LINEAR SPACES

1. Definition and basic properties of linear spaces

Definition 1.1. A linear space X over a field F is a set whose elements are called *vectors* and where two operations, *addition* and *scalar multiplication*, are defined:

(1) addition, denoted by $+$, such that to every pair $x, y \in X$ there correspond a vector $x + y \in X$, and

$$x + y = y + x, \quad x + (y + z) = (x + y) + z, \qquad x, y, z \in X; \tag{1.1}$$

$(X, +)$ is a group, with neutral element denoted by 0 and inverse denoted by $-$, $x + (-x) = x - x = 0$.

(2) scalar multiplication of $x \in X$ by elements $k \in F$, denoted by $kx \in X$, and

$$k(ax) = (ka)x, \quad k(x + y) = kx + ky, \quad (k + a)x = kx + ax, \qquad x, y \in X, \ k, a \in F. \tag{1.2}$$

Moreover $1x = x$ for all $x \in X$, 1 being the unit in F.

It follows from the definition that

$$0x = 0, \qquad (-1)x = -x.$$

Example 1.2. Several spaces of functional analysis have the structure of linear spaces on $\mathbb{R}$ (real vector spaces), $\mathbb{C}$ (complex vector spaces):

- $C(\Omega)$, Ω open set in $\mathbb{R}^n$.
- $H(\omega)$, holomorphic functions on ω open set of $\mathbb{C}$.
- all solutions of a linear ODE or linear PDE.
- $X = \{x = (a_1, a_2, \dots), a_i \in \mathbb{R}\}$.

In the linear spaces we can define several construction and concepts, based only on linearity.
Given $S, T \subset X$, define

$$S + T = \Big\{x = y + z, \ y \in S, \ z \in T\Big\}, \quad -S = \Big\{x = -y, \ y \in S\Big\}, \quad kS = \Big\{x = ky, \ y \in S\Big\}. \tag{1.3}$$

If Z, U are linear spaces over the same field, then

$$Z \oplus U = \Big\{(z, u), \ z \in Z, \ u \in U\Big\}. \tag{1.4}$$

A subset $Y \subset X$ is a *linear subspace* of X if Y is a itself a linear space, i.e.

$$aY + bY \subset Y, \quad a, b \in F.$$

If $S \subset X$, the *linear span* of S is the intersection of all linear subspaces Y_σ containing S, i.e. it is the smallest linear subspace of X containing S. Given the points $x_1, \dots, x_n$, the element

$$x = \sum_{i=1}^n a_i x_i, \qquad a_i \in F,$$

is called *linear combination* of $\{x_1, \dots, x_n\}$.

If X is generated by the linear combination of a finite number of points, we say that it is *finite dimensional*, otherwise it is *infinite dimensional*.

Proposition 1.3. *The linear span of S is the the set of all linear combinations of elements of S.*

Proof. Clearly the linear span is a vector space which contains S, and it is contained in all subspace Y containing S. $\square$

If Y is a linear subspace of X, then

$$x_1 \equiv x_2 \text{ if } x_1 - x_2 \in Y \tag{1.5}$$

is an equivalence relation: in fact

(1) $x \equiv x$ since $x - x = 0 \in Y$;
(2) $x \equiv y$ implies $x - y \in Y$, i.e. $-(x - y) = y - x \in Y$, i.e. $y \equiv x$;
(3) $x \equiv y$, $y \equiv z$ implies $x - y \in Y$, $y - z \in Y$, i.e. $(x - y) + (y - z) = x - z \in Y$, i.e. $x \equiv z$.

