Chapter 1

Definition of hyperbolic systems of balance laws

Consider the PDE
\[
\frac{\partial}{\partial t} H(t,x,U(t,x)) + \text{div} F(t,x,U(t,x)) = G(t,x,U(t,x)),
\] (1.0.1)

where
\[
U : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^N, \quad G,H : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad F_i : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad i = 1, \ldots, d.
\]

The divergence should be intended as
\[
\text{div} F = \sum_{i=1}^d \partial_x F_i.
\]

This is a system of \(N\) first order equations of balance laws in \(\mathbb{R}^d\), because at the formal level integrating in smooth regions \(\Omega \subset \mathbb{R}^d\)
\[
\frac{d}{dt} \int_\Omega H(t,x,U(t,x))dx = -\int_{\partial\Omega} F(t,x,U(t,x)) \cdot n_\Omega(x)H^{d-1}(dx) + \int_\Omega G(t,x,U(t,x))dx,
\] (1.0.2)

where \(n\) is the outer normal and the scalar product should be intended as
\[
F \cdot n = \sum_{i=1}^d F_i n_i, \quad n_\Omega = (n_1, \ldots, n_d).
\] (1.0.3)

If \(U \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)\), then we consider (1.0.2) in the weak sense: for all \(\phi \in C_c(\mathbb{R}^+ \times \mathbb{R}^d)\)
\[
\int_{\mathbb{R}^+ \times \mathbb{R}^d} \left\{ \frac{\partial}{\partial t} \phi(t,x)H(t,x,U(t,x)) + F(t,x,U(t,x) : D_x \phi(t,x) + \phi(t,x)G(t,x,U(t,x)) \right\}dxdt = 0. \] (1.0.4)

with the same convention as in (1.0.3). In this formulation an initial data \(U_0 : \mathbb{R}^d \rightarrow \mathbb{R}^N\) appears as
\[
\int_{\mathbb{R}^+ \times \mathbb{R}^d} \left\{ \frac{\partial}{\partial t} \phi H + F \cdot D_x \phi + \phi G \right\}dxdt + \int_{\mathbb{R}^d} \phi(0,x)H(0,x,U_0(x))dx = 0.
\]

The smoothness of the functions \(H, F\) and \(G\) will be not of our concern, so we assume them as smooth as needed. Instead we are interested in the relations among these functions and \(U\) in order to develop an existence/uniqueness theory.

1.1 Hyperbolicity

As a thumb-rule, one studies the system (1.0.1) when \(H, F\) and \(G\) are linear independent on \((t,x)\) (by smoothness it is natural for a local analysis): thus
\[
A \frac{\partial}{\partial t} U(t,x) + \sum_{i=1}^d B_i \partial_x_i U(t,x) = CU(t,x),
\] (1.1.1)
for some $N \times N$ matrices $A$, $C$ and $B_i$, $i = 1, \ldots, d$. The relation with the original equation is that

$$A = D_U H(\bar{t}, \bar{x}, \bar{U}), \quad B_i = D_U F_i(\bar{t}, \bar{x}, \bar{U}), \quad C = D_U G(\bar{t}, \bar{x}, \bar{U}).$$

(1.1.2)

By taking the Fourier transform

$$\hat{U}(t, \xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} U(t, x) dx$$

we obtain the system of first order ODEs

$$A \frac{d}{dt} \hat{U}(t, \xi) - i \left( \sum_{i=1}^d \xi_i B_i \right) \hat{U}(t, \xi) = C \hat{U}(t, \xi).$$

At a macro level ($|\xi| \to 0$) the evolution is controlled by $A d\hat{U}/dt = C \hat{U}$, while at a micro level ($|\xi| \to \infty$) is controlled by

$$A \frac{d}{dt} \hat{U}(t, \xi) = i \left( \sum_{i=1}^d \nu_i B_i \right) \hat{U}(t, \xi), \quad \nu := \frac{\xi}{|\xi|}.$$  

The requirement that the solution does not blow up is thus that for any $\nu \in S^{d-1}$ there are no eigenvalues with real part and no generalized eigenvectors (remember that we rescaled by $|\xi|$), i.e. the eigenvalue problem

$$\left[ \lambda A - \left( \sum_{i=1}^d \nu_i B_i \right) \right] R = 0$$  

(1.1.3)

has $N$ linearly independent eigenvectors. Clearly the matrix $A$ should be invertible.

Following (1.1.2), we given the following definition:

**Definition 1.1.1.** The system (1.0.1) is called hyperbolic in the $t$-direction if for all $(\bar{t}, \bar{x}, \bar{U}) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^N$, $\nu \in S^{d-1}$ the matrix $DH(\bar{t}, \bar{x}, \bar{U})$ is invertible and the eigenvalue problem

$$\left[ \lambda DH(\bar{t}, \bar{x}, \bar{U}) - \sum_{i=1}^d \nu_i B_i(\bar{t}, \bar{x}, \bar{U}) \right] R = 0$$

(1.1.4)

has $N$ real eigenvalues and $N$ linearly independent eigenvectors

$$\lambda_1(\bar{t}, \bar{x}, \bar{U}), \ldots, \lambda_N(\bar{t}, \bar{x}, \bar{U}), \quad R_1(\bar{t}, \bar{x}, \bar{U}), \ldots, R_N(\bar{t}, \bar{x}, \bar{U}),$$

Since $H$ is invertible at least locally, in the following we will use $H(t, x, U)$ as a new variable $U$, thus we will consider the canonical form of the PDE

$$U_t + \text{div } F(t, x, U) = G(t, x, U).$$

(1.1.5)

In this case the hyperbolicity condition (1.1.4) becomes

$$\left[ \lambda I - \sum_{i=1}^d \nu_i B_i(\bar{t}, \bar{x}, \bar{U}) \right] R = 0.$$  

**1.1.1 Entropy conditions and symmetric hyperbolic systems**

A particularly simple class of weak solutions to (1.1.5) are shock fronts:

$$U(t, x) = \begin{cases} U^-(t, x) & (t, x) \in \Gamma, \\ U^+(t, x) & (t, x) \notin \Gamma, \end{cases} \quad \Gamma \subset \mathbb{R}^{d+1} \text{ with smooth boundary.}$$

By taking the sequence of functions

$$\phi^\epsilon(t, x) := \psi(t, x) \left( \rho^\epsilon \ast \mathcal{H}^{d-1}_{\nu \cdot \Gamma} \right)(t, x)$$

the weak balance (1.0.4) becomes

$$\int_{\partial \Gamma} \psi(t, x) \left( U^+ - U^-, F(t, x, U^+) - F(t, x, U^-) \right) \cdot n_\Gamma(t, x) d\mathcal{H}^{d-1}_{\nu \cdot \Gamma} = 0,$$
This is a relation which must be satisfied also by piecewise smooth solutions, i.e. solutions which are $C^1$ outside a locally finite number of smooth hypersurfaces: in this case $U^-, U^+$ are replaced by the limits on the two sides of the discontinuity.

