SBV-LIKE REGULARITY FOR HAMILTON-JACOBI EQUATIONS WITH A CONVEX HAMILTONIAN

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Abstract. In this paper we consider a viscosity solution u of the Hamilton-Jacobi equation

$$\partial_t u + H(D_x u) = 0 \qquad \text{in } \Omega \subset [0,T] \times \mathbb{R}^n,$$

where H is smooth and convex. We prove that when $d(t,\cdot) := H_p(D_x u(t,\cdot))$, $H_p := \nabla H$, is BV for all $t \in [0,T]$ and suitable hypotheses on the Lagrangian L hold, the Radon measure $\mathrm{div} d(t,\cdot)$ can have Cantor part only for a countable number of t's in $[0,T]$. This result extends a result of Robyr for genuinely nonlinear scalar balance laws and a result of Bianchini, De Lellis and Robyr for uniformly convex Hamiltonians.

1. Introduction

We consider the Hamilton-Jacobi equation

$$\partial_t u + H(D_x u) = 0 \quad \text{in } \Omega \subset [0,T] \times \mathbb{R}^n,$$

where H is a smooth convex Hamiltonian and Ω an open set of $[0,T]\times\mathbb{R}^n$. A viscosity solution of such an equation is locally Lipschitz but in general it doesn't have any additional regularity. The structure of the non-differentiability set of viscosity solutions has been studied by several authors, see for example Fleming [14], Cannarsa and Soner [10]. The majority of the existing results are in the case of a strictly convex Hamiltonian. Under this assumption the viscosity solution u is semiconcave, this implies in particular that Du belongs to BV and D^2u is a matrix of Radon measures. It is therefore of interest to see when Du belongs to SBV. The first result in this direction was proven by Cannarsa, Mennucci and Sinestrari in [8]. There, the authors were able to prove the SBV regularity of Du as a corollary to a more general result on the rectifiability of the singular set of Du. Therefore they needed a strongly regular initial datum $u(0,x) = u_0(x)$ in $W^{1,\infty}(\mathbb{R}^n) \cap C^{R+1}(\mathbb{R}^n)$, $R \geq 1$. Less regularity can be asked to the initial datum when attempting directly to the Cantor part of D^2u. In [5], Bianchini, De Lellis and Robyr proved that, when the Hamiltonian is uniformly convex and the initial datum is bounded Lipschitz, $D_x u(t,\cdot)$ belongs to $[SBV(\Omega_t)]^n$, $\Omega_t := \{x \in \mathbb{R}^n |\ (t,x) \in \Omega\}$, out of a countable number of t's in $[0,T]$. This means that $D_x^2 u(t,\cdot)$ can have Cantor part only for a countable number of t's in $[0,T]$. In particular Du belongs to $[SBV(\Omega)]^{n+1}$. This result was first obtained in the one-dimensional case by Ambrosio and De Lellis in [2].

When H is just convex, $D_x u(t,\cdot)$ looses in general its BV regularity, an example can be found in Remark 3.7 in Bianchini [4]. However, in this paper, we show that an SBV-like regularity result can be proven for the vector field

$$d(t,x) := H_p(D_x u(t,x)),$$

defined on the set U of points (t,x) where $u(t,x)$ is differentiable in x. Here H_p is the gradient of the Hamiltonian $H(p)$. Indeed, the divergence $\mathrm{div} d(t,\cdot)$ is in general a locally finite Radon measure. Moreover when the vector field $d(t,\cdot)$ is BV and suitable hypotheses are made on the Lagrangian L, the Legendre transform of H, the measure $\mathrm{div} d(t,\cdot)$ has Cantor part only for a countable number of t's in $[0,T]$.

More precisely let H be $C^2(\mathbb{R}^n)$, convex and superlinear, i.e. such that $\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty$.
(HYP(0)) Suppose the vector field $d(t,\cdot)$ belongs to $[BV(\Omega_t)]^n$ for every $t \in [0,T]$.

Define V_{π_n} as

$$V_{\pi_n} := \{v \in \mathbb{R}^n |\ L(\cdot) \text{ is not twice differentiable in } v\},$$

and

$$\Sigma_{\pi_n} := \{(t,x) \in U |\ d(t,x) \in V_{\pi_n}\} \quad \text{and} \quad \Sigma^c_{\pi_n} := U \setminus \Sigma_{\pi_n}.$$

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(HYP(n)) We suppose V_{π_n} to be contained in a finite union of hyperplanes Π_{π_n}.