Proposition 1.4. *The quotient set X/Y made of the equivalence classes mod Y is a linear space (quotient space).*

Proof. Denote by $[x]$ the equivalent class of x. Define addition $+$ by $[x] + [y] = [x + y]$ and scalar multiplication by $k[x] = [kx]$. This definition does not depend on the particular representative chosen: in fact, if $x' \equiv x$, $y' \equiv y$, then

$$\begin{aligned} [x' + y'] &= \Big\{z \colon z - x' - y' \in Y\Big\} = \Big\{z \colon z - x - y \in Y + (x' - x) + (y' - y)\Big\} \\ &= \Big\{z \colon z - x - y \in Y\Big\} = [x + y], \end{aligned}$$

and similarly for the scalar product

$$[kx'] = \Big\{z \colon z - kx' \in Y\Big\} = \Big\{z \colon z - kx \in Y + k(x' - x)\Big\} = \Big\{z \colon z - kx \in Y\Big\} = [kx].$$

Then it is clear that $+$ is commutative and associative, and the inverse of $[x]$ is $[-x] = -[x]$, and the scalar product satisfies (1.2). Moreover $1[x] = [1x] = [x]$ □

For *real* vector spaces one can define the notion of *convexity*: if $K \subset X$, K is called convex if

$$aK + (1 - a)K \subset K, \quad 0 \le a \le 1, \tag{1.6}$$

or equivalently $ax + (1 - a)y \in K$. An immediate consequence is that if $x_1, \dots, x_n \subset K$, then all *convex combinations*

$$x = \sum_{i=1}^{n} a_i x_i, \qquad a_i > 0, \ \sum_{i=1}^{n} a_i = 1, \tag{1.7}$$

belong to K.

For complex vector spaces, we can extend the definition of $K \subset X$ convex if

$$aK + (1 - a)K \subset K, \quad a \in \mathbb{R}, \ 0 \le a \le 1. \tag{1.8}$$

or one can introduce the notion of *balanced sets*: we define $K \subset X$ balanced if

$$aK \subset K \text{ for all } |a| \le 1. \tag{1.9}$$

If $S \subset X$, then the *convex hull* of S is the intersection of all convex set containing S. It can be characterized equivalently as the smallest convex set containing S or the set of all convex combination of elements of S.

Proof. Clearly each convex sets containing S must contain its convex hull. Conversely the convex hull is a convex set containing S. □

A convex set $E \subset K$, K convex, is called *extreme set* if $E \neq \emptyset$ and if $(y + z)/2 \in E$, then $y, z \in E$. Also in finite dimension one can construct convex sets without extreme points.

2. Linear maps

If X U are two linear spaces, a mapping $\mathbf{M} : X \mapsto U$ is a *linear map* iff

$$\mathbf{M}(x + y) = \mathbf{M}x + \mathbf{M}y, \quad \mathbf{M}(kx) = k\mathbf{M}x. \tag{2.1}$$

An isomorphism of linear spaces is a map $\mathbf{M}$ which is one to one and onto.

Proposition 2.1. *If $\mathbf{M} : X \mapsto U$ linear map, $K \subset U$ convex, $E \subset K$ extreme set, then*
$$\mathbf{M}^{-1}(E) = \Big\{x \in X : \mathbf{M} \in E\Big\},$$
when non empty, is an extreme subset of $\mathbf{M}^{-1}(K)$.

Proof. Clearly $\mathbf{M}^{-1}(K)$ is a convex set: in fact if $\mathbf{M}(x), \mathbf{M}(y) \in K$, then
$$\mathbf{M}(\lambda x + (1-\lambda)y) = \lambda \mathbf{M}x + (1-\lambda)\mathbf{M}y \in K,$$
since K is convex.

Let $x \in \mathbf{M}^{-1}(E)$, and assume that there exists $y, z \in \mathbf{M}^{-1}(K)$ such that $x = (y+z)/2$. Then
$$\mathbf{M}x = \mathbf{M}\left(\frac{y+z}{2}\right) = \frac{1}{2}\Big(\mathbf{M}y + \mathbf{M}z\Big),$$
so that $\mathbf{M}y, \mathbf{M}z \in E$, because E is extremal. □

Remark 2.2. In particular, if $U = \mathbb{R}$ and denoting the linear operator by ℓ, if $H \subset X$ convex, then the extreme subset are $H_{\max} = \max_{x\in H} \ell x$, $H_{\min} = \min_{x \in H} \ell x$. Thus if $\ell^{-1}(H_{\max}) \neq \emptyset$, then this is an extreme subset, and the same for $\ell^{-1} H_{\min}$.

