9 Non-perturbative effects and holomorphy

In this lecture we will start our program regarding the study of the non-perturbative regime of supersymmetric theories. The main point of this first lecture will be to introduce holomorphy, or better put holomorphy, which is an intrinsic property of supersymmetric theories, at work. Before doing that, however, there are a few standard non-perturbative field theory results we need to review.

9.1 Instantons and anomalies in a nutshell

Gauge theories might contain a $\theta$-term, which is

$$\mathcal{S}_\theta = \frac{\theta_{\text{YM}}}{32\pi^2} \int d^4x \, \text{Tr} \, F_{\mu\nu} \tilde{F}^{\mu\nu} \quad \text{where} \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} . \quad (9.1)$$

This term is a total derivative. Indeed

$$\frac{1}{2} \int d^4x \, \text{Tr} \, F_{\mu\nu} \tilde{F}^{\mu\nu} = \int d^4x \, \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \left[ A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma \right] , \quad (9.2)$$

which implies that the $\theta$-term does not have any effect on the classical equations of motion. However, when one quantizes a theory one has to average over all fluctuations, not just those satisfying the classical equations of motions and therefore the $\theta$-term can in fact be relevant in some cases.

One such configurations are instantons, classical solutions of the Euclidean action that approach pure gauge for $|x| \to \infty$ (so to make the action (9.1) finite). In this case the relevant integral is in fact an integer number, the so-called instanton number of the configuration (aka winding number)

$$\mathcal{S}_\theta = \frac{\theta_{\text{YM}}}{32\pi^2} \int d^4x \, \text{Tr} \, F_{\mu\nu} \tilde{F}^{\mu\nu} = n \, \theta_{\text{YM}} \quad \text{where} \quad n \in \mathbb{Z} . \quad (9.3)$$

The instanton number is a topological quantity, in the sense that it does not change upon continuous deformations of the gauge field configuration. Moreover, since the action enters the path integral as $\int \mathcal{D}\phi e^{i\mathcal{S}_\theta}$, the $\theta$-angle indeed behaves as an angle, in the sense that the shift

$$\theta_{\text{YM}} \to \theta_{\text{YM}} + 2\pi , \quad (9.4)$$

is a symmetry of the theory.

An instanton field configuration interpolates between different vacua of the gauge theory. Both vacua are gauge equivalent to the usual vacuum with zero gauge potential but the corresponding gauge transformation cannot be deformed to the identity.
(these are known as large gauge transformations). If this were the case it would have been possible to let the field strength being vanishing in all space-time, contradicting eq. (9.3). The fact that \( F_{\mu\nu} \) cannot vanish identically for configurations with \( n \neq 0 \) implies that there is a field energy associated with the gauge field configuration interpolating between the different vacua. An energy barrier and an associated quantum mechanical tunneling amplitude proportional to \( e^{-S_E} \) where \( S_E \) is the Euclidean action of a field configuration which interpolates between the different vacua. Instantons are nothing but just such interpolating field configurations. Notice that configurations related by large gauge transformations are weighted differently in the action, because of eq. (9.3), implying that they should not be identified as physically equivalent configurations.

Instantons have an intrinsic non-perturbative nature. Recall that the RG-equation for the gauge coupling \( g \) reads

\[
\mu \frac{\partial g}{\partial \mu} = -\frac{b_1}{16\pi^2}g^3 + \mathcal{O}(g^5),
\]

(9.5)

where \( b_1 \) is a numerical coefficient which depends on the theory. The solution of this equation at one loop is

\[
\frac{1}{g^2(\mu)} = -\frac{b_1}{8\pi^2} \log \frac{\Lambda}{\mu}.
\]

(9.6)

where the scale \( \Lambda \) is defined as the scale where the one-loop coupling diverges. It sets the scale where higher-loop and non-perturbative effects should be taken into account. For any scale \( \mu_0 \) we have that

\[
\Lambda \equiv \mu_0 e^{-\frac{\pi^2}{b_1\mu_0^2}}.
\]

(9.7)

It is important to stress that \( \Lambda \) does not depend on the energy scale: it is a RG-invariant quantity. Indeed

\[
\frac{\partial \Lambda}{\partial \mu_0} = e^{-\frac{8\pi^2}{b_1\mu_0^2}} + \mu_0 \left[ -\frac{8\pi^2}{b_1g^2(\mu_0)} \left( \frac{2}{16\pi^2} g^3(\mu_0) + \mathcal{O}(g^5) \right) \right] e^{-\frac{8\pi^2}{b_1g^2(\mu_0)}}
\]

\[
= e^{-\frac{8\pi^2}{b_1g^2(\mu_0)}} + \mu_0 (-\frac{1}{\mu_0}) e^{-\frac{8\pi^2}{b_1g^2(\mu_0)}} = 0.
\]

(9.8)

up to higher-order corrections. This can be reiterated order by order in perturbation theory, getting the same result, namely that \( \partial \Lambda / \partial \mu_0 = 0 \).

Let us now consider an instanton field configuration. There exists a lower bound on the Euclidean action of an instanton. Indeed,

\[
0 \leq \int d^4x \text{Tr} \left( F_{\mu\nu} \pm \tilde{F}_{\mu\nu} \right)^2 = \int d^4x \left[ 2 \text{Tr} F_{\mu\nu} F^{\mu\nu} \pm 2 \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \right]
\]

(9.9)
which implies
\[ \int d^4x \text{Tr} F_{\mu\nu} F^{\mu\nu} \geq \left| \int d^3x \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \right| = 32\pi^2 n \tag{9.10} \]
where the last equality holds for an instanton configuration with instanton number \( n \). This implies that there is a lower bound to an instanton action: instanton contributions to amplitudes are suppressed at least by (multiply above equation by \( 1/4g^2 \))
\[ e^{-S_{\text{inst}}} = \left( e^{-\frac{g^2}{2\pi} n} \right) = \left( \frac{\Lambda}{\mu} \right)^{n b_1} \tag{9.11} \]
where in the last step we have used eq. (9.6). This shows that instantons are inherently non-perturbative effects, since they vanish for \( \Lambda \to 0 \), and are very weak, if not negligible, in the perturbative regime.