Under the assumption of hyperbolicity and for small (moderate) jumps, it follows that $n\Gamma$ and $U^+ - U^-$ have a direction close to $(\lambda, \nu)$ and $R$ satisfying (1.1.4). Notice here that we can exchange $U^- \leftrightarrow U^+$ without altering the relation.

### 1.1.2 Entropy

The quasilinear form of (1.1.5)

$$
\partial_t U + \sum_{i=1}^{d} D U F_i \partial_{x_i} U = \left( G - \sum_{i=1}^{d} (\partial_{x_i} F_i)(t, x, U) \right)
$$

(1.1.7)

can be multiplied for the $U$-derivative of any function $\eta : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^N \mapsto \mathbb{R}^N$ obtaining

$$
\partial_t \eta(t, x, U) + \sum_{i=1}^{d} D U \eta D U F_i \partial_{x_i} \eta + \sum_{i=1}^{d} (\partial_{x_i} \eta)(t, x, U) (\partial_t U) + \sum_{i=1}^{d} (\partial_{x_i} \eta)(t, x, U) (\partial_i U) = \left( G - \sum_{i=1}^{d} (\partial_{x_i} F_i)(t, x, U) \right) + (\partial_t \eta)(t, x, U) + \sum_{i=1}^{d} (\partial_{x_i} \eta)(t, x, U).
$$

(1.1.8)

In this latter case, we have an additional balance law for smooth solutions to (1.1.5)

$$
\partial_t \eta + \sum_{i=1}^{d} \partial_{x_i} q_i = \left( G - \sum_{i=1}^{d} (\partial_{x_i} F_i)(t, x, U) \right) + (\partial_t \eta)(t, x, U) + \sum_{i=1}^{d} (\partial_{x_i} q_i)(t, x, U).
$$

(1.1.9)

### Entropy dissipation

For non smooth solutions, in general the weak formulation is not satisfied. This is easily seen on the Rankine-Hugoniot conditions (1.1.6) if the functions $H, F_i$ are nonlinear (see Exercise 4).

In many systems coming from physics, there exists a convex entropy $\eta$, and moreover the balance laws (1.1.5) are the formal limits of more complicated systems which takes into account other effects (viscosity, heat transmission, ...). A paradigm is the diagonal viscous perturbation

$$
\partial_t U^\varepsilon(t, x) + \text{div} F (t, x, U^\varepsilon(t, x)) = G(t, x, U^\varepsilon(t, x)) + \varepsilon \Delta U^\varepsilon,
$$

By multiplying by $D U \eta$ and assuming (1.1.8), one obtains

$$
\partial_t \eta + \sum_{i=1}^{d} \partial_{x_i} q_i = \left( G - \sum_{i=1}^{d} \partial_{x_i} F_i(t, x, U^\varepsilon) \right) + (\partial_t \eta)(t, x, U^\varepsilon) + \sum_{i=1}^{d} (\partial_{x_i} q_i)(t, x, U^\varepsilon) + \varepsilon D U \eta \Delta U^\varepsilon
$$

$$
= \left( G - \sum_{i=1}^{d} \partial_{x_i} F_i(t, x, U^\varepsilon) \right) + (\partial_t \eta)(t, x, U^\varepsilon) + \sum_{i=1}^{d} (\partial_{x_i} q_i)(t, x, U^\varepsilon) + \varepsilon \text{div} (D U \eta D_x U^\varepsilon) - \varepsilon D U \eta : D_x U^\varepsilon \times D_x U^\varepsilon - \varepsilon (\partial_x D U \eta) D_x U^\varepsilon,
$$

because

$$
D U \eta \Delta U^\varepsilon = \text{div} (D U \eta D_x U^\varepsilon) - \varepsilon (D_x D U \eta) \cdot D_x U^\varepsilon.
$$

Under the assumptions

$$
\|\epsilon D_x U\|_{L^2_{loc}} \cdot \|U\|_{L^\infty} \leq C,
$$
one obtains for all test functions $\phi \in C_c(\mathbb{R}^+ \times \mathbb{R}^d)$

\[
\left| \int \phi \, \varepsilon \, \text{div} \left( D_U \eta D x U^r \right) \, dt \, dx \right| = \varepsilon \left| \int D_U \eta \nabla \phi \cdot D x U^r \, dt \, dx \right|
\leq \left| \int (\sqrt{\varepsilon} \nabla \phi) \cdot (\sqrt{\varepsilon} D x U^r) \, dt \, dx \right|
\leq \sqrt{\varepsilon} \| \nabla \phi \|_{L^\infty} \sqrt{L^{d+1}(\text{supp } \phi)} \| \varepsilon D x U \|_{L^2(\text{supp } \phi)} \to 0
\]
as $\varepsilon \to 0$, and similarly

\[
\varepsilon (\partial_x D_U \eta) D x U^r \to 0 \quad \text{in } L^1_\text{loc}(\mathbb{R}^{d+1}),
\]

and since $\eta$ is convex

\[
\varepsilon D_U^2 \eta : D x U^r \times D x U^r \geq 0.
\]

Then the weak balance of the entropy should be actually a dissipation

\[
\partial_t \eta + \text{div} \, \partial_x q \leq G'(t, x, U) \quad \text{in distribution,}
\]

with the notation $q := (q_1, \ldots, q_d)$ and $G'$ the right hand side of (1.1.9).

\[
G'(t, x, U) := \left( G - \sum_{i=1}^d (\partial_x F_i) \right) + (\partial_t \eta)(t, x, U) + \sum_{i=1}^d (\partial_x q_i).
\]

The balance inequality (1.1.10)

\[
\forall \phi \in C^1_c(\mathbb{R}^{d+1}; \mathbb{R}^+) \left( \int_{\mathbb{R}^+ \times \mathbb{R}^d} \left( \phi t \eta + q \cdot D x \phi + G' \phi \right) \, dx \, dt + \int_{\mathbb{R}^d} \eta(t = 0) \phi(t = 0) \, dx \geq 0 \right)
\]
is correct if we prove the $L^1_\text{loc}$-convergence $U^r \to U$.

