TRANSPORT RAYS AND APPLICATIONS TO HAMILTON-JACOBI EQUATIONS

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1. Introduction

2. Settings

We consider a σ-generated probability space (X, Σ, μ) and a partition $X = \cup_{\alpha \in A} X_\alpha$, where $A = X/\sim$ is the quotient space, and $h : X \to A$ the quotient map. We give to A the structure of a probability space by introducing the σ-algebra $\mathcal{A} = h_\sharp \Omega$, where Ω are the saturated sets in Σ (unions of fibers of h), and $m = h_\sharp \mu$ the image measure such that $m(S) = \mu(h^{-1}(S))$.

Remark 2.1. $\mathcal{A}$ is the largest σ-algebra such that h is measurable.

The following example shows that even though Σ is σ-generated, $\mathcal{A}$ in general is not.

Example 2.2. Consider in $([0,1], \mathcal{B})$ (Borel) the equivalence relation

$$x \sim y \quad \text{iff} \quad x - y = 0 \mod \alpha,$$

for some $\alpha \in [0,1]$. If $\alpha = p/q$, with $p, q \in \mathbb{N}$ relatively prime, then we can take

$$(A, \mathcal{A}) = \left(\left[0, \frac{1}{q}\right], \mathcal{B} \right).$$

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then A is a Vitali set. If $\mu = \mathcal{L}^1|_{[0,1]}$, then $m = h_\sharp \mathcal{L}^1|_{[0,1]}$ has only sets of either full or negligible measure. Assume by contradiction that $\{a_n\}_{n \in \mathbb{N}}$ generates $\mathcal{A}$. Since $h^{-1}(x) = \{x + n\alpha \mod 1 : n \in \mathbb{N}\} \in \mathcal{B}$, it follows that each $a \in A$ belongs to a generating set of measure 0. But this leads to a contradiction:

$$1 = m(A) = m \left(\bigcup_{m(a_n)=0} a_n \right) \leq \sum_{m(a_n)=0} m(a_n) = 0.$$

We now define the measure algebra $(\hat{\mathcal{A}}, \hat{m})$ by the following equivalence relation on $\mathcal{A}$:

$$a_1 \sim a_2 \quad \text{iff} \quad m(a_1 \triangle a_2) = 0.$$

It is easy to check that $\hat{\mathcal{A}}$ is a σ-algebra and $\hat{m}$ is a measure on $\hat{\mathcal{A}}$.

Proposition 2.3. $(\hat{\mathcal{A}}, \hat{m})$ *is σ-generated.*

$\hat{\mathcal{A}}$ is isomorphic to a sub-σ-algebra of Σ.

Remark 2.4. More generally, if (X, Σ, μ) is generated by a family of cardinality ω_α, then each sub-σ-algebra $\mathcal{A} \subset \Sigma$ is essentially generated by a family of sets of cardinality ω_α or less.

This is a particular case of a deep result, *Maharam's Theorem* ([7], 332T(b)), which describes isomorphisms between probability spaces: if $(\hat{\Sigma}, \hat{\mu})$ is a probability algebra, then

$$(\hat{\Sigma}, \hat{\mu}) \simeq \prod_i c_i \left[\bigotimes_{J_i} \{0,1\} \right], \quad \sum_i c_i = 1,$$

where $\bigotimes_{J_i} \{0,1\}$ is the measure space obtained by throwing the dice J_i times.

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3. Disintegration

Definition 3.1. We introduce the following relation on A:

$$a_1 \sim a_2 \text{ iff the following holds:}$$
$$\text{for all } \hat{A} \in \hat{\mathcal{A}}, a_1 \in \hat{A} \text{ iff } a_2 \in \hat{A}.$$

The equivalence classes of this relation are the *atoms* of the measure m. In particular, we can define the measure space

$$(\Lambda = A/\sim, \hat{\mathcal{A}}, m).$$

The σ-algebra is isomorphic to the σ-generated $\hat{\mathcal{A}}$ constructed in the previous section.

Definition 3.2. The *disintegration* of the measure μ with respect to the partition $X = \bigcup_\alpha X_\alpha$ is a map

$$A \to P(X), \quad \alpha \mapsto \mu_\alpha,$$

where $P(X)$ is the class of probability measures on X, such that the following properties hold:

(1) for all $B \in \Sigma$, the map $\alpha \mapsto \mu_\alpha(B)$ is m-measurable;
(2) for all $B \in \Sigma$, $A \in \mathcal{A}$,
$$\mu(B \cap h^{-1}(A)) = \int_A \mu_\alpha(B) \,\mathrm{d}m(\alpha).$$

It is unique if μ_α is determined m-a. e.