Anomalies are classical symmetries of the action which are broken by quantum effects. In other words, we have
\[ \partial_{\mu} j_{A}^\mu = 0 \quad \text{quantum corrections} \quad \partial_{\mu} j_{A}^\mu \neq 0 \tag{9.12} \]
where \( j_A \) is the current associated to the anomalous symmetry.

In what follows we will focus on chiral anomalies, that is anomalies associated to chiral currents. These arise in field theories in which fermions with chiral symmetries are coupled to gauge fields. Recall that local currents cannot be anomalous, since they would imply violation of unitarity of the theory (we only know how to couple spin-1 fields in a way respecting unitarity to conserved currents). Hence a quantum field theory, in order to make sense, should not have any gauge anomaly. On the contrary, global chiral currents can be anomalous. This is what is usually meant as chiral anomaly.

Let us review a few basic facts about anomalies. Anomalies get contribution at one loop, only, by evaluation of triangle diagrams like those reported in Figure 9.1. Let us first consider a set of global currents \( j_A \). The one-loop three-point function corresponding to diagram \( a \) of Figure 9.1 will be
\[ \langle j^\mu_A(x_1) j^\nu_B(x_2) j^\rho_C(x_3) \rangle = \text{Tr}(t_{AB} t_{AC}) f_{\mu\nu\rho}(x_i) \tag{9.13} \]
where the trace comes from contraction of the group generators around the loop. This correlator has important properties, as we will see momentarily, but it does not provide by itself any anomaly: the corresponding classical conservation law is not violated quantum mechanically.
Figure 9.1: One-loop diagrams contributing to correlators of one global current with two global or local currents. Diagram \( a \) does not provide any anomaly. Diagrams \( b \) and \( c \), instead, contribute to the anomaly of the global current \( j_A \).

Suppose now to gauge some (or all) global currents, by coupling the original Lagrangian with gauge fields as

\[
\mathcal{L} = \mathcal{L}_{\text{free}} + \sum_B V^B_\mu j^\mu_B ,
\]

(9.14)

and let us compute the correlator \( \langle j_A V_B V_C \rangle \). The one-loop diagrams contributing to such correlator are diagrams \( b \) and \( c \) of Figure 9.1. By differentiating the result one gets

\[
\partial^\mu j^\mu_A \sim \text{Tr} (t_A \{ t_B, t_C \}) F^\nu_B \tilde{F}^\mu_{C,\nu} ,
\]

(9.15)

and we do have an anomaly now. We clearly see here that \( j_A \) should be a global current, since if this were not the case we would have had a violation of unitarity in the quantum theory.

From the above formula we see that the anomaly coefficient is proportional to \( \text{Tr} (t_A \{ t_B, t_C \}) \). This vanishes for real and pseudoreal representations. Indeed, for real or pseudoreal representations we have that \( t_A = - (t_A)^T \) and it then easily follows that

\[
\text{Tr}_r (t_A \{ t_B, t_C \}) = - \text{Tr}_r (t_A \{ t_B, t_C \}) .
\]

(9.16)

Therefore, only massless chiral fermions can contribute to the anomaly coefficient.

This result, once applied to local currents, which cannot be anomalous, provides severe restrictions on the massless fermion content of a quantum field theory.

Suppose to have a theory with gauge group \( G \) with generators \( t_A \), a global symmetry group \( \tilde{G} \) with generators \( \tilde{t}_A \), and a set of Weyl fermions \( \psi_i \), transforming in the representations \( (r_i, \tilde{r}_i) \) of the gauge and global symmetry groups, respectively. In this case, the ABJ anomaly computation gives

\[
\partial^\mu j^\mu_A \sim \sum_i \text{Tr}_{\tilde{r}_i} \tilde{t}_A \text{Tr}_{r_i} (t_B t_C) F^\nu_B \tilde{F}^\mu_{C,\nu} .
\]

(9.17)
Since $\text{Tr}_\hat{G} \tilde{L}_A = 0$ for any simple algebra, only abelian factors $U(1) \subset \hat{G}$ can be anomalous. On the other hand, $\text{Tr}_{r_i} (t_B t_C) = C(r_i) \delta_{BC}$, where $C(r_i)$ is the quadratic invariant (Casimir) of the representation $r_i$. Working everything out, paying attention to numerical coefficients, one finally gets for an abelian group

$$\partial_\mu j^\mu = \frac{A}{16\pi^2} F_B^{\mu\nu} \tilde{F}^B_{\mu\nu} ,$$

(9.18)

where $A = \sum_i q_i C(r_i)$ is the anomaly coefficient, $q_i$ being the $U(1)$ global charges of fermion fields $\psi_i$.

This result shows the connection between anomalies, instantons and the $\theta$-angle. First, we see that the anomaly is proportional to the instanton number we have previously defined. Indeed, integrating eq. (9.18) in space-time, and comparing with eq. (9.3), we get

$$\Delta Q = 2An ,$$

(9.19)

where $n$ is the instanton number and $\Delta Q$ the amount of charge violation due to the anomaly. So we see that anomalous symmetries are violated by a specific amount, given by eq. (9.19), in an instanton background. This also shows that anomalies are IR effects, since the violation is very mild at weak coupling.

As far as equation (9.18) is concerned, the effect of the anomalous $U(1)$ symmetry corresponds to a shift in the $\theta$-angle as

$$\psi_i \rightarrow e^{i\alpha} \psi_i \implies \theta_{YM} \rightarrow \theta_{YM} + 2\alpha A .$$

(9.20)

That is, the anomalous breaking can be seen as an explicit breaking: a term in the action, the $\theta$-term, in fact, is not invariant under the anomalous symmetry.

Notice that if we perform a $U(1)$ transformation but assign transformation properties to $\theta_{YM}$ as to compensate for the shift, then the anomalous $U(1)$ is promoted to an actual symmetry of a larger theory (where the complexified gauge coupling is promoted to a dynamical field). This symmetry, however, is spontaneously broken by the coupling constant VEV ($\theta_{YM}$ in this case). As we will later see, this way of looking at anomalous symmetries can be efficiently used to put constraints on the construction of low-energy effective Lagrangians.

9.2 ’t Hooft anomaly matching condition

The usefulness of correlators between global currents has been pointed out by ’t Hooft. As we will review below, they compute scale independent information about
a quantum theory (hence providing a powerful tool to understand some of its non-perturbative properties).

