7 Supersymmetry breaking

If supersymmetry is at all realized in Nature, it must be broken at low enough energy: we do not see any mass degeneracy in the elementary particle spectrum, at least at energies of order $10^2$ GeV or lower. The idea is then that supersymmetry is broken at some scale $M_s$, such that at energies $E > M_s$ the theory behaves as a supersymmetric theory, while at energies $E < M_s$ it does not. On general ground, there are two ways supersymmetry can be broken, either spontaneously or explicitly.

- **Spontaneous supersymmetry breaking**: the theory is supersymmetric but has a scalar potential admitting (stable, or metastable but sufficiently long-lived) supersymmetry breaking vacua. In such vacua one or more scalar fields acquire a VEV of order $M_s$, which then sets the scale of supersymmetry breaking.

- **Explicit supersymmetry breaking**: the Lagrangian contains terms which do not preserve supersymmetry by themselves. In order for them not to ruin the nice and welcome UV properties of supersymmetric theories, these terms should have positive mass dimension, in other words they should be irrelevant in the far UV. In this case we speak of *soft* supersymmetry breaking. In such scenario, the scale $M_s$ enters explicitly in the Lagrangian.

As we will show later, soft supersymmetry breaking models can (and typically do) actually arise as low energy effective descriptions of models where supersymmetry is broken spontaneously. Therefore, we will start focusing on spontaneous supersymmetry breaking. Only after we will discuss supersymmetry breaking induced by soft terms.

7.1 Vacua in supersymmetric theories

We have already seen that supersymmetric vacua are in one-to-one correspondence with the zero’s of the scalar potential. In other words, the vacuum energy is zero if and only if the vacuum preserves supersymmetry. Hence, non-supersymmetric vacua correspond to minima of the potential which are not zero’s. In this case supersymmetry is broken in the perturbative theory based on these positive energy vacua.

Notice how different is spontaneous supersymmetry breaking with respect to spontaneous breaking of ordinary internal symmetries (being them global or local).
There, what matters is the location of the minima of the potential in field space, while here it is the absolute value of the potential at such minima. This implies that in, e.g. a supersymmetric gauge theory there can be minima which preserve both gauge symmetry and supersymmetry, others which break both, and others which preserve gauge symmetry and break supersymmetry, or vice versa. A schematic picture of these different situations is reported in figure 7.1.

Figure 7.1: A schematic picture of possible patterns of spontaneous gauge symmetry and supersymmetry breakings. The potential on the upper left does not admit any symmetry breaking vacuum. The one on the lower right, instead, admits two vacua breaking both gauge symmetry and supersymmetry. The other two represent mixed situations where either supersymmetry or gauge symmetry are broken.

While non-supersymmetric vacua can be either global or local minima of the potential (corresponding to stable or metastable vacua, respectively), supersymmetric vacua, if present, are obviously global minima of the potential, since in a supersymmetric theory the scalar potential is a semi-positive definite quantity.
Recall the expression (5.82), that is
\[ V(\phi, \bar{\phi}) = FF + \frac{1}{2} D^2 , \] (7.1)
where
\[ F_i = \frac{\partial W}{\partial \phi^i} , \quad D^a = -g \left( \bar{\phi}_i (T^a)_j \phi^j + \xi^a \right) \] (7.2)
(we focus here on models with canonical Kähler potential; later we will also discuss situations where the Kähler potential is not canonical).

Supersymmetric vacua are described by all possible set of scalar field VEVs satisfying the D and F-term equations
\[ F^i(\phi) = 0 , \quad D^a(\bar{\phi}, \phi) = 0 . \] (7.3)
If there exist more than one solution, it means there are more supersymmetric vacua, generically a moduli space of vacua, if these are not isolated. If there does not exist a set of scalar field VEVs for which eqs. (7.3) are satisfied, then supersymmetry is broken and the minima of the potential are all necessarily positive, \( V_{\text{min}} > 0 \).

Notice, on the contrary, that on any vacuum, supersymmetric or not, global or local, the following equations always hold
\[ \frac{\partial V(\phi, \bar{\phi})}{\partial \phi^i} = 0 , \quad \frac{\partial V(\phi, \bar{\phi})}{\partial \phi_i} = 0 , \] (7.4)
which simply say that vacua sit at extrema of the scalar potential.

An equivalent statement about supersymmetric vacua is that on supersymmetric vacua the supersymmetric variations of fermion fields vanish. This can be seen as follows. Due to Lorentz invariance, on a vacuum any field’s VEV or its derivative should vanish, but scalar fields. Recalling how the different field components of a chiral or vector superfield transform under supersymmetry transformations, it follows that on a vacuum state we have
\[ \delta(\phi^i) = 0 , \quad \delta(F^i) = 0 , \quad \delta(\psi_\alpha^i) \sim \epsilon_\alpha \langle F^i \rangle \\
\delta(F^a_{\mu \nu}) = 0 , \quad \delta(D^a) = 0 , \quad \delta(\chi_\alpha^a) \sim \epsilon_\alpha \langle D^a \rangle . \] (7.5)
Therefore, in a generic vacuum the supersymmetric variations of the fermions is not zero: it is actually proportional to the vacuum expectation values of the auxiliary fields. A supersymmetric vacuum state is by definition supersymmetric invariant (!). Hence, from the above equations it follows that on a supersymmetric vacuum also the supersymmetric variations of the fermions should be zero, the latter being equivalent to the D and F-term equations (7.3), as anticipated.
7.2 Goldstone theorem and the goldstino

When a global symmetry is spontaneously broken, Goldstone theorem says that there is a massless mode in the spectrum, the Goldstone field, whose quantum numbers should be related to the broken symmetry. We should expect this theorem to work also for spontaneously broken supersymmetry. In fact, given that supersymmetry is a fermionic symmetry, the Goldstone field should be in this case a Majorana spin $1/2$ fermion, the so-called goldstino.

Consider the most general supersymmetric Lagrangian with gauge and matter fields, eq. (5.81), and suppose it admits some vacuum where supersymmetry is broken. In this vacuum eqs. (7.4) hold, while (some of) eqs. (7.3) do not. Recalling eqs. (5.80) and (5.82) we have in this vacuum that

$$\frac{\partial V(\phi_i, \bar{\phi}_p)}{\partial \phi^i} = F_j^i(\phi) \frac{\partial^2 W}{\partial \phi^i \partial \bar{\phi}^j} - g D^a \bar{\phi}_j (T^a)_i^j = 0 .$$

(7.6)

On the other hand, since the superpotential is gauge invariant, we have that

$$\delta^a W = \frac{\partial W}{\partial \phi^i} \delta^a \phi^i = \bar{F}_i (T^a)_i^j \phi^j = 0 .$$

(7.7)

Combining the former equation with the complex conjugate of the latter evaluated in the vacuum, we easily get a matrix equation

$$M \begin{pmatrix} \langle F_j^i \rangle \\ \langle D^a \rangle \end{pmatrix} = 0 \quad \text{where} \quad M = \begin{pmatrix} \langle \frac{\partial^2 W}{\partial \phi^i \partial \bar{\phi}^j} \rangle & -g \langle \bar{\phi}_j (T^a)_i^j \rangle \\ -g \langle \bar{\phi}_i (T^b)_j^i \rangle & 0 \end{pmatrix} .$$

(7.8)

The above equation implies that the matrix $M$ has an eigenvector with zero eigenvalue. Now, this matrix is nothing but the fermion mass matrix of the Lagrangian (5.81)! This can be seen looking at the non-derivative fermion bilinears of (5.81), which on the vacuum get contributions also from the cubic coupling between scalar fields, their superpartners and gauginos, and which can be written as

$$\cdots - \frac{1}{2} \left( \psi^i , \sqrt{2i} \lambda^b \right) M \left( \frac{\psi^j}{\sqrt{2i}} \lambda^a \right) + h.c. + \cdots$$

(7.9)

Hence, on the supersymmetry breaking vacuum the spectrum necessarily admits a massless fermion, the goldstino. It is easy to see that in terms of spin $1/2$ particles belonging to the different multiplets, the goldstino $\psi^G_\alpha$ corresponds to the following linear combination

$$\psi^G_\alpha \sim \langle F^i \rangle \psi^\dagger_\alpha + \langle D^a \rangle \lambda^a_\alpha .$$

(7.10)
The proof we have provided of the goldstino theorem has been based on the Lagrangian (5.81). In fact, one can provide a similar proof using properties of the supercurrent and Ward identities, which does not rely on the existence of an explicit classical Lagrangian. The supersymmetry Ward identity reads
\[ \langle \partial^\mu S_{\mu} (x) S_{\nu \beta} (0) \rangle = - \delta^4 (x) \langle \delta_{\alpha} S_{\nu \beta} \rangle = - 2 \sigma^\mu_{\alpha \beta} \langle T_{\mu \nu} \rangle \delta^4 (x) , \] (7.11)
where the last equality follows from the current algebra, see eq. (4.71) (Schwinger terms cannot have a non-vanishing VEV in a Lorentz invariant vacuum, while this is possible for the energy-momentum tensor, \( T_{\mu \nu} \sim \eta_{\mu \nu} \)). Integrating eq. (7.11), one gets for the two-point function of the supercurrent that
\[ \langle S_{\mu \alpha} (x) S_{\nu \beta} (0) \rangle = \cdots + (\sigma_\mu \bar{\sigma}_\nu)_{\alpha \beta} \frac{x_\nu}{x^4} \langle T \rangle , \] (7.12)
where \( \langle T \rangle = \eta_{\mu \nu} \langle T_{\mu \nu} \rangle \) and the \( \cdots \) are terms which are not relevant to the present discussion. Upon Fourier transforming we finally get
\[ \langle S_{\mu \alpha} (k) S_{\nu \beta} (-k) \rangle = \cdots + (\sigma_\mu \bar{\sigma}_\nu)_{\alpha \beta} k_\nu \frac{\langle T \rangle}{k^2} , \] (7.13)
which shows the presence of a massless pole (the goldstino), proportional to the vacuum energy density, in the supercurrent two-point function. The above equation shows that the goldstino is the lowest energy excitation of the supercurrent, and it is so if and only if the vacuum energy is non-vanishing. This shows, as anticipated, that the goldstino theorem holds universally, \( \text{i.e.} \) also for vacua where supersymmetry is broken in a strongly coupled phase of the theory, where classical arguments may not apply.

