar{\phi}(x) - \frac{1}{4} \theta \theta \theta \theta \square \bar{\phi}(x) . \]

Let us now try and see how does a chiral (or anti-chiral) superfield transform under supersymmetry transformations. This amounts to compute
\[ \delta_{\epsilon, \epsilon} \Phi(y; \theta) = (i \epsilon Q + i \epsilon \overline{Q}) \Phi(y; \theta) \quad (4.56) \]
(and similarly for \( \Phi \)). To compute eq. (4.56) it is convenient to write the differential operators \( Q_\alpha, \overline{Q}_\dot{a} \) in the \( (y^\mu, \theta_\alpha, \bar{\theta}_\dot{a}) \) coordinate system. This amounts to trade the
partial derivatives taken with respect to \((x^\mu, \theta_\alpha, \bar{\theta}_\dot{\alpha})\) for those taken with respect to the new system \((y^\mu, \theta_\alpha, \bar{\theta}_\dot{\alpha})\) and plug this into eqs. (4.36). The final result reads

\[
\begin{align*}
Q^\text{new}_\alpha &= -i \partial_\alpha \\
\overline{Q}^\text{new}_\dot{\alpha} &= i \partial_\dot{\alpha} + 2 \theta^{\alpha} \sigma^{\mu}_{\alpha \dot{\alpha}} \frac{\partial}{\partial y^\mu}
\end{align*}
\] (4.57)

Plugging these expressions into eq. (4.56) one gets

\[
\delta_{\epsilon, \bar{\epsilon}} \Phi(y; \theta) = \left( \epsilon^\alpha \partial_\alpha + 2i \theta^{\alpha} \sigma^{\mu}_{\alpha \dot{\alpha}} \bar{\epsilon}^\dot{\alpha} \frac{\partial}{\partial y^\mu} \right) \Phi(y; \theta)
\]
\[
= \sqrt{2} \epsilon \psi - 2 \epsilon \theta F + 2i \theta \sigma^{\mu} \bar{\epsilon} \left( \frac{\partial}{\partial y^\mu} \phi + \sqrt{2} \theta \frac{\partial}{\partial y^\mu} \psi \right)
\]
\[
= \sqrt{2} \epsilon \psi + \sqrt{2} \theta \left( -\sqrt{2} \epsilon F + \sqrt{2} i \sigma^{\mu} \bar{\epsilon} \frac{\partial}{\partial y^\mu} \phi \right) - i \theta \left( -\sqrt{2} \epsilon \sigma^{\mu} \frac{\partial}{\partial y^\mu} \psi \right).
\]
(4.58)

Therefore, the final expression for the supersymmetry variation of the different field components of the chiral superfield \(\Phi\) reads

\[
\begin{align*}
\delta \phi &= \sqrt{2} \epsilon \psi \\
\delta \psi_\alpha &= \sqrt{2} i (\sigma^{\mu} \bar{\epsilon})_\alpha \partial_\mu \phi - \sqrt{2} \epsilon \sigma^{\mu} \bar{\epsilon} \\
\delta F &= i \sqrt{2} \partial_\mu \psi_\alpha \sigma^{\mu} \bar{\epsilon}
\end{align*}
\] (4.59)

It is left to the reader to derive the corresponding expressions for an anti-chiral superfield. In this case, one should write the generators \(Q_\alpha, \overline{Q}_\dot{\alpha}\) in the \((\bar{y}^\mu, \theta_\alpha, \bar{\theta}_\dot{\alpha})\) coordinate system.

### 4.5 Real (aka vector) superfields

In order to have gauge interactions we clearly need to find a supersymmetric invariant projection which saves the vector field \(v^\mu\) in the general expression (4.14) and makes it real (this was not the case for the constraint (4.49), for which the vector component is \(\sim \partial_\mu \phi\)). The right thing to do is to impose a reality condition on the general superfield \(Y\). Indeed, under hermitian conjugation, \(Y \rightarrow \overline{Y}\), one has that \(v^\mu \rightarrow \overline{v}^\mu\); so imposing a reality condition, the vector component not only survives as an independent degree of freedom, but becomes real.