One can find examples of maps so that the image of extremal set are not extremal, also in finite dimension.

Since we can define
$$(\mathbf{M} + \mathbf{N})x = \mathbf{M}x + \mathbf{N}x, \qquad (k\mathbf{M})x = k\mathbf{M}x, \tag{2.2}$$
then the set of linear maps of X into U , denoted as $L(X, U)$, is a linear space. If $\mathbf{M} : X \to U$, $\mathbf{N} : U \to W$, the composition $\mathbf{NM} : X \to W$ is
$$\mathbf{NM}x = \mathbf{N}(\mathbf{M}x).$$
This "product" is distributive, i.e. if $\mathbf{P} : X \mapsto U$, $\mathbf{Q} : Z \mapsto X$ are linear maps,
$$\mathbf{N}(\mathbf{M} + \mathbf{P})x = \mathbf{NM}x + \mathbf{NP}x, \quad (\mathbf{M} + \mathbf{P})\mathbf{Q}x = \mathbf{MQ}x + \mathbf{PQ}x.$$
If we have a third linear space Z ans a linear operator $\mathbf{P} : W \mapsto Z$, then by associativity of composition of maps
$$(\mathbf{PN})\mathbf{M}x = \mathbf{P}(\mathbf{NM})x.$$

A map $\mathbf{M} : X \mapsto U$ is *invertible* if it maps X one to one and onto U, its inverse is denoted by $\mathbf{M}^{-1}$ and
$$\mathbf{M}^{-1}\mathbf{M} = I \in L(X, X), \quad \mathbf{M}\mathbf{M}^{-1} = I \in L(U, U).$$
The *nullspace* $N_{\mathbf{M}}$ of $\mathbf{M}$ is the set
$$N_{\mathbf{M}} = \Big\{x \in X : \mathbf{M}x = 0\Big\}, \tag{2.3}$$
the *range* $R_{\mathbf{M}}$ is
$$R_{\mathbf{M}} = \Big\{u \in U : \exists x \in X, \mathbf{M}x = u\Big\}. \tag{2.4}$$
Clearly both are linear subspaces, and $\mathbf{M} : X \mapsto U$ is invertible iff $N_{\mathbf{M}} = \{0\}$, $R_{\mathbf{M}} = U$. Moreover, $\mathbf{M} : X/N_{\mathbf{M}} \mapsto R_{\mathbf{M}}$ is one to one and onto.

Composition of invertible maps is invertible, but the opposite is false in general, see exercises.

We now consider maps of X into itself, i.e. $\mathbf{M} \in L(X, X)$. We can define in this case the j-th power $\mathbf{M}^j$, with null space $N_j = N_{\mathbf{M}^j}$. Clearly $N_j \subset N_{j+1}$, because if $\mathbf{M}^j x = 0$ then
$$\mathbf{M}^{j+1}x = \mathbf{M}(\mathbf{M}^j x) = \mathbf{M}0 = 0.$$

Let Y be a linear subspace of X, and assume that Y is invariant for $\mathbf{M}$,
$$\mathbf{M}Y \subset Y.$$
Then the operator $\mathbf{L} : X/Y \mapsto X/Y$ defined by
$$\mathbf{L}[x] = [\mathbf{M}x]$$

is a linear operator. In fact, if $x_1 \equiv x_2$, then $x_2 = x_1 + y$, $y \in Y$, and

$$\mathbf{M}x_1 \equiv \mathbf{M}x_2$$

because Y is invariant. The linearity follows easily. Similar definitions can be naturally done in the case

$$\mathbf{L} : X/Y_1 \mapsto X/Y_2, \quad \mathbf{L}[x]_{Y_1} = [\mathbf{M}x]_{Y_2}, \quad [x]_{Y_i} \in X/Y_i, \tag{2.5}$$

Y_1, Y_2 linear subspace of X, Y_1 invariant for $\mathbf{M}$ and $Y_1 \subset Y_2$.

In the following we will denote $\mathbf{L}$ simply as $\mathbf{M}$. Similarly we denote with $\mathbf{M}$ the restriction of $\mathbf{M}$ from Y to Y.