Let us consider a Lagrangian $L$ defined at some scale $\mu$, with some non-anomalous global symmetry group $G$ generated by currents $j^\mu_A$. Compute the triangle diagram for three global currents (which is not an anomaly!) and call $A_{UV}$ the result. Weakly gauge the global symmetry group $G$ by adding new gauge fields $V^A_\mu$ and define a new Lagrangian

$$L' = L - \frac{1}{4g^2} \text{Tr} F^{\mu\nu} F_{\mu\nu} + j^\mu_A V^A_\mu .$$

This theory is inconsistent since it has a gauge anomaly, $A_{UV}$, because we have gauged $G$. Let us then add some spectator free massless fermion fields $\psi_s$ (spectator in the sense that they couple only through the $G$ gauge coupling) transforming in representations of $G$ so to exactly cancel the anomaly, i.e. $A_s = -A_{UV}$. The resulting theory

$$L'' = L - \frac{1}{4g^2} \text{Tr} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}_s \phi \psi_s + (j^\mu_s A + j^\mu_A) V^A_\mu ,$$

where $j_{s,A}$ are the currents associated to the spectator fermions $\psi_s$, is non-anomalous. Consider this anomaly-free theory at some scale $\mu' < \mu$. Since the spectator fermion fields and gauge fields can be made arbitrarily weakly coupled by taking $g \to 0$, the IR dynamics of the enlarged theory (9.22) is just the IR dynamics of the original theory plus the arbitrarily weakly coupled spectator theory. Therefore, $A_s$ should be the same and since the theory is anomaly free, we should have that $A_{IR} + A_s = 0$, which implies

$$A_{IR} = A_{UV} .$$

Taking $g \to 0$ spectators fields completely decouple and (9.23) should still hold. The punchline is that in a quantum field theory, anomaly coefficients associated to global currents are scale independent quantities, and their UV and IR values should match. This is known as ’t Hooft anomaly matching condition. As we will see, a simple equation such as (9.23) puts severe constraints on the IR dynamics of a quantum field theory, in particular on its IR (maybe composite) fermion content.

As an immediate example of the powerfulness of ’t Hooft anomaly matching condition let us notice that it implies that a theory with global conserved currents but with ’t Hooft anomaly (that is a non vanishing triangular anomaly associated to these global currents), does not have a mass gap. This remarkable result can be claimed without doing any sort of non-perturbative computation; just a one-loop one!
9.3 Holomorphy

In what follows, we want to discuss a bit further a property of supersymmetric theories, known as holomorphy, which plays a crucial role when it comes to understand the quantum properties of supersymmetric theories and to what extent they differ from non-supersymmetric ones. Let us first briefly recall the concept of Wilsonian effective action.

When dealing with effective theories we deal with effective actions. The transition from a fundamental (bare) Lagrangian down to an effective one, involves integrating out high-momentum degrees of freedom. The effective action (aka Wilsonian action) is defined from the bare action \( S_{\mu_0} \) defined at some UV scale \( \mu_0 \), as

\[
e^{iS_\mu} = \int_{\phi(p), p > \mu} D\phi e^{iS_{\mu_0}}
\]

where \( S_\mu \) is the effective action, of which we review below few basic properties.

The Wilsonian action correctly describes a theory’s degrees of freedom at energies below a given scale \( \mu \) (the cut-off). It is local on length scales larger than \( 1/\mu \), and describes in a unitary way physical processes involving energy-momentum transfers less than \( \mu \). As far as processes are concerned:

- at energies \( E \approx \mu \), the effective action couplings and masses are given by the tree-level couplings in the effective action (effects of all higher energy degrees of freedom have already been integrated out),

- at energies \( E \ll \mu \) there will be quantum corrections due to fluctuations of modes of the fields in \( S_\mu \) with energies between \( E \) and \( \mu \).

The upshot is that the Wilsonian action \( S_\mu \) is the action which describes the physics at the scale \( \mu \) by its classical couplings.

As we have already noticed, any parameter in a supersymmetric Lagrangian can be thought of as a VEV of a superfield. In particular, if we focus on the F-term part of the Lagrangian (i.e. the superpotential) it follows that each coupling, which is complex in general, can be thought of as the lowest component VEV of a (very heavy) chiral superfield. The theory can then be viewed as an effective theory of a bigger theory where these heavy fields have been integrated out and they act as spurions at low energy.

This makes it manifest that the (F-term) bare Lagrangian is not only holomorphic in the fields but also in the couplings. This property is important since, as
we will see, promoting coupling constants to chiral superfields one can often ex-
tend symmetries of the superpotential and put severe constraints on the form (and
sometime the very existence) of quantum corrections. This important result can
also be proven by means of a (supersymmetric) Ward identity which implies that
all coupling constants appearing in the tree level superpotential must only appear
holomorphically in quantum corrections to the superpotential.

The crucial observation, which is a consequence of such Ward identities and,
more in general, of holomorphy, is that the Wilsonian action is also holomorphic
in the coupling constants. In other words, the couplings appearing in $S_\mu$ should
be holomorphic functions of the UV couplings. Basically, holomorphy makes the
restrictions on possible quantum corrections allowed by supersymmetry apparent.
It provides a supersymmetric version of selection rules.

Let us recall that the Wilsonian effective action is not the effective action $\Gamma$ which
is obtained by integrating out all degrees of freedom down to $\mu = 0$. The latter is
the generating functional of 1PI graphs and calculates the Green functions of the
original, UV theory. It is not holomorphic in the coupling constants and suffers from
holomorphic anomalies. It is not the correct thing to look at in asymptotically free
gauge theories since it is not well defined. The two effective actions are one and the
same only if there are no interacting massless particles, which make the 1PI effective
action suffer from IR divergences.

In order to make this rather abstract discussion more explicit, let us consider a
concrete example. Suppose we have a given supersymmetric theory and, following
the logic outlined previously, let us consider an enlarged symmetry group which
includes a spurious $U(1)$ symmetry, associated to a coupling constant $\lambda$, which
breaks this symmetry spontaneously (a spurion) and has unit charge with respect
to it, $Q(\lambda) = 1$. This simply means that we have in the tree-level superpotential a
term like

$$W_{\text{tree}} \supset \lambda \mathcal{O}_{-1}, \quad (9.25)$$

where the operator $\mathcal{O}$ has charge -1 with respect to the spurious $U(1)$ symmetry.