A real (aka vector) superfield \(V\) is a superfield such that

\[
V = \overline{V}.
\] (4.60)
Looking at the general expression (4.14) this leads to the following expansion for \( V(x, \theta, \bar{\theta}) \)

\[
V(x, \theta, \bar{\theta}) = C(x) + i \theta \chi(x) - i \bar{\theta} \bar{\chi}(x) + \theta \sigma^\mu \bar{\theta} v_\mu + \frac{i}{2} \theta \theta (M(x) + iN(x))
- \frac{i}{2} \bar{\theta} \bar{\theta} (M(x) - iN(x)) + i \theta \theta \bar{\theta} \left( \bar{\chi}(x) + \frac{i}{2} \bar{\sigma}^\mu \partial_\mu \chi(x) \right)
- i \bar{\theta} \theta \theta \left( \lambda(x) + \frac{i}{2} \sigma^\mu \partial_\mu \chi(x) \right) + \frac{1}{2} \theta \theta \bar{\theta} \bar{\theta} \left( D(x) - \frac{1}{2} \partial^2 C(x) \right).
\]

(4.61)

Notice that, as such, this superfield has 8\(B\) + 8\(F\) degrees of freedom. The next step is to introduce the supersymmetric version of gauge transformations. As we shall see, after gauge fixing, this will reduce the number of off-shell degrees of freedom to 4\(B\) + 4\(F\), which become 2\(B\) + 2\(F\) on-shell (for a massless representation), as it should be the case for a massless vector multiplet of states, see eq. (3.15).

First notice that if \( \Phi \) is a chiral superfield, then \( \Phi + \Phi \) is a (very special) vector superfield. So, under the following transformation on \( V \)

\[
V \rightarrow V + \Phi + \bar{\Phi}
\]

(4.62)

one gets a a superfield \( V \) which is still real. Under the shift (4.62) the vector component \( v_\mu \) in (4.61) transforms as \( v_\mu \rightarrow v_\mu - \partial_\mu (2 \text{Im} \phi) \). This is precisely how an ordinary (abelian) gauge transformation acts on a vector field. Therefore, eq. (4.62) is a natural definition for the supersymmetric version of a gauge transformation.

Under eq. (4.62) the component fields of \( V \) transform as

\[
\begin{align*}
C & \rightarrow C + 2 \text{Re} \phi \\
\chi & \rightarrow \chi - i \sqrt{2} \psi \\
M & \rightarrow M - 2 \text{Im} F \\
N & \rightarrow N + 2 \text{Re} F \\
D & \rightarrow D \\
\lambda & \rightarrow \lambda \\
v^\mu & \rightarrow v^\mu - 2 \partial_\mu \text{Im} \phi
\end{align*}
\]

(4.63)

where the components of \( \Phi \) have been dubbed \((\phi, \psi, F)\). From the transformations above one sees that properly choosing \( \Phi \), namely choosing

\[
\text{Re} \phi = -\frac{C}{2} , \quad \psi = -\frac{i}{\sqrt{2}} \chi , \quad \text{Re} F = -\frac{N}{2} , \quad \text{Im} F = \frac{M}{2}.
\]

(4.64)

one can gauge away (namely put to zero) \( C, M, N, \chi \). The choice above is called Wess-Zumino gauge. In this gauge a vector superfield can be written as

\[
V_{WZ} = \theta \sigma^\mu \bar{\theta} v_\mu(x) + i \theta \theta \bar{\theta} \bar{\lambda}(x) - i \bar{\theta} \theta \theta \lambda(x) + \frac{1}{2} \theta \theta \bar{\theta} \bar{\theta} D(x).
\]

(4.65)
Therefore, taking into account gauge invariance (that is, the redundancy of one of the vector degrees of freedom, the one associated to the transformation $v_\mu \rightarrow v_\mu - \partial_\mu (2 \text{ Im}\phi)$, which the WZ gauge does not fix), we end-up with $4_B + 4_F$ degrees of freedom off-shell. As we shall see later, $D$ will turn out to be an auxiliary field; therefore, by imposing the equations of motion for $D$, the spinor $\lambda$ and the vector $v^\mu$, one will end up with $2_B + 2_F$ degrees of freedom on-shell. Since we like to formulate gauge theories keeping gauge invariance manifest off-shell, the WZ gauge is defined as a gauge where $C = M = N = \chi = 0$, but no restrictions on $v^\mu$. This way, while remaining in the WZ gauge, we still have the freedom to do ordinary gauge transformations. In other words, once in the WZ gauge, we can still perform a supersymmetric gauge transformation (4.62) with parameters $\phi = -\bar{\phi}, \psi = 0, F = 0$.

