Regularity of solutions to Hamilton-Jacobi, Hyperbolic Conservation Laws and Monge transportation problem

L. Caravenna, C. De Lellis, M. Gloyer, R. Robyr, S. B.

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Outline

Structure of solutions to Hamilton-Jacobi

Consider the Hamilton-Jacobi equation

$$u_t + H(\nabla u) = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^m,$$

with uniformly convex Hamiltonian H and Lipschitz initial data.

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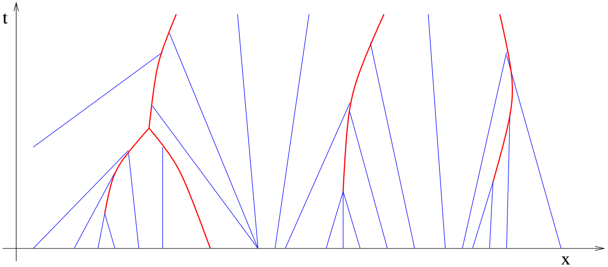

Structure of solutions to Hyperbolic conservation laws

For strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \ u \in \mathbb{R}^m,$$

one can decompose the derivative u_x in waves

$$u_x = \sum_i v_i \tilde{r}_i, \quad u_t = \sum_i w_i \tilde{r}_i,$$

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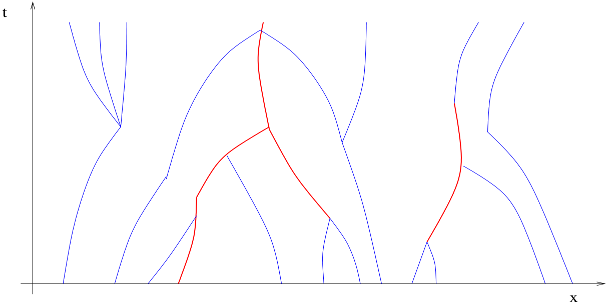
3. the continuous part of v_i satisfies the equation

$$(v_i)_t + (\lambda_i v_i)_x = J_i, \quad J_i \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}).$$

Hence each v_i should have a similar structure to the second derivative of the solution to a Hamilton-Jacobi equation, but the functions λ_i depends on u and there is a source J_i.



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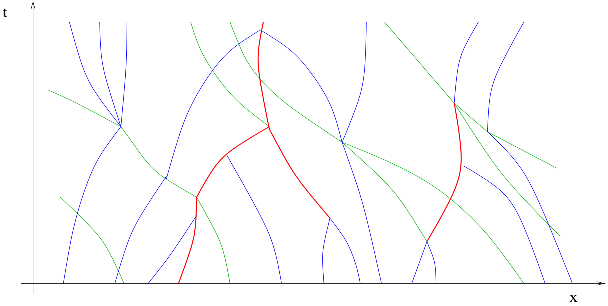

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The first question is an easy application of a well known rectifiability criteria.

Outline

Area estimate

The solution to HJ equation

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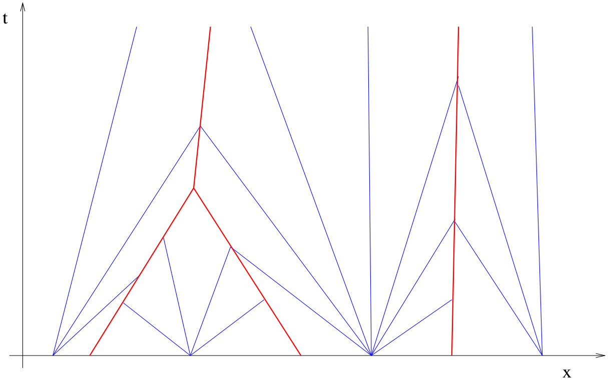
In particular, it is uniformly approximated by the sequence of functions

$$u_n(t,x) = \min \left\{ u(0,y) + tL\left(\frac{x-y}{t}\right), y \in \{y_1, \dots, y_n\} \right\}. \quad (1)$$