The balance for a piecewise smooth solution yields

\[
\forall (t, x) \in \partial \Gamma \left[ \left( q(t, x, U^+) - \eta(t, x, U^-), q(t, x, U^+) - q(t, x, U^-) \right), n_\Gamma(t, x) \leq 0 \right] \quad (1.1.11)
\]

In general this implies that, moving in the direction $n_\Gamma$, one of the jumps $U^- \to U^+$ or $U^+ \to U^-$ is not entropic. The weak solutions $U$ which satisfy the dissipation inequality (1.1.10) are called dissipative.

### 1.1.3 Symmetrization

The entropy relation (1.1.8)

\[
D_U \eta(t, x, U) D_U F_i(t, x, U) = \sum_{\gamma=1}^N \partial_U \gamma \eta \partial_U \gamma, F_{\gamma,i} = D_U q_i(t, x, U) = \partial_U q_i
\]

implies that the second derivatives of $q_i$ are symmetric: for $U = (U_1, \ldots, U_N)$,

\[
\partial_U^2 U_{\beta} q_i(t, x, U) = \partial_U^2 U_{\alpha} q_i(t, x, U)
\]

\[
\downarrow
\]

\[
\partial_U^2 \left( \sum_{\gamma=1}^N \partial_U \gamma \eta \partial_U \gamma, F_{\gamma,i} \right) = \partial_U \left( \sum_{\gamma=1}^N \partial_U \gamma \eta \partial_U \gamma, F_{\gamma,i} \right)
\]

\[
\downarrow
\]

\[
\sum_{\gamma=1}^N \partial_U^2 U_{\gamma} \eta \partial_U \gamma, F_{\gamma,i} + \sum_{\gamma=1}^N \partial_U \gamma \eta \partial_U^2 U_{\gamma} \eta \partial_U \gamma, F_{\gamma,i} = \sum_{\gamma=1}^N \partial_U^2 U_{\gamma} \eta \partial_U \gamma, F_{\gamma,i} + \sum_{\gamma=1}^N \partial_U \gamma \eta \partial_U^2 U_{\gamma} \eta \partial_U \gamma, F_{\gamma,i}
\]

\[
\downarrow
\]

\[
\sum_{\gamma=1}^N \partial_U^2 U_{\gamma} \eta \partial_U \gamma, F_{\gamma,i} = \sum_{\gamma=1}^N \partial_U^2 U_{\gamma} \eta \partial_U \gamma, F_{\gamma,i} \quad (1.1.12)
\]

We have proved that multiplying the quasilinear form of the PDE (1.1.7) by the symmetric matrix $D_U^2 \eta$ we have

\[
D_U^2 \eta \partial_t U + \sum_{i=1}^d D_U^2 \eta D_U F_i \partial_x U = D^2 \eta \left( G - \sum_{i=1}^d (\partial_x F_i) \right),
\]

with $D_U^2 \eta D_U F_i$ symmetric matrices for all $i = 1, \ldots, d.$
Definition 1.1.2. The system of balance laws (1.1.5) is called symmetric hyperbolic if there is a definite positive matrix $S = S(U)$ (the symmetrizer) such that

$$\forall i = 1, \ldots, d \quad \text{the matrix } S(U)D_U F_i(t, x, U) \text{ is symmetric}$$

We thus conclude that

Lemma 1.1.3. If the system (1.1.5) has a strictly convex smooth entropy then it is symmetric hyperbolic.

We can further observe that defining the entropy coordinates $W$ by

$$U = (D_U \eta)^{-1}(W)$$

one obtains the conservation form

$$\partial_t (D_U \eta)^{-1} + \sum_{i=1}^{d} F(t, x, (D_U \eta)^{-1}) = G(t, x, (D_U \eta)^{-1}).$$

(1.1.14)

Letting $\eta'$ be the Legendre transform of $\eta$,

$$\eta'(W) := \sup \{ W \cdot U - \eta(U) \},$$

so that $D_W \eta' \circ D_U \eta = D_U \eta \circ D_W \eta' = I$ then the quasilinear form of (1.1.14) takes the form

$$G(t, x, D_W \eta') = \partial_t D_W \eta' + \sum_{i=1}^{d} F_i(t, x, D_W \eta')$$

$$= D^2_W \eta' \partial_t W + \sum_{i=1}^{d} D_U F D^2_W \eta' \partial x_i W + \sum_{i=1}^{d} (\partial x_i F_i)(t, x, D_W \eta')$$

and by the symmetry of $D^2_W \eta D_U F$ and the relation

$$D^2_W \eta(U) = (D^2_W \eta')^{-1}(D_U \eta(U))$$

we have

$$D_U F(t, x, D_W \eta') D^2_W \eta'(W) = D^2 \eta'(W) \left[ (D^2 \eta'(W))^{-1} D_U F(t, x, D_W \eta') \right] D^2_W \eta'(W)$$

$$= D^2 \eta'(W) \left[ D^2 \eta(\eta'(U)) D_U F(t, x, \eta'(U)) \right] D^2 \eta'(W).$$

symmetric symmetric by (1.1.12)

We have proved that

Proposition 1.1.4. For a system in conservation form with a strictly convex entropy we thus conclude that the system is symmetric hyperbolic in conservation form when using the entropy coordinates (1.1.13).