Let us end this section with two important comments. First notice that in the WZ gauge each term in the expansion of $V_{WZ}$ contains at least one $\bar{\chi}$. Therefore

$$V_{WZ}^2 = \frac{1}{2} \bar{\phi} \tilde{D} v_\mu v^\mu, \quad V_{WZ}^n = 0 \quad n \geq 3.$$ (4.66)

These identities will simplify things a lot when it comes to construct supersymmetric gauge actions. Second, it should be remarked that the WZ gauge does not commute with supersymmetry. Acting with a supersymmetry transformation on a vector superfield in the WZ gauge, one obtains a new superfield which is not in the WZ gauge. Hence, when working in this gauge, after a supersymmetry transformation, one has to do a compensating supersymmetric gauge transformation (4.62), with a properly chosen $\Phi$, to come back to the WZ gauge.

### 4.6 (Super)Current superfields

The two superfields described above are what we need to describe matter and radiation in a supersymmetric theory, if we are not interested in gravitational interactions. However, in a supersymmetric theory, also composite operators should sit in superfields. There are at least two other types of superfields which accommodate important composite operators, i.e. conserved currents and the supersymmetry current (supercurrent for short), the latter being ubiquitous in a supersymmetric QFT, as this is the current associated to the supersymmetry charge itself. Both these superfields turn out to be real superfields, as the superfield described in the previous section, but current conservation implies extra supersymmetric invariant conditions they should satisfy which make them a particular class of real superfields. In what follows, we will briefly describe both of them.

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4.6.1 Internal symmetry current superfields

Because of Noether theorem, in a local QFT any continuous symmetry is associated to a conserved current $j_\mu$ satisfying $\partial^\mu j_\mu = 0$, and to the corresponding conserved charge $Q$ defined as $Q = \int d^3x j^0$. Here we are referring to non-R symmetries. R-symmetry will be discussed later.

As any other operator, in a supersymmetric theory a conserved current should sit in a superfield. It turns out that this is a real scalar superfield $\mathcal{J}$ satisfying the following extra constraint

$$D^2 \mathcal{J} = \overline{D}^2 \mathcal{J} = 0.$$  \hspace{1cm} (4.67)

A real superfield satisfying the constraint above is called *linear* superfield. Working a little bit one can show that a real superfield subject to the conditions (4.67) has the following component expression

$$\mathcal{J} = J(x) + i\theta j(x) - i\bar{\theta} \bar{j}(x) + \theta \sigma^\mu \bar{\theta} \partial_\mu j(x) + \frac{1}{2} \theta^2 \bar{\theta} \sigma^\mu \partial_\mu \bar{j}(x) - \frac{1}{2} \bar{\theta}^2 \theta \sigma^\mu \partial_\mu j(x) + \frac{1}{4} \theta^2 \bar{\theta}^2 \Box J(x), \hspace{1cm} (4.68)$$

where $J$ is a real scalar and $j_\alpha$ a spinor. By imposing eq. (4.67) on the above expression one easily sees that the current $j_\mu$ satisfies $\partial^\mu j_\mu = 0$, i.e. is a conserved current. So the constraint (4.67) is indeed the correct supersymmetric generalization of current conservation. Note that while the condition (4.67) is compatible with supersymmetry (both $D^2$ and $\overline{D}^2$ commute with supersymmetry transformations), it stands on a slightly different footing with respect to the conditions (4.49), (4.50) and (4.60). The latter constrain the dependence of a superfield as a function of the fermionic coordinates $(\theta_\alpha, \bar{\theta}_\dot{\alpha})$, but they do not say anything about space-time dependence. On the contrary, eq. (4.67) constrains also the space-time dependence of some of the fields imposing differential equations in $x$-space, one obvious example being the conservation equation $\partial^\mu j_\mu = 0$. In this sense, (4.67) is an on-shell constraint.

A few comments are in order. First notice that, as compared to a general real superfield (4.61), a linear superfield has less independent components. This is due to the extra condition (4.67) a linear superfield has to satisfy. Another comment regards the spin content of $\mathcal{J}$. One condition that $\mathcal{J}$ should (and does) satisfy is that it should not contain fields with spin higher than one. If this were the case, one could not gauge the current $j_\mu$ without introducing higher-spin gauge fields, something which is expected not to be consistent in a local interacting QFT with rigid supersymmetry (recall our discussion in the previous lecture). This implies
that $\mathcal{J}$ should be a real scalar superfield, namely its lowest component $J$ should be a scalar. Finally, it is worth notice that the detailed structure of $\mathcal{J}$ is not uniquely fixed, but in fact defined up to Schwinger terms entering the current algebra.