Given a vector space X, we say that X has *finite dimension* if there is a finite number of elements x_i, $i = 1, \dots, n$, such that X is the linear span of $\{x_1, \dots, x_n\}$:

$$X = \left\{ \sum_i a_i x_i \colon a_i \in F, x_i \in X, i = 1, \dots, n \right\}. \tag{2.6}$$

In this case we define the *dimension* of X, $\dim(X)$, as the minimal number of elements needed so that (2.6) holds.

Proposition 2.3. *We have*

$$\dim\Big(N_j/N_{j-1}\Big) \geq \dim\Big(N_{j+1}/N_j\Big). \tag{2.7}$$

Note the opposite inequality w.r.t. $N_j \subset N_{j+1}$.

Proof. In fact, if $[z] \in N_{j+1}/N_j$, then $z = x + N_j$, where $\mathbf{M}^{j+1}x = 0$. Thus

$$\mathbf{M}z = \mathbf{M}x + \mathbf{M}N_j \subset y + N_{j-1},$$

where $\mathbf{M}^j y = 0$. We have proved that $\mathbf{M}z \in N_j/N_{j-1}$.

We prove now that $\mathbf{M} : N_{j+1}/N_j \mapsto N_j/N_{j-1}$ is injective. In fact, if $\mathbf{M}([z_1] - [z_2]) = 0$ in N_j/N_{j-1}, with $[z_1], [z_2] \in N_{j+1}/N_j$, then $\mathbf{M}(z_1 - z_2) \in N_{j-1}$, i.e. $z_1 - z_2 \in N_j$, or $[z_1] = [z_2]$. Thus $\mathbf{M}$ maps N_{j+1}/N_j one to one into N_j/N_{j-1}. □

In particular, if for some j it happens $N_j = N_{j+1}$, then $N_j = N_k$ for all $k \geq j$.

Proposition 2.4. *If* $\mathbf{M} : Y \mapsto Y$ *and* $\mathbf{M} : X/Y \mapsto X/Y$ *are invertible, then* $\mathbf{M} : X \mapsto X$ *is invertible.*

Proof. We first show that $\mathbf{M}$ is injective. In fact, if $\mathbf{M}z = 0$, then $\mathbf{M}[z] = 0$, and from $\mathbf{M} : X/Y \mapsto X/Y$ invertible it follows that $z \in Y$, and from $\mathbf{M} : Y \mapsto Y$ invertible it follows $z = 0$.

Next, to prove surjectivity, we look for $\mathbf{M}x_0 = u_0$. We can solve the above equation modulo Y, i.e. there is $x_1 \in X$ such that $\mathbf{M}x_1 = u_0 + z$, $z \in Y$. From the first condition there is $y \in Y$ such that $\mathbf{M}y = z$, so that $x_0 = x_1 - y$. □

Note that in general if $\mathbf{M}$ is invertible and Y is invariant, $\mathbf{M} : Y \mapsto Y$, $\mathbf{M} : X/Y \mapsto X/Y$ need not to be invertible, see exercises.

3. Index of a linear map

A linear map is called *degenerate* if its range is finite dimensional,

$$\dim(R_{\mathbf{M}}) < \infty. \tag{3.1}$$

If $\mathbf{A} : X \mapsto U$ is degenerate, and $\mathbf{L} : Z \mapsto X$, $\mathbf{R} : U \mapsto V$ are linear maps, then

$$\mathbf{AL}, \ \mathbf{RA}$$

are degenerate. Moreover the set of degenerate maps from X to U is a linear subspace of $L(X, U)$. For the special case of maps from X into itself, the space of degenerate maps forms an ideal with respect to the composition of maps.

We say that $\mathbf{M} : X \mapsto U$, $\mathbf{L} : U \mapsto X$ are pseudoinverse iff

$$\mathbf{LM} = \mathbf{I} + \mathbf{G}, \qquad \mathbf{ML} = \mathbf{I} + \mathbf{G}, \tag{3.2}$$

with $\mathbf{G}$ degenerate (from X to X or $U \mapsto U$, respectively).