1.1.4 Scale invariance and finite speed of propagation

It is quite simple to see that the homogeneous PDE

$$\partial_t U + \text{div } F(U) = 0$$

is invariant for the rescaling

$$t, x \mapsto \rho t, \rho x.$$ 

Often we will call this the hyperbolic rescaling.

For the symmetric linear system

$$U_t + \sum_{i=1}^{d} B_i \partial x_i U = 0$$

the assumption of hyperbolicity yields the definition of the functions

$$\{1, \ldots, N\} \times \mathbb{S}^{d-1} \ni (\alpha, \nu) \mapsto (\lambda_\alpha(\nu), R_\alpha(\nu)) \in \mathbb{R} \times \mathbb{R}^N,$$

and by symmetrization we can assume $R_\alpha(\nu)$ to be orthogonal.
We can thus write the $L^2$-Green Kernel $G(t, x)$ by

$$\hat{G}(t, \xi) := \sum_{\alpha=1}^{N} e^{it|\lambda_{\alpha}(\xi/|\xi|)|} R_\alpha(\xi/|\xi|) \times R_\alpha^T(\xi/|\xi|),$$

and it is fairly easy to prove that $G(t, x)$ is an $L^2$-multiplier:

$$|\hat{G}(t, \xi)| \leq 1 \implies \|G(t) \ast U\|_{L^2} = \|\hat{G}(t)\hat{U}\|_{L^2} \leq \|\hat{U}\|_{L^2}.$$  

Notice that by the hyperbolic rescaling one has $G(t, x) = G(1, x/t)$ in distribution, which corresponds exactly to the $\hat{G}(1, x/t) = \hat{G}(1, \xi \xi)$ above.

In the general hyperbolic case, one can prove (exercise 7)

$$\hat{G}(t, \xi) = \frac{1}{2\pi i} \oint_{\gamma} e^{itz} \left( zA - \sum_{i=1}^{d} \xi_i B_i \right)^{-1} dz, \quad z \in \mathbb{Z}. \quad (1.1.15)$$

where $\gamma$ is a closed path containing all the roots of

$$\det \left( zA - \sum_{i=1}^{d} \xi_i B_i \right) = 0.$$

It is fairly easy to deduce that also in this case $G(t, x)$ is a Fourier multiplier (just observe that it is enough to study the case $\xi \in \mathbb{S}^{d-1}$), and that $\hat{G}(t, \xi)$ is analytic.

The solution to the Cauchy problem $U(0, x) = \hat{U}_0(x)$ can then be written as

$$U(t, x) = \int_{\mathbb{R}_d} G(t, x - y)U_0(y)dy = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{G}(t, \xi)\hat{U}_0(\xi)d\xi.$$  

Take as initial data

$$\hat{U}_0^\rho(x) = \frac{1}{(2\pi\rho^2)^{d/2}} e^{-|x/\rho|^2/2} \hat{U}_0, \quad \hat{U}_0^\rho(\xi) = \frac{1}{(2\pi)^{d/2}} e^{-|\rho\xi|^2/2} \hat{U}_0, \quad \hat{U}_0 \in \mathbb{R}^N,$$

and then

$$\hat{U}_0^\rho(t, x) = \int_{\mathbb{R}^d} G(t, x - y)U_0^\rho(y)dy = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{G}(t, \xi)\hat{U}_0^\rho(\xi)d\xi.$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{1}{2\pi i} \left( \oint_{\gamma} e^{i(t\xi - x + tz)} \left( zA - \sum_{i=1}^{d} \xi_i B_i \right)^{-1} dz \right) \hat{U}_0 d\xi$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{1}{2\pi i} \left( \oint_{\gamma} e^{i(|\xi| t)} \left( zA - \sum_{i=1}^{d} \xi_i B_i \right)^{-1} dz \right) \hat{U}_0 d\xi,$$

where now $\gamma$ can be chosen independently of $\xi$.

We thus obtain

$$|\hat{U}_0^\rho(t, x)| = \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{2\pi i} \left( \oint_{\gamma} e^{i(t\xi - x + tz)} \left( zA - \sum_{i=1}^{d} \xi_i B_i \right)^{-1} dz \right) \hat{U}_0 d\xi \right|$$

$$= \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{2\pi i} \left( \oint_{\gamma} e^{i(t\xi - x + tz) - \rho|\xi|^2/2} \left( zA - \sum_{i=1}^{d} \xi_i B_i \right)^{-1} dz \right) \hat{U}_0 d\xi \right|$$

$$= \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{2\pi i} \left( \oint_{\gamma} e^{-|x + tz|/|\xi|} e^{-\rho|\xi|^2/(2\rho^2)} \left( zA - \sum_{i=1}^{d} \xi_i B_i \right)^{-1} dz \right) \hat{U}_0 d\xi \right|,$$

and for

$$|x| \geq d + \max_{\nu \in \mathbb{S}^{d-1}} \left\{ |\lambda|, \lambda \text{ eigenvalue of } \lambda A - \sum_{i=1}^{d} \nu_i B_i \right\}$$

one has

$$|\hat{U}_0^\rho(t, x)| \leq O(\rho^{-d}) e^{-d^2/(2\rho^2)}.$$  

$$\quad (1.1.16)$$
In particular, if $U_0$ is an initial data with support inside $B^d(0,\delta)$, then from the diagram

$$
\begin{array}{ccc}
U_0 & \xrightarrow{G(t)\ast} & U(t) = G(t) \ast U_0 \\
\downarrow & & \downarrow \\
\overline{U^\rho} \ast & = & \overline{U^\rho \ast} \ast U_0
\end{array}
$$

and the estimate consequence of (1.1.1)

$$
|U^\rho(t,x)| \leq O(1) e^{-d^2/(2\rho^2)} \quad \text{for} \quad |x| \geq \delta + \max_{\nu \in S^{d-1}} \left\{ |\lambda|, \lambda \text{ eigenvalue of } \lambda A - \sum_{i=1}^d \nu_i B_i \right\}
$$

it follows that

$$
U(t,x) = 0 \quad \text{for} \quad |x| \geq \delta + \max_{\nu \in S^{d-1}} \left\{ |\lambda|, \lambda \text{ eigenvalue of } \lambda A - \sum_{i=1}^d \nu_i B_i \right\}.
$$

Hence we deduce that

**Proposition 1.1.5.** The support of the distribution $G(t,x)$ is contained in the light cone

$$
\left\{ \frac{|x|}{t} \leq \max_{\nu \in S^{d-1}} \left\{ |\lambda|, \lambda \text{ eigenvalue of } \lambda A - \sum_{i=1}^d \nu_i B_i \right\} \right\}.
$$

### 1.2 Balance across surfaces

In many situations, the balance law of the form

$$
\text{div} \, F(U) = G \tag{1.2.1}
$$

is verified by $U \in L^\infty(\mathbb{R}^d, \mathbb{R}^N)$, $G \in M\text{loc}(\mathbb{R}^d, \mathbb{R}^N)$.