This can be understood as follows. Because the conserved charge $Q$ is a non-R symmetry charge, it commutes with supersymmetry generators, $[Q, Q] = 0$. This implies that the operator on the right hand side of the current algebra

$$[Q_\alpha, j_\mu] = \mathcal{O}_{\alpha\mu} ,$$

should be an operator which vanishes when acting with $\partial_\mu$, because so is $j_\mu$, and it should also be a total space-time derivative for $\mu = 0$, say $\mathcal{O}_{\alpha 0} = \partial^\nu A_{\alpha\nu}$, so that it integrates to zero, because so happens to the left hand side given that

$$\int d^4x [Q_\alpha, j_0] = \int dt [Q_\alpha, Q] = 0 .$$

An operator of this kind is known as Schwinger term. Different Schwinger terms provide different versions of the superfield $\mathcal{J}$, which is hence not univocally defined. The superfield defined in eq. (4.68) is one possible such completions, for which $\mathcal{O}_{\alpha\mu} = -2i(\sigma_\mu)_{\alpha}^\beta \partial^\nu j_\beta$. This can be easily checked using eqs. (4.34)-(4.35). This said, the first terms in the expansion of $\mathcal{J}$, namely $J + i\theta j - i\bar{\theta} \bar{j} + \theta \sigma^\mu \bar{\theta} j_\mu(x)$, are universal in the sense that they turn out not to depend on the specific Schwinger term appearing in eq. (4.69).

### 4.6.2 Supercurrent superfields

While currents associated to internal symmetries might or might not be there, in any supersymmetric theory there always exists, by definition, a conserved current, the supersymmetry current $S_\mu$, associated to the conservation of the supercharge $Q_\alpha$. In terms of the supercurrent, the supersymmetry charge is $Q_\alpha = \int d^4x S^0_\alpha$. Such supercurrent should sit in a superfield.

An equation analogous to eq. (4.69) is imposed by the supersymmetry algebra, which reads

$$\{Q_\alpha, S_{\alpha\nu}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} T_{\mu\nu} + \mathcal{O}_{\alpha\dot{\alpha}\nu} ,$$

where $T_{\mu\nu}$ is the (conserved) energy-momentum tensor and $\mathcal{O}_{\alpha\dot{\alpha}\nu}$ is again a Schwinger term. Note that, unlike eq. (4.69), the $\nu = 0$ component of the left hand side of eq. (4.71) does not integrate to zero now but is proportional to $\int dt P_\mu$ by the supersymmetry algebra, namely to $\int d^4x T^0_\mu$. This is why, on top of a Schwinger
term, the energy-momentum tensor appears on the right hand side of eq. (4.71). This also shows that the supercurrent and the energy-momentum tensor sit in the same superfield, \( T_{\mu} \) being the highest spin field of the representation (otherwise, it would be problematic coupling supersymmetry with gravity avoiding higher spin currents). This is the current operators counterpart of the fact that the graviton and the gravitino sit in the same multiplet. These properties are enough to fix the universal structure of the supercurrent superfield which reads

\[
J_\mu = j_\mu + \theta (S_\mu + \ldots) + \bar{\theta} (\bar{S}_\mu + \ldots) + \theta \sigma^\nu \bar{\theta} (2T_{\mu\nu} + \ldots) + \ldots .
\]

(4.72)

The arbitrariness of the Schwinger term gives rise to different possible completions of \( J_\mu \). The most known such completions is due to Ferrara and Zumino. The FZ multiplet can be described by a pair of superfields \((J_\mu, X)\) satisfying the relation

\[
2 \bar{D}^\alpha \sigma^\mu_{\alpha\dot{\alpha}} J_\mu = D_\alpha X ,
\]

(4.73)

with \( J_\mu \) being a real vector superfield, and \( X \) a chiral superfield, \( \bar{D}_\alpha X = 0 \). The same comment we made on the on-shell nature of the condition (4.67) holds also in this case. From the defining equation above one can work out the component expression of these two superfields. They read

\[
J_\mu = j_\mu + \theta \left( S_\mu - \frac{1}{3} \sigma_\mu \bar{S} \right) + \bar{\theta} \left( \bar{S}_\mu + \frac{1}{3} \bar{\sigma}_\mu S \right) + \frac{i}{2} \theta^2 \partial_\mu x^* - \frac{i}{2} \bar{\theta}^2 \partial_\mu x
\]

\[
+ \theta \sigma^\nu \bar{\theta} \left( 2T_{\mu\nu} - \frac{2}{3} \eta_{\mu\nu} T + \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \partial^\rho j^\sigma \right) + \ldots
\]