### 7.3 F-term breaking

From our discussion it is clear that given a generic Lagrangian, there are two a priori independent ways we can break supersymmetry: either giving a non vanishing expectation value to (some) F-terms or to (some) D-terms. We will consider both options in turn.

In this section we will start considering F-term breaking and therefore we assume, for the time being to deal with a theory with chiral superfields, only.

The most general renormalizable Lagrangian of this sort reads
\[ \mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \bar{\Phi}_i \Phi^i + \int d^2 \theta W (\Phi^i) + \int d^2 \bar{\theta} \bar{W} (\bar{\Phi}_i) , \] (7.14)
where
\[ W(\Phi^i) = a_i \Phi^i + \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{3} g_{ijk} \Phi^i \Phi^j \Phi^k. \] (7.15)

The equations of motions for the auxiliary fields read
\[ \mathcal{F}_i(\phi) = \frac{\partial W}{\partial \phi^i} = a_i + m_{ij} \phi^j + g_{ijk} \phi^j \phi^k, \] (7.16)
and the potential is
\[ V(\phi, \bar{\phi}) = \sum_i |a_i + m_{ij} \phi^j + g_{ijk} \phi^j \phi^k|^2. \] (7.17)

Supersymmetry is broken if and only if there does not exist a set of scalar field VEVs such that all F-terms vanish, \( \langle F^i \rangle = 0 \). This implies that in order for supersymmetry to be broken, it is necessary some \( a_i \) to be different from zero. If not, the trivial solution \( \langle \phi^i \rangle = 0 \) solves all F-equations. So, any model of F-term supersymmetry breaking needs a superpotential admitting linear terms.

Notice that this conclusion applies also for a superpotential with higher non-renormalizable couplings. In fact, it does also in presence of a non-canonical Kähler potential! This can be seen recalling the expression of the scalar potential when a non-canonical Kähler metric is present
\[ V(\phi, \bar{\phi}) = (K^{-1})_i^j \frac{\partial W}{\partial \phi^i} \frac{\partial \bar{W}}{\partial \phi^j}. \] (7.18)

From this expression it is clear that unless it were singular (something signalling, as already discussed, an inconsistency of the effective theory analysis), a non-trivial Kähler metric could not influence the existence/non existence of supersymmetric vacua, which is still dictated by the possibility/impossibility to make the first derivatives of the superpotential vanish. What gets modified by a non-trivial Kähler potential, instead, is the value of the vacuum energy (for non-supersymmetric vacua, only!) and the particle spectrum around a given vacuum (for both supersymmetric and non-supersymmetric vacua).

In what follows we will consider several examples of F-term breaking.

**Example 1**: The Polonyi model.

Let us consider the theory of a single chiral superfield with canonical Kähler potential and a linear superpotential
\[ K = \overline{X} X, \quad W = \lambda X. \] (7.19)
This is the most minimal set-up one can imagine for a F-term supersymmetry breaking model. The potential reads

\[ V = \left| \frac{\partial W}{\partial X} \right|^2 = |\lambda|^2, \]  

(7.20)

Supersymmetry is clearly broken for any $|X|$, and the latter is in fact a flat direction. The supersymmetry breaking scale is set by the modulus of $\lambda$ itself, $|\lambda| = M_s^2$. In Figure 7.2 we report the (trivial) shape of the scalar potential.

Figure 7.2: The potential of the Polonyi model.

A few comments are in order. First notice that the theory possesses an R-symmetry, the R-charge of $X$ being $R(X) = 2$. At a generic point of the moduli space, then, both supersymmetry and R-symmetry are broken. Second, notice that although supersymmetry is broken, the spectrum is degenerate in mass: $|X|$, its phase $\alpha$, and $\psi_X$ are all massless. The fermion field has a good reason to be massless: it is the goldstino predicted by Goldstone theorem. Seemingly, the phase of $X$ is expected to be massless: it is nothing but the goldstone boson associated to the broken R-symmetry. Finally, the modulus of the scalar field $|X|$ is massless since it parametrizes the (non-supersymmetric) moduli space. However, there are no reasons to expect this moduli space to be protected, in principle, against quantum corrections, since supersymmetry is broken. Hence, generically, one would expect it to be lifted at the quantum level and $|X|$ to get a mass. This is not the case in this simple theory, since it is a non-interacting theory, and there are no quantum corrections whatsoever. In general, however, things are different: a non-supersymmetric moduli space gets typically lifted at one or higher loops, and the putative moduli get a mass. For this reason, non-supersymmetric moduli spaces are dubbed pseudo-moduli spaces, and the moduli parametrizing them, pseudo-moduli. We will see examples of this sort soon.
Let us now consider the following innocent-looking modification of the model above. Let’s add a mass term to $X$,

$$\Delta W = \frac{1}{2} mX^2 .$$

(7.21)

Things drastically change, since supersymmetry is now restored. Indeed, the F-term equation now reads

$$F(X) = mX + \lambda = 0 ,$$

(7.22)

which admits the solution $\langle X \rangle_{SUSY} = -\frac{\lambda}{m}$. Hence, there is a choice of scalar field VEV which makes the potential $V = |\lambda + mX|^2$ vanish, as illustrated in figure 7.3.

![Figure 7.3: The potential of the massive Polonyi model.](image)

The spectrum is supersymmetric and massive: all fields have mass $m$. This agrees with physical expectations: $\psi_X$ is no more the goldstino, since this is not expected to be there now; $|X|$ is no more a (pseudo)modulus since the supersymmetric vacuum is isolated (the VEV of $|X|$ is not a flat direction); finally, $\alpha$ is not anymore the goldstone boson associated to the broken R-symmetry since the superpotential term $\Delta W$ breaks the R-symmetry explicitly. More precisely, $W = \lambda X + \frac{1}{2} mX^2$ does not admit any R-charge assignment for $X$ such that $R(W) = 2$, meaning that the theory does not admit a R-symmetry to start with.

Things might also change (both qualitatively and quantitatively) if one allows the Kähler potential not being canonical. Suppose we keep $W = \lambda X$ but we let the Kähler metric be non-trivial, that is

$$V = (K_{XX})^{-1} |\lambda|^2 \quad \text{with} \quad K_{XX} = \frac{\partial^2 K}{\partial X \partial X} \neq 1 .$$

(7.23)
A non-trivial Kähler metric can deform sensibly the pseudo-moduli space of figure 7.2. A sample of possible different behaviors, which depend on the asymptotic properties (or singularities) of the Kähler metric, is reported in Figure 7.4.

![Graphs showing different vacua and behavior](image)

Figure 7.4: Qualitatively different potentials of non-canonical Polonyi-like models.

Physically, the different behaviors reported in figure 7.4 should be understood as follows. In presence of a classical pseudomoduli space like the one in figure 7.2, the lifting of the pseudomoduli at quantum level occurs because the massless particles which bring from a vacuum to another get a mass at one loop. Sometime, such effect can be mimicked by a non-canonical Kähler potential. In fact, these seemingly ad-hoc theories can and sometime do arise at low energies as effective theories of more complicated UV-renormalizable theories: the mass scale entering the non-canonical Kähler potential is nothing but the UV cut-off of these low energy effective theories.

Let us try to make the above discussion more concrete by considering an explicit example.
**Example 2:** A Polonyi model with quartic Kähler potential.