For $j = n, \ldots, 3$ for every $(j-1)$-dimensional plane π_{j-1} in Π_{π_j}, let $L_{\pi_{j-1}} : \mathbb{R}^{j-1} \to \mathbb{R}$ be the $(j-1)$-dimensional restriction of L to π_{j-1} and

$$V_{\pi_{j-1}} := \{v \in \mathbb{R}^{j-1} | \ L_{\pi_{j-1}}(\cdot) \text{ is not twice differentiable in } v\}.$$

Define

$$\Sigma_{\pi_{j-1}} := \{(t,x) \in \Sigma_{\pi_j} | \ d(t,x) \in V_{\pi_j}\} \quad \text{and} \quad \Sigma^c_{\pi_{j-1}} := \Sigma_{\pi_j} \setminus \Sigma_{\pi_{j-1}}.$$

(HYP(j-1)) We suppose $V_{\pi_{j-1}}$ to be contained in a finite union of $(j-2)$-dimensional planes $\Pi_{\pi_{j-1}}$, for every $\pi_{j-1} \in \Pi_{\pi_j}$.

Let us note that the BV regularity of the vector field $d(t,\cdot)$ is automatically satisfied by a viscosity solution whose initial datum is semiconcave, as a consequence of Proposition 2.16. However, Remark 3.7 in [4] shows an example of an Hamilton-Jacobi equation with a convex Hamiltonian in which the related vector field $d(t,\cdot)$ does not belong to BV. Therefore the BV regularity is a property which is not always satisfied by the vector field $d(t,\cdot)$.

In Example 5.6 we show an Hamilton-Jacobi equation for which the hypotheses (HYP(n)),...,(HYP(2)) are satisfied.

Theorem 1.1. *Under the assumptions (HYP(0)),(HYP(n)),...,(HYP(2)), the Radon measure* $\mathrm{div} d(t,\cdot)$ *has Cantor part on* Ω_t *only for a countable number of* t*'s in* $[0,T]$.

The theorem above can be seen as the multi-dimensional version of a result proven by Robyr in [19]. In that paper Robyr studied entropy solutions of the genuinely nonlinear scalar balance laws

$$\partial_t v(t,x) + D_x(f(t,x,v(t,x))) + g(t,x,v(t,x)) = 0 \quad \text{in an open set } \Omega \subset \mathbb{R}^2,$$

where the source term g belongs to $C^1(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})$, f belongs to $C^2(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})$ and f is such that the set $\{p_i \in \mathbb{R} | \ f_{pp}(t,x,p_i) = 0\}$ is at most countable for every fixed $(t,x) \in \Omega$. His main result states that BV entropy solutions of such equations belong to $SBV_{loc}(\Omega)$. In the one-dimensional case genuinely nonlinear scalar balance laws and Hamilton-Jacobi equations are equivalent when $f(t,x,p) = H(p)$ and $g(t,x,p) = 0$. The entropy solution of the first can be seen as the gradient of the viscosity solution of the second, $v(t,x) = D_x u(t,x)$. With this consideration our result is a kind of generalization of Robyr's result to the multi-dimensional case. Furthermore, we prove that, in the one-dimensional case, when the Hamiltonian is convex and smooth, the BV regularity of $d(t,x)$ follows automatically and there is no need to add further hypotheses to prove its SBV regularity out of a countable number of t's. Since the BV regularity of $d(t,x)$ does not imply in general the same regularity for $\frac{\partial}{\partial x}u(t,\cdot)$, the BV regularity of $\frac{\partial}{\partial x}u(t,\cdot)$ has yet to be required to prove the SBV regularity, see Remark 4.3.

Remark 1.2. Theorem 1.1 is sharp. Indeed at the end of Remark 3.3 in [2] we can find an example of a viscosity solution to the Hamilton-Jacobi equation

$$\partial_t u + \frac{(D_x u)^2}{2} = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}$$

whose related vector field $d(t,x) = D_x u(t,x)$ is not SBV in an infinite set of times in $[0,1]$.