First of all, if $\mathcal{H}^{d-1}(\Omega) = 0$, then $G(\Omega) = 0$: in fact, for the closed ball $\text{clos} \, B(x,r)$ it follows

$$
|G(\text{clos} \, B(x,r))| = \lim_{\epsilon \to 0} G\left((1 - \text{dist}(x, \text{clos} \, B(x,r)))/\epsilon^+\right) = \lim_{\epsilon \to 0} \int F \cdot \nabla (1 - \text{dist}(x, \text{clos} \, B(x,r))/\epsilon^+) \, dx \leq \lim_{\epsilon \to 0} \|F\|_\infty \left(1 + \frac{\epsilon}{r}\right)^{d-1} = \|F\|_\infty r^{d-1}.
$$

Hence if $\mathcal{H}^{d-1}(\Omega) = 0$, then for all $\epsilon > 0$ we can find a countable covering of $\Omega$ made of closed balls $\text{clos} \, B(x_i, r_i)$ such that

$$
\sum_{i \in \mathbb{N}} r_i^{d-1} < \epsilon \quad \text{which implies} \quad |G(\Omega)| \leq \|F\|_\infty \epsilon.
$$

Hence $|G(\Omega)| = 0$.

Let now $\Omega$ be an open domain with smooth boundary, $\psi$ a smooth function and take the Lipschitz test function

$$
\phi^\varepsilon(x) := \psi(x) \left[1 - \left(1 - \text{dist}(x, \mathbb{R}^d \setminus \Omega)/\varepsilon^+\right)\right].
$$

Then the balance reads as

$$
-\int_{\mathbb{R}^d} \phi^\varepsilon \, dG = \int_{\mathbb{R}^d} F(U) \cdot D_x \phi^\varepsilon \, dx \leq \int_{\mathbb{R}^d} \left[1 - \left(1 - \text{dist}(x, \mathbb{R}^d \setminus \Omega)/\varepsilon^+\right)\right] F(U) \cdot D_x \psi \, dx + \int_{\mathbb{R}^d} \psi F(U) \cdot D_x \left[1 - \left(1 - \text{dist}(x, \mathbb{R}^d \setminus \Omega)/\varepsilon^+\right)\right] \, dx.
$$

When $\epsilon \to 0$ we obtain

$$
\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \phi^\varepsilon \, dG = G(\Omega),
$$

where $G(\Omega)$ is the distribution associated with $\Omega$. 

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \left[ 1 - (1 - \text{dist}(x, \mathbb{R}^d \setminus \Omega)/\epsilon) \right]^+ F(U) \cdot D_x \psi \, dx = \int_{\Omega} F(U) \cdot D_x \psi \, dx,
\]
and for the last term we can use the estimates
\[
\mathcal{L}^d \left( \{ D_x \left[ 1 - (1 - \text{dist}(x, \mathbb{R}^d \setminus \Omega)/\epsilon) \right]^+ \neq 0 \} \right) \leq (1 + \mathcal{O}(1)) \mathcal{H}^{d-1}(\partial \Omega) \epsilon,
\]
\[
\|D_x \left[ 1 - (1 - \text{dist}(x, \mathbb{R}^d \setminus \Omega)/\epsilon) \right]^+ \|_{\infty} \leq \frac{1}{\epsilon}
\]
to evaluate
\[
\left| \int_{\mathbb{R}^d} \psi F(U) \cdot D_x \left[ 1 - (1 - \text{dist}(x, \mathbb{R}^d \setminus \Omega)/\epsilon) \right]^+ \right| \leq \mathcal{O}(1) \|\psi\|_{\infty} \mathcal{H}^{d-1}(\partial \Omega).
\]
Hence up to subsequences
\[
\left[ F(U) \cdot D_x \left[ 1 - (1 - \text{dist}(x, \mathbb{R}^d \setminus \Omega)/\epsilon) \right]^+ \right] \mathcal{L}^d \rightarrow F \in \mathcal{M}(\partial \Omega, \mathbb{R}^N).
\]
Since however the other two parts of the equation converges, then this is an actual limit.

If we use a local system of coordinates such that \( \partial \Omega \) is flat, then a more precise estimate yields
\[
\left| \int_{\mathbb{R}^d} \psi F(U) \cdot D_x \left[ 1 - (1 - \text{dist}(x, \mathbb{R}^d \setminus \Omega)/\epsilon) \right]^+ \right| \leq \mathcal{O}(1) \int_{\partial \Omega} |\psi| \mathcal{H}^{d-1},
\]
which means that we can represent \( F \) as
\[
F = f(x; \Omega) \mathcal{H}^{d-1}.
\]
More generally, the formula holds for Lipschitz boundaries.

**Proposition 1.2.1.** For all \( \Omega \subset \mathbb{R}^d \) smooth, the vector field \( F \in L^\infty(\mathbb{R}^d) \) satisfying \((1.2.1)\) has an inner and outer trace in \( L^\infty(\mathcal{H}^{d-1}_{\text{ext}}) \).