Let us consider the following model

\[ K = \bar{X}X - \frac{c}{\Lambda^2} (\bar{X}X)^2 \quad \text{and} \quad W = \lambda X , \quad \text{(7.24)} \]

where \( c > 0 \). Notice that the R-symmetry is not broken by the non-canonical Kähler potential (7.24), which is R-symmetry invariant. So the \( U(1)_R \) symmetry is a symmetry of the theory. The Kähler metric and the scalar potential now read

\[ K_{XX} = 1 - \frac{4c}{\Lambda^2} \bar{X}X \quad \text{and} \quad V = K_{XX} |\lambda|^2 = \frac{|\lambda|^2}{1 - \frac{4c}{\Lambda^2} \bar{X}X} . \quad \text{(7.25)} \]

The Kähler potential is an instance of models like those in the upper-left diagram of Figure 7.4: the Kähler metric \( K_{XX} \) vanishes for large enough \(|X|\), \(|X| \to |\Lambda|/2\sqrt{c}\), which is order the natural cut-off of the theory. The potential, which is depicted in figure 7.5, admits a (unique) minimum at \( \langle X \rangle = 0 \). Therefore, there is an isolated vacuum now and it is a supersymmetry breaking one. One can compute the spectrum around such vacuum and find that \( \psi_X \) is consistently massless (it is the goldstino), while now the scalar field is massive, \( m_X^2 \sim c|\lambda|^2/\Lambda^2 \).

**Example 3:** Supersymmetry restoration by new degrees of freedom.

Let us now deform the basic Polonyi model by adding a new superfield, \( Y \), while keeping the Kähler potential canonical. The superpotential reads

\[ W = \lambda X + \frac{1}{2} hXY^2 . \quad \text{(7.26)} \]

Notice that this model has an R-symmetry, with R-charge assignment \( R(X) = 2 \quad \text{and} \quad R(Y) = 0 \). From the F-equations one can compute the potential which reads

\[ V = |hXY|^2 + \frac{1}{2} hY^2 + |\lambda|^2 , \quad \text{(7.27)} \]
implying that there are two supersymmetric vacua at
\[ \langle X \rangle_{\text{SUSY}} = 0 \quad , \quad \langle Y \rangle_{\text{SUSY}} = \pm \sqrt{-\frac{2\lambda}{h}} . \] (7.28)

So we see that the additional degrees of freedom have restored supersymmetry. Interestingly, there are other local minima of the potential, a pseudo-moduli space in fact, where supersymmetry is broken
\[ \langle X \rangle_{\text{SB}} = \text{any} \quad , \quad \langle Y \rangle_{\text{SB}} = 0 \quad \text{where} \quad V = |\lambda|^2 . \] (7.29)

The physical interpretation is as follows. For large \( \langle X \rangle \), the superfield \( Y \) gets a large mass and affects the low energy theory lesser and lesser. The theory reduces effectively to the original Polonyi model, which breaks supersymmetry and whose vacuum energy is indeed \( V = |\lambda|^2 \). It is a simple but instructive exercise to compute the mass spectrum around the non-supersymmetric minima. The chiral superfield \( X \) is obviously massless while \( Y \) gets a mass. There is a first (obvious) contribution to both the scalar and the fermion components of \( Y \) from \( h \langle X \rangle \), and a second contribution which affects only the scalar component of \( Y \) coming from \( F_X \), which is non-vanishing. The end result is
\[ m_Y^2 = |h \langle X \rangle|^2 \pm |h\lambda| \quad , \quad m_{\psi_Y} = h \langle X \rangle . \] (7.30)

From the above expressions, we see that the supersymmetry breaking pseudomoduli space has a tachyonic mode which develops (and destabilizes the vacuum) for
\[ |X|^2 < |\lambda/h| \equiv |X_c|^2 . \] (7.31)

In such region the potential decreases along the \( \langle Y \rangle \) direction towards the supersymmetry vacua. A qualitative picture of the potential is reported in Figure 7.6.

**Example 4**: Runaway behavior.

A minimal modification of the above theory gives a completely different dynamics. Let us suppose that the cubic term of the superpotential (7.26) has the square shifted from \( Y \) to \( X \). In this case we would have for the superpotential the following expression
\[ W = \lambda X + \frac{1}{2} h X^2 Y . \] (7.32)

The F–equations are
\[ F_X = \lambda + h X Y \quad , \quad F_Y = \frac{1}{2} h X^2 , \] (7.33)
and the scalar potential

\[ V = \frac{1}{2} hX^2 + |hXY + \lambda|^2. \]  

(7.34)

Differently from previous example, it is not possible to satisfy both F-equations and so there are no supersymmetric ground states now. Notice, in passing, that the R-symmetry is preserved by the superpotential as in the previous example, with charge assignment \( R(X) = 2 \) and \( R(Y) = -2 \) this time.

Now the question is: where is the minimum of this supersymmetry breaking potential? An analysis of \( V \) shows that the global minimum is reached for \( Y \to \infty \).

A quick way to see it is to set \( X = -\frac{\lambda}{hY} \), which kills the second contribution to the potential, the \( F_X \)-equation. By plugging this back into \( V \) one gets

\[ V \underset{Y \to \infty}{\longrightarrow} 0. \]  

(7.35)

In other words, there is no stable vacuum but actually a \textit{runaway} behavior and supersymmetry is restored at infinity in field space.

A more physical way to reach the same conclusion is as follows. For large \( |Y| \) the amount of supersymmetry breaking gets smaller and smaller and \( X \) mass larger and larger. Hence the theory can be described by a theory where \( X \) is integrated out solving its equation of motion, which for large enough mass reduces to \( \partial W/\partial X = 0 \).

Figure 7.6: The potential of the model of Example 3: supersymmetry restoration.
This sets \( X = -\lambda / (hY) \) and the \( Y \)-dependent only superpotential becomes

\[
W_{\text{eff}} = -\frac{\lambda^2}{2hY},
\]

which gives the runaway behavior described by the potential (7.35).

Notice that within all models discussed so far, the only renormalizable one which breaks supersymmetry in a stable vacuum is the original Polonyi model (in fact there is an all pseudo-moduli space). This model is however rather uninteresting per sé, since it describes a non-interacting theory. One might wonder whether there exist reasonably simple models which are renormalizable, interacting and break supersymmetry in stable vacua. The simplest such model is the re-known O’Raifeartaigh model, which we now describe.

**Example 5**: The O’Raifeartaigh model.

Let us consider the theory of three chiral superfields with canonical Kähler potential and a superpotential given by

\[
W = \frac{1}{2} hX \Phi_1^2 + m\Phi_1 \Phi_2 - \mu^2 X.
\]

The superpotential respects the R-symmetry, the R-charge assignment being \( R(X) = 2, R(\Phi_1) = 0 \) and \( R(\Phi_2) = 2 \). The F-term equations read

\[
\begin{align*}
\mathcal{F}_X &= \frac{1}{2} h\phi_1^2 - \mu^2 \\
\mathcal{F}_1 &= hX\phi_1 + m\phi_2 \\
\mathcal{F}_2 &= m\phi_1
\end{align*}
\]

Clearly the first and the third equations cannot be solved simultaneously. Hence supersymmetry is broken. Let us try to analyze the theory a bit further. There are two dimensionful scales, \( \mu \) and \( m \). Let us choose in what follows \( |\mu| < |m| \) (nothing crucial of the following analysis would change choosing a different regime). In this regime one can show that the minimum of the potential is at

\[
\phi_1 = \phi_2 = 0 \quad , \quad X = \text{any}
\]

and the vacuum energy is \( V = |\mu|^2 \). Again, we find a pseudo-moduli space of vacua since \( X \) is not fixed by the minimal energy condition. In Figure 7.7 we depict the potential as a function of the scalar fields.

Let us compute the (classical) spectrum around the supersymmetry breaking vacua. The full chiral superfield \( X \) is massless, right in the same way as for the
Polonyi model (notice that for larger and larger $|X|$ the model gets closer and closer to the Polonyi model since all other fields get heavier and heavier). The massless fermion mode $\psi_X$ is nothing but the goldstino. The only non vanishing F-term in the vacuum is $F_X$, so that the goldstino gets contribution only from $\psi_X$ agrees with eq. (7.10). The phase $\alpha$ of the scalar field $X = |X|e^{i\alpha}$ is the Goldstone boson associated to R-symmetry, which is spontaneously broken in the vacuum. Note that while $\Phi_2$ is charged under the R-symmetry, the phase of $\phi_2$ does not contribute to the R-axion since the VEV of $\phi_2$ is vanishing in the supersymmetry breaking vacua (7.39). Finally, $|X|$ is massless since it is a modulus (at least at classical level).

One can easily compute the ($|X|$-dependent) mass spectrum of all other fields and get

$$m_0^2(|X|) = |m|^2 + \frac{1}{2} \eta |h\mu|^2 + \frac{1}{2} |hX|^2$$
$$\pm \frac{1}{2} \sqrt{|h\mu|^2 + 2\eta |h\mu|^2 |hX|^2 + 4|m|^2 |hX|^2 + |hX|^4}$$

$$m_{1/2}^2(|X|) = \frac{1}{4} \left(|hX| \pm \sqrt{|hX|^2 + 4|m|^2} \right)^2.$$ (7.40)

where $\eta = \pm 1$, giving different masses to the four real scalar modes belonging to $\Phi_1$ and $\Phi_2$. As expected, the spectrum is manifestly non-supersymmetric. Notice that $m_{1/2}^2(|X|) = m_0^2(|X|)|_{\mu^2=0}$. Note that these infinitely many (degenerate in energy) vacua are in fact physically inequivalent, since the mass spectrum depends on $|X|$.