Note that if $\mathbf{M}$, $\mathbf{L}$ are pseudoinverse, then adding degenerate maps we obtain again pseudoinverse maps. In fact, if $\mathbf{A} : X \mapsto U$ is degenerate, then

$$\mathbf{L}(\mathbf{M} + \mathbf{A}) = \mathbf{I} + \mathbf{G} + \mathbf{LA} = \mathbf{I} + \mathbf{G}',$$

with $\mathbf{G}'$ degenerate. Similarly for $\mathbf{L}$.

Moreover if $\mathbf{M}$, $\mathbf{L}$ and $\mathbf{A}$, $\mathbf{B}$ are couple of pseudoinverse maps, then the pseudinverse of $\mathbf{AM}$ is $\mathbf{LB}$: in fact

$$\mathbf{LBAM} = \mathbf{L}(\mathbf{I} + \mathbf{G})\mathbf{M} = \mathbf{LM} + \mathbf{G}' = \mathbf{I} + \mathbf{G}''.$$

We define the codimension of the linear subspace $R \subset U$ by

$$\text{codim} R = \dim(U/R). \tag{3.3}$$

Proposition 3.1. *A linear map $\mathbf{M} : X \mapsto U$ has a pseudoinverse iff*

$$\dim N_{\mathbf{M}} < \infty, \qquad \mathit{codim} R_{\mathbf{M}} < \infty. \tag{3.4}$$

Proof. For the "only if" part, let $\mathbf{L} : U \mapsto X$ be a pseudoinverse of X. If $x \in N_{\mathbf{M}}$, then

$$\mathbf{LM}x = x + \mathbf{G}x = 0,$$

i.e. $x \in R_{\mathbf{G}}$ which is finite dimensional. Similarly, if $y \in R_{\mathbf{ML}} = R_{\mathbf{I}+\mathbf{G}}$, then $y \in R_{\mathbf{M}}$, since $R_{\mathbf{L}} \subset X$. This implies that $R_{\mathbf{M}} \supset R_{\mathbf{I}+\mathbf{G}}$. It follows that

$$\text{codim} R_{\mathbf{M}} \leq \text{codim} R_{\mathbf{I}+\mathbf{G}}.$$

Since for $x \in N_{\mathbf{G}}$, then $(\mathbf{I} + \mathbf{G})x = x$, then $N_{\mathbf{G}} \subset R_{\mathbf{I}+\mathbf{G}}$, so that

$$\text{codim} R_{\mathbf{I}+\mathbf{G}} \leq \text{codim} N_{\mathbf{G}}.$$

Since $\mathbf{G} : X/N_{\mathbf{G}} \mapsto R_{\mathbf{G}}$ is one to one, then $\text{codim} N_{\mathbf{G}} = \dim R_{\mathbf{G}}$, in particular is finite dimensional. This concludes the "only if" part.

For the if part, we recall first Zorn's Lemma.

Let A be a partially ordered set, i.e. there is an order relation $\leq$, defined for some pairs of elements $a, b \in A$ which is

- transitive:
$$a \leq b,\ b \leq c \quad \Longrightarrow \quad a \leq c;$$
- reflexive:
$$a \leq a \quad \forall a \in A;$$
- antisymmetric:
$$a \leq b,\ b \leq a \quad \Longrightarrow \quad a = b.$$

A subset B of A is *totally ordered* if for all $a, b \in B$ either $a \leq b$ or $b \leq a$. An element $u \in A$ is an *upper bound* of a subset $B \subset A$ if $b \leq u$ for all b in the subset B. A element m is *maximal* if every element a of the set A such that $a \leq m$ or $m \leq a$, satisfies $a \leq m$.

Lemma 3.2 (Zorn's Lemma)**.** *If every totally ordered subset of a partially ordered set has an upper bound, then the partially ordered set has a maximal element.*

By Zorn's lemma for any subspace N of X there is a complementary (not unique!) subspace such that $X = N \oplus Y$, i.e. every $x \in X$ can be written as

$$x = n + y, \qquad n \in N,\ y \in Y.$$

In fact, let A be the set of subspaces Y of X such that $N \cap Y = \{0\}$, ordered by inclusion. The set A is clearly partially ordered, and if $\{Y_\alpha\}_\alpha$ is a totally ordered subset of A, then $\cup_\alpha Y_\alpha$ is an upper bound. Thus there is a maximal element Y.