1.3 Examples

1.3.1 Equations for material science

The balance laws for continuum thermodynamics in Eulerian coordinates are the following:

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho v) &= 0 & \text{conservation of mass} \\
\partial_t (\rho v) + \text{div}(\rho v \otimes v) &= \text{div} T + \rho b & \text{balance of momentum} \\
\partial_t \left( \rho \varepsilon + \frac{1}{2} \rho |v|^2 \right) + \text{div} \left( \left( \rho \varepsilon + \frac{1}{2} \rho |v|^2 \right) v \right) &= \text{div}(Tv + Q) + \rho b \cdot v + \rho r & \text{balance of energy}
\end{align*}
\]

where

\[
T \quad \text{is the stress matrix,} \\
b \quad \text{external force,} \\
\varepsilon \quad \text{the internal energy,} \\
Q \quad \text{the heat flux,} \\
r \quad \text{is the heat supply per unit mass.}
\]

To this \((d+2) \times (d+2)\)-system the additional dissipation of entropy \( s \) (per unit mass) is given by

\[
\partial_t (\rho s) + \text{div}(\rho sv) \geq \text{div} \left( \frac{Q}{\Theta} \right) + \rho \frac{r}{\Theta},
\]

where \( \Theta \) is the temperature. The opposite sign in the above entropy relation is that the thermodynamics entropy is concave.
Lagrangian coordinates

If we consider that the functions \((\rho, v, \Theta)\) are images in \((t, x)\) of a reference body \((\rho_0, X, \Theta)\), where \(X : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^D\) is a bi-Lipschitz homeomorphism (set \(Y = X^{-1}\)), then the balance of mass, momentum, energy and the dissipation of entropy can be obtained in Lagrangian coordinates. We will use the variable \((t, y) \in \mathbb{R}^+ \times \mathbb{R}\) for the reference configuration, and

\[(t, y) \to (t, x = X(t, y))\]

is the transformation of the reference configuration into the actual configuration.

With these variables, it is natural to define the quantities

1. the speed of the particle at \(y\) is \(v(t, y) := \dot{X}(t, y)\),

2. the deformation of the body

\[F(t, y) := D_y X(t, y), \quad \det F \geq a > 0\]

to avoid singularities in the body motion.

The equations in these coordinates becomes:

**conservation of mass:** in strong formulation,

\[
0 = \partial_t \rho + \text{div}(\rho v) \\
= (\partial_t + v \cdot D_x) \rho + (\text{div} v) \rho \\
= \dot{\rho} + \frac{\det(D_y X)}{\det(D_y X)} \rho \\
= \det(D_y X) (\dot{\rho} \det(D_y X) + \rho \det(D_y X)) \\
= \det(D_y X) (\rho \det(D_y X)) = \det(D_y X) \dot{\rho}_0
\]

\[
\dot{\rho}_0 = 0;
\]

**balance of momentum:** in the weak formulation, with \(\tilde{\phi} = \phi \circ X\),

\[
0 = \int_{\mathbb{R}^{d+1}} \left( \rho v \partial_t \phi(t,x) + \rho vv \cdot D_x \phi(t,x) - T \cdot D_x \phi(t,x) - \rho b \phi(t,x) \right) dx dt \\
= \int_{\mathbb{R}^{d+1}} \left( \rho v (\partial_t \phi) + \rho vv \cdot D_x \phi - T \cdot D_x \phi - \rho b \phi \right) \det(D_y X) dy dt \\
= \int_{\mathbb{R}^{d+1}} \left( \rho_0 \dot{\phi}(t, y) - (\det(D_y X) TY^T_x) \cdot D_y \phi(t, y) - \rho_0 b \phi(t, y) \right) dy dt
\]

\[
(\rho_0 v) = \text{div}_y S + \rho_0 b
\]

with \(S := \det(D_y X) TY^T_x\) being Piola-Kirchhoff stress;

**balance of energy:** any of the above methods

\[
(\rho_0 \varepsilon + \frac{1}{2} \rho_0 |v|^2) = \text{div}_y \left( Sv + \det(D_y X) QY_x^T \right) + \rho_0 b \cdot v + \rho_0 r,
\]

and we set \(Q := \det(D_y X) QY_x^T\);

**dissipation of entropy:** any of the above methods

\[
(\rho_0 s) \geq \text{div}_y \left( \frac{Q}{\Theta} \right) + \rho_0 \frac{r}{\Theta},
\]
If we check carefully (1.3.2), one observes that \( \phi \mapsto \phi \circ X \) is a homeomorphism from \( \text{Lip}(\mathbb{R}^d) \) into itself, and the change of variable is ok under the assumptions on \( X \). We have thus proved that the Lagrangian and Eulerian weak formulations are equivalent in our setting (bi-Lipschitz transformation).

In the case we are in \( \mathbb{R}^3 \) and we require the conservation of angular momentum, then the balance reads as

\[
(\rho_0 v \wedge X) = \text{div}_y (S \wedge X) + \rho_0 b \wedge X
\]

and by (1.3.3) one obtains

\[
0 = \sum_{i=1}^3 \partial_{y_i} X_j S_{ik} - \partial_{y_i} X_k S_{ij} \quad \implies \quad D_y XS = S^T D_y X^T
\]

i.e. \( T \) is symmetric.

**Compatibility conditions**

For smooth solutions it is required that the process is thermodynamically admissible, i.e. the entropy \( s \) increases:

\[
\begin{align*}
\rho_0 \dot{v} &= \text{div}_y S + \rho_0 b \\
\rho_0 \dot{\varepsilon} + \rho_0 v \cdot \dot{v} &= \text{div}_y S v + S : D_y v + \text{div}_y \mathcal{Q} + \rho_0 \dot{b} + \rho_0 \dot{r} \\
\rho_0 \dot{s} &= \text{div}_y \mathcal{Q} - \frac{2 \text{div}_y \Theta}{\Theta^2} + \rho_0 \frac{r}{\Theta} \\
\rho_0 \dot{s} &= \text{div}_y \mathcal{Q} - \frac{2 \cdot D_y \Theta}{\Theta^2} + \rho_0 \frac{r}{\Theta} \\
\rho_0 \dot{s} - \rho_0 \Theta \dot{s} &\leq S : D_y v + \frac{2 \cdot D_y \Theta}{\Theta}. \quad (1.3.4)
\end{align*}
\]

This is a relation among the constitutive relations.