\textit{Example 6}: A modified O’Raifeartaigh model.

![Figure 7.7: The (classical) potential of the O’Raifeartaigh model.](image)
Let us end this overview by considering a modification of the previous model. Let us add a (small) mass perturbation for $W$

\[ \Delta W = \frac{1}{2} \epsilon m \Phi_2^2 \text{ with } \epsilon << 1 . \]  

(7.41)

Notice that this term breaks the R-symmetry enjoyed by the original O’Raifeartaigh model. The only F-equation which gets modified is the one for $\Phi_2$ which now reads

\[ \overline{F}_2 = m \phi_1 + \epsilon m \phi_2 . \]  

(7.42)

The presence of the second term removes the conflict we had before between this equation and the F-equation for $X$. Hence we can solve all F-term equations simultaneously and supersymmetry is not broken anymore. The (two) supersymmetric vacua are at

\[ X = \frac{m}{h \epsilon} , \quad \phi_1 = \pm \sqrt{\frac{2 \mu^2}{h}} , \quad \phi_2 = \mp \frac{1}{\epsilon} \sqrt{\frac{2 \mu^2}{h}} . \]  

(7.43)

For $\epsilon << 1$ these vacua are far away, in field space, from where the supersymmetry breaking vacua of the O’Raifeartaigh model sit (the VEV of $\phi_2$ becomes larger and larger and hence very far from $\phi_2 = 0$, the value of $\phi_2$ in O’Raifeartaigh model supersymmetry breaking vacua). In fact, near the origin of field space the potential of the present model is practically identical to the one of the original O’Raifeartaigh model and one can show by direct computation that a classically marginal pseudo-moduli space of supersymmetry breaking minima is present, there.

Computing the mass spectrum near the origin one gets now

\[ m_0^2(|X|) = \frac{1}{2} \left\{ |hX|^2 + |m|^2 (2 + |\epsilon|^2) + \eta |h \mu|^2 \right\} \]  
\[ \pm \sqrt{\left[ |hX|^2 + |m|^2 (2 + |\epsilon|^2) + \eta |h \mu|^2 \right]^2 - 4 |m|^2 [ |hX| \epsilon - m |^2 + \eta |h \mu|^2 (1 + |\epsilon|^2)]} \]  

(7.44)

\[ m_{1/2}^2(|X|) = m_0^2(|X|)|_{\mu^2=0} . \]

A close look to the above spectrum shows that in order for mass squared eigenvalues being all positive, the following inequality should be satisfied

\[ \left| 1 - \frac{e h X}{m} \right|^2 > (1 + |\epsilon|^2) \frac{h \mu^2}{m^2} \]  

(7.45)

For small $\epsilon$ and $\mu/m$, the marginally stable region described by the above inequality includes a large neighborhood around the origin, and the tachyonic mode develops only for $|X|$ (parametrically) larger than a critical value $|X_c|$

\[ |X| < |X_c| . \]  

(7.46)
Notice that this is quite the opposite of what we got in Example 4, where the marginally stable region was above a critical value; as we will see, this difference has crucial consequences at the quantum level. For $\epsilon \to 0$ one gets that $X_c, X_{SUSY} \to \infty$ and the supersymmetric vacua are pushed all the way to infinity. This is consistent with the fact that for $\epsilon = 0$ one recovers the original O’Raifeartaigh model where supersymmetric vacua are not present. A rough picture of the potential is given in Figure 7.8.

Figure 7.8: The (classical) potential of the modified O’Raifeartaigh model.

For future reference, let me notice the following interesting fact. In all models we have been considering so far, the existence of (stable) supersymmetry breaking vacua was always accompanied by the existence of an R-symmetry in the theory (think of the original Polonyi model in Example 1, the model in Example 2, the O’Raifeartaigh model of Example 5 and, to some extent, the model of Example 4). Its presence, however, does not seem to be a sufficient condition for supersymmetry breaking: think of the model of Example 3 which does possess an R-symmetry but does not break supersymmetry. On the contrary, whenever superpotential terms explicitly breaking the R-symmetry were introduced (the massive Polonyi model of Example 1 or Example 6), supersymmetric vacua were found. Finally, every time we found locally stable supersymmetry breaking vacua (again Example 6), in the vicinity of such vacua an approximate R-symmetry, which the theory does not possess as an exact symmetry, was recovered (essentially, in Examples 6 the superpotential perturbation responsible for the explicit breaking of the R-symmetry becomes negligible near the marginally stable supersymmetry breaking vacua). All this suggests
some sort of relation between R-symmetry and supersymmetry breaking. We will
discuss this issue later in this lecture, and put such apparent connection on a firm
ground.

7.4 Pseudomoduli space: quantum corrections

In most supersymmetry breaking models we discussed, we found a pseudo-moduli
space of non supersymmetric vacua. Associated to this, we found a massless scalar
mode $|X|$. While the masslessness of the goldstino and of the Goldstone boson
associated to R-symmetry breaking are protected by symmetries, there are no sym-
metries protecting the pseudo-modulus from getting a mass There isn’t any sym-
metry relating the (degenerate in energy) non supersymmetric vacua. Therefore, by
computing quantum corrections, one might expect this field to get a mass, some-
how. Let us stress the difference with respect to a moduli space of supersymmetric
vacua. Think about the harmonic oscillator. When we quantize the bosonic har-
cmonic oscillator, the energy of the ground state gets a $\frac{1}{2}\hbar\omega$ contribution. On the
contrary, if the ground state is fermionic, the contribution is the same in modulus
but with opposite sign (fermions tend to push the energy down). In a supersymmet-
ric situation, the mass degeneracy between bosonic and fermionic degrees of freedom
provides equal but opposite contribution to the vacuum energy and the total energy
hence remains zero. In a non-supersymmetric vacuum the degeneracy is not there
anymore (think about the spectrum we computed in the O’Raifeartaigh model) so
one expects things to change. In what follows we will try to make this intuition con-
crete by computing the one-loop Coleman-Weinberg effective potential for both the
O’Raifeartaigh and the modified O’Raifertaigh models. In practice, what we have
to do is to compute corrections in the coupling $h$ at one loop in the background
where the pseudo-modulus $|X|$ has a non-vanishing VEV.

For a supersymmetric theory the Coleman-Weinberg potential reads

$$V_{\text{eff}} = \frac{1}{64\pi^2} \text{STr} \mathcal{M}^4 \log \frac{\mathcal{M}^2}{\Lambda^2} = \frac{1}{64\pi^2} \left( \text{Tr} m_B^4 \log \frac{m_B^2}{\Lambda^2} - \text{Tr} m_F^4 \log \frac{m_F^2}{\Lambda^2} \right), \quad (7.47)$$

where $\mathcal{M} = \mathcal{M}(|X|)$ is the full tree level mass matrix, $m_B$ and $m_F$ correspond to
boson and fermion masses respectively, and $\Lambda$ is a UV cut-off.

There are a few terms missing in the expression (7.47) of the effective potential,
if compared to a generic non-supersymmetric theory. Let us consider them in turn.
First, we miss the cosmological constant term

\[ \sim \Lambda^4. \]  \hspace{1cm} (7.48)

This term is missing since it only depends on the spectrum, and not on the masses of the different modes. In a supersymmetric theory the spectrum admits an equal number of bosonic and fermionic degrees of freedom, no matter whether one is in a supersymmetric or non-supersymmetric vacuum. Since bosons and fermions contribute opposite to this term, this degeneracy ensures this term to be vanishing.

A second term which is missing is the one proportional to

\[ \sim \Lambda^2 \text{STr} \mathcal{M}^2. \]  \hspace{1cm} (7.49)

This is not expected to vanish in our supersymmetry breaking vacuum since particles have different masses. In other words, the mass spectrum is not supersymmetric along the pseudo-moduli space, recall for instance eqs. (7.40) describing the non-supersymmetric mass spectrum of the O’Raifeartaigh model. However, an explicit computation shows that also this term is vanishing. This is not specific to this model. As we will show later, every time supersymmetry is broken spontaneously at tree level, provided the Kähler potential is canonical and in the absence of FI terms, cancellations occur so to give \( \text{STr} \mathcal{M}^2 = 0. \)

The only divergent term present in the expression (7.47) is proportional to

\[ \sim \log \Lambda^2 \text{STr} \mathcal{M}^4. \]  \hspace{1cm} (7.50)

This term does not vanish in general but it does not depend on \( |X| \). As such, as we will see momentarily, it can be reabsorbed in the renormalization of the tree level vacuum energy \( |\mu|^2 \). The upshot is that the only non-trivial \( |X| \)-dependent term in (7.47) is the finite term

\[ \sim \text{STr} \mathcal{M}^4 \log \mathcal{M}^2. \]  \hspace{1cm} (7.51)

