6 Theories with extended supersymmetry

Until now we have discussed theories with $\mathcal{N} = 1$ supersymmetry. In this lecture we will discuss the structure of theories with extended supersymmetry. This will also let us emphasize the basic differences which arise at the quantum level between theories with different number of supersymmetries.

6.1 $\mathcal{N} = 2$ supersymmetric actions

In this section we would like to construct the most general $\mathcal{N} = 2$ supersymmetric action in four dimensions. We will follow the same logic of the previous lecture, but we will not develop the corresponding $\mathcal{N} = 2$ superspace approach, whose formulation is beyond our present scope. Rather, we will use the (by now familiar) $\mathcal{N} = 1$ superspace formalism and see which specific properties does more supersymmetry impose on an otherwise generic $\mathcal{N} = 1$ Lagrangian.

We have two kinds of $\mathcal{N} = 2$ multiplets we have to deal with, vector multiplets and hypermutiplets. What we noticed at the level of representations of the supersymmetry algebra on states, lecture 3, holds also at the field level. In particular, using a $\mathcal{N} = 1$ language, a $\mathcal{N} = 2$ vector superfield can be seen as the direct sum of a vector superfield V and a chiral superfield Φ (with same internal quantum numbers, of course). Similarly, in terms of degrees of freedom a hypermultiplet can be constructed out of two $\mathcal{N} = 1$ chiral superfields, H_1 and H_2. Schematically, we have

$$[\mathcal{N} = 2 \text{ vector multiplet}] \;\; : \;\; V = (\lambda_\alpha, A_\mu, D) \;\; \oplus \;\; \Phi = (\phi, \psi_\alpha, F)$$

$$[\mathcal{N} = 2 \text{ hypermultiplet}] \;\; : \;\; H_1 = (H_1, \psi_{1\alpha}, F_1) \;\; \oplus \;\; \bar{H}_2 = \left(\bar{H}_2, \bar{\psi}_{2\dot{\alpha}}, \bar{F}_2\right)$$

(notice that H_1 and $\bar{H}_2$ transform in the same representations of internal symmetries, while H_2 transforms in the complex conjugate representation).

Let us start considering pure SYM. There are two minimal requirements we should impose. The chiral superfield Φ should transform in the same gauge group representation as the vector superfield V, that is the adjoint representation. Moreover, we have now a larger R-symmetry group, whose compact component, $SU(2)_R$, should be a symmetry of the Lagrangian. All bosonic fields A_μ, D, F and ϕ are singlets under $SU(2)_R$, but $(\lambda_\alpha, \psi_\alpha)$ transform as a doublet. This is because (Q^1_α, Q^2_α) transform under the fundamental representation of $SU(2)_R$, and the same should hold for λ_α and ψ_α (recall that they are obtained acting with the two supersymmetry generators on the Clifford vacuum).

The Lagrangian reads

$$\begin{aligned}
\mathcal{L}_{SYM}^{\mathcal{N}=2} &= \frac{1}{32\pi}\,\mathrm{Im}\left(\tau\int d^2\theta\,\mathrm{Tr}\,W^\alpha W_\alpha\right)+\int d^2\theta d^2\bar\theta\,\mathrm{Tr}\,\bar\Phi e^{2gV}\Phi = \\
&= \mathrm{Tr}\,\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-i\lambda\sigma^\mu D_\mu\bar\lambda-i\psi\sigma^\mu D_\mu\bar\psi+\overline{D_\mu\phi}D^\mu\phi+\frac{\theta}{32\pi^2}g^2F_{\mu\nu}\tilde F^{\mu\nu}+\right. \\
&+ \left.\frac{1}{2}D^2+\bar FF+i\sqrt{2}g\bar\phi\{\lambda,\psi\}-i\sqrt{2}g\{\bar\psi,\bar\lambda\}\phi+gD\left[\phi,\bar\phi\right]\right\}
\end{aligned} \tag{6.1}$$

where

$$\phi=\phi^aT_a \quad , \quad \psi_\alpha=\psi_\alpha^aT_a \quad , \quad F=F^aT_a$$

$$\lambda_\alpha=\lambda_\alpha^aT_a \quad , \quad D=D^aT_a \quad , \quad v_\mu=v_\mu^aT_a$$

with $a=1,2,\ldots,\dim G$. The reason why commutators and anti-commutators appear in the Lagrangian (6.1) is just that all fields transform in the adjoint representation of G. Indeed, given that $(T^a_{adj})_{bc}=-if_{abc}$, we have that, e.g. $\bar\phi\lambda\psi$ really is

$$\bar\phi^b\lambda^a(T^a_{adj})_{bc}\psi^c=-i\bar\phi^b\lambda^af_{abc}\psi^c=i\bar\phi^b\lambda^af_{bac}\psi^c=\bar\phi^b\lambda^a\psi^c\;T_b\left[T_a,T_c\right]=\mathrm{Tr}\;\bar\phi\{\lambda,\psi\}\ , \tag{6.2}$$

and similarly for all other contributions in eq. (6.1).