Suppose we are interested in the appearance of a given operator $\mathcal{O}_{-10}$ among
quantum corrections. In general, we expect it to appear at tenth or higher order as

$$\Delta W \sim \lambda^{10} \mathcal{O}_{-10} + \lambda^{11} \lambda \mathcal{O}_{-10} + \cdots + \lambda^{10} e^{-1/|\lambda|^2} \mathcal{O}_{-10}, \quad (9.26)$$

where we have assumed that the classical limit, $\lambda \to 0$ is well defined and so we
do not expect any negative powers of $\lambda$ to appear. Holomorphy implies that only
the first term appears. All other terms cannot be there since are non-holomorphic in nature (both $\lambda$ and $\bar{\lambda}$ appear). A corollary of the above discussion is that any operator with positive charge with respect to the $U(1)$ is also disallowed. For one thing, we cannot have negative powers of $\lambda$ because we are supposing the theory is well defined in the classical limit (in other words, we are assuming that the physics is smooth for $\lambda \to 0$), while any power of $\bar{\lambda}$ is forbidden by holomorphy. Notice that the latter property is due to supersymmetry, and it is not shared by an ordinary field theory.

The basic message we want to convey here is that holomorphicity in the coupling constants (and usual selection rules for symmetries under which coupling constants may transform) and the requirement of smoothness of physics in various weak-coupling limits, provide severe constraints on the structure of the effective superpotential of a supersymmetric quantum field theory.

### 9.4 Holomorphy and non-renormalization theorems

Using holomorphy one can prove many non-renormalization theorems (and go beyond them, as we will see).

*Example 1:* the WZ model (and its many siblings). The tree level superpotential of the WZ model has the following structure

$$W_{\text{tree}} = \frac{1}{2} m \Phi^2 + \frac{1}{3} \lambda \Phi^3.$$  \hfill (9.27)

The question one might ask is: what is the form of the effective superpotential $W_{\text{eff}}$, once quantum corrections (both perturbative and non-perturbative) are taken into account? Let us try to answer this question using holomorphy. First, promote $m$ and $\lambda$ to spurion superfields. This makes the theory enlarging its symmetries by a flavor $U(1)$ symmetry and a R-symmetry, according to the table below

$$
\begin{array}{c|cc}
U(1)_R & U(1) \\
\hline
\Phi & 1 & 1 \\
m & 0 & -2 \\
\lambda & -1 & -3
\end{array}
$$  \hfill (9.28)

The superpotential has (correctly) R-charge 2 and flavor $U(1)$ charge 0. Notice that both symmetries are spontaneously broken whenever the spurion superfields, $m$ and $\lambda$ have a non-vanishing lower component VEV.
Because of what we discussed in the previous section, the effective (that is, exact) superpotential should be some holomorphic function of the three superfields $\Phi, m$ and $\lambda$, with R-charge equal to 2 and flavor charge equal to 0. Its most general form can be written as a function of $\lambda \Phi/m$ as follows

$$W_{\text{eff}} = m \Phi^2 f \left( \frac{\lambda \Phi}{m} \right) = \sum_{n=-\infty}^{\infty} a_n \lambda^n m^{1-n} \Phi^{n+2}, \quad (9.29)$$

where $f_{\text{tree}} = \frac{1}{2} + \frac{1}{3} \lambda \Phi/m$, and $a_n$ are arbitrary coefficients.

The form of $f$ can be fixed as follows. First, in the classical limit, $\lambda \to 0$, we should recover the tree level result. This implies that there cannot appear negative powers of $\lambda$; hence $n \geq 0$ and, in order to agree with (9.27) at tree-level, $a_0 = \frac{1}{2}$ and $a_1 = \frac{1}{3}$. Taking also the massless limit at the same time, $m \to 0$, restricts $n$ further, i.e. $n \leq 1$. The upshot is that the effective superpotential should be nothing but the tree level one: holomorphy (plus some obvious physical requirements, more below) tells us that the superpotential of the WZ model is not renormalized at any order in perturbation theory and non-perturbatively!

The requirement about finiteness in the massless limit requires a few more comments. Taking the massless limit at finite $\lambda$ does not lead to a weakly-coupled theory, so one could not use smoothness arguments so naively. However, taking both $m, \lambda \to 0$ such that $m/\lambda \to 0$ we do achieve the result above, since the theory is free in this case. One may still wonder whether this conclusion is correct since in this limit there is a massless particle and so the effective theory should have some IR divergences. This is not the case since we do not run the RG-flow down to $\mu = 0$: there are no IR divergences in the Wilsonian effective action, as opposed to the 1PI effective action.

Another, equivalent way to see the absence of negative powers of $m$ in the effective superpotential is to observe that all terms with $n \geq 0$ are generated by tree-level diagrams only, in the UV theory (it is a matter of number of vertices and propagators), see Figure 9.2 below. All diagrams of the kind of the one depicted in Figure 9.2 are not 1PI for $n > 1$; they cannot be produced from loops, and they should not be included in the effective action for finite $m$. So the integer $n$ in eq. (9.29) is indeed restricted to be either 0 or 1.

What we have just proven, namely that the superpotential of the WZ model is not renormalized at any order in perturbation theory and non-perturbatively, is not specific to the WZ model. It actually applies to all models where only chiral
superfields are present: in these cases, that is in the absence of gauge interactions, the tree-level superpotential is an exact quantity.