In the multi-dimensional case the question on the SBV regularity of $d(t,\cdot)$ without any additional hypothesis is still open.

The paper is organized as follows. In Section 2 we recall preliminary results on Hamilton-Jacobi equations and viscosity solutions. In Section 3 we extend the definition of the vector field d to the all Ω, we prove that $\mathrm{div} d(t,\cdot)$ is a locally finite Radon measure on Ω_t, for all $t \in [0,T]$, and present the general strategy used to prove that $\mathrm{div} d(t,\cdot)$ has a Cantor part only for a countable number of t's in $[0,T]$. In Section 4 we study the one-dimensional case and we prove that $\mathrm{div} d(t,\cdot)$ belongs to $SBV(\Omega_t)$, out of a countable number of t's in $[0,T]$, without any additional hypothesis. In Section 5 we study the multi-dimensional case and prove Theorem 1.1. We also state some easy corollaries.

2. Preliminaries

2.1. **Generalized differentials.** We begin with the definition of generalized differential, see Cannarsa and Sinestrari [9] and Cannarsa and Soner [10].

Let Ω be an open subset of $\mathbb{R}^n$.

Definition 2.1. Let $u : \Omega \to \mathbb{R}$, for any $x \in \Omega$ the sets

$$D^-u(x) = \left\{ p \in \mathbb{R}^n | \ \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\},$$

$$D^+u(x) = \left\{ p \in \mathbb{R}^n | \ \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\},$$

are called, respectively, the *subdifferential* and *superdifferential* of u at x.

2.2. **Decomposition of a Radon measure.** Given an $[L^\infty(\mathbb{R}^n)]^n$ vector field $d(x)$ such that $\mathrm{div} d(x) =: \mu(x)$ is a Radon measure on $\mathbb{R}^n$, we can decompose μ into three mutually singular measures:

$$\mu = \mu_a + \mu_c + \mu_j.$$

μ_a is the absolutely continuous part with respect to the Lebesgue measure. μ_j is the singular part of the measure which is concentrated on a $\mathcal{H}^{n-1}$-rectifiable set. μ_c, the Cantor part, is the remaining part.

2.3. **BV and SBV functions.** A detailed description of the spaces BV and SBV can be found in Ambrosio, Fusco and Pallara [3], Chapters 3 and 4. For the reader convenience, we briefly recall that, given $u \in BV(\mathbb{R}^n)$, the distributional derivative of u, which by definition must be a measure with bounded total variation, is decomposable into three mutually singular measures:

$$Du = D_a u + D_c u + D_j u.$$

$D_a u$ is the absolutely continuous part with respect to the Lebesgue measure. $D_j u$ is the part of the measure which is concentrated on the rectifiable $(n-1)$-dimensional set J, where the function u has jump discontinuities, thus for this reason it is called jump part. $D_c u$, the Cantor part, is the singular part which satisfies $D_c u(E) = 0$ for every Borel set E with $\mathcal{H}^{n-1}(E) < \infty$. If this part vanishes, i.e. $D_c u = 0$, we say that $u \in SBV(\mathbb{R}^n)$. When $u \in [BV(\mathbb{R}^n)]^k$ the distributional derivative Du is a matrix of Radon measures and the decomposition can be applied to every component of the matrix.

We recall here some properties of BV functions which will be useful later on.

Definition 2.2. Let u in $[L^1_{loc}(\mathbb{R}^n)]^k$, we say that u has an *approximate limit* at $x \in \mathbb{R}^n$ if there exists $z \in \mathbb{R}^k$ such that

$$\lim_{\rho \to 0} ⨍_{B_\rho(x)} |u(y) - z| dy = 0.$$

The set S_u of points where this property does not hold is called the *approximate discontinuity set.* For any $x \in \mathbb{R}^n \setminus S_u$ the vector z is called *approximate limit of* u at x and is denoted by $\tilde{u}(x)$.

Proposition 2.3. *Let u and v belong to $[BV(\mathbb{R}^n)]^k$. Let*

$$L := \{x \in \mathbb{R}^n \setminus (S_u \cup S_v) | \ \tilde{u}(x) = \tilde{v}(x)\},$$

where $\tilde{u}(x), \tilde{v}(x)$ are the approximate limit of u and v respectively. Then Du and Dv are equal when restricted to L.