(4.74)

and

\[
X = x + \frac{2}{3} \theta S + \theta^2 \left( \frac{2}{3} T + i \partial_\mu j_\mu \right) + \ldots ,
\]

(4.75)

where \( \ldots \) stand for the supersymmetric completion and we have defined the trace operators \( T \equiv T^\mu_\mu \) and \( S_\alpha \equiv \sigma^\mu_{\alpha\dot{\alpha}} \bar{S}^\dot{\alpha}_\mu \). All in all, the FZ superfield contains a (in general non-conserved) R-current \( j_\mu \), a symmetric and conserved \( T_{\mu\nu} \), a conserved \( S_{\alpha\mu} \), and a complex scalar \( x \). That \( j_\mu \) is a R-current follows from the fact that it sits in the same superfield where \( S_{\alpha\mu} \) sits and so it does not commute with supersymmetry. From the above expression one can also see that whenever \( X \) vanishes the current \( j_\mu \) becomes conserved and all trace operators vanish. In this case the theory is conformal and \( j_\mu \) becomes the always present (and conserved) superconformal R-current.

We have seen that the FZ multiplet contains a non-conserved R-current. What if a theory admits a \( U(1)_R \) R-symmetry? For theories with an R-symmetry there exists
an alternative supermultiplet accommodating the energy-momentum tensor and the supercurrent, the so-called $\mathcal{R}$ multiplet. It turns out this is again defined in terms of a pair of superfields $(\mathcal{R}_\mu, \chi_\alpha)$ which now satisfy a different on-shell condition

$$2 \overline{D}^\alpha \sigma^{\mu\alpha}_\alpha \mathcal{R}_\mu = \chi_\alpha ,$$

(4.76)

where $\mathcal{R}_\mu$ is a real vector superfield and $\chi_\alpha$ a chiral superfield which, besides $\overline{D}_\alpha \chi_\alpha = 0$, also satisfies the identity $\overline{D}_\alpha \chi_\alpha = 0$. This implies that $\partial^\mu \mathcal{R}_\mu = 0$, from which it follows that the lowest component of $\mathcal{R}_\mu$ is now a conserved current, the $R$-current $j^R_\mu$. The component expression of the superfields making-up the $\mathcal{R}$ multiplet reads

$$\mathcal{R}_\mu = j^R_\mu + \theta S_\mu + \bar{\theta} \overline{S}_\mu + \theta \sigma^\nu \bar{\theta} \left( 2 T_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} (\partial^\rho j^\sigma + C^\rho) \right) + \ldots$$

(4.77)

and

$$\chi_\alpha = -2 S_\alpha - \left( 4 \delta^\alpha_\alpha T + 2i (\sigma^\rho \bar{\sigma}^\sigma) \delta^\alpha_\alpha C_{\rho\sigma} \right) \theta \beta + 2i \theta^2 \sigma^\nu_\alpha \partial_\nu \overline{S}^\alpha + \ldots$$

(4.78)

where again $\ldots$ stand for the supersymmetric completion, and $C_{\mu\nu}$ is a closed two-form. That $j^R_\mu$ is an $R$-current can be easily seen noticing that the current algebra now reads $[\mathcal{Q}_\alpha, j^R_\mu] = S_{\alpha\mu}$. Taking the time-component and integrating, this implies that $\int dt [\mathcal{Q}_\alpha, Q^R] = \int dt Q_\alpha$, which is what is expected for an $R$-symmetry, recall eq. (2.80). Notice, finally, that when $X = 0$, the FZ multiplet (4.74) becomes a (special instance of an) $R$-multiplet, one for which $\chi_\alpha = 0$. Indeed, its lowest component $j_\mu$ becomes now the conserved superconformal $R$-current.