Example 2: As a second example, we want to illustrate what holomorphy can tell us about the running of gauge coupling in supersymmetric gauge theories. Let us focus, for definiteness, on SQCD. Recall that this is a supersymmetric gauge theory with gauge group $SU(N)$ and $F$ flavors, described by $F$ pairs of chiral superfields $(Q, \tilde{Q})$ transforming in the fundamental respectively anti-fundamental representation of the gauge group and vanishing superpotential, $W(Q, \tilde{Q}) = 0$. At the classical level, the global symmetries are as detailed below

$$
\begin{array}{cccccc}
SU(F)_L & SU(F)_R & U(1)_B & U(1)_A & U(1)_{R_0} \\
\begin{array}{cccc}
Q^i_a & F & \bullet & 1 & 1 & a \\
\tilde{Q}_j^b & \bullet & \overline{F} & -1 & 1 & a \\
\lambda & \bullet & \bullet & 0 & 0 & 1 \\
\end{array}
\end{array}
$$

where the convention on indices is the same as in lecture 5, cf the discussion below eq.(5.103), and the R-charges of $Q$ and $\tilde{Q}$ are the same since under charge conjugation (which commutes with supersymmetry) $Q \leftrightarrow \tilde{Q}$. For later convenience, we have also written down the charges of the gaugino field. The axial current and the $R_0$ current are anomalous, the anomaly coefficients being

$$
A_A = \frac{1}{2}(+1)F + \frac{1}{2}(+1)F = F
$$

$$
A_{R_0} = \frac{1}{2}[(a - 1)F + (a - 1)F] + N = N + (a - 1)F
$$

These two anomalous symmetries admit an anomaly-free combination (which is obviously an R-symmetry)

$$
j^R_\mu = j^R_{0\mu} + \frac{(1-a)F - N}{F} j^A_\mu , \quad (9.30)
$$
under which the matter fields have the following charges $R(Q^i_a) = R(\tilde{Q}^i_j) = \frac{F-N_F}{F}$ (note that while the anomaly-free R-charge of matter fields does not depend on $a$, while the gaugino has always R-charge equal to 1). Therefore, the group of continuous global symmetries at the quantum level is $G_F = SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R$. Notice that for $F = 0$, namely for pure SYM, there do not exist an axial current and in turn the R-symmetry is anomalous. This difference will play a crucial role later on.

What we are interested in is the gauge coupling running, namely the $\beta$ function which, to leading order in the gauge coupling, is

$$\beta = \mu \frac{\partial g}{\partial \mu} = -\frac{b_1}{16\pi^2} g^3 + \mathcal{O}(g^5).$$  \hspace{1cm} (9.31)

The one-loop coefficient $b_1$ can be easily computed from the field content of the classical Lagrangian and reads $b_1 = 3N - F$. The question we would like to answer is whether holomorphy can tell us something about higher-loop (and non-perturbative) corrections to the gauge coupling running.

Let us consider pure SYM, first, whose action is

$$\mathcal{L} = \frac{1}{16\pi^2} \int d^2 \theta \, \tau \, \text{Tr} \, W^a W_a + h.c.,$$  \hspace{1cm} (9.32)

where $\tau$ is the complexified gauge coupling, $\tau = \frac{\theta \sigma}{2\pi} + \frac{4\pi i}{g^2}$. Notice that $\tau$ appears holomorphically in the action above, but the gauge fields are not canonically normalized (to go to a basis where gauge fields are canonically normalized, instead, one should transform $V \rightarrow gV$, as we did already when constructing matter-coupled SYM actions).

We want to discuss quantum corrections due to gauge coupling running, using holomorphy. To this purpose, since we want to treat all parameters as complex ones, we can also trade the dynamical generated scale $\Lambda$ for a complex parameter. For $G = SU(N)$ the one-loop running of the gauge coupling is

$$\frac{1}{g^2(\mu)} = -\frac{3N}{8\pi^2} \log \left( \frac{|\Lambda|}{\mu} \right), \hspace{1cm} |\Lambda| = \mu_0 e^{-\frac{8\pi^2}{3Ng_0^2}},$$  \hspace{1cm} (9.33)

where $3N$ is the one-loop coefficient of pure SYM $\beta$-function and $|\Lambda|$ what we previously called $\Lambda$. We can then define a holomorphic intrinsic scale $\Lambda$ as

$$\Lambda = |\Lambda| e^{\frac{\theta_{YM}}{3N}} = \mu e^{\frac{2\pi i}{3N}},$$  \hspace{1cm} (9.34)
in terms of which the one-loop complexified gauge coupling reads

\[ \tau_{\text{1-loop}} = \frac{3N}{2\pi i} \log \frac{\Lambda}{\mu}. \]  

(9.35)

What about higher order corrections? Suppose we integrate down to a scale \( \mu \), then

\[ W_{\text{eff}} = \frac{\tau(\Lambda; \mu)}{16\pi i} \text{Tr} W^{\alpha} W_{\alpha}. \]  

(9.36)

Since physics is periodic under \( \theta_{\text{YM}} \to \theta_{\text{YM}} + 2\pi \), the following rescaling

\[ \Lambda \to e^{\frac{2\pi i}{3N}} \Lambda , \]  

(9.37)

is a symmetry of the theory, under which \( \tau \to \tau + 1 \). The most general form for \( \tau \) transforming this way under (9.37) is

\[ \tau(\Lambda; \mu) = \frac{3N}{2\pi i} \log \frac{\Lambda}{\mu} + f(\Lambda; \mu) \]  

(9.38)

with \( f \) a holomorphic function of \( \Lambda \) having the following properties:

- \( f \) should have a positive Taylor expansion in \( \Lambda \) in such a way that in the limit \( \Lambda \to 0 \), which is a classical limit, we get back the one-loop result.

- Plugging the transformation (9.37) into the expression (9.35) shows that \( \tau_{\text{1-loop}} \) already accounts for the shift of \( \theta \)-angle by \( 2\pi \). Hence, the function \( f \) should be invariant under (9.37).

These two properties imply that the effective coupling (9.38) should have the following form

\[ \tau(\Lambda; \mu) = \frac{3N}{2\pi i} \log \frac{\Lambda}{\mu} + \sum_{n=1}^{\infty} a_n \left( \frac{\Lambda}{\mu} \right)^{3Nn}. \]  

(9.39)

Recalling that the instanton action is

\[ e^{-S_{\text{inst}}} = \left( \frac{\Lambda}{\mu} \right)^{3N} , \]  

(9.40)

we conclude that the function \( f \) receives only non-perturbative corrections and these corrections come from \( n \)-instantons contributions. The upshot is that \( \tau \) is one-loop exact, in perturbation theory.