Proof. See Remark 3.93 in [3]. □

Proposition 2.4. *Let u belongs to $[BV(\mathbb{R}^n)]^k$. Then $D_c u$ vanishes on sets which are σ-finite with respect to $\mathcal{H}^{n-1}$ and on sets of the form $\tilde{u}^{-1}(E)$ with $E \subset \mathbb{R}^k$ and $\mathcal{H}^1(E) = 0$.*

Proof. See Proposition 3.92 in [3]. □

Proposition 2.5. *Let u belongs to $[BV(\mathbb{R}^n)]^k$. For $j = 1, \dots, n-1$ define the $(n-j)$-dimensional restriction $u_{x_1,\dots,x_j}(\cdot) : \mathbb{R}^{n-j} \to \mathbb{R}^k$ as $u_{x_1,\dots,x_j}(\hat{x}) = u(x_1, \dots, x_j, \hat{x})$ for fixed $(x_1, \dots, x_j) \in \mathbb{R}^j$. Then $u_{x_1,\dots,x_j}(\cdot)$ is $[BV(\mathbb{R}^{n-j})]^k$ for $\mathcal{H}^j$-a.e. $(x_1, \dots, x_j)$ in $\mathbb{R}^j$.*

Proof. This is a well known result. The proof in the case $j = n-1$ can be found in [3] Section 3.11, in the other cases is similar. □

2.4. **Semiconcave functions.** For a complete introduction to the theory of semiconcave functions we refer to Cannarsa and Sinestrari [9], Chapter 2 and 3 and Lions [18]. For our purpose we define semiconcave functions with a linear modulus of semiconcavity. In general this class is considered only as a particular subspace of the class of semiconcave functions with general semiconcavity modulus. The proofs of the following statements can be found in the mentioned references.

Definition 2.6. We say that a function $u:\Omega\to\mathbb{R}$ is *semiconcave* and we denote with $SC(\Omega)$ the space of functions with such a property, if $\exists C>0$ such that for any $x,z\in\Omega$ such that the segment $[x-z,x+z]$ is contained in Ω

$$u(x+z)+u(x-z)-2u(x)\le C|z|^2.$$

Proposition 2.7. *Let $u:\Omega\to\mathbb{R}$ belongs to $SC(\Omega)$ with semiconcavity constant $C\ge 0$. Then the function*

$$w: x\mapsto u(x)-\frac{C}{2}|x|^2$$

is concave, i.e. for any x,y in Ω such that the whole segment $[x,y]$ is contained in Ω, $\lambda\in[0,1]$

$$w(\lambda x+(1-\lambda)y)\ge \lambda w(x)+(1-\lambda)w(y).$$

Within all the properties of a semiconcave function let us recall that when u is semiconcave Du is a BV map, hence its distributional Hessian D^2u is a symmetric matrix of Radon measures and can be split into the three mutually singular parts $D^2_a u, D^2_j u, D^2_c u$. Moreover the following proposition holds.

Proposition 2.8. *Let u be a semiconcave function. If D denotes the set of points where D^+u is not single-valued, then $|D^2_c u|(D)=0$.*

Proof. Indeed, the set of points where D^+u is not single-valued, i.e. the set of singular points, is a $\mathcal{H}^{n-1}$-rectifiable set. □

Definition 2.9. We say that a function $v:\Omega\to\mathbb{R}$ is *semiconvex* if $u:=-v$ is semiconcave.

2.5. **Viscosity solutions.** A concept of generalized solution to the equation

$$\partial_t u+H(D_x u)=0 \qquad \text{in } \Omega\subset[0,T]\times\mathbb{R}^n, \tag{2.1}$$

was found to be necessary since classical solutions break down and solutions which satisfy (2.1) almost everywhere are not unique. Crandall and Lions introduced in [12] the notion of viscosity solution to solve both these problems, see also Crandall, Evans and Lions [11].