Assume now that there is an $x \notin N + Y$. Then $Y' = Y + \text{span}\{x\}$ is again in A, $Y \subsetneq Y'$, contradicting that Y is maximal. Thus for every $x = n + y$, and since $Y \cap N = \{0\}$, the decomposition is unique.

We can thus define the projection of x onto N by $\mathbf{P}x = n$. Note that if N has finite codimension then $\dim Y = \text{codim} N$. In fact, the map

$$X/Y \ni [x] = [n + y] = [n] \mapsto n \in N$$

is surjective and injective.

Decompose now
$$X = N_{\mathbf{M}} \oplus Y, \quad U = R_{\mathbf{M}} \oplus V.$$
From the fact that $\mathbf{M} : X/N_{\mathbf{M}} \mapsto R_{\mathbf{M}}$ one to one and onto, then $\mathbf{M} : Y \mapsto R_{\mathbf{M}}$ is invertible: one in fact can again use the identification
$$X/N_{\mathbf{N}} \ni [x] = [y] \mapsto y \in Y,$$
so that
$$z \in R_{\mathbf{M}} \implies \exists ! [x] \in X/N_{\mathbf{M}} : \mathbf{M}[x] = z \implies \exists ! y \in Y : \mathbf{M}y = z.$$
Denote its inverse by $\mathbf{M}^{-1}$. Define now $\mathbf{K} = \mathbf{M}^{-1}\mathbf{P}_{R_{\mathbf{M}}}$. Clearly
$$\mathbf{KM} = \begin{cases} \mathbf{I} & \text{on } Y \\ 0 & \text{on } N_{\mathbf{M}} \end{cases} = \mathbf{I} - \mathbf{P}_{N_{\mathbf{M}}}, \qquad \mathbf{MK} = \begin{cases} \mathbf{I} & \text{on } R_{\mathbf{M}} \\ 0 & \text{on } V \end{cases} = \mathbf{I} - \mathbf{P}_V.$$
By constructions the projections have finite dimensional range. □

We now introduce the *index* of a linear map with pseudoinverse:

(3.5) $$\mathrm{ind}\mathbf{M} = \dim N_{\mathbf{M}} - \mathrm{codim} R_{\mathbf{M}}.$$

Theorem 3.3 (Stability of the index)**.** *If $\mathbf{M} : X \mapsto U$, $\mathbf{L} : U \mapsto W$ are linear maps with pseudoinverse, then*

(3.6) $$\mathrm{ind}\mathbf{LM} = \mathrm{ind}\mathbf{L} + \mathrm{ind}\mathbf{M}.$$

Proof. By considering the linear spaces $X/N_{\mathbf{M}}$, $R_{\mathbf{L}}$, we reduce to prove that
$$\mathrm{ind}\mathbf{LM} = \dim N_{\mathbf{L}} - \mathrm{codim} R_{\mathbf{M}},$$
with $\mathbf{M}$ one to one and $\mathbf{L}$ onto.

Decompose $U = R_{\mathbf{M}} + Y$, $\mathbb{R}_{\mathbf{M}} \cap Y = \{0\}$ such that

(3.7) $$N_{\mathbf{L}} = (N_{\mathbf{L}} \cap R_{\mathbf{M}}) + (N_{\mathbf{L}} \cap Y).$$

This can be done using again Zorn's Lemma, or just by finite dimensional arguments. In fact, we consider $N_{\mathbf{L}} \cap R_{\mathbf{M}} \subset N_{\mathbf{L}}$. Then, we decompose first (we are in finite dimension)
$$N_{\mathbf{L}} = (N_{\mathbf{L}} \cap R_{\mathbf{M}}) + \tilde{Y}, \quad R_{\mathbf{M}} \cap \tilde{Y} = \{0\}.$$
Then we find a maximal subspace Y containing $\tilde{Y}$ and such that $R_{\mathbf{M}} \cap Y = \{0\}$. As in the proof of Proposition 3.1 one concludes that $U = R_{\mathbf{M}} + Y$, and (3.7) holds.