Remark 3.3. (1) Since we are not requiring the elements of the partition X_α to be measurable, in general $\mu_\alpha(X_\alpha) \neq 1$ for those X_α which are measurable. In this case we say that the disintegration is not *strongly consistent* with h.
(2) For general spaces which are not σ-generated, sometimes a disintegration nonetheless exists, but in general there is no uniqueness.
(3) The disintegration formula can easily be extended to measurable functions:
$$\int_X f \,\mathrm{d}\mu = \int_A \left(\int_X f \,\mathrm{d}\mu_\alpha \right) \mathrm{d}m(\alpha).$$

We now state the general disintegration theorem.

Theorem 3.4. *Assume that (X, Σ, μ) is a σ-generated probability space, $X = \bigcup_{\alpha \in A} X_\alpha$ a partition of X, $h : X \to A$ the quotient map, and $(A, \mathcal{A}, m)$ the quotient measure space. Then the following holds:*

(1) *There is a unique disintegration $\alpha \mapsto \mu_\alpha$.*
(2) *If $(\Lambda, \hat{\mathcal{A}}, m)$ is the σ-generated algebra equivalent to $(A, \mathcal{A}, m)$, and $p : A \to \Lambda$ the quotient map, then the sets*
$$X_\lambda = (p \circ h)^{-1}(\lambda)$$
are μ-measurable, the disintegration
$$\mu = \int_\Lambda \mu_\lambda \,\mathrm{d}m(\lambda)$$
is strongly consistent $p \circ h$, and
$$\mu_\alpha = \mu_{p(\alpha)} \quad \text{for } m\text{-a. e. } \alpha.$$

Definition 3.5. $R \subset X$ is a *rooting set* for $X = \bigcup_{\alpha \in A} X_\alpha$, if for each $\alpha \in A$ there exists exactly one $x \in R \cap X_\alpha$.

R is a *μ-rooting set* if there exists a set $\Gamma \subset X$ of full μ-measure such that R is a rooting set for

$$\Gamma = \bigcup_{\alpha \in A} \Gamma_\alpha = \bigcup_{\alpha \in A} \Gamma \cap X_\alpha.$$

Proposition 3.6. *If $\mu = \int_A \mu_\alpha \,\mathrm{d}m(\alpha)$ is srongly consistent with the quotient map, then there exists a Borel μ-rooting set.*

Example 3.7. Consider again

$$x \sim y \quad \text{iff} \quad x - y = 0 \mod \alpha.$$

If $\alpha = p/q$ with p, q relatively prime, then the rooting set is $[0, 1/q)$, so that we know that the disintegration is strongly consistent with h. One can check that

$$\mu = \int_0^{\frac{1}{q}} \mathrm{d}\alpha \sum_{n=0}^{q-1} \delta\left(x - \alpha - \frac{n}{q}\right).$$

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then we know that

$$(\Lambda, \hat{\mathcal{A}}, m) \simeq \Big(\{\lambda\}, \{\emptyset, \{\lambda\}\}, \delta_\lambda\Big),$$

so that

$$\mu = \int \mathrm{d}m\, \mu.$$

4. Hamilton-Jacobi equation and monotonicity

In the following we consider the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(\nabla u) = 0, \\ u(0, x) = \bar{u}(x), \end{cases}$$

where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$ and $\bar{u} \in L^\infty(\mathbb{R}^d)$. We assume that $H : \mathbb{R}^d \to \mathbb{R}$ is C^1 and convex. We denote by $L = H^*$ the Legendre transform of H and assume that it has at least linear growth,

$$L(x) \geq \frac{1}{c}(|x| - c),$$

and is locally Lipschitz. By the properties of the Legendre transform, L is strictly convex.

For example, we can consider $H(x) = L(x) = \frac{1}{2}|x|^2$.