As compared to $\mathcal{N}=1$ Lagrangians describing matter coupled SYM theory, the above Lagrangian is special in many respects. A necessary and sufficient condition for $\mathcal{N}=2$ supersymmetry is the existence of a $SU(2)_R$ rotating the two generators Q^1_α, Q^2_α into each other. This has several consequences in the structure of the Lagrangian. The kinetic terms for λ and ψ have the same normalization. Moreover, and more importantly, the Lagrangian has no superpotential, $W=0$. Indeed, a superpotential would give ψ interactions and/or mass terms, that are absent for λ. This is clearly forbidden by the $SU(2)_R$ symmetry. While we do not have a superpotential, we do have a potential, which comes from D-terms. Indeed, the auxiliary fields equations of motion are in this case

$$F^a=0 \quad , \quad D^a=-g\left[\phi,\bar\phi\right]^a \tag{6.3}$$

(the auxiliary fields F^a appear only in the non-dynamical kinetic term $\bar F_aF^a$ and therefore are trivial). The potential hence reads

$$V(\phi,\bar\phi)=\frac{1}{2}D^aD_a=\frac{1}{2}g^2\;\mathrm{Tr}\left[\phi,\bar\phi\right]^2\ . \tag{6.4}$$

The above expression shows that pure $\mathcal{N}=2$ SYM enjoys a huge moduli space of supersymmetric vacua. Indeed, the potential vanishes whenever the fields ϕ belong to the Cartan subalgebra of the gauge group G. At a generic point of the moduli space the scalar field matrix can be diagonalized and the gauge group is broken to $U(1)^r$, with r the rank of G. The low energy effective dynamics is that of r massless vector multiplets and $dim(G)-r$ massive vector multiplets whose masses depend on the scalar fields VEVs. The theory is said to be in a Coulomb phase, since charged external sources will feel a Coulomb-like potential. The (classical) moduli space is a r-dimensional complex manifolds, parametrized by r massless complex scalars. Singularities arise whenever some VEVs become degenerate and the theory gets partially un-higgsed (in particular, at the origin of the moduli space one recovers the full G gauge symmetry). In some later lectures we will see how this classical picture gets modified once (non-perturbative) quantum corrections are taken into account.

The $SU(2)_R$ symmetry (in fact just the center of the group, $\mathbb{Z}_2$) can be used also to check that the Lagrangian (6.1) is invariant under two independent supersymmetries, as it should. Eq. (6.1) is written in terms of two $\mathcal{N}=1$ superfields and, correspondingly, it is obviously $\mathcal{N}=1$ invariant. Acting now with a $\mathbb{Z}_2$ R-symmetry rotation which acts as $\psi_\alpha \to \lambda_\alpha$ and $\lambda_\alpha \to -\psi_\alpha$, while leaving the bosonic fields invariant, one sees that the same Lagrangian shows an invariance under an *independent* $\mathcal{N}=1$ supersymmetry acting on two different superfields with entries $(A_\mu, -\psi_\alpha, D)$ and $(\phi, \lambda_\alpha, F)$. So we conclude that the Lagrangian is indeed $\mathcal{N}=2$ supersymmetric invariant.

Let us now consider the addition of hypermultiplets. In this case the scalar fields, H_1 and $\bar{H}_2$ form a $SU(2)_R$ doublet (again, recall how they are constructed from the ground state of the corresponding $\mathcal{N}=2$ supersymmetry representation, lecture 3). Hypermultiplets cannot interact between themselves since no cubic $SU(2)$ invariant is possible. Therefore, for renormalizable theories a superpotential is not allowed and interactions turn out to be all gauge interactions.

Let us now suppose that matter transforms under some non-trivial representation of the gauge group. We get for the corresponding $\mathcal{N}=2$ hypermultiplet Lagrangian

$$\mathcal{L}^{\mathcal{N}=2}_{Matter} = \int d^2\theta d^2\bar{\theta}\left(\bar{H}_1 e^{2gV_R} H_1 + \bar{H}_2 e^{-2gV_R} H_2\right) + \int d^2\theta \sqrt{2} g H_1 \Phi H_2 + \text{ h.c. }, \tag{6.5}$$

where the suffix R on the vector superfield V refers to the representation of G carried by the hypermultiplets. The F-term coupling the hypermultiplets with the chiral multiplet Φ belonging to the $\mathcal{N}=2$ vector multiplet is there because of $\mathcal{N}=2$ supersymmetry (it is somehow the supersymmetric partner of the kinetic terms which couple the hypermultiplet to V). So we see that eventually a cubic interaction does arise, but it is a gauge interaction, in the sense that it vanishes once the gauge coupling $g \to 0$.