Let us start considering the O’Raifertaigh model. We should simply plug the tree level masses (7.40) into formula (7.47). A lengthy but straightforward computation shows that \( V_{\text{eff}} \) is a monotonic increasing function of \( |X| \) and can hence be expanded in a power series in \( |X|^2 \). For small \( |X| \) we get

\[ V_{\text{eff}}(|X|) = V_0 + m_X^2 |X|^2 + \mathcal{O}(|X|^4) \]  \hspace{1cm} (7.52)

where

\[ V_0 = |\mu|^2 \left[ 1 + \frac{h^2}{32\pi^2} \left( \log \frac{|m|^2}{\Lambda^2} + v(y) + \frac{3}{2} \right) + \mathcal{O}(h^4) \right], \]  \hspace{1cm} (7.53)
with
\[ y = \left| \frac{h \mu^2}{m^2} \right| < 1 \quad \text{and} \quad v(y) = -\frac{y^2}{12} + \mathcal{O}(y^4), \] (7.54)

and
\[ m_X^2 = \frac{1}{32\pi^2} \left| \frac{h^4 \mu^4}{m^2} \right| z(y) \quad \text{where} \quad z(y) = \frac{2}{3} + \mathcal{O}(y^2). \] (7.55)

The minimum of the potential is at \(|X| = 0\), and besides the tree level contribution \(|\mu^2|^2\) it gets a contribution proportional \(\sim \mu^2\) which is just a constant, \(|X|\)-independent shift. As anticipated, the UV cut-off dependence can be reabsorbed in a renormalization of the vacuum energy. Indeed, we can define a running coupling
\[ \mu^2(E) \equiv \mu^2_{\text{bare}} \left[ 1 + \frac{|h|^2}{64\pi^2} \left( \log \frac{E^2}{\Lambda^2} + \frac{3}{2} \right) + \mathcal{O}(h^4) \right] \] (7.56)
geetting
\[ V_0 = |\mu^2(E = m)|^2 \left( 1 + \frac{|h|^2}{32\pi^2} v(y) + \mathcal{O}(h^4) \right) \] (7.57)
and the \(\Lambda\)-dependence has disappeared from the potential.

The upshot of this analysis is that loop corrections have lifted the classical pseudo-moduli space, leaving just one isolated non supersymmetric vacuum. In this vacuum the scalar field \(X\) gets a (one-loop) mass while \(\psi_X\), which is the goldstino, remains massless (notice, in passing, that in the unique supersymmetry breaking vacuum R-symmetry is preserved, as in Example 2). The shape of the potential in the \(X\)-direction becomes at all similar to that of Figure 7.5.

Let us now see what quantum corrections say about the marginally stable supersymmetry breaking vacua of the modified O’Raifeartaigh model, the one including the superpotential perturbation (7.41). We will just briefly sketch the main results. The interested reader could try to work out all data in detail. One should again evaluate the Coleman-Weinberg potential using the tree level spectrum computed near the origin of field space, where the putative marginally stable vacua live, eqs. (7.44). Plugging the latter into formula (7.47), what one finds is that, again, the vacuum degeneracy is lifted and a (locally) stable non-supersymmetric vacuum survives at \(|X| = |X_{\text{min}}|\) where in our regime, \(\epsilon << 1\), \(X_{\text{min}}\) is near the origin and very far, in field space, from the two supersymmetric vacua sitting at \(|X| = |X_{\text{SUSY}}|\). More precisely we get
\[ V_{\text{eff}}(|X|) = V_0 + m_X^2 |X - X_{\text{min}}|^2 + \mathcal{O}(\epsilon^2, |X - X_{\text{min}}|^4), \] (7.58)
where \(X_{\text{min}} \sim \frac{\omega}{\mu} f(y) + \mathcal{O}(\epsilon^3)\). The spectrum in the supersymmetry breaking vacuum enjoys a massless fermion, \(\psi_X\), the goldstino, while the \(X\)-field gets a \(\epsilon\)-
independent) mass, as in the original O’Raifeartaigh model. The effective potential, once projected into $X$-direction, looks roughly like that in Figure 7.9.

![Figure 7.9: The effective potential of the modified O’ Raifeartaigh model project onto the $|X|$ direction.](image)

One might ask whether such locally stable supersymmetry breaking minimum is of any physical relevance. An estimate of its lifetime $\tau$ can be given looking at the decay rate

$$\Gamma \sim e^{-S_B}$$

(recall that $\tau \sim 1/\Gamma$) where $S_B$ is the so-called bunch action, the difference between the Euclidean action of the tunneling configuration and that of remaining in the metastable vacuum. Its exact form depends on the details of the potential, but a simple estimate can be given in the so-called thin wall approximation, which is justified when $|X_{SUSY} - X_{Max}|^4 \gg V_{Meta}$, and which is the case here. In this approximation the bunch action reads

$$S_B \sim \frac{\langle \Delta X \rangle^4}{\Delta V} \text{ where } \langle \Delta X \rangle = \langle X \rangle_{SUSY} - \langle X \rangle_{Meta}, \Delta V = V_{Meta} - V_{SUSY} = V_{Meta}. \quad (7.60)$$

An explicit computation shows that $S_B \sim \epsilon^{-\alpha}$ where $\alpha > 0$. This implies that for $\epsilon << 1$ the bunch action can be made arbitrary large so, in this limit, the locally stable vacuum can be made parametrically long-lived. The upshot of this analysis is that at the quantum level the classically marginal pseudo-moduli space of the modified O’Raifertaigh model is lifted but a local, parametrically long-lived supersymmetry breaking vacuum survives. It is an instructive exercise, which is left to the reader, to repeat this quantum analysis for the classically marginal pseudo-
moduli space of Example 3. In this case, the pseudo-moduli space gets completely lifted, and no locally stable supersymmetric minimum survives quantum corrections.

Let us close this section stressing that nothing we said (and computed) about quantum corrections, pseudomoduli lifting, etc... affects the supersymmetry breaking mechanism itself. All models we have been discussing so far, if breaking supersymmetry, were doing it at tree level. We have not encountered examples where supersymmetry was unbroken at tree level and one-loop quantum corrections induced supersymmetry breaking. Everything coming from the one-loop potential can and does modify the classical supersymmetry breaking vacua (which are not protected by supersymmetry), while it leaves completely unaffected the supersymmetric ones, if any. This agrees with non-renormalization theorems and our claims about the robustness of supersymmetric moduli spaces against (perturbative) quantum corrections.

We will have more to say about F-term breaking in due time. Let us pause a bit now, and consider the other possibility we have alluded to, i.e. spontaneous supersymmetry breaking induced at tree level by D-terms.

### 7.5 D-term breaking

In a generic theory, where chiral and vector superfields are present, in absence of FI terms it is F-term dynamics which governs supersymmetry breaking. This is because it so happens that whenever one can set all F-terms to zero, using (global) gauge invariance acting on the scalar fields one can set to zero all D-terms, too. So, if one wants to consider genuine D-term breaking, one should consider FI terms, hence abelian gauge factors. In what follows, we will review the most simple such scenario, where two massive chiral superfields with opposite charge are coupled to a single \( U(1) \) factor, and a FI term is present in the Lagrangian. The Lagrangian reads

\[
\mathcal{L} = \frac{1}{32\pi} \text{Im} \left( \tau \int d^2\theta W^\alpha W_\alpha \right) + \int d^2\theta d^2\bar{\theta} \left( \xi V + \Phi_+ e^{2eV} \Phi_+ + \Phi_- e^{-2eV} \Phi_- \right) \\
+ m \int d^2\theta \Phi_+ \Phi_- + h.c. ,
\]

(7.61)
where under a gauge transformation the two chiral superfields transform as $\Phi_{\pm} \rightarrow e^{\pm ie\alpha} \Phi_{\pm}$. The equations of motion for the auxiliary fields read

$$\begin{cases}
F_\pm = m\phi_\mp \\
D = -\frac{1}{2} \left[ \xi + 2e (|\phi_+|^2 - |\phi_-|^2) \right]
\end{cases}$$  \hspace{1cm} (7.62)

It is clearly impossible to satisfy all auxiliary fields equations, due to the presence of the FI parameter $\xi$. Hence supersymmetry is broken, as anticipated. The scalar potential reads

$$V = \frac{1}{8} \left[ \xi + 2e (|\phi_+|^2 - |\phi_-|^2) \right]^2 + m^2 \left( |\phi_+|^2 + |\phi_-|^2 \right)$$

$$= \frac{1}{8}\xi^2 + \left( m^2 - \frac{1}{2}e\xi \right) |\phi_-|^2 + \left( m^2 + \frac{1}{2}e\xi \right) |\phi_+|^2 +$$

$$+ \frac{1}{2}e^2 (|\phi_+|^2 - |\phi_-|^2)^2.$$  \hspace{1cm} (7.63)

The vacuum structure and the low energy dynamics depends on the sign of $m^2 - \frac{1}{2}e\xi$.

There is a qualitative difference between the depending on such sign

- $m^2 > \frac{1}{2}e\xi$. All terms in the potential are positive and the minimum of $V$ is at $\langle \phi_\pm \rangle = 0$, where $V = \frac{1}{8}\xi^2$. Supersymmetry is broken but gauge symmetry is preserved. The only auxiliary field which gets a non-vanishing VEV is $D$, so in this case one speaks of pure D-term breaking. We are in a situation like the one depicted in the upper right diagram of Figure 7.1.