Moreover, it is required the invariance for changes of coordinates, i.e. the functions \( \varepsilon, S, Q, s \) are independent on translations and rotations. In the reference coordinates, this means that they are invariant for rotations \( O \in \text{SO}^d \): if \( \varepsilon', S', Q', s' \) are the new functions, then

\[
\varepsilon' = \varepsilon, \quad S' = S O^T, \quad Q' = Q, \quad s' = s.
\]

Depending on the choice of \( \varepsilon, S, Q \) and \( s \) as functions of \( \rho_0, X, \Theta \), one obtains different systems of balance laws.

**Thermoelasticity**

It is assumed the following functional dependence:

\[
(\varepsilon, S, \Theta, \mathcal{Q}) = (\varepsilon, S, \Theta, \mathcal{Q})(D_y X, s, D_y \Theta), \quad \Theta \geq 0.
\]

We assume the medium homogeneous (invariant for translations w.r.t. \( y \)), otherwise a dependence on \( y \) may appear.

The entropy compatibility conditions (1.3.3) becomes

\[
0 \geq \rho_0 \dot{\varepsilon} - \rho_0 \Theta \dot{s} - S : D_y v - \frac{2 \cdot D_y \Theta}{\Theta}
\]

\[
= (\rho_0 D_{y X} \varepsilon - S) : D_y X + \rho_0 (\partial_s \varepsilon - \Theta) \dot{s} + \rho_0 D_{y \Theta} \varepsilon D_y \Theta - \frac{2 \cdot D_y \Theta}{\Theta}
\]

\[
\downarrow \quad \text{by varying } D_y X, s, D_y \Theta, D_y \Theta
\]

\[
S = \rho_0 D_{y X} \varepsilon, \quad \Theta = \partial_s \varepsilon, \quad D_{y \Theta} \varepsilon = 0, \quad 2 \cdot D_y \Theta \geq 0.
\]

We are thus left with two functions \( \varepsilon(D_y X, \Theta), \mathcal{Q}(D_y X, s, \Theta) \), and from material frame indifference the dependence on \( D_y X \) is invariant w.r.t. rotations.

Additional symmetries generates simples systems.
Equations of fluid dynamics

If we assume that $\varepsilon$, $\mathcal{D}$ are invariant for all matrices with positive determinant of the $y$-coordinates (i.e. the material is an isotropic fluid), then by changing local coordinates in order to have $D_y X = \det(D_y X)^{1/d} = (\rho_0/\rho)^{1/d}$ one obtains that

$$\varepsilon = \varepsilon(\det(D_y X)^{1/d}, s) = \varepsilon(\rho, s), \quad \mathcal{D} = \mathcal{D}^2(\det(D_y X)^{1/d}, s, D_x \Theta) = \mathcal{D}(\rho, s, D_x \Theta),$$

and then

$$T = \frac{\rho}{\rho_0} S \left( \frac{\rho_0}{\rho} \right)^{1/d} \left( \frac{\rho_0}{\rho} \right)^{1/d-1} (\rho_0 \partial_{D_y X} \varepsilon) = -\left( \rho^2 \partial_{\rho} \varepsilon \right) \frac{\Pi}{d} = -\rho A, \quad p = p(\rho, s) = \frac{1}{d} \rho^2 \partial_{\rho} \varepsilon(\rho, s) \text{ pressure.}$$

The rotations in the $x$-variable implies now that

$$Q(\rho, s, D_x \Theta) = O Q(\rho, s, OD_x \Theta) \quad Q = k(\rho, s, |D_x \Theta|) D_x \Theta$$

where $k$ is the thermal conductivity.

In the simplest case where

$$p = R_\rho \Theta, \quad \varepsilon = \varepsilon \Theta, \quad \gamma = 1 + \frac{R}{c},$$

$R$ being the universal gas constant, $c$ the specific heat and $\gamma$ is the adiabatic constant, one obtains

$$\Theta = \frac{\varepsilon}{c} = \partial_{\varepsilon} \implies \varepsilon = f(\rho) e^{s/c},$$

$$p = R_\rho f e^{s/c} = \rho^2 \partial_\varepsilon f \implies f(\rho) = c \rho^{\gamma - 1},$$

and finally

$$\varepsilon = c \rho^{\gamma - 1} e^{s/c}, \quad p = R_\rho \gamma e^{s/c}, \quad \Theta = \rho^{\gamma - 1} e^{s/c}.$$

To verify hyperbolicity, one observes that by the invariance w.r.t. rotations it is enough to consider the one dimensional system

$$\begin{cases}
\partial_t \rho + \partial_x (\rho v) = 0 \\
\partial_t (pv) + \partial_x (pv^2 + p) = 0 \\
\partial_t (\rho e + \rho v^2 / 2) + \partial_x (\rho e v + \rho v^3 / 2 + pv - Q) = 0
\end{cases}$$

so that by subtracting the first multiplied by $v$ to the second and the first multiplied by $\varepsilon$ and the second multiplied by $v$ to the third one gets

$$-\partial_x p = \partial_t (pv) + \partial_x (pv^2) - v (\partial_t \rho + \partial_x (pv))$$

$$\partial_x Q = \partial_t (\rho e + \rho v^2 / 2) + \partial_x (\rho e v + \rho v^3 / 2 + pv) - (\varepsilon + v^2 / 2)(\partial_t \rho + \partial_x (pv)) - v (\rho \partial_t v + \rho v \partial_x v + \partial_x p)$$

$$\partial_x Q = \rho \partial_t \varepsilon + \rho v \partial_x \varepsilon + \rho v^2 \partial_x v + p \partial_x v.$$