Eliminating the auxiliary fields F_1 and F_2, the scalar potential for the hypermultiplets can be recast as a D-term contribution only and reads

$$V(H_1, H_2) = \frac{1}{2}D^2 = \frac{1}{2}g^2|\bar{H}_1 T_R^a H_1 - \bar{H}_2 T_R^a H_2|^2 \ , \ \ D^a = g \ \mathrm{Tr}\left(\bar{H}_1 T_R^a H_1 - \bar{H}_2 T_R^a H_2\right) \ . \tag{6.6}$$

Notice finally that a mass term can be present and has the form

$$mH_1H_2 \ . \tag{6.7}$$

However, a term of this sort can be there only for BPS hypermultiplets (which as discussed time ago are short enough to close the algebra within maximal spin 1/2 particle states).

6.1.1 Non-linear sigma model III

It is also possible to get the $\mathcal{N}=2$ version of the σ-model, once renormalizability is relaxed. $\mathcal{N}=2$ supersymmetry will make it special, as compared to the $\mathcal{N}=1$ case. We do not want to enter into much details and will just sketch the end result.

Let us start with pure SYM. Differently from the $\mathcal{N}=1$ case, this is a meaningful thing to do, since scalar fields are present in a $\mathcal{N}=2$ vector multiplet. To write down the $\mathcal{N}=2$ σ-model it is sufficient to take the $\mathcal{N}=1$ σ-model Lagrangian (5.120), set the superpotential to zero, and take into account that the chiral superfield Φ transforms in the adjoint representation. On general grounds, one expects that the Kähler potential K should be related, in a $\mathcal{N}=2$ consistent way, to the generalized complexified gauge coupling $\mathcal{F}_{ab}$, since the scalars spanning the manifold $\mathcal{M}$ sit in the same multiplets where the vectors sit (in particular, one would expect that an isometry transformation on $\mathcal{M}$ should have effects on the vectors, too). Equivalently, one can notice from (5.120) that the imaginary part of the generalized complexified gauge coupling multiplies the gaugino kinetic term while the Kähler metric that of the matter fermion fields. These should transform as a doublet under $SU(2)_R$ and then one would expect $\mathcal{F}_{ab}$ and K_{ab} to be exchanged under a $\mathbb{Z}_2$ R-symmetry rotation

$$\mathrm{Im}\ \mathcal{F}_{ab} \times \lambda^a \sigma^\mu D_\mu \bar{\lambda}^b \quad \longleftrightarrow \quad K_a^b \times \psi^a \sigma^\mu D_\mu \bar{\psi}_b \ . \tag{6.8}$$

What one finds is that $\mathcal{F}_{ab}$ and K can be written in terms of one and the same holomorphic function $\mathcal{F}(\Phi)$, dubbed *prepotential* and read

$$\mathcal{F}_{ab}(\Phi) = \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi^a \partial \Phi^b} \tag{6.9}$$

$$K(\Phi, \bar{\Phi}) = -\frac{i}{32\pi} \bar{\Phi}_a \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi^a} + \text{ h.c.} = -\frac{i}{32\pi} \bar{\Phi}_a \mathcal{F}^a(\Phi) + \frac{i}{32\pi} \overline{\mathcal{F}_a(\Phi)} \Phi^a \ , \tag{6.10}$$

which is the very non-trivial statement that the full $\mathcal{N} = 2$ σ-model action is uniquely determined by a single holomorphic function, the prepotential $\mathcal{F}(\Phi)$. The end result for the σ-model action reads

$$\begin{aligned} \mathcal{L}_{eff}^{\mathcal{N}=2} &= \frac{1}{64\pi i} \int d^2\theta \mathcal{F}_{ab}(\Phi) W^{\alpha a} W^b_\alpha + \frac{1}{32\pi i} \int d^2\theta d^2\bar{\theta} \left(\bar{\Phi} e^{2gV}\right)^a \mathcal{F}_a(\Phi) + \text{ h.c.} \\ &= \frac{1}{32\pi} \text{ Im } \left[\int d^2\theta \mathcal{F}_{ab}(\Phi) W^{\alpha a} W^b_\alpha + 2 \int d^2\theta d^2\bar{\theta} \left(\bar{\Phi} e^{2gV}\right)^a \mathcal{F}_a(\Phi) \right] . \end{aligned} \tag{6.11}$$

Using eqs. (6.9)-(6.10) we can compute the Kähler metric and see its relation with the complexified gauge coupling which is

$$K_a^b(\phi, \bar{\phi}) = \frac{\partial^2 K(\phi, \bar{\phi})}{\partial \phi^a \partial \bar{\phi}_b} = -\frac{i}{32\pi} \left(\frac{\partial^2 \mathcal{F}(\phi)}{\partial \phi^a \partial \phi^b} - \frac{\partial^2 \overline{\mathcal{F}(\phi)}}{\partial \bar{\phi}_a \partial \bar{\phi}_b} \right) = \frac{1}{16\pi} \text{ Im}\mathcal{F}_{ab}(\phi) \ . \tag{6.12}$$