These solutions have a very simple structure:



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By (1), one can show that this is the worst case, i.e.

$$c(t,s,x) \begin{cases} \leq \left(\frac{t-s}{t}\right)^m & t \leq s \\ \geq \left(\frac{t-s}{t}\right)^m & t \geq s \end{cases}$$

Since along optimal rays we have the dual solution

$$u(s,x) = \max \left\{ u(t,y) - (t-s) L\left(\frac{y-x}{t-s}\right) \right\},$$

we obtain the bound on the Jacobian

$$\min \left\{ \left(\frac{t-s}{t}\right)^m, \left(\frac{T-s}{T-t}\right)^m \right\} \le c \le \max \left\{ \left(\frac{t-s}{t}\right)^m, \left(\frac{T-s}{T-t}\right)^m \right\},$$

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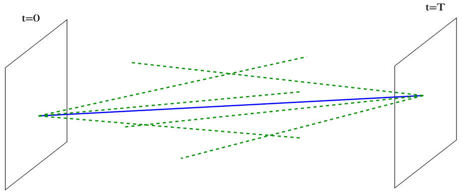

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The above estimate implies that the change of variable

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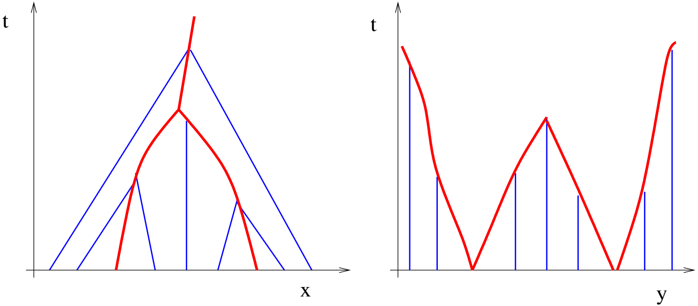

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$$\mathcal{L}^{d+1}_{\mathbb{R}^+\times\mathbb{R}^m} = \int c(t, y)dtm(dy),$$

i.e. $\forall\phi \in C^c(\mathbb{R}^+ \times \mathbb{R}^m)$

$$\int \phi\mathcal{L}^{d+1} = \int \bigg(\int \phi(t, y + td(y))c(t, y)dt \bigg) m(dy).$$

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Remark. The proof depends only on the convexity of H.

Riemannian manifolds

Instead of the cost $\|\cdot\|$, we can use the distance cost on a Riemannian manifold (M, m)

$$d(x,y) = \inf\left\{ \int_0^1 \sqrt{m_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)}dt, \gamma(0) = x, \gamma(1) = y \right\}.$$

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In the smooth case, the study of the evolution of the Jacobian J reduces to the system

$$\begin{cases} \dot J &= UJ \\ \dot U &= -U^2 - R(\dot\gamma, \dot\gamma) \end{cases}$$

where R is a symmetric tensor whose trace is Ric.

In particular, if $\dot{J}J^{-1}(0)$ is symmetric (ok for optimal transportation where a potential is present) we obtain that $\det J$ satisfies

$$\frac{d^2}{dt^2}\log\det J \leq -\mathrm{Ric}(\dot{\gamma},\dot{\gamma}),$$

so that if the Ricci curvature is bounded from below, the function

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Hence if the two measures μ, ν are absolutely continuous, the Jacobian never vanish and we obtain again an absolutely continuous disintegration along rays:

$$\frac{d^2}{dt^2}\log c(t,y) = \frac{d^2}{dt^2}\log\left(\frac{1}{\det J}\right) \geq \mathrm{Ric}(\dot\gamma,\dot\gamma).$$

General geodesic spaces

The semiconvexity estimate is stable under a weak convergence notion of metric spaces. Recall that a geodesic space is a metric space (X, d_L) such that

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Definition

(X, d, d_L, η) is a *measure geodesic space* if (X, d) is a Polish space, (X, d_L) is a geodesic space with $d_L : X \times X \to [0, +\infty]$ d-l.s.c., and η is a probability measure on (X, d).