One can compute the mass spectrum and find agreement with expectations. The two fermions belonging to the two chiral multiplets have (supersymmetric) mass $m$ and hence form a massive Dirac fermion. The two scalar fields $\phi_+$ and $\phi_-$ have masses $\sqrt{m^2 + 1/2e\xi}$ and $\sqrt{m^2 - 1/2e\xi}$, respectively. Finally, both the photon $A_\mu$ and the photino $\lambda$ remain massless. The former, because gauge symmetry is preserved, the latter because supersymmetry is broken and a massless fermionic mode, the goldstino, is expected (in this case the goldstino gets contribution from the photino only, since the only non-vanishing auxiliary field VEV is that of the D-field, $\psi_\alpha^G \sim \langle D \rangle \lambda_\alpha$).

- $m^2 < \frac{1}{2}e\xi$. Now the sign of the mass term for $\phi_-$ is negative. The minimum of the potential is at $\langle \phi_+ \rangle = 0$, $\langle \phi_- \rangle = \sqrt{\frac{\xi}{2e} - \frac{m^2}{e^2}} \equiv h$. Hence both supersymmetry and gauge symmetry are broken. Both the D-field and $F_+$ get a VEV: in this case we have a so-called mixed D and F-term breaking. The value of
the potential at its minimum is \( V = \frac{1}{8} \xi^2 - \frac{1}{2} e^2 h^4 \). We are in a situation of the type depicted in the lower right diagram of Figure 7.1.

In order to compute the mass spectrum one should expand the potential around \( \langle \phi_+ \rangle = 0 \) and \( \langle \phi_- \rangle = h \). A lengthy but simple computation gives the following answer. The complex scalar field \( \phi_+ \) has mass \( m_{\phi_+} = \sqrt{2} m \). The real part of \( \phi_- \) gets a mass \( m_{\phi_-} = \sqrt{2} e h \), while the imaginary part disappears from the spectrum (in fact, it is eaten by the photon, which becomes massive, with mass \( m_{A_{\mu}} = \sqrt{2} e h \)). The three fermions mix between themselves (there is a mixing induced from Yukawa couplings). One eigenfunction is massless, and is nothing but the goldstino \( \psi^G_\alpha \sim \langle D \rangle \lambda_\alpha + \langle F_+ \rangle \psi^a_+ \). The other two, \( \tilde{\psi}_\pm \), get equal mass \( m_{\tilde{\psi}_\pm} = \sqrt{2} e^2 h^2 + m^2 = \sqrt{e} \xi - m^2 \).

Figure 7.10 gives a summary of the mass spectrum of the FI model as a function of \( \frac{1}{2} e \xi \) which, at fixed \( m \), is the order parameter of the supersymmetry breaking transition.

![Figure 7.10: The mass spectrum of the Fayet-Iliopoulos model as a function of the FI parameter \( \xi \).](image)

One of the most attractive features of supersymmetric fields theories is the stability of masses under quantum corrections. In models where the FI mechanism plays a role, the physical mass spectrum depends on \( \xi \) which is not protected, a priori, since it appears in a D-term. This is different from models where F-terms are responsible for the supersymmetry breaking dynamics, since these are superpotential terms and protected by non-renormalization theorems. Therefore, it is important to
investigate the circumstances under which the FI term does not get renormalized. It can be shown that the contribution renormalizing the FI term is proportional to the trace of the $U(1)$ generator taken over all chiral superfields present in the model. This trace is proportional to the gravitational anomaly. Therefore, we can conclude that the FI term does not renormalize for theories free of gravitational anomalies.

7.6 Indirect criteria for supersymmetry breaking

We have already alluded to some possible relation between supersymmetry breaking and $R$-symmetry. In what follows, we will try to make this intuition precise and, more generally, present a few general criteria one can use to understand whether a theory might or might not break supersymmetry, without having a precise knowledge of the details of the theory itself. These criteria might be useful as guiding principles when trying to construct models of supersymmetry breaking in a bottom-up approach and, at the same time, they allow to have a handle on theories which are more involved than the simple ones we analyzed in previous sections. Finally, having some general criteria, possibly being valid also beyond the realm of perturbative physics might also be useful when one has to deal with theories in strongly coupled phases, where a perturbative, semi-classical approach is not possible, and where the direct study of the zero’s of the potential is not easy or even not possible.

7.6.1 Supersymmetry breaking and global symmetries

Let us consider a supersymmetric theory which has a spontaneously broken global symmetry and which does not admit (non compact) classical flat directions. This theory, generically, breaks supersymmetry. This can be easily proven as follows. Since there is a broken global symmetry, the theory admits a goldstone boson (a massless particle with no potential). If supersymmetry were unbroken then one should expect a scalar companion of this goldstone boson, which, being in the same multiplet of the latter, would not admit a potential either. But then, the theory would admit a flat direction, contrary to one of the hypotheses. This is a sufficient condition for supersymmetry breaking.

In the above reasoning we have assumed that the second massless scalar corresponds to a non-compact flat direction. This is typically the case since the Goldstone boson is the phase of the order parameter, and its scalar companion corresponds to a dilation of the order parameter, and therefore represents indeed a non-compact
flat direction.

Consider now a theory of $F$ chiral superfields $\Phi^i$ with superpotential $W$. Supersymmetry is unbroken if

$$ F_i = \frac{\partial W}{\partial \phi^i} = 0 \quad \forall i = 1, 2, \ldots, F. \quad (7.64) $$

These are $F$ holomorphic conditions on $F$ complex variables. Therefore, if the superpotential $W$ is generic, one expects (typically distinct) solutions to exist. Hence, supersymmetry is unbroken. By the superpotential being generic we mean the following. The superpotential is, in general, a function of the $\Phi^i$'s of degree, say, $n$. It is generic if all possible polynomials of degree $n$ or lower compatible with the symmetries of the theory are present.

Suppose now that $W$ preserves some global non-R symmetry. Hence, $W$ is a function of singlet combinations of the $\Phi^i$'s. It is easy to see that in terms of these reduced number of variables, eqs. (7.64) impose an equal number of independent conditions. Suppose, for definiteness, that the global symmetry is a $U(1)$ symmetry and call $q_i$ the corresponding charge of the $i$-th chiral superfield $\Phi^i$. Hence, we can rewrite the superpotential as e.g.

$$ W = W(X_i) \quad \text{where} \quad X_i = \Phi_i \Phi_1^{-q_i/q_1}, \quad i = 2, 3, \ldots, F. \quad (7.65) $$

If we now consider eqs. (7.64) we have

$$ \begin{cases} j \neq 1 \quad \frac{\partial W}{\partial \phi^j} = \Phi_1^{-q_j/q_1} \frac{\partial W(X_i)}{\partial X_j} = 0 \\ j = 1 \quad \frac{\partial W}{\partial \phi^1} = \frac{\partial W(X_i)}{\partial X_k} \frac{\partial X_k}{\partial \phi^1} = 0 \quad , \quad k = 2, \ldots, F. \quad (7.66) \end{cases} $$

We see that the equation for $F_1$ is automatically satisfied if the others $F - 1$ are satisfied. Hence, having a system of $F - 1$ holomorphic equations in $F - 1$ variables, generically the system allows for solutions. The same reasoning holds for a generic global symmetry. A global symmetry (under which the superpotential is a singlet) diminishes the number of independent variables, but it diminishes also the number of independent $F$-equations by the same amount. Hence, again, if $W$ is generic, eqs. (7.64) can be solved and supersymmetry is unbroken.

Suppose now that the global symmetry under consideration is a R-symmetry. The crucial difference here is that the superpotential is charged under this symmetry, $R(W) = 2$. Let us call $r_i$ the R-charge of the $i$-th superfield $\Phi^i$. We can now rewrite the superpotential as

$$ W = \Phi_1^{2/r_1} f(X_i) \quad \text{where} \quad X_i = \Phi_i \Phi_1^{-r_i/r_1}, \quad i = 2, 3, \ldots, F. \quad (7.67) $$
If we now compute eqs. (7.64) we get

\[
\begin{align*}
\forall j \neq 1 & \quad \frac{\partial W}{\partial \phi_j} = \phi_1^{2-t_j} \frac{\partial f(X_i)}{\partial X_j} = 0 \\
\forall j = 1 & \quad \frac{\partial W}{\partial \phi_1} = 2 \phi_1^{2-t_1-1} f(X_i) + \phi_1^{2-t_1} \frac{\partial f(X_i)}{\partial X_k} \frac{\partial X_k}{\partial \phi_1} = 0 .
\end{align*}
\]

(7.68)

Once the first \( F - 1 \) equations are satisfied, the remaining one reduces to \( f(X_i) = 0 \), which is not at all trivial. So now we have \( F \) independent equations in \( F - 1 \) variables and hence, generically, solutions do not exist. So we conclude that supersymmetry is broken, generically. The upshot is that the existence of an R-symmetry is a necessary condition for supersymmetry breaking, if the potential is generic (and, if it is then spontaneously broken, it is a sufficient condition if there are no classical flat directions).