Therefore, we finally get for the potential (recall that $W = 0$)

$$V(\phi, \bar{\phi}) = -\frac{1}{2\pi} \left(\text{Im } \mathcal{F}_{ab}(\phi)\right)^{-1} \left[\bar{\phi}, \mathcal{F}_c(\phi) T^c\right]^a \left[\bar{\phi}, \mathcal{F}_d(\phi) T^d\right]^b \ . \tag{6.13}$$

A Kähler manifold where the Kähler potential can be written in terms of a holomorphic function as in eq. (6.10) is called *special* Kähler manifold. From a geometric point of view this corresponds to a Kähler manifold endowed with a symplectic structure (a $2n_v$ symplectic bundle, where n_v is the number of vector multiplets). Note that the renormalizable Lagrangian (6.1) is recovered taking $\mathcal{F}(\Phi) = \frac{1}{2}\tau \, \text{Tr}\Phi^2$.

This is not the end of the story, though. To the σ-model action we have constructed one can add hypermultiplets. We refrain to present its structure here and just make two comments. Hypermultiplets contain two complex scalars. What one finds is that the corresponding σ-model is defined on a quaternionic manifold, known as HyperKähler manifold, which is, essentially, the quaternionic extension of a Kähler manifold (in particular, there are three rather than just one complex structures). So in $\mathcal{N} = 2$ supersymmetry, due to the existence of two sets of scalars, those belonging to matter multiplets and those belonging to gauge multiplets, the most general scalar manifold is (classically) of the form

$$\mathcal{M} = \mathcal{M}^V \otimes \mathcal{M}^H \ , \tag{6.14}$$

where $\mathcal{M}^V$ is a special Kähler manifold and $\mathcal{M}^H$ a HyperKähler manifold. Notice that, once renormalizability is relaxed, quartic (and higher, if $SU(2)_R$ singlets) superpotential couplings are possible. We will have much more to say about $\mathcal{N}=2$ σ-models in later lectures.

6.2 $\mathcal{N}=4$ supersymmetric actions

Let us now discuss the structure of the $\mathcal{N}=4$ Lagrangian. We have only one kind of multiplet in this case, the vector multiplet. So, from a $\mathcal{N}=4$ perspective, we can only have pure SYM theories. The decomposition of the $\mathcal{N}=4$ vector superfield in terms of $\mathcal{N}=1$ representations is as follows

$$[\mathcal{N}=4 \text{ vector multiplet}] \;\;:\;\; V=(\lambda_\alpha, A_\mu, D) \;\oplus\; \Phi_A=\left(\phi^A,\psi^A_\alpha,F^A\right) \quad A=1,2,3\;.$$

The propagating degrees of freedom are a vector field, six real scalars (two for each complex scalar ϕ_A) and four gauginos. The Lagrangian is very much constrained by $\mathcal{N}=4$ supersymmetry. First, the chiral superfields Φ_A should transform in the adjoint representation of the gauge group, since internal symmetries commute with supersymmetry. Moreover, we have now a large R-symmetry group, $SU(4)_R$ (the fact that the scalar fields transform under the fundamental representation of $SO(6)$, which is real, makes the R-symmetry group of the $\mathcal{N}=4$ theory being at most $SU(4)$ and not $U(4)$). The four Weyl fermions transform in the fundamental of $SU(4)_R$, while the six real scalars in the two times anti-symmetric representation, which is nothing but the fundamental representation of $SO(6)$. The auxiliary fields are singlets under the R-symmetry group. Using $\mathcal{N}=1$ superfield formalism the Lagrangian reads

$$\begin{aligned}\mathcal{L}^{\mathcal{N}=4}_{SYM} &= \frac{1}{32\pi}\,\mathrm{Im}\,\left(\tau\int d^2\theta\;\mathrm{Tr}\;W^\alpha W_\alpha\right)+\int d^2\theta d^2\bar\theta\;\mathrm{Tr}\;\sum_A \bar\Phi_A\,e^{2gV}\,\Phi_A\\ &- \int d^2\theta\sqrt{2}g\;\mathrm{Tr}\;\Phi_1\left[\Phi_2,\Phi_3\right]+\;\text{h.c.}\end{aligned}\tag{6.15}$$

where the commutator in the third term appears for the same reason as for the $\mathcal{N}=2$ Lagrangian (6.1). Notice that the choice of a single $\mathcal{N}=1$ supersymmetry generator breaks the full $SU(4)_R$ R-symmetry to $SU(3)\times U(1)_R$. The three chiral superfields transform in the $\underline{3}$ of $SU(3)$ and have R-charge $R=2/3$ under the $U(1)_R$. It is an easy but tedious exercise to perform the integration in superspace and get an explicit expression in terms of fields. Finally, one can solve for the auxiliary fields and get an expression where only propagating degrees of freedom are present, and where $SU(4)_R$ invariance is manifest. We refrain to perform the calculation here. We would only like to point out that the scalar potential can be written in a rather compact form in terms of the six real scalars X_i making up the three complex scalars ϕ_A and reads

$$V = \frac{1}{2}g^2 \ \mathrm{Tr} \sum_{i,j=1}^{6} [X_i, X_j]^2 \ . \tag{6.16}$$