The viscosity solution is given explicitly by

$$u(t, x) = \inf\left\{\bar{u}(y) + tL\left(\frac{x - y}{t}\right) : y \in \mathbb{R}^d\right\}.$$

Remark 4.1. The following properties can easily be checked:

(1) Finite speed of propagation: $u(t, x)$ depends only on the values of $\bar{u}(y)$ for $|x - y| \leq c$.
(2) Uniform Lipschitz continuity: For fixed y, the function $\bar{u}(y) + tL((x-y)/t)$ is uniformly Lipschitz in x for $|x - y| \leq c$. Hence $u(t, x)$ is uniformly Lipschitz in x for all $t > 0$.
(3) Semigroup property: For $t > s > 0$, we have that

$$u(t, x) = \min\left\{u(s, z) + (t - s)L\left(\frac{x - z}{t - s}\right) : z \in \mathbb{R}^d\right\}.$$

(4) If $D^2 H \in [1/c, c]\mathbb{I}$, then $D^2 L \in [1/c, c]\mathbb{I}$, and thus $u(t, x) - c|x|^2/2t$ is concave in x. Hence $u(t, \cdot)$ is quasi-concave for $t > 0$.

We can solve the backward problem

$$\begin{cases} v_t + H(\nabla v) = 0, \\ v(1, x) = \bar{u}(1, x). \end{cases}$$

Then v has the same properties as above for $t < 1$, and the following duality holds:

$$u(1, x) = \min\Big\{v(0, y) + L(x - y)\Big\},$$
$$v(0, y) = \max\Big\{u(1, x) - L(x - y)\Big\}.$$

We say that $u(1, x)$ and $v(0, y)$ are *L-conjugate functions.*

Definition 4.2. The couple $[y, x] \in \mathbb{R}^d \times \mathbb{R}^d$ such that

$$u(1, x) = v(0, y) + L(x - y)$$

is called an *optimal couple*. The corresponding segment

$$\Big\{(1 + t)y + tx : 0 \leq t \leq 1\Big\}$$

is called an *optimal ray*.

Remark 4.3. Due to the strict convexity of L, two rays cannot intersect anywhere except at their end points.

Remark 4.4. In general, the duality of $u(1, x)$ and $v(0, y)$ does *not* imply that $u(t, x) = v(t, x)$ for $0 < t < 1$.

Example 4.5. In the case $H(x) = L(x) = |x|^2/2$, the optimal rays are the graph of the maximal monotone operator

$$x \mapsto y(x) = \partial_x \left(\frac{|x|^2}{2} - u(1, x) \right).$$

It follows from Minty's Theorem ([2], p.142) that the map

$$x \mapsto tx + (1 - t)y(x)$$

is surjective. Since the rays do not intersect, it follows that for all $0 < t < 1$ and all $z \in \mathbb{R}^d$, there exists a unique optimal couple $[y, x]$ such that

$$z = (1 - t)y + tx.$$

Hence, using the explicit formula for the optimal ray, we obtain

$$u(t, x) = v(t, x) \qquad \text{for all } t \in [0, 1],\ x \in \mathbb{R}^d.$$

Hence the solution is both $(2t)^{-1}$-concave and $(2(1 - t))^{-1}$-convex, thus $u \in C^{1,1}$.

The set $y(x)$ is convex, therefore the rays departing from a given point form a convex set.

The example can be generalized to the case $H(x) = 1/2\langle x, Ax\rangle$ by a linear change of variable.

Example 4.6. A more difficult case is $\mathrm{D}^2 H, \mathrm{D}^2 L \in [1/c, c]\mathbb{I}$. One can use that $v(0, y) + |y|^2/(2c)$ is convex to compute the optimal rays for $t \ll 1$:

$$\begin{aligned} u(t, z) &= \min \left\{ v(0, y) + tL\left(\frac{z - y}{t}\right) \right\} \\ &= \min \left\{ v(0, y) + \frac{|y|^2}{2c} + tL\left(\frac{z - y}{t}\right) - \frac{|y|^2}{2c} \right\}. \end{aligned}$$

The last two terms together are convex for $t < c^{-2}$, so that the minimizer is given by

$$\nabla L\left(\frac{z - y}{t}\right) = \partial^- v(0, y),$$

where $\partial^- v(0, y)$ denotes the subdifferential of v at y:

$$\partial^- f(x) = \left\{ p : \liminf \frac{f(x + h) + f(x) - ph}{|h|} \geq 0 \right\}.$$

Similarly, we can introduce the superdifferential

$$\partial^+ f(x) = \left\{ p : \limsup \frac{f(x + h) + f(x) - ph}{|h|} \leq 0 \right\}.$$

These sets are convex, but in general they are empty. We thus obtain

$$z = y + t\nabla H(\partial^- v(0, y)) \qquad \text{for } 0 \leq t \ll 1.$$

The strict convexity implies that the map $z \mapsto y$ is single-valued and Lipschitz. For $t \ll 1$, the projections of the rays $z(y)$ are still convex sets. However, in general the rays do not extend to $t = 1$.