This is known as the Nelson-Seiberg criterium. The O’Raifeartaigh model meets this criterium. It possesses an R-symmetry (which is then spontaneously broken along the pseudo-moduli space), the superpotential is generic, and it breaks supersymmetry. The modified O’Raifeartaigh model instead admits supersymmetry preserving vacua. Indeed, the R-symmetry is absent since the mass perturbation \( \Delta W \) breaks it explicitly. So, one would expect the model not to break supersymmetry. And in fact it doesn’t. We have noticed, though, that somewhere else in the space of scalar field VEVs this model admits non-supersymmetric vacua which, if the mass perturbation is small enough, we have proven to be long-lived. In this region the R-breaking perturbation is negligible and an approximate R-symmetry (the O’Raifeartaigh model’s original one) is recovered. This property is in fact not specific to the modified O’Raifeartaigh model, but is a generic feature of supersymmetry breaking metastable vacua.

Summarizing, a rough guideline in the quest for supersymmetry breaking theories can be as follows:

<table>
<thead>
<tr>
<th>R-symmetry</th>
<th>SUSY status</th>
</tr>
</thead>
<tbody>
<tr>
<td>No R-symmetry</td>
<td>SUSY unbroken</td>
</tr>
<tr>
<td>R-symmetry</td>
<td>SUSY (maybe) broken</td>
</tr>
<tr>
<td>Approximate R-symmetry</td>
<td>SUSY (maybe) broken locally, restored elsewhere</td>
</tr>
</tbody>
</table>

Since necessary conditions are quite powerful tools, let me stress again one important point. The existence of an R-symmetry is a necessary condition for supersymmetry breaking under the assumption that the superpotential is generic. If this is not the case, supersymmetry can be broken even if the R-symmetry is absent. Another possibility, which typically occurs when gauge degrees of freedom are present
in the Lagrangian, is that R-symmetry is absent, but then it arises as an accidental symmetry in the low energy effective theory. Also in this case supersymmetry can be broken even if R-symmetry was absent in the UV Lagrangian. We will see examples of this sort later in this course.

7.6.2 Topological constraints: the Witten Index

Another powerful criterium exists which helps when dealing with theories with complicated vacuum structure and for which it is then difficult to determine directly whether supersymmetry is broken, i.e. to find the zero’s of the potential. This criterium, which provides a necessary condition for supersymmetry breaking, has to do with the so-called Witten index, which, for a supersymmetric theory, is a topological invariant quantity.

The Witten index, let us dub it $I_W$, is an integer number which measures the difference between the number of bosonic and fermionic states, for any given energy level. In a supersymmetric theory, for any positive energy level there is an equal number of bosonic and fermionic states. This is obvious if supersymmetry is unbroken, but it also holds if supersymmetry is broken: every state is degenerate with the state obtained from it by adding a zero-momentum goldstino (which is certainly there, if supersymmetry is broken). In other words, a state $|\Omega\rangle$ is paired with $|\Omega + \{p_\mu = 0 \text{ goldstino}\}\rangle$. On the contrary, zero energy states can be unpaired since, due to the supersymmetry algebra, such states are annihilated by the supercharges. Therefore, in a supersymmetric theory the Witten index can get contribution from the zero energy states only, regardless the vacuum one is considering preserves supersymmetry or it does not.

Strictly speaking the above argument holds only if we put the theory in a finite volume. In an infinite volume, when supersymmetry is broken one has to deal with IR singularity issues. In particular, the (broken) supercharge diverges and acting with it on a physical state gives a non-normalizable state. This can be seen as follows. From the current algebra (4.71) one sees that

$$E \eta^{\mu \nu} = \langle T^{\mu \nu} \rangle = \frac{1}{4} \bar{\sigma}^{\mu \dot{\alpha}} \langle \{Q_{\alpha}, \bar{\Sigma}_{\dot{\alpha}} \} \rangle .$$

(7.69)

This shows that if the vacuum energy is non vanishing, the vacuum is transformed into one goldstino states if acted with $Q_{\alpha}$ (recall that the supercurrent creates a goldstino when acting on the vacuum). In an infinite volume a non-vanishing vacuum energy density corresponds to an infinite total energy, implying that the supercharge
diverges and that the zero-momentum goldstino state is not defined (the corresponding state does not exist in the Hilbert space). Putting the theory in a finite volume is a way to regularize (translational invariance can be maintained imposing periodic boundary conditions on all fields). Therefore, in what follows we will start considering the theory in a finite volume $V$ and only later take the infinite volume limit. As we will see, what is relevant for the argument we want to convey is not affected by these issues and holds true also when $V \to \infty$.

A theory in a finite volume has a discrete energy spectrum, all states in the Hilbert space are discrete and normalizable and can be counted unambiguously. Our goal is to compute the Witten index in such finite volume theory. To this end, we can restrict to the zero-momentum subspace of the Hilbert space. In a supersymmetric theory the energy of any state is semi-positive definite hence, from the relativistic equation $m^2 = E^2 - |\vec{p}|^2$, it follows that the energy is larger or equal than the momentum, so zero energy states have $\vec{p} = 0$. Setting $|\vec{p}| = 0$ we are excluding from the counting massless states with $E > 0$. These states, though, do not contribute to the Witten index. Massive states, on the contrary, never contribute to it since they necessarily have $E > 0$, regardless the value of $|\vec{p}|$. So only massless states with $|\vec{p}| = 0$ contribute to the Witten index. The upshot is that restricting the Hilbert space to the subspace $|\vec{p}| = 0$ does not hurt and so this is what we will do in what follows. Nicely, in such subspace the supersymmetry algebra simplifies. In particular, using four-component spinor notation in which the supercharge $Q$ is a Majorana spinor, the supersymmetry algebra in the subspace $|\vec{p}| = 0$ is just

$$\{Q, \bar{Q}\} = 2\gamma^0 P_0 . \tag{7.70}$$

This implies that $Q_1^2 = Q_2^2 = Q_3^2 = Q_4^2 = H$, where $H$ is the Hamiltonian of the system and $Q_i$ are the four components of $Q$.

Suppose to have a bosonic state $|b\rangle$ for which $Q^2 |b\rangle = E |b\rangle$, where $Q$ is one of the $Q_i$’s. Then the fermionic state obtained from $|b\rangle$ as

$$|f\rangle \equiv \frac{1}{\sqrt{E}} Q |b\rangle , \tag{7.71}$$

has also energy $E$. This does not apply to zero-energy states since they are annihilated by $Q$, and hence are not paired. So, as anticipated, the Witten index receives contributions only from zero energy states. In Figure 7.11 we report the general form of the spectrum of a supersymmetric theory.

In order to appreciate its topological nature, let us define the index a bit more rigorously. A supersymmetric theory is a unitary representation of the Poincaré
Figure 7.11: The spectrum of a supersymmetric theory in a finite volume. Circles indicate bosons, squares indicate fermions. The zero energy level is the only one where there can exist a different number of circles and squares.

superalgebra on some Hilbert space $\mathcal{H}$. Let us assume that

$$\mathcal{H} = \bigoplus_{E \geq 0} \mathcal{H}_E.$$  \hfill (7.72)

The Witten index is defined as

$$I_W(\beta) = \text{S} \text{Tr}_{\mathcal{H}} e^{\beta H} \equiv \text{Tr}_{\mathcal{H}} (-1)^F e^{\beta H}, \quad \beta \in \mathbb{R}^+.$$  \hfill (7.73)

It follows that

$$I_W(\beta) = \sum_{E \geq 0} e^{\beta E} \text{Tr}_{\mathcal{H}_E} (-1)^F = \sum_{E \geq 0} e^{\beta E} [n_B(E) - n_F(E)] =$$

$$= n_B(0) - n_F(0) = \text{Tr}_{\mathcal{H}_0} (-1)^F = I_W(0).$$  \hfill (7.74)

We have been rewriting what we have already shown to hold. The point is that this way it is clear that the index does not depend on $\beta$: its value does not vary if we vary $\beta$. More generally, one can prove that the Witten index does not depend on any parameter, like in particular coupling constants, and can then be computed in appropriate corners of the parameter space (say at weak coupling) and the result one gets is exact. In other words, the Witten index is a topological invariant.

Suppose one starts from a situation like the one depicted in Figure 7.11. Varying the parameters of the theory, like masses, couplings, etc..., it may very well be that some states move around in energy. The point is that they must do it in pairs, in a supersymmetric theory. Hence, it can happen that a pair of non-zero energy states moves down to zero energy; or, viceversa, that some zero energy states may acquire non-zero energy. But again, this can only happen if an equal number of bosonic and
fermionic zero energy states moves towards a non-zero energy level. The upshot is that the Witten index does not change. This is summarized pictorially in Figure 7.12.

![Diagram](image)

Figure 7.12: Supersymmetric theory dynamics. Upon modifications of parameters of the theory, the number of zero energy states can change, but the Witten index remains the same.

What is this useful for? The crucial point is that the Witten index measures the difference between zero-energy states only. Suppose it is different from zero, $I_W \neq 0$. This means that there exists some zero-energy state, hence supersymmetry is unbroken. But, because of the topological nature of $I_W$, this conclusion holds at any order in perturbation theory and even non-perturbatively! A theory with non vanishing Witten index cannot break supersymmetry. Suppose instead that $I_W = 0$. Now one cannot conclude anything, just that the number of bosonic and fermionic zero-energy state is the same; but one cannot tell whether this number is zero (broken supersymmetry) or different from zero (unbroken supersymmetry).