From the above expression we see that $\mathcal{N} = 4$ SYM enjoys a large moduli space of vacua. Except for the origin, where all X_i VEVs vanish, the gauge group is generically broken along the moduli space and the theory is in a Coulomb phase, very much like pure $\mathcal{N} = 2$ SYM. At a generic point of the moduli space the gauge group breaking pattern $G \to U(1)^r$, where r is the rank of G, and the dynamics is that of r copies of $\mathcal{N} = 4$ $U(1)$ theory.

One might ask whether a σ-model action is possible for $\mathcal{N} = 4$ models. After all, we are plenty of scalar fields, actually $3n$ complex scalars, if n is the dimension of G. The answer is that there is only one possible σ-model compatible with $\mathcal{N} = 4$ supersymmetry (the stringent constraint comes from the large R-symmetry group), the trivial one: $\mathcal{M} = \mathbb{R}^{6n}$. So, the Lagrangian (6.15) is actually the only possible $\mathcal{N} = 4$ Lagrangian one can build. This also implies that, unlike pure $\mathcal{N} = 2$ SYM, the moduli space of vacua has a trivial topology.

6.3 On non-renormalization theorems

One of the advantages, in fact *the* advantage of supersymmetry is that it makes quantum corrections much better behaved with respect to ordinary field theories.

Many relevant results about UV properties of supersymmetric field theories were obtained back in the 1980's and can be summarized in terms of powerful non-renormalization theorems. At that time, a very efficient approach was developed to deal with supersymmetric quantum field theories, a version of Feynman rules, known as *supergraphs* techniques, which let one work directly with superfields in superspace with no need to expand into component fields. Most non-renormalization theorems were proved using such techniques whose description, however, is beyond the scope of these lectures. Here I just want to mention what is possibly the main result so obtained: in a supersymmetric quantum field theory containing chiral and vector superfields, the most general term that can be generated by loop diagrams has only one Grassman integral over *all* superspace

$$\int d^4x_1 \dots d^4x_n \, d^2\theta d^2\bar{\theta} \; G(x_1, \dots, x_n) F_1(x_1, \theta, \bar{\theta}) \dots F_n(x_n, \theta, \bar{\theta}) \tag{6.17}$$

where $G(x_1, \dots, x_n)$ is a translationally invariant function and the F_i's are products of superfields and their covariant derivatives. Such term is a D-term and does not contribute to superpotential terms, which are F-terms, implying that the superpotential is tree-level exact, i.e. it is not renormalized at any order in perturbation theory! The only possible corrections may arise at the non-perturbative level (and in some cases, namely when only chiral superfields are present, the latter also vanish, as we will see later). On the contrary, D-terms, that is the Kähler potential, are renormalized, and generically do receive corrections at any order in perturbation theory (and non-perturbatively).

Let us try and see what non-renormalization theorems imply for theories with different number of supersymmetries.

Let us first focus on a renormalizable $\mathcal{N} = 1$ action describing a chiral superfield Φ (the generalization to many chiral superfields is straightforward and does not present any relevant difference). There are three supersymmetric contributions to the action. One, the kinetic term, is a D-term, and undergoes renormalizations. Two are F-terms (the mass term and the cubic term) and are hence exact, perturbatively. Concretely

$$\int d^4x d^2\theta d^2\bar{\theta} \; \bar{\Phi}\Phi \quad \rightarrow \quad Z_\Phi \int d^4x d^2\theta d^2\bar{\theta} \; \bar{\Phi}\Phi \tag{6.18}$$

$$m \int d^4x d^2\theta \; \Phi^2 + h.c. \quad \rightarrow \quad m \int d^4x d^2\theta \; \Phi^2 + h.c. \tag{6.19}$$

$$\lambda \int d^4x d^2\theta \; \Phi^3 + h.c \quad \rightarrow \quad \lambda \int d^4x d^2\theta \; \Phi^3 + h.c \; . \tag{6.20}$$

This means that m and λ do get renormalized but only logarithmically at one loop, instead of quadratically and linearly, respectively. Hence

$$Z_m Z_\Phi = 1 \quad , \quad m \rightarrow Z_\Phi^{-1} m$$

$$Z_\lambda Z_\Phi^{3/2} = 1 \quad , \quad \lambda \rightarrow Z_\Phi^{-3/2} \lambda \; .$$