Since $\mathbf{L}$ is surjective, then the only points which are not mapped by $\mathbf{L}$ are the points in Y, or
$$W = (\mathbf{L}Y) + R_{\mathbf{LM}}.$$
The number of different points in Y which are mapped by $\mathbf{L}$ are exactly $Y/(Y \cap N_{\mathbf{L}})$.

Since $\mathbf{M}$ is injective, then $N_{\mathbf{LM}}$ are the points of X which are mapped by $\mathbf{M}$ into $N_{\mathbf{L}}$, i.e. $N_{\mathbf{L}} \cap R_{\mathbf{M}}$.

We conclude
$$\begin{aligned} \dim N_{\mathbf{LM}} - \mathrm{codim} R_{\mathbf{LM}} &= \dim(N_{\mathbf{L}} \cap R_{\mathbf{M}}) - \dim(Y/(Y \cap N_{\mathbf{L}})) \\ &= \dim(N_{\mathbf{L}} \cap R_{\mathbf{M}}) - \mathrm{codim}(R_{\mathbf{M}}) + \dim(Y \cap N_{\mathbf{L}}) \\ &= \dim(N_{\mathbf{L}}) - \mathrm{codim}(R_{\mathbf{M}}). \end{aligned}$$
□

4. Exercises

(1) On the one dimensional vector space $\mathbb{C}$, consider the convex set $C = \{z = t, t \in [0,1]\}$ and the point $\bar{z} = i/2$. Clearly there are not linear map $\mathbf{M}(z) = \alpha z$ such that $\sup_C |\mathbf{M}(z)| \leq |\mathbf{M}(\bar{z})|$. Show that instead this happens when $\mathbb{C}$ is considered with the linear structure of the real vector space $\mathbb{R}^2$.

(2) Prove by means that the axioms on linear spaces that:
- every intersection and union of balanced sets is balanced;
- every intersection of convex sets is convex;
- A is convex iff $sA + tA = (s+t)A$ for all $s, t > 0$.

(3) Consider the linear space

$$X = \Big\{ x = (a_1, a_2, \dots), a_i \in \mathbb{R} \Big\}.$$

Define the linear maps $\mathbf{R}$, $\mathbf{L}$ (right and left shift respectively) as

$$\mathbf{R}(a_1, a_2, \dots) = (0, a_1, a_2, \dots), \quad \mathbf{L}(a_1, a_2, \dots) = (a_2, a_3, \dots).$$

- Prove that $\mathbf{LR}$ is invertible, but $\mathbf{RL}$ not.
- Show that are pseudoinverse of each other.
- Compute the null spaces, ranges and index and verify the index theorem.

(4) Prove that if $\mathbf{L}_1$, $\mathbf{L}_2$ are pseudinverse of $\mathbf{M}$, then $\mathbf{L}_1 - \mathbf{L}_2$ is degenerate.

(5) Consider X as the space of bounded function on $\mathbb{R}$, and let $\mathbf{S}x(t) = x(t-1)$ (right shift of 1). Let Y be the subspace of functions which vanishes on $x < 0$.
- Prove that Y is invariant, but $\mathbf{S}$ is not invertible on either Y or X/Y.
- Compute the null space on X/Y.

(6) In the vector space $C([0,1], \mathbb{R})$ find the extremal subsets of

$$B = \Big\{ u \in C([0,1], \mathbb{R}) : |u(x)| \leq 1 \Big\}.$$

(7) Let $\mathbf{M} \in L(X, Y)$ be a linear map. Show that

$$\dim\big(X/N_{\mathbf{M}}\big) = \dim R_{\mathbf{M}}.$$

In particular, if $Y = \mathbb{R}$ or $\mathbb{C}$, then $\dim(X/N_{\mathbf{M}}) = 1$.