Thus in the coordinates $(\rho, v, s)$ we obtain

$$A = \begin{bmatrix}
1 & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & \rho
\end{bmatrix}, \quad B = \begin{bmatrix}
v & \rho & 0 \\
\partial_p \rho & pv & \partial_p v \\
0 & pv^2 + p & pv
\end{bmatrix},$$

because the heat transmission is a second order terms in the equation. The determinant is

$$\det \begin{bmatrix}
v - \lambda & \rho & 0 \\
0 & v - \lambda & \partial_p \rho / \rho \\
0 & 0 & v - \lambda
\end{bmatrix} = (\lambda - v)^3 - (\lambda - v) [\partial_x p(v^2 + p/\rho)/\rho + \partial_\rho p],$$

which has roots

$$\lambda = v, \quad \lambda = v \pm \sqrt{\partial_x p(v^2 + p/\rho)/\rho + \partial_\rho p}.$$ The system is thus strictly hyperbolic (i.e. the eigenvalues are distinct) if

$$\partial_x p(v^2 + p/\rho)/\rho + \partial_\rho p > 0.$$

In the degenerate case $\partial_x p(v^2 + p/\rho)/\rho + \partial_\rho p = 0$, then the eigenvectors are given by the system

$$\begin{bmatrix}
0 & \rho & 0 \\
\partial_p \rho / \rho & 0 & \partial_p \rho / \rho \\
0 & v^2 + p/\rho & 0
\end{bmatrix} R = 0,$$

which does not have 3 eigenvectors, i.e. it is not hyperbolic.
Isentropic gas dynamics

In the case one considers the isentropic process \( s = \bar{s} \), then

\[
\varepsilon = \frac{1}{\gamma - 1} \rho^{\gamma - 1}, \quad p = \rho^\gamma,
\]

and the system becomes the \( 2 \times 2 \) \( p \)-system

\[
\begin{cases}
\partial_t \rho + \text{div}(\rho v) = 0, \\
\partial_t (\rho v) + \text{div}(\rho v \times v) + D_x p = 0.
\end{cases}
\]

The requirement of hyperbolicity reduces to \( \partial_\rho p > 0 \).

Thermoviscoelasticity

In this case we assume that the internal energy \( \varepsilon \), the stress \( S \), the temperature \( \Theta \) and the heat flux \( \mathcal{Q} \) depends also on the time derivative of the deformation \( D_y X \), i.e. the spatial derivative of the speed \( D_y v \). The compatibility [13.3] is now

\[
0 \geq \rho_0 \dot{\varepsilon} - \rho_0 \Theta \dot{s} - S : D_y v - \mathcal{Q} \cdot D_y \Theta = (\rho_0 D_{D_y X} \varepsilon - S) : D_y v + \rho_0 D_{D_y v} \varepsilon : D_y \dot{v} + \rho_0 (\varepsilon_s - \Theta) \dot{s} + \rho_0 \varepsilon_{D_y} D_y \dot{\Theta} - \mathcal{Q} \cdot D_y \Theta \\
\text{by varying } D_y X, \dot{s}, D_y \Theta, D_y \Theta
\]

\[
S = \rho_0 D_{D_y X} \varepsilon + Z(D_y X, D_y v, s, D_y \Theta), \quad \Theta = \varepsilon_s, \quad \varepsilon_{D_y} = 0, \quad Z : D_y v + \frac{1}{\Theta} \mathcal{Q} \cdot D_y \Theta \geq 0.
\]

In the case of thermoelastic fluids in \( N = 3 \), the invariance for all matrices with determinant 1 and for changes of coordinates gives that the constitutive relations

\[
\varepsilon = \varepsilon(\rho, s), \quad T = -p(\rho, s) I + \lambda(\rho, s) \text{tr} \left( \frac{D_y v + D_y v^T}{2} \right) + 2\mu(\rho, s) \frac{D_y v + D_y v^T}{2}, \quad Q = k(\rho, s) D_x \Theta,
\]

and the entropy increase implies

\[
\lambda \left( \text{tr} \left( \frac{D_y v + D_y v^T}{2} \right) \right)^2 + 2\text{tr} \left( \frac{D_y v + D_y v^T}{2} \right)^2 + \frac{k}{\Theta} |D_x \Theta|^2 \geq 0, \\\n\text{by varying } \lambda, \mu, k \geq 0.
\]
1.4 Exercises

1. Prove the blow up for the linear system if it is not hyperbolic.

2. Consider the Lagrangian form of the $p$-system in one space dimension,

\[
\begin{align*}
\partial_t \rho + \partial_x u &= 0, \\
\partial_t u + \partial_x p(\rho) &= 0.
\end{align*}
\]

Prove that the system is hyperbolic or elliptic at $(\bar{\rho}, \bar{u})$ depending on the sign of $p'(\bar{\rho})$.

3. Prove that a scalar balance laws (i.e. $u \in \mathbb{R}$) has a large family of entropies.

4. Consider the one dimensional scalar system

\[
\partial_t u + \partial_x \left( \frac{u^n}{n} \right) = 0.
\]

Prove that the Rankine-Hugoniot conditions reads as

\[
\sigma = \frac{1}{n} \frac{(u^+)^n - (u^-)^n}{u^+ - u^-}, \quad \sigma \text{ speed of the jump.}
\]

In particular deduce that the jumps for different $n$ are incompatible, even if the compatibility condition \[1.1.8\] holds.

5. Prove that by \[1.1.10\] the following quantity is a measure if $\eta$ is a convex entropy and $U$ is a dissipative solution:

\[
\partial_t \eta + \text{div} \partial_x q - G'(t, x, U).
\]

6. Write explicitly the Green kernel for the homogeneous hyperbolic system in one space dimension

\[
\partial_t U + A \partial_x U = 0.
\]

7. Prove \[1.1.15\] and deduce the bound

\[
\exists C > 0 \forall \xi (|\hat{G}(t, \xi)| \leq C).
\]

8. A weak shock is a discontinuity in the derivative of $U$ across a surface for the balance law

\[
\text{div } F((t, x, U) = G(t, x, U).
\]

Verify that the jump conditions becomes

\[
\sum_i D_\omega(n_i F_i) (\partial_n U^+ - \partial_n U^-) = 0.
\]

(Hint: pass to the limit to the quasilinear form of the PDE on both sides of the surface.)

9. Prove that for the linear equation \[1.1.1\] with $C = 0$ (homogeneous equation) the jump conditions reduce to the eigenvalue/eigenvector equation \[1.1.3\].

10. Prove \[1.2.2\].