Something similar happens for the renormalization of the factor e^{2gV}. However, things are more subtle, here. Notice that

$$\int d^2\theta d^2\bar{\theta} \bar{\Phi} e^{2gV} \Phi \tag{6.21}$$

is a D-term and hence it renormalizes. However, what one can see using the background field method to renormalize gauge theories is that gV does not renormalize. An independent renormalization for g and V leading to a kinetic term of the form

$$\bar{\Phi} e^{2 Z_g g Z_V^{1/2} V} \Phi \tag{6.22}$$

would correspond to counterterms of the form

$$\bar{\Phi} V e^{gV} \Phi \tag{6.23}$$

which are not gauge invariant and cannot be generated by loop diagrams. This implies that the integral (6.21) should renormalize as the kinetic term (6.18) (not because of supersymmetry, but just due to gauge invariance!), meaning that g and V compensate each other upon renormalization. In other words, gV is not renormalized. Another way to see this is the following. Consider pure SYM. In this theory the only possible counterterm would correspond to something proportional to the action itself

$$\int d^2\theta \ \mathrm{Tr}\ W^\alpha W_\alpha + h.c. \ , \tag{6.24}$$

which would then correspond to a wave-function renormalization of the full Lagrangian (this is certainly there since the integral above is not a F-term, but rather a D-term, as already noticed). This means that one should multiply by the same function both the kinetic terms $dVdV$ as well as the interaction terms $gVVdV$ and g^2V^4 to keep gauge invariance (recall we are considering a non-abelian gauge group). In order for this to be the case one needs that if

$$V \rightarrow Z_V^{1/2} V \tag{6.25}$$

then

$$g \rightarrow Z_V^{-1/2} g \ , \tag{6.26}$$

which implies that gV is not renormalized, as anticipated, $Z_g = Z_V^{-1/2}$. The conclusion is that in renormalizable theories with $\mathcal{N} = 1$ supersymmetry there are only two independent renormalization, Z_Φ and Z_V, which are just logaritmically divergent at one-loop, and correspond to wave-function renormalization of chiral and vector superfields, respectively.

In passing, notice that the fact that $Z_V \neq 1$ means that the integral (6.24) is renormalized. This does not contradict non-renormalization theorems, since, as already observed, (6.24) is not a F-term, really, but actually a D-term, and then it does renormalize. That $Z_V \neq 1$ also implies that $Z_g \neq 1$, meaning that in $\mathcal{N} = 1$ SYM theories the gauge coupling runs, and can get corrections at all loops, in general.

What about higher supersymmetry? All what we said, so far, still applies, since any extended supersymmetry model is also $\mathcal{N} = 1$. However, extended supersymmetry imposes further constraints. In what follows, we stick to the notation we have used in sections 6.1 and 6.2 when discussing $\mathcal{N} = 2$ and $\mathcal{N} = 4$ Lagrangians, respectively.

Let us start from $\mathcal{N} = 2$ supersymmetry. For one thing, since V and Φ belong now to the same multiplet, we have that

$$Z_V = Z_\Phi \ . \tag{6.27}$$

As for the hypermultiplets, from the cubic interaction $gH_1\Phi H_2$ which appears in the superpotential (and which is then tree level exact, being a F-term) we get the following condition

$$Z_g^2 Z_\Phi Z_{H_1} Z_{H_2} = 1 \ . \tag{6.28}$$

The first two contributions cancel since $Z_V = Z_\Phi$ and, as for $\mathcal{N} = 1$ supersymmetry, gV is not renormalized, meaning $Z_g = Z_V^{-1/2}$. So we get

$$Z_{H_1} Z_{H_2} = 1 \ . \tag{6.29}$$

Hence the wave-functions of the two chiral superfields making up an hypermultiplet are not independent. All in all, we have then only two independent renormalizations in $\mathcal{N} = 2$ supersymmetry, Z_V and, say, Z_{H_1}. In fact, for massless representations there is the $SU(2)_R$ symmetry rotating the scalar components of H_1 and H_2 into each other. Hence, they should have the same renormalization, which means, using eq. (6.29), that $Z_{H_1} = Z_{H_2} = 1$. Actually, the same holds for massive (BPS) representations. In this case the existence of a non-trivial central charge does break the R-symmetry group to $USp(2)$; however, the algebra of such group is the same as that of $SU(2)$ and one can again conclude that $Z_{H_1} = Z_{H_2} = 1$.

It turns out that, because of the relation between Z_Φ and Z_g, not only $\mathcal{N} = 2$ SYM has a unique renormalization but it is one-loop exact in perturbation theory. In other words, the gauge coupling β-function gets only one-loop contributions, perturbatively. We will derive this important result in a later lecture, when discussing the dynamics of supersymmetric gauge theory. There, we will use a very powerful approach which is based on a crucial property of supersymmetry, known as *holomorphy*. For the time being let us just stress that this one-loop exactness of $\mathcal{N}=2$ SYM gauge coupling does not hold for $\mathcal{N}=1$ SYM, whose physical gauge coupling receives corrections at all orders in perturbation theory.