Example 4.7. Taking

$$L(x) = 3|x| + x_1|x|,$$

$$u(1,x) = \min\left\{L\left(x - \binom{0}{1}\right), L\left(x - \binom{0}{1}\right)\right\},$$

and computing the corresponding $v(0,y)$, one can prove that there is a gap between the rays, i. e. there exists a region inside of which one has

$$u(t,x) > v(t,x).$$

Example 4.8. In the general case, when L is only strictly convex, there is no notion of subdifferential or superdifferential. One then has no quasi-concavity or quasi-convexity, hence no BV regularity. The graph of the optimal rays $[y,x]$ is in general full of holes, and there is no interval where one could prove $C^{1,1}$ regularity.

5. Regularity properties of L-conjugate functions and optimal rays

We consider a pair of L-conjugate functions $u, v \in L^\infty \cap \mathrm{Lip}$,

$$u(x) = \min\left\{v(y) + L(x-y) : y \in \mathbb{R}^d\right\},$$

$$v(y) = \max\left\{u(x) - L(x-y) : x \in \mathbb{R}^d\right\},$$

where as before $L : \mathbb{R}^d \to \mathbb{R}$ is strictly convex and has at least linear growth.

We define the set

$$F = \left\{[y,x] \in \mathbb{R}^d \times \mathbb{R}^d : [x,y] \text{ optimal couple }\right\}$$

and its projection

$$F(t) = \left\{z : z = (1-t)y + tx \text{ for some } [y,x] \in F\right\}.$$

By the duality, we know that $F(0) = F(1) = \mathbb{R}^d$. On the set $F(t)$ we define the vector field

$$\begin{aligned} \mathbf{p}_t(z) &= (1, p_t(z)) \\ &= (1, x-y), \qquad \text{where } [y,x] \in F \text{ such that } z = (1-t)y + tx. \end{aligned}$$

We also introduce the set-valued functions

$$y(x) = \left\{y : [y,x] \in F\right\},$$

$$x(y) = \left\{x : [y,x] \in F\right\}.$$

From the fact that u and v are L-conjugate, we obtain the following lemma.

Lemma 5.1. *The set F and its projections $F(t)$ are closed, and the set-valued functions $x(y)$, $y(x)$ have locally compact images.*

In particular $y(x)$, $x(y)$ are Borel measurable, because the inverse of compact sets is compact.

Example 5.2. When ∇u, ∇v exist, then they are related to p by

$$\nabla v(y) = \partial L(p_0(y)),$$

$$\nabla u(x) = \partial L(p_1(x)).$$

If we consider for example

$$L(x) = \frac{1}{2}|x|^2 + |x_1|,$$

$$u(x) = \min\left\{L\left(x - \binom{1}{0}\right), L\left(x - \binom{-1}{0}\right)\right\},$$

then it is easy to check that $u(0)$ has no subdifferential or superdifferential.

5.1. **Rectifiability property of jumps.** Define the sets

$$J_m = \Big\{ x \in \mathbb{R}^d : \text{ there exist } y_1, y_2 \in y(x) \text{ such that } |y_1 - y_2| \geq \frac{1}{m} \Big\}$$

and

$$J = \bigcup_{m \in \mathbb{N}} J_m.$$

Lemma 5.3. *J_m is closed and countably $(d-1)$-rectifiable, i. e. it can be covered with a countable number of images of Lipschitz functions $\mathbb{R}^{d-1} \to \mathbb{R}^d$.*

The lemma can be proved by applying the rectifiability criterion [1], Theorem 2.61. The proof is analogous to Lemma 3.4 in [4].

In a similar way we obtain the following proposition:

Proposition 5.4. *The set*

$$\begin{aligned} J^k &= \bigcup_{m \in \mathbb{N}} J^k_m \\ &= \bigcup_{m \in \mathbb{N}} \Big\{ x \in \mathbb{R}^d : \text{ there exist } y_1, \dots, y_{k+1} \in y(x) \text{ s. t. } B\Big(0, \frac{1}{m}\Big) \subset \mathrm{co}\{y_1, \dots, y_{k+1}\} \Big\} \end{aligned}$$

is countably $(d-k)$-rectifiable, i. e. it can be covered with a countable number of images of Lipschitz functions $\mathbb{R}^{d-k} \to \mathbb{R}^d$.