The one-loop exactness of the SYM gauge coupling can be equivalently proven as follows. The \( \theta \)-term is a topological term so it does not get renormalized perturbatively. Therefore the \( \beta \)-function, \( \beta = \beta(\tau) \) can only involve \( \text{Im} \, \tau \). If \( \beta \) should be
a holomorphic function of $\tau$ this implies that it can only be an imaginary constant (a holomorphic function $f(z)$, which is independent of $\text{Re} \, z$, is an imaginary constant). Therefore

$$\beta(\tau) \equiv \mu \frac{d}{d\mu} \tau = ia \ ,$$

(9.41)

which implies

$$\mu \frac{d}{d\mu} \theta_{YM} = 0 \ , \quad \mu \frac{d}{d\mu} g = -\frac{a}{8\pi} g^2 + 0 \ .$$

(9.42)

So we see that, indeed, the gauge coupling does not receive corrections beyond one-loop, in perturbation theory (for the theory at hand $a = 3N/2\pi$).

All what we said above applies identically to SQCD (again, working in the basis where gauge fields are not canonically normalized and the complexified gauge coupling enters holomorphically in the action), the only difference being that the one-loop coefficient of the $\beta$-function is now proportional to $3N - F$, with $F$ being the number of flavors.

Remarkably, in some specific cases one can show that also non-perturbative corrections are absent. One such instances is pure SYM, and the argument goes as follows. As already noticed, the R-symmetry of pure SYM is anomalous

$$\partial_{\mu} j_R^{\mu} = 0 \quad \text{corrections} \quad \partial_{\mu} j_R^{\mu} = 2N \frac{1}{32\pi^2} F_{\mu\nu}^{\alpha} \tilde{F}_{\alpha}^{\mu\nu} \ .$$

(9.43)

The $U(1)_R$, however, is not fully broken. This can be seen as follows. A R-symmetry transformation with parameter $\alpha$, under which the gaugino transforms as

$$\lambda \rightarrow e^{i\alpha} \lambda$$

(9.44)

is equivalent to a shift of the $\theta$-angle

$$\theta_{YM} \rightarrow \theta_{YM} + 2N\alpha \ .$$

(9.45)

The point is that the transformation $\theta_{YM} \rightarrow \theta_{YM} + 2\pi k$ where $k \in \mathbb{Z}$, is a symmetry of the theory. So, whenever the $U(1)_R$ parameter $\alpha$ equals $\pi k/N$, the theory is unchanged also at the quantum level. This implies that a discrete subgroup of the original continuous abelian symmetry is preserved,

$$U(1)_R \rightarrow \mathbb{Z}_{2N} \ .$$

(9.46)

Treating the complexified gauge coupling $\tau$ as a spurion field, we can define a spurious symmetry given by

$$\lambda \rightarrow e^{i\alpha} \lambda \ , \quad \tau \rightarrow \tau - \frac{N\alpha}{\pi} \ .$$

(9.47)
This constrains the coefficients $a_n$ in the expansion (9.39). Indeed, under the spurious symmetry the holomorphic scale $\Lambda = \mu e^{\frac{2\pi i}{3N}}$ transforms as

$$\Lambda \rightarrow e^{-i\frac{2Nn}{3N}}\Lambda .$$

(9.48)

Hence we have

$$\tau(\Lambda; \mu) \rightarrow -\frac{N\alpha}{\pi} + \frac{3N}{2\pi i} \log \frac{\Lambda}{\mu} + \sum_{n=1}^{\infty} a_n \left(\frac{\Lambda}{\mu}\right)^{3Nn} e^{-2iNan} .$$

(9.49)

Since $\forall n \neq 0$ certainly $e^{-2Nn} \neq 0$, it follows that to match the spurious symmetry we need to have

$$a_n = 0 \quad \forall n > 0 .$$

(9.50)

Hence in pure SYM also non-perturbative corrections to the gauge coupling are absent! This does not hold in presence of matter, namely for SQCD, since there the R-symmetry is not anomalous and running the above argument one would not get any constraint on the coefficients $a_n$ (the r.h.s. of eq. (9.43) is zero in this case and $\theta_{YM}$ would be insensitive to R-symmetry transformations).

If we collect all what we have learned about the SQCD gauge coupling we might have the feeling that something wrong is going on. There are three apparently incompatible results regarding the running of the gauge coupling.

- **Due to holomorphy**, the supersymmetric gauge coupling runs only at one-loop in perturbation theory, and the full perturbative $\beta$-function hence reads

$$\beta = -\frac{g^3}{16\pi^2}(3N - F) ,$$

(9.51)

- **There exists a well known result** in the literature which claims that the exact, all-loops $\beta$-function of SQCD is

$$\beta = -\frac{g^3}{16\pi^2} \left[ 3N - \sum_{i=1}^{F}(1 - \gamma_i) \right]$$

(9.52)

where $\gamma_i = d\log Z_i(\mu)/d\log \mu$ are matter fields anomalous dimensions. This result gets contribution at all loops and is in clear contradiction with the previous result. Eq. (9.52) is sometime called the NSVZ $\beta$-function.

- **Another piece of knowledge** we have about the $\beta$-function of SQCD (and, in general, of any gauge theory) is that its one and two-loop coefficients are
universal, in the sense that are renormalization scheme independent. This can be easily proven as follows. Changing renormalization scheme amounts to define a new coupling $g'$ which is related to $g$ as

$$g' = g + ag^3 + \mathcal{O}(g^5) .$$

(9.53)

Suppose that the $\beta$-function for $g$ is

$$\beta_g = b_1g^3 + b_2g^5 + \mathcal{O}(g^7) .$$

(9.54)

We get for the $\beta$-function for $g'$

$$\beta_{g'} = \beta_g \frac{\partial g'}{\partial g} = \beta_g (1 + 3ag^2 + \mathcal{O}(g^4)) = b_1g^3 + (b_2 + 3ab_1)g^5 + \mathcal{O}(g^7) .$$

(9.55)

We can invert the relation between $g$ and $g'$ and get

$$g = g' - ag'^3 + \mathcal{O}(g'^5) ,$$

(9.56)

and finally

$$\beta_{g'} = b_1g'^3 - 3ab_1g'^5 + (b_2 + 3ab_1)g'^3 + \mathcal{O}(g'^7) = b_1g'^3 + b_2g'^5 + \mathcal{O}(g'^7) ,$$

(9.57)

which shows that the first two coefficients of the $\beta$-function are indeed universal. Given the universality of the $\beta$-function up to two loops, the discrepancy between the two expressions for the SQCD $\beta$-function, (9.51) and (9.52), cannot just be a matter of renormalization scheme.