So we conclude that having a non-vanishing Witten index is a sufficient condition for the existence of supersymmetric vacua, and that having it vanishing is a necessary condition for supersymmetry breaking. And a robust one, since $I_W$ is an exact
quantity.

Few comments are in order at this point.

First, the fact that we have been working in a finite volume does not question our main conclusions. If supersymmetry is unbroken in an arbitrary finite volume it means the ground state energy $E(V)$ is zero for any $V$. Since the large-$V$ limit of zero is still zero, supersymmetry is unbroken also in the infinite volume limit. If one can explicitly compute the Witten index at finite volume and find that it is not vanishing, one can safely conclude that supersymmetry is not broken even in the actual theory, i.e. at infinite volume. On the contrary, the converse is not necessarily true. It might be that supersymmetry is broken at finite volume and restored in the infinite volume limit. Suppose that $I_W = 0$ and that one knows that supersymmetry is broken, that is the minimal energy states have positive energy. The energy density goes as $E(V)/V$ and it may very well be that for $V \to \infty$ the increase of $E$ is not enough to compensate for the larger and larger volume. So the energy density can very well become zero in the infinite volume limit and supersymmetry restored. But this does not hurt much, since all what the vanishing of the Witten index provides is a necessary condition for supersymmetry breaking, not a sufficient one.

A second comment regards the relation between classical and quantum results. Suppose one can explicitly check at tree level that a given theory has non-vanishing Witten index, $I_W \neq 0$. For what we said above, this implies that supersymmetry is unbroken classically and that it cannot be broken whatsoever, neither perturbatively nor non-perturbatively. On the contrary, if $I_W = 0$ at tree level and we know that supersymmetry is unbroken classically, it can very well be that (non-perturbative) quantum effects may break it.

The theorem we have discussed may find very useful applications. For one thing, it turns out that pure SYM theories have non-vanishing Witten index (for SYM with gauge group $G$, the index equals the dual Coxeter number of $G$, which for $SU$ and $Sp$ groups is just $r + 1$, where $r$ is the rank of $G$). So pure SYM theories cannot break supersymmetry. As a corollary, SYM theories with massive matter (like massive SQCD) cannot break supersymmetry either. This is because for low enough energy all massive fields can be integrated out and the theory flows to pure SYM, which has non-vanishing index. More generally, non-chiral theories, for which a mass term can be given to all matter fields, are not expected to break supersymmetry.

What about chiral theories, instead? Chiral theories behave differently. In this case some chiral superfield cannot get a mass anyway, and so these theories cannot
be obtained from deformation of vector-like theories, as massive SQCD. Hence, one cannot conclude that these theories cannot break supersymmetry. As we will see when discussing models of dynamical supersymmetry breaking, most known examples of theories breaking supersymmetry are, in fact, chiral theories.

There is a subtlety in all what we said so far, which is sort of hidden in some of our claims. We said that the Witten index is robust against any continuous change of parameters. But it turns out that a perturbation that changes the asymptotic behavior of the potential may induce a change in $I_W$. This is related to the topological nature of the index, which makes it depending on boundary effects. Consider the following simple potential for a scalar field $\phi$,

$$V(\phi) = (m\phi - g\phi^3)^2.$$  \hspace{1cm} (7.75)

For $g = 0$ low energy states correspond to $\phi \sim 0$. For $g \neq 0$ low energy states may correspond to $\phi = 0$ but also to $\phi \sim m/g$ (no matter how small $g$ is). So we see here that $I_W(g = 0) \neq I_W(g \neq 0)$. What is going on? The point is that switching on and off $g$ changes the asymptotic behavior of $V$ for large $\phi$ (that is, at the boundary of field space): in the large $\phi$ region, for $g = 0$ $V \sim \phi^2$ while for $g \neq 0$ $V \sim \phi^4$. The punchline is that the Witten index is invariant under any change in the parameters of a theory in which, in the large field regime, the potential changes by terms no bigger than the terms already present. If this is not the case, the Witten index can change discontinuously. In other words, $I_W$ is independent of numerical values of parameters as long as these are non-zero. When sending a set of parameters to zero, or switching on some couplings which were absent, one should check that the asymptotic behavior of the potential is unchanged, in order to avoid new states coming in from (or going out to) infinity. Coming back to our SQCD example, we see that for massive SQCD the potential in the large field regime is quadratic, while for massless SQCD is flat: the two theories do not have a priori the same Witten index. Therefore, while massive SQCD is expected to be in the same equivalence class of pure SYM (as far as supersymmetry breaking is concerned), this is not guaranteed for massless SQCD. In other words, no conclusions can be drawn for the massless regime by the analysis in the massive regime. We will see explicit examples of this phenomenon in later lectures.

Let us finally notice that Witten index argument limits a lot the landscape of possible supersymmetry breaking theories; for instance, non-chiral gauge theories most likely cannot break supersymmetry.
7.6.3 Genericity and metastability

Before concluding this section, there is yet another important conclusion we can draw from all what we have learned. Both the Nelson-Seiberg criterium and Witten index argument seem to favor, at least statistically, supersymmetry breaking into metastable vacua.

For one thing, thinking about R-symmetry one might have the impression to fall into a vicious circle. Having an R-symmetry (which is a necessary condition for supersymmetry breaking, if the superpotential is generic) forbids a mass term for the gaugino which, being a fermion in a real representation and having R-charge $R(\lambda) = 1$, would have a R-symmetry breaking mass term. But we do not see any massless gauginos around, so gauginos should be massive. If we have an R-symmetry which is spontaneously broken, we could generate a mass for gauginos but we should also have an R-axion, which is not observed. This suggests that R-symmetry should be broken explicitly. But then, generically, we cannot break supersymmetry! The conclusion is that asking for stable vacua compatible with phenomenological observations implies one should look for non-generic theories, which are obviously much less than generic ones. If one accepts, instead, that supersymmetry might be broken in metastable vacua, then R-symmetry would not be an exact symmetry but only an approximate one. In this case gaugino mass and supersymmetry breaking would be compatible, at least in the metastable vacuum (it is worth noticing that in concrete models there is, not unexpectedly, some tension between the magnitude of gaugino mass and the lifetime of such supersymmetry breaking vacua).

Regardless the R-axion problem mentioned above (which in fact can also get a mass by gravitational effects), there exists another argument favoring metastability. This is related with the computations we performed in section 7.4 when we studied one-loop corrections of the O’Raifeartaigh model. We have seen that quantum corrections lift the classical pseudomoduli space, leaving one unique supersymmetry breaking vacuum. However, such vacuum is (the only) one where R-symmetry is in fact not broken, so gauginos cannot get a mass! This is not a specific feature of the O’Raifeartaigh model but applies to any model where the R-charges of superfields are either 0 or 2. It has been proven that a necessary condition for having the true vacuum to break the R-symmetry is to have fields in the Lagrangian having R-charge different from 0 and 2. Models of this kind exist and have been constructed. However, in all such models supersymmetry preserving vacua also exist. Hence, supersymmetry breaking vacua where also R-symmetry is in the end broken, are
actually metastable.

Also Witten index argument favors metastability, statistically. If accept we leave in a metastable vacuum, it means we allow for the existence of supersymmetry preserving vacua elsewhere in field space. Hence, all theories with non-vanishing Witten index would not be anymore excluded from the landscape of possible supersymmetry breaking and phenomenologically sensible theories. For example, non-chiral theories would be back in business. A notable such example will be presented in section 11.

The punchline is that, generically, it may be more likely we leave in a metastable vacuum rather than in a fully stable one. Just... we need to ensure that its lifetime is long enough to be safe!

7.7 Exercises

1. Consider a theory of $n$ chiral superfields $\Phi^i$ with superpotential (7.15). Prove that, for an interacting theory (that is, some $g_{ijk}$ should be non-vanishing), in order to have spontaneous supersymmetry breaking one needs at least three chiral superfields. Derive the generic form of the corresponding three-superfield superpotential.

2. Compute the one-loop effective potential on the classically marginal non-supersymmetric vacua of the model in Example 3. What is the fate of these vacua after quantum corrections are taken into account?

3. Compute the mass spectrum of the FI model both in the pure D-term as well as in the mixed D and F-terms breaking phases, and check explicitly that the spectrum satisfies the so-called supertrace mass formula, that is $\text{STr} \mathcal{M}^2 = 0$.

4. Consider a theory of three chiral superfields with canonical Kähler potential and superpotential

$$W = \frac{1}{2} h_1 X \Phi_1^2 + \frac{1}{2} h_2 \Phi_2 \Phi_1^2 + f X$$

Show the existence of a classical moduli space of supersymmetry breaking vacua. Compute the one-loop corrections to the tree-level result and show that the moduli space is not lifted at one-loop. Can you find a simple reason to explain such a behavior?

5. Consider all models of F-term breaking of section 7.3 and discuss whether and how the Nelson-Seiberg criterium applies or not.
References