5.2. **Some approximations.** To prove the estimates on the vector field $\mathbf{p}_t$ or p_t, we need an approximation technique. The following proposition will be an essential tool.

Proposition 5.5. *Assume that*

$$\bar{u}_n(y) \to \bar{u}(y), \qquad L_n(x) \to L(x)$$

locally uniformly, and that we have the uniform bound

$$L_n(x) \geq \frac{1}{c}(|x| - c) \quad \text{for all } n \in \mathbb{N},$$

where c does not depend on n.

Then the conjugate functions $u_n(1, x)$, $v_n(0, y)$ converge uniformly to $u(1, x)$, $v(0, y)$, and the graph F_n converges locally in Hausdorff distance fo a closed subset of F.

5.3. **Fundamental example.** Let $\{y_i : i \in \mathbb{N}\}$ be a dense sequence in $\mathbb{R}^d$, and define

$$u_N(x) = \min \Big\{ u(y_i) + L(x - y_i) : i = 1, \dots, N \Big\}.$$

We can split $\mathbb{R}^d$ into at most N open regions Ω_i (Voronoi-like cells), inside which we have

$$u_N(x) = u(y_i) + L(x - y_i), \quad x \in \Omega_i,$$

together with the negligible set

$$\bigcup_{i \neq j} \Big(\bar{\Omega}_i \cap \bar{\Omega}_j \Big).$$

The boundary of each region is Lipschitz, and inside each region the corresponding directional field p_N is given by

$$p_N(x) = x - y_i, \quad x \in \Omega_i.$$

5.4. **Divergence estimate.** In the points x where the field $p(x)$ is single valued, the approximate $p_n(x)$ converges to $p(x)$. This implies that

$$p_n(x) \to p(x) \quad \mathcal{L}^d\text{-a.e.}$$

Using this fact we can prove the following proposition:

Proposition 5.6. *$\operatorname{div} p$ is a locally bounded measure satisfying*

$$\operatorname{div} p - d\mathcal{L}^d \leq 0.$$

Proof. The approximating fields satisfy the bound, thus by the above convergence we get the bound for $\operatorname{div} p$. It is a measure because positive distributions are measures. □

6. Jacobian estimates

As in the previous section, we take a dense sequence $\{y_i : i \in \mathbb{N}\}$ in $\mathbb{R}^d$. For a fixed time $t \in (0,1)$, we consider the approximation with finitely many points at $t = 0$,

$$u_N(t,x) = \min \left\{ u(0,y_i) + tL\left(\frac{x - y_i}{t}\right) : i = 1, \dots, N \right\}.$$

Take a compact subset $A(t) \subset F(t)$. We denote by $A_N(s)$ the push-forward of the set $A(t)$ along the approximating rays $p_N(t,x)$. Then we get

$$\mathcal{L}^d(A_N(s)) \geq \left(\frac{s}{t}\right)^d \mathcal{L}^d(A(t)) \quad \text{for } s \leq t.$$

Up to a set of measure ϵ, we can assume that $p_N(t)$, $p(t)$ are continuous and $p_N(t) \to p(t)$ uniformly on $A(t)$. Then $A_N(s)$ is compact for $s \leq t$, and it converges to $A(s)$ in Hausdorff distance. Since $\mathcal{L}^d$ is upper semicontinuous with respect to the Hausdorff distance, this implies that

$$\mathcal{L}^d(A(s)) \geq \left(\frac{s}{t}\right)^d \mathcal{L}^d(A(t)) \quad \text{for } s \leq t.$$

By repeating the above approximation with finitely many points at $t = 1$, one obtains the corresponding estimate

$$\mathcal{L}^d(A(s)) \geq \left(\frac{1-s}{1-t}\right)^d \mathcal{L}^d(A(t)) \quad \text{for } s \geq t.$$

We thus obtain the following estimate for the push-forward of the Lebesgue measure.