Let us finally consider $\mathcal{N}=4$ supersymmetry. Here we have a single superfield, the vector superfield which, in $\mathcal{N}=1$ language, can be seen as one vector superfield V and three chiral superfields Φ_A (all transforming in the adjoint representation of the gauge group). The $SU(3)$ symmetry rotating the three chiral superfields implies that the latter should have all and the same wave-function renormalization

$$Z_{\Phi_1} = Z_{\Phi_2} = Z_{\Phi_3} = Z \ . \tag{6.30}$$

On the other hand, due to eq. (6.27), this Z should equal Z_V, the wave-function of the vector superfield. Plugging this into eq. (6.28), which for the $\mathcal{N}=4$ Lagrangian (6.15) is

$$Z_g^2 Z_{\Phi_1} Z_{\Phi_2} Z_{\Phi_3} = 1 \ , \tag{6.31}$$

and recalling that $Z_g = Z_V^{-1/2}$ it follows that

$$Z_V = Z_{\Phi_1} = Z_{\Phi_1} = Z_{\Phi_1} = 1 \ , \tag{6.32}$$

which means that $\mathcal{N}=4$ SYM is perturbatively finite; in other words, the theory does not exhibit ultraviolet divergences in the correlation functions of canonical fields! Though we are not going to prove it here, it turns out that in fact $\mathcal{N}=4$ is finite also once non-perturbative corrections are taken into account. Indeed, the latter give finite contributions, only, and therefore the theory is believed to be UV finite.

There is yet another important property of $\mathcal{N}=4$ SYM we would like to mention. The theory is superconformal invariant, and it is so at the full quantum level. Let us see how this goes. A theory whose Lagrangian contains only dimension four operators, like the $\mathcal{N}=4$ Lagrangian (and many others, in fact) is classically scale invariant. For any relativistic field theory this implies a larger symmetry algebra, the conformal Poincaré algebra which, besides Poincaré generators, includes also dilations and special conformal transformations, the corresponding group being $SO(2,4) \simeq SU(2,2)$. The generators associated to dilations and special conformal transformations, D and K^μ, respectively, act as follows

$$\begin{aligned} D \ &: \ x^\mu \longrightarrow \lambda\, x^\mu \\ K^\mu \ &: \ x^\mu \longrightarrow \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2} \ , \end{aligned}$$

and have the following commutation relations between themselves and with the generators of the Poincaré algebra

$$[P_\mu, D] = iP_\mu \quad , \quad [D, M_{\mu\nu}] = 0 \quad , \quad [K_\mu, D] = -iK_\mu \quad , \quad [K_\mu, K_\nu] = 0$$
$$[P_\mu, K_\nu] = 2i\left(M_{\mu\nu} - \eta_{\mu\nu}D\right) \quad , \quad [K_\mu, M_{\rho\sigma}] = i\left(\eta_{\mu\rho}K_\sigma - \eta_{\mu\sigma}K_\rho\right) \; . \qquad (6.33)$$

Supersymmetry enlarges further the symmetry group. A conformal invariant supersymmetric theory enjoys an even larger algebra, the *superconformal* algebra, which includes, besides dilations and special conformal transformations, also conformal supersymmetry transformations $S^I_\alpha, \bar{S}^I_{\dot\alpha}$ (which appear in the commutator of the supersymmetry charges Q^I_α with the generators of special conformal transformations K^μ), and the generators associated to R-symmetry transformations, T^I_J (where $I, J = 1, \ldots, N$), which are now part of the algebra and do not act just as external automorphisms (they appear in the anti-commutator of the supersymmetry charges with the S^I_α's). The associated supergroup is $SU(2,2|N)$. The non-vanishing (anti)commutators involving the new generators are

$$\left[K_\mu, Q^I_\alpha\right] = 2i\sigma_{\mu\alpha\dot\beta}\bar{S}^{\dot\beta I} \; , \; \{S^I_\alpha, \bar{S}_{\dot\beta\, J}\} = 2\sigma^\mu_{\alpha\dot\beta}K_\mu\delta^I_J \quad , \quad \left[D, Q^I_\alpha\right] = -\frac{i}{2}Q^I_\alpha \; , \; \left[D, S^I_\alpha\right] = \frac{i}{2}S^I_\alpha$$
$$\{Q^I_\alpha, S_{\beta J}\} = \epsilon_{\alpha\beta}\left(\delta^I{}_J D + T^I_J\right) + \frac{1}{2}\delta^I{}_J\sigma^{\mu\nu}_{\alpha\beta}M_{\mu\nu} \quad , \quad \left[P_\mu, S^I_\alpha\right] = \sigma_{\mu\alpha\dot\beta}\bar{Q}^{\dot\beta I} \; . \qquad (6.34)$$

The $\mathcal{N} = 4$ SYM action is invariant under this larger symmetry algebra, $SU(2,2|4)$ in this case, but it is certainly not the only theory having this property, at the classical level. Classical superconformal invariance is shared by any supersymmetric Lagrangian made solely by dimension four operators (in other words, with dimensionless, hence classically marginal couplings), well-known examples being the massless WZ model, and in fact any SYM theory, like $\mathcal{N} = 1$ SQCD discussed in the previous lecture.