The FZ and $\mathcal{R}$ multiplets are the more common supercurrent multiplets. However, there are instances in which a theory does not admit a $R$-symmetry (and hence the $\mathcal{R}$ multiplet cannot be defined) and the FZ multiplet is not a well-defined operator, e.g. it is not gauge invariant. In these cases, one should consider yet another multiplet where the supercurrent can sit, the so-called $\mathcal{S}$ multiplet, which is bigger than the two above. We will not discuss the $\mathcal{S}$ multiplet here, and refer to the references given at the end of this lecture. On the contrary, there exist theories in which both the FZ and the $\mathcal{R}$ multiplets can be defined. In such cases it turns out that the two are related by a so-called shift transformation defined as

$$\mathcal{R}_\mu = J_\mu + \frac{1}{4} \delta^{\alpha\beta}_{\mu} \left[ D_\alpha, \overline{D}_\beta \right] U , \quad X = -\frac{1}{2} \overline{D}^2 U , \quad \chi_\alpha = \frac{3}{2} \overline{D}^2 D_\alpha U ,$$

(4.79)

where $U$ is a real superfield associated to a non-conserved (and non-$R$) current.
4.7 Exercises

1. Prove identities (4.20).

2. Check that the differential operators $Q_\alpha$ and $\overline{Q}_\dot{\alpha}$ (4.36) close the supersymmetry algebra (4.37).

   \textit{Hint:} recall that all $\theta$'s and $\bar{\theta}$'s anti-commute between themselves, and that
   \begin{equation}
   \{a_i, a_j\} = 0 \implies \frac{\partial}{\partial a_i} a_j = \frac{\partial a_j}{\partial a_i} - a_j \frac{\partial}{\partial a_i} , \tag{4.80}
   \end{equation}

   which implies that, e.g.
   \begin{equation}
   \{\partial_\alpha, \bar{\theta}\} = 0 , \{\partial_\alpha, \theta^\beta\} = \delta^\beta_\alpha , \{\bar{\partial}_\dot{\alpha}, \bar{\theta}\dot{\beta}\} = \delta^{\dot{\beta}}_\dot{\alpha} . \tag{4.81}
   \end{equation}

3. Check that the covariant derivatives $D_\alpha$ and $\overline{D}_\dot{\alpha}$ (4.44) anticommute between themselves and with the supercharge operators (4.36).

4. Compute how the field components of an anti-chiral superfield $\Psi$ transform under supersymmetry transformations. Show that if $\Psi = \overline{\Phi}$ one gets the hermitian conjugate of the transformations (4.59).

5. Compute the supersymmetric variation of a vector superfield in the WZ gauge, and find the explicit form of the chiral superfield $\Phi$ which, via a compensating supersymmetric gauge transformation, brings the vector superfield back to the WZ gauge.

References


5 Supersymmetric actions: minimal supersymmetry

In the previous lecture we have introduced the basic superfields one needs to construct $N=1$ supersymmetric theories, if one is not interested in describing gravitational interactions. We are now ready to look for supersymmetric actions describing the dynamics of these superfields. We will first concentrate on matter actions and construct the most general supersymmetric action describing the interaction of a set of chiral superfields. Then we will introduce SuperYang-Mills theory which is nothing but the supersymmetric version of Yang-Mills. Finally, we will couple the two sectors with the final goal of deriving the most general $N=1$ supersymmetric action describing the interaction of radiation with matter. In all these cases, we will consider both renormalizable as well as non-renormalizable theories, the latter being relevant to describe effective low energy theories.

Note: in what follows we will deal with gauge theories, and hence gauge groups, like $SU(N)$ and alike. In order to avoid confusion, in the rest of these lectures we will use calligraphic $N$ when referring to the number of supersymmetry, $N = 1, 2$ or $4$.

5.1 $\mathcal{N}=1$ Matter actions

Following the general strategy outlined in §4.3 we want to construct a supersymmetric invariant action describing the interaction of a (set of) chiral superfield(s). Let us first notice that a product of chiral superfields is still a chiral superfield and a product of anti-chiral superfields is an anti-chiral superfield. Conversely, the product of a chiral superfield with its hermitian conjugate (which is anti-chiral) is a (very special, in fact) real superfield.

Let us start analyzing the theory of a single chiral superfield $\Phi$. Consider the following integral

$$\int d^2\theta \ d^2\bar{\theta} \ \Phi \Phi^* .$$

(5.1)

This integral satisfies all necessary conditions to be a supersymmetric Lagrangian. First, it is supersymmetric invariant (up to total space-time derivative) since it is the integral in superspace of a superfield. Second, it is real and a scalar object. Indeed, the first component of $\Phi$ is $\phi$ which is real and a scalar. Now, the $\theta^2\bar{\theta}^2$ component of a superfield, which is the only term contributing to the above integral, has the same tensorial structure as its first component since $\theta^2\bar{\theta}^2$ does not have any free