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It can be shown that under this convergence Ricci bounds are preserved.

Example. The optimal transport in Wiener space is equivalent to the optimal transport in

$$\left(\ell^2, \| \cdot \|_{\ell^2}, \| \cdot \|_{h^1}, \prod_i \frac{i}{\sqrt{\pi}} e^{-i^2 x_i^2} dx_i \right),$$

and it is possible to show that the curvature is 1 (Sturm).

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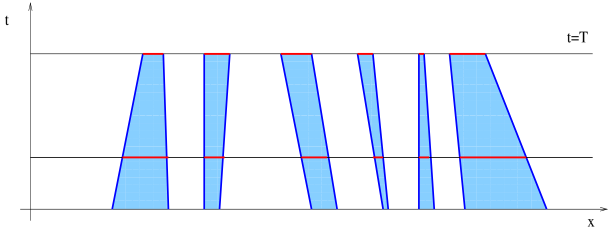

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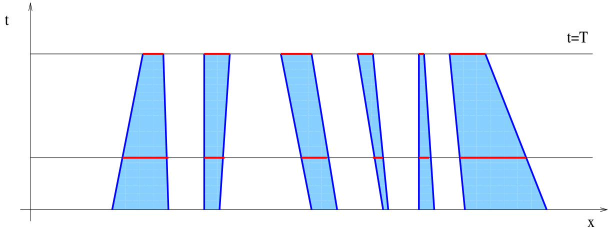

In particular, if there is a Cantor part (hence single rays), the area is strictly positive.

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Since $\mathrm{div} d(t,x) = \mathrm{tr}(D^2 H(\nabla u) D^2 u)$, $D^2 H \geq c\mathbb{I}$ and $D^2 u \leq 0$ in the Cantorian part, we obtain that $\mathrm{tr}(D^2 u)$ has not Cantor parts, hence $D^2 u$ has not Cantor parts.

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2. equation for v_i^s:

$$(v_i^s)_t + (\tilde{\lambda}_i v_i^s)_x = J_i^s,$$

$$|J_i^s|\big((s,t] \times \mathbb{R}\big) \leq \mathrm{Tot.Var.}(v_i - v_i^s(s)) - \mathrm{Tot.Var.}(v_i - v_i^s(t)) \\ + C\big(Q(s) - Q(t)\big).$$

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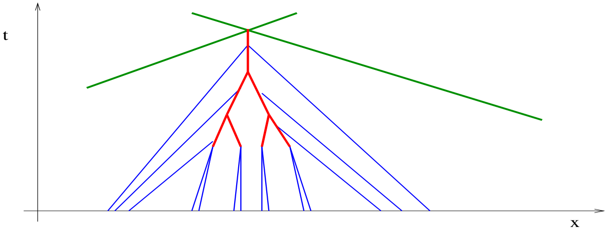

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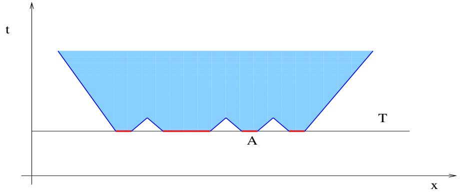

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Outline

L. Ambrosio and C. De Lellis.
A note on admissible solutions of 1d scalar conservation laws and 2d Hamilton-Jacobi equations.

S.B. and L. Caravenna.
SBV regularity for hyperbolic systems.

S.B. and F. Cavalletti.
The Monge problem for distance cost in geodesic spaces

S.B., C. De Lellis and R. Robyr.
SBV regularity for Hamilton-Jacobi equations in $\mathbb{R}^n$.

S.B. and M. Gloyer.
Euler equation for a singular variational problem.