Lemma 6.1. *Let*

$$\mu(s) = [z + (s-t)p]_\sharp \mathcal{L}^d.$$

Then

$$\mu(s) = c(s,t,z)\mathcal{L}^d|_{F(t)},$$

with

$$c(s,t,z) \in \left[\left(\frac{s}{t}\right)^d, \left(\frac{1-s}{1-t}\right)^d \right] \quad \textit{for } s \leq t,$$

$$c(s,t,z) \in \left[\left(\frac{1-t}{1-s}\right)^d, \left(\frac{t}{s}\right)^d \right] \quad \textit{for } t \leq s.$$

Proof. By the previous estimates, we have for $s \geq t$,

$$\left(\frac{1-t}{1-s}\right)^d \mathcal{L}^d(A(s)) \leq \mathcal{L}^d(A(t)) \leq \left(\frac{t}{s}\right)^d \mathcal{L}^d(A(s)).$$

By the definition of the image measure,

$$\mathcal{L}^d(A(t)) = \mu(s)(A(s)).$$

Thus the result follows. □

The function $c(s,t,z)$ is the Jacobian of the transformation.

6.1. **Disintegration of the Lebesgue measure.** Using Lemma 6.1, we now apply the Fubini-Tonelli theorem to a measurable set $A = \bigcup_t \{t\} \times A(t) \subset \bigcup_t \{t\} \times F(t)$ to obtain

$$\begin{aligned} \int_A \mathrm{d}t \times \mathcal{L}^d &= \int \mathrm{d}t \int_{A(t)} \mathcal{L}^d \\ &= \int \mathrm{d}t \int_{A(t,s)} c(t,s) \mathcal{L}^d \\ &= \int \mathcal{L}^d \int \mathrm{d}t \, c(t,s) \chi_{A(t,s)}, \end{aligned}$$

where $A(t,s)$ is the image of the set $A(t)$ by

$$A(t,s) = (z + (s-t)p(z))(A(t)).$$

Remark 6.2. In the new coordinates, $\mathrm{d}t\, c(t,s)$ is concentrated on a single optimal ray.

Since the rays do not intersect, we can disintegrate the Lebesgue measure along rays,

$$\mathcal{L}^d \times \mathrm{d}t|_F = \int \mathrm{d}m(\alpha) \mu_\alpha.$$

We can parameterize the rays by the points of the plane $t = 1/2$, then the support of μ_α is the optimal ray passing through $\alpha \in F(1/2)$. Using the previous formula, we obtain the following theorem:

Theorem 6.3. *The disintegration of the Lebesgue measure on the set of optimal rays F is*

$$\int \mathrm{d}m(\alpha)\mu_\alpha,$$

with

$$\begin{aligned} m(\alpha) &= \mathcal{L}^d \int_0^1 c\left(t, \frac{1}{2}\right) \mathrm{d}t, \\ \mu_\alpha &= \left(\int_0^1 c\left(t, \frac{1}{2}\right) \mathrm{d}t \right)^{-1} c\left(t, \frac{1}{2}\right) \mathrm{d}t, \end{aligned}$$

where $c\left(t, \frac{1}{2}\right)$ is the Jacobian along the ray $\alpha + (t - 1/2)p(\alpha)$.

Remark 6.4. By Fubini's theorem,

$$\int_0^1 c\left(t, \frac{1}{2}\right) \mathrm{d}t < +\infty \quad \mathcal{L}^d\text{-a. e.},$$

therefore the formula makes sense.

In the following we denote $c(t,\alpha) = c(t, 1/2, \alpha)$.

6.2. **Regularity of the Jacobian and applications.**

Lemma 6.5. *$c(t,\alpha) \in W^{1,1}_t$, and there exists a $K_d > 0$ such that*

$$\int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}t} c(t,\alpha) \right| \mathrm{d}t \le K_d.$$

Proof. Since

$$\frac{\mathrm{d}}{\mathrm{d}t} c(t,\alpha) + \frac{d}{1-t} c(t,\alpha) \ge 0,$$

we can estimate

$$\begin{aligned} \int_0^{\frac{1}{2}} \left| \frac{\mathrm{d}}{\mathrm{d}t} c(t,\alpha) \right| \mathrm{d}t &\le \int_0^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}t} c(t,\alpha) + 2\frac{d}{1-t} c(t,\alpha)\, \mathrm{d}t \\ &\le c\left(\frac{1}{2}, \alpha\right) + 4d \int_0^{\frac{1}{2}} c(t,\alpha)\, \mathrm{d}t, \end{aligned}$$

and similarly

$$\int_{\frac{1}{2}}^{1} \left| \frac{\mathrm{d}}{\mathrm{d}t} c(t,\alpha) \right| \mathrm{d}t \leq c\left(\frac{1}{2},\alpha\right) + 4d \int_{\frac{1}{2}}^{1} c(t,\alpha)\,\mathrm{d}t,$$