How can we reconcile this apparent contradiction? The answer turns out to be surprisingly simple. Let us first consider pure SYM whose action is

$$\mathcal{L} = \frac{1}{16\pi i} \int d^2\theta \tau \mathrm{Tr} W^\alpha W_\alpha + h.c .$$

(9.58)

As we already noticed, if one integrates in superspace one gets a space-time action where gauge fields are not canonically normalized

$$\mathcal{L} = -\frac{1}{4g^2} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} + \ldots .$$

(9.59)

Let us call the gauge coupling defined in this frame holomorphic gauge coupling $g_h$, defined via the complexified gauge coupling as $\tau = 4\pi i / g_h^2$. In order to get a Lagrangian in terms of canonically normalized fields one should rescale $V \rightarrow gV$. In
other words, we should perform the change of variables $V_h = g_p V_p$. In terms of this physical gauge coupling $g_p$ the Lagrangian reads

$$\mathcal{L} = \frac{1}{4} \int d^2 \theta \left( \frac{1}{g_p^2} - i \frac{\theta c}{8\pi^2} \right) \text{Tr} W^\alpha(g_p V_p) W_\alpha(g_p V_p) + h.c. \right), \quad (9.60)$$

Notice that the Lagrangian above is not holomorphic in the physical coupling since $g_p$ is real as $g_p V_p$ should also be real. The crucial point now is that the two Lagrangians (9.58) and (9.60) are not equivalent under the change of variables $V_h = g_p V_p$ in the path integral, since there is a rescaling anomaly (there is an anomalous Jacobian in passing from $V_h$ to $V_p$), that is $D(g_p V_p) \neq D V_p$. In particular, one can show that

$$D(g_p V_p) = D V_p \exp \left[ -i \frac{1}{4} \int d^2 \theta \left( \frac{2T(\text{Adj})}{8\pi^2} \log g_p \right) \text{Tr} W^\alpha(g_p V_p) W_\alpha(g_p V_p) + h.c. \right].$$

(9.61)

Hence we get for the partition function

$$Z = \int D V_h \exp \left[ i \frac{1}{4} \int d^2 \theta \frac{1}{g_h^2} \text{Tr} W^\alpha(V_h) W_\alpha(V_h) + h.c. \right] = [V_h = g_p V_p]$$

$$= \int D V_p \exp \left[ i \frac{1}{4} \int d^2 \theta \left( \frac{1}{g_p^2} - \frac{2T(\text{Adj})}{8\pi^2} \log g_p \right) \text{Tr} W^\alpha(g_p V_p) W_\alpha(g_p V_p) + h.c. \right],$$

(9.62)

which implies

$$\frac{1}{g_p^2} = \text{Re} \left( \frac{1}{g_h^2} \right) - \frac{2T(\text{Adj})}{8\pi^2} \log g_p = \text{Re} \left( \frac{1}{g_h^2} \right) - \frac{2N}{8\pi^2} \log g_p.$$ \quad (9.63)

where in the last equality we used the fact that for $SU(N)$ the Dynkin index for the adjoint $T(\text{Adj}) = N$. Differentiating with respect to $\log \mu$, and using the expression (9.51) for the holomorphic gauge coupling $\beta$-function (setting $F = 0$), one gets for the physical gauge coupling $g_p$ precisely the (pure SYM) NSVZ $\beta$-function (9.52).

One can repeat an identical reasoning for SQCD where the relation between the physical and the holomorphic gauge couplings reads

$$\frac{1}{g_p^2} = \text{Re} \left( \frac{1}{g_h^2} \right) - \frac{2T(\text{Adj})}{8\pi^2} \log g_p - \sum_i \frac{T(r_i)}{8\pi^2} \log Z_i.$$ \quad (9.64)

Differentiating with respect to $\log \mu$ (using again $T(\text{Adj}) = N$ and taking matter to be in the fundamental, for which $T(r) = 1/2$), one gets for the physical gauge coupling exactly the expression (9.52).
We now see why there is no contradiction with two-loops universality of the $\beta$-function. The point is simply that the relation between the holomorphic and the physical gauge coupling is not analytic. In other words, one cannot be Taylor-expanded in the other, because of the log-term (it is a singular change of renormalization scheme: the so-called holomorphic scheme is not related continuously to any other physical renormalization scheme). Furthermore, we also now understand why the physical $\beta$-function gets contribution at all loops. This is just because of wave function renormalization (both of the vector superfield as well as of matter superfields): the physical gauge coupling differs from the holomorphic gauge coupling by effects coming from wave-function renormalization, which get contribution at all loops. Consistently, the physical $\beta$-function can be expressed exactly in terms of anomalous dimension of fields, once the one-loop coefficient (which agrees with that of the holomorphic $\beta$-function) has been calculated.

One can repeat the same kind of reasoning for gauge theories with extended supersymmetry which, after all, are just (very) special cases of $\mathcal{N} = 1$ theories. In doing so one immediately gets the result we anticipated when discussing non-renormalization theorems, namely that $\mathcal{N} = 2$ is one-loop exact and $\mathcal{N} = 4$ tree-level exact. Let us start from $\mathcal{N} = 2$ pure SYM. Using $\mathcal{N} = 1$ language we have a vector and a chiral multiplet, the latter transforming in the adjoint of the gauge group. As we have already seen, due to $\mathcal{N} = 2$ supersymmetry, the kinetic terms of $V$ and $\Phi$ are both changed according to the holomorphic gauge coupling. Hence, going to canonical normalization for all fields we must rescale them the same way, $V_h = g_p V_p$ and $\Phi_h = g_p \Phi_p$. The crucial point is that the Jacobian for $V$ cancels exactly that from $\Phi$! In other words, $D(g_p V_p) D(g_p \Phi_p) = D V_p D \Phi_p$, implying that the holomorphic and physical gauge couplings coincide. Adding matter, nothing changes since, as we have already noticed, kinetic terms for hypermultiplets do not renormalize. Hence, $\mathcal{N} = 2$ is (perturbatively) one-loop finite. Applying this result to $\mathcal{N} = 4$ we conclude that the latter is tree-level exact, in fact, since the $\mathcal{N} = 4$ one-loop $\beta$-function vanishes.

9.5 Holomorphic decoupling

Holomorphy helps also in getting effective superpotentials when one has to integrate out some massive modes and study the theory at scales lower than the corresponding mass scale.