What makes $\mathcal{N} = 4$ SYM special is that, as we have observed above, the Lagrangian does not renormalize (recall that essential to this proof was the use of the $SU(3)$ subgroup of the R-symmetry group rotating the three scalar superfields Φ^A). In particular, as we have seen before, $Z_g = 1$. In other words, the $\mathcal{N} = 4$ β-function vanishes identically: the theory remains scale invariant at the quantum level, and the superconformal symmetry $SU(2,2|4)$ is then an exact symmetry of the theory. An equivalent conclusion can be reached by observing that $\mathcal{N} = 4$ SYM is a (very special) $\mathcal{N} = 2$ theory. The $\mathcal{N} = 2$ gauge coupling is one-loop exact and since in $\mathcal{N} = 4$ SYM this is the only coupling appearing in the Lagrangian, it is enough to compute the one-loop β-function for g. One can easily see that such one-loop coefficient vanishes, concluding that the theory is superconformal also at the quantum level. In fact, the equivalence of these proofs lies in the fact that the gauge coupling β-function and the R-symmetry are in the same supermultiplet, the ($\mathcal{N} = 4$) supercurrent multiplet.

This non-renormalization property is not shared by other theories, in general: typically, the superconformal algebra is broken by quantum corrections and couplings run. For instance, in the massless WZ model, the coupling, which is classically marginal, becomes irrelevant quantum mechanically (i.e., it flows to zero and the theory becomes free in the IR). On the contrary, UV-free supersymmetric gauge theories, like pure $\mathcal{N} = 1, 2$ SYM, enjoy dimensional transmutation and a dynamically scale is generated at the quantum level.

What we said above about the finiteness of $\mathcal{N} = 4$ does not mean that any operator has protected dimension. The scaling dimension of canonical fields (gauge fields, gauginos and adjoint scalars) is unaffected by quantum corrections, but this does not happen, in general, to composite gauge invariant operators. Yet, in a superconformal theory there are special operators whose dimension is protected. To see how this comes, let us start considering the conformal algebra (6.33). In unitary theories there is a lower bound for the scaling dimension Δ of a field (e.g. $\Delta \geq 1$ for a scalar field in four dimensions). Since K_μ lowers the scaling dimension of a field, any representation of the conformal algebra should admit an operator with minimal dimension Δ which is annihilated by K_μ (at $x_\mu = 0$). Such states are called conformal primary operators. Since the conformal algebra is a subalgebra of the superconformal algebra, representations of the latter in general decompose into representations of the former. By definition, a superconformal primary operator is an operator which is annihilated (at $x_\mu = 0$) both by K_μ and $S^I_\alpha, \bar{S}^I_{\dot{\alpha}}$. From the commutator $\left[K_\mu, Q^I_\alpha\right]$ in (6.34) it also follows that any operator which is obtained from a superconformal primary by the action of Q^I_α, and hence sits in the same supermultiplet, is a primary operator of the conformal algebra. Superconformal primary operators which are annihilated by some of the supercharges are called *chiral primaries* and, most importantly, their dimension is fixed by their R-symmetry representation, and as such are protected against quantum corrections (this can be proven by playing a bit with the superconformal algebra, which lets express the scaling dimension Δ of a chiral primary operator in terms of Lorentz and R-symmetry representations). By supersymmetry, this implies that in a superconformal theory operators belonging to supersymmetry representations which include a chiral primary operator do not renormalize. For instance, in $\mathcal{N}=4$ SYM, a class of superconformal primaries are all operators made of symmetric traceless products of the scalar fields X_i's, e.g. $\mathrm{Tr}\,(X^{\{i}X^{j\}}) = \mathrm{Tr}\,(X^iX^j) - \frac{1}{6}\delta^{ij}\mathrm{Tr}\,(X^kX^k)$.

As a final comment, let us just notice that in $\mathcal{N}=4$ SYM superconformal invariance is/is not realized depending on the point of the moduli space one is sitting. The phase where all scalar field VEVs $\langle X_i \rangle$ vanish is called superconformal phase since at the origin of the moduli space the gauge group remains unbroken and superconformal invariance is preserved. In other words, physical states are not only gauge invariant, but carry unitary representations of $SU(2,2|4)$. On the contrary, on the Coulomb branch, where gauge symmetry is broken, also superconformal symmetry is broken since scalar VEVs $\langle X_i \rangle$ set a dimensionful scale in the theory.

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