Hence

$$\text{Tot.Var.}(c(\cdot,\alpha)) \leq 4d + 2c\left(\frac{1}{2},\alpha\right). \tag{6.1}$$

In particular the limits

$$\lim_{t\to 0^+,1^-} c(t,\alpha)$$

exist. From the normalization

$$\int_0^1 c(t,\alpha)\,\mathrm{d}t = 1$$

and the estimate

$$c(t,\alpha) \geq \min\left\{2^d|t|^d, 2^d|1-t|^d\right\} c\left(\frac{1}{2},\alpha\right),$$

it follows that there is K'_d such that

$$c\left(\frac{1}{2},\alpha\right) \leq K'_d,$$

so that by (6.1) there is K_d such that

$$\text{Tot.Var.}(c(\cdot,\alpha)) \leq K_d$$

□

Corollary 6.6.

$$\frac{1}{c}\left|\frac{\mathrm{d}}{\mathrm{d}t}c\right| \in L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x).$$

6.3. **Divergence formulation.**

Proposition 6.7. *We have the following relation between c and the divergence of the vector field p:*

$$\mathrm{div}(1, p\chi_F) = \frac{1}{c}\frac{\mathrm{d}c}{\mathrm{d}t}\mathrm{d}t\,\mathrm{d}z\bigg|_F.$$

From

$$\frac{1}{c}\frac{\mathrm{d}c}{\mathrm{d}t} \in \left(-\frac{d}{1-t}, \frac{d}{t}\right)$$

it follows that it is an absolutely continuous measure.

Proof. Take a test function $\phi \in C^1_c(F)$. Applying the disintegration along the rays, one obtains

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_0^1 \phi(t,z)\mathrm{div}(1, p_t\chi_{F(t)})\,\mathrm{d}t\,\mathrm{d}z \\
&= -\int_{\mathbb{R}^d}\int_0^1 \chi_{F(t)}(z)\phi_t(t,z) + p_t(z)\cdot\nabla\phi(t,z)\,\mathrm{d}t\,\mathrm{d}z \\
&= -\int \mathrm{d}m(\alpha)\int_0^1 \mathrm{d}t\, c(t,\alpha)\Big[\phi_t(t,(1-t)y+tx) + (x-y)\cdot\nabla\phi(t,(1-t)y+tx)\Big] \\
&= -\int \mathrm{d}m(\alpha)\int_0^1 \mathrm{d}t\, c(t,\alpha)\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,(1-t)y+tx) \\
&= \int \mathrm{d}m(\alpha)\int_0^1 \mathrm{d}t\,\frac{\mathrm{d}c}{\mathrm{d}t}(t,\alpha)\phi(t,(1-t)y+tx) \\
&= \int_{\mathbb{R}^d}\int_0^1 \left(\frac{1}{c}\frac{\mathrm{d}c}{\mathrm{d}t}\right)\phi(t,z)\mathrm{d}t\,\mathrm{d}z.
\end{aligned}$$

□

References

[1] L. Ambrosio and N. Fusco and D. Pallara. Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs, 2000.

[2] F. Aubin and A. Cellina. *Differential inclusions, set-valued maps and viability theory.* Springer, 1984.

[3] S. Bianchini, C. De Lellis and R. Robyr. SBV regularity for Hamilton-Jacobi equations in $\mathbb{R}^d$. In preparation.

[4] S. Bianchini and M. Gloyer. On the Euler-Lagrange equation for a variational problem: the general case II. To appear in *Math. Z.*

[5] P. Cannarsa, A. Mennucci and C. Sinestrari. Regularity Results for Solutions of a Class of Hamilton-Jacobi Equations. *Arch. Rational Mech. Anal.*, 140:197–223, 1997.

[6] D.H. Fremlin. Measure Theory, Vol. 3: Measure Algebras. Torres Fremlin, 2004.

[7] D.H. Fremlin. Measure Theory, Vol. 4: Topological Measure Spaces. Torres Fremlin, 2006.

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