Let us consider a model of two chiral superfields, $L$ and $\Phi$, interacting via the
following superpotential

\[ W = \frac{1}{2} M \Phi^2 + \frac{\lambda}{2} L^2 \Phi, \quad (9.65) \]

The above superpotential does not suffer from quantum corrections, because of holomorphy (neither perturbatively nor non-perturbatively). The spectrum is that of a massless chiral superfield and a massive one. If we want to study the system at energies \( \mu < M \), we have to integrate \( \Phi \) out. In fact, in order to do so we can use holomorphicity arguments, and proceed as we did when proving the exactness of the WZ superpotential. Let us first promote the couplings to spurion fields and, consequently, enlarge the global symmetries as follows

\[
\begin{array}{ccc}
U(1)_a & U(1)_b & U(1)_R \\
L & 0 & 1 & 1 \\
\Phi & 1 & 0 & 0 \\
M & -2 & 0 & 2 \\
\lambda & -1 & -2 & 0 \\
\end{array}
\quad (9.66)
\]

The low energy effective superpotential should be a dimension-three function of \( \lambda, M \) and \( L \) respecting the above symmetries. This implies that

\[ W_{\text{eff}} = a \frac{\lambda^2 L^4}{M}, \quad (9.67) \]

where \( a \) is an undetermined constant of order one.

The same result can be obtained by using the ordinary integrating out procedure. At scales well below \( M \), the chiral superfield \( \Phi \) is frozen at its VEV (we do not have enough energy to make it fluctuate). Therefore, we can integrate the field out by solving its equation of motion, which is just an algebraic one, involving only the F-term, since the kinetic term (the D-term) is trivially zero. The equation of motion is

\[ \frac{\partial W}{\partial \Phi} = M \Phi + \frac{\lambda}{2} L^2 = 0 \quad \Rightarrow \quad \Phi = -\frac{\lambda}{2M} L^2. \quad (9.68) \]

Substituting back into the superpotential we get

\[ W_{\text{eff}} = -\frac{1}{8} \frac{\lambda^2 L^4}{M}, \quad (9.69) \]

which is the same as (9.67) (with the undetermined coefficient being fixed). Notice that the superpotential we have just obtained is the effective superpotential one generates in perturbation theory, in the limit of small \( \lambda \), see Figure 9.3.

Let us emphasize that in this case, differently from the case discussed in the previous section, we have allowed for negative powers of \( M \). In other words, we
have not required any smoothness in the $M \to 0$ limit. The reason is that the effective theory we are considering is valid only at energies lower than $M$, which is indeed a UV cut-off for the theory. Hence, we can accept (and actually do expect) singularities as we send $M$ to zero: new massless degrees of freedom are expected to arise; those associated to $\Phi$, the superfield we have integrated out.

As a final instructive example, let us consider a perturbation of the previous model. The superpotential we would like to analyze is

$$W = \frac{1}{2} M \Phi^2 + \frac{\lambda}{2} L^2 \Phi + \frac{\epsilon}{6} \Phi^3.$$  \hspace{1cm} (9.70)

Again, if we want to study the system at energies $\mu < M$, we have to integrate the massive field out. The equation of motion for $\Phi$ gives

$$\Phi = -\frac{M}{\epsilon} \left( 1 \mp \sqrt{1 - \frac{\epsilon \lambda L^2}{M^2}} \right).$$  \hspace{1cm} (9.71)

We have now two possible solutions, hence two different vacua. Consistently, as we send $\epsilon$ to zero one of the two vacua approaches the one of the unperturbed model while the second is pushed all the way to infinity.

The effective superpotential now reads

$$W_{\text{eff}} = \frac{M^3}{3\epsilon^2} \left[ 1 - \frac{3 \epsilon \lambda L^2}{2 M^2} \mp \left( 1 - \frac{\epsilon \lambda L^2}{M^2} \right) \sqrt{1 - \frac{\epsilon \lambda L^2}{M^2}} \right].$$  \hspace{1cm} (9.72)

There are again singularities in the effective superpotential, both in parameter space as well as in field $L$ space, now. Comparing to the unperturbed case one can suspect that at these points extra massless degrees of freedom show up. Indeed, computing
the (effective) mass for the field we have integrated out, we get
\[
\frac{\partial^2 W}{\partial \Phi^2} = M + \epsilon \Phi = \text{(on the solution)} = \pm M \sqrt{1 - \frac{\epsilon L^2}{M^2}}. \tag{9.73}
\]

The field $\Phi$ becomes massless at $\langle L \rangle = \pm M / \sqrt{\epsilon \lambda}$, precisely the two singularities of the effective superpotential (9.72). In the limit $\epsilon \to 0$, keeping $M$ fixed, one recovers, again, the result of the unperturbed theory.

Again, the same result could have been obtained just using holomorphicity arguments. Promoting also the coupling $\epsilon$ to a spurion field with charges
\[
\begin{array}{ccc}
U(1)_a & U(1)_b & U(1)_R \\
\epsilon & -3 & 0 & 2
\end{array}
\tag{9.74}
\]
and repeating the same argument as in the previous example, one could conclude that the effective superpotential should have the following structure
\[
W_{\text{eff}} = \frac{M^3}{\epsilon^2} f \left( \frac{\epsilon \lambda L^2}{M^2} \right), \tag{9.75}
\]
which has precisely the structure of the exact expression (9.72). Taking various limits one can actually fix also the form of the function $f$ (modulo an overall numerical coefficient, as before).

This way of integrating out in supersymmetric theories, which preserves holomorphy, is called holomorphic decoupling. We will heavily use holomorphic decoupling when studying the quantum properties of SQCD in our next lecture. For instance, using this technique it is possible to get the effective superpotential for SQCD with an arbitrary number of flavors once the exact expression (including numbers like $a$ in eq. (9.67)) for a given number of flavors is known. Everything amounts to integrate flavors in and out (more later).

### 9.6 Exercises

1. Using holomorphy and (spurious) symmetries, show that the superpotential
\[
W = \mu_1 \Phi + \mu_2 \Phi^2 + \cdots + \mu_n \Phi^n, \tag{9.76}
\]

is not renormalized at any order in perturbation theory and non-perturbatively.
References