Definition 2.10. A bounded uniformly continuous function $u:\Omega\to\mathbb{R}$ is called a *viscosity solution* of (2.1) provided that

i) u is a viscosity subsolution of (2.1): for each $v\in C^\infty(\Omega)$ such that $u-v$ has a maximum at $(t_0,x_0)\in\Omega$,
$$\partial_t v(t_0,x_0)+H(D_x v(t_0,x_0))\le 0;$$
ii) u is a viscosity supersolution of (2.1): for each $v\in C^\infty(\Omega)$ such that $u-v$ has a minimum at $(t_0,x_0)\in\Omega$,
$$\partial_t v(t_0,x_0)+H(D_x v(t_0,x_0))\ge 0.$$

2.6. **Properties of the viscosity solution of Hamilton-Jacobi equations.** We introduce a locality property, whose proof can be found in [5].

Proposition 2.11. *Let u be a viscosity solution of (2.1) in Ω. Then u is locally Lipschitz. Moreover for any $(t_0,x_0)\in\Omega$, there exists a neighborhood $\mathcal{U}$ of (t_0,x_0), a positive number δ and a Lipschitz function v_0 on $\mathbb{R}^n$ such that*

(Loc) u coincides on $\mathcal{U}$ with the viscosity solution of

$$\begin{cases} \partial_t v+H(D_x v)=0 & \textit{in } [t_0-\delta,\infty)\times\mathbb{R}^n\\ v(t_0-\delta,x)=v_0(x). \end{cases}$$

Motivated by the above proposition, let us consider the Cauchy problem

$$\begin{cases} \partial_t u + H(D_x u) = 0 & \text{in } \Omega \subset [0,T] \times \mathbb{R}^n, \\ u(0,x) = u_0(x) & \text{for all } x \in \Omega_0, \end{cases} \tag{2.2}$$

where $u_0(x)$ is a bounded Lipschitz function on Ω_0.

The proofs of the following statements can be found in Evans [13], Section 3.3 and Chapter 10. See also Cannarsa and Sinestrari [9], Fleming [14], Fleming and Rishel [15], Fleming and Soner [16] and Lions [18].

The convexity of the Hamiltonian in the p-variable relates Hamilton-Jacobi equations to variational problems.

Let L be the Lagrangian of our system, i.e. the Legendre transform of the Hamiltonian H

$$L(v) = \sup_p \{\langle v, p\rangle - H(p)\}.$$

When we consider a smooth convex Hamiltonian the corresponding Lagrangian is strictly convex but non smooth in general. In the case of a smooth uniformly convex Hamiltonian instead, the Lagrangian inherits the same properties of H, i.e. L is itself smooth and uniformly convex.

Theorem 2.12. *There exist a sufficiently big $M > 0$ and a sufficiently small $\varepsilon > 0$ such that on the set $\Omega^{\varepsilon,M} := \{(t,x) \in \Omega |\ t \in [0,\varepsilon], \mathrm{dist}((t,x), \partial\Omega) > M\}$ the unique viscosity solution of the Cauchy problem (2.2) is the Lipschitz continuous function $u(t,x)$ defined as*

$$u(t,x) = \min_{y \in \Omega_0^{\varepsilon,M}} \left\{ u(0,y) + tL\left(\frac{x-y}{t}\right)\right\}. \tag{2.3}$$

From now on we restrict to the set $\Omega^{\varepsilon,M}$. To simplify the notations we will call it Ω and we will study the equation on the time interval $[0,T]$ implying that $T < \varepsilon$.

Theorem 2.13. *Let $u(t,x)$ be a viscosity solution of the Cauchy problem (2.2) defined on Ω as in the previous theorem.*

i) *The minimum point y for $(t,x) \in \Omega$ in (2.3) is unique if and only if $u(t,x)$ is differentiable in x. Moreover in this case $y = x - tH_p(D_x u(t,x))$.*
ii) *(Dynamic Programming Principle) Fix $(t,x) \in \Omega$, then for all $t' \in [0,t]$*

$$u(t,x) = \min_{z \in \Omega_{t'}} \left\{ u(t',z) + (t-t')L\left(\frac{x-z}{t-t'}\right)\right\}. \tag{2.4}$$

iii) *Let $0 < s < t$, let $(t,x) \in \Omega$ and y be a minimum point in (2.3). Let $z = \frac{s}{t}x + \left(1 - \frac{s}{t}\right)y$. Then y is the unique minimum point for*

$$u(s,z) = \min_{w \in \Omega_0} \left\{ u(0,y) + sL\left(\frac{z-w}{s}\right)\right\}.$$

Definition 2.14. Let $y \in \Omega_0$ be a minimizer for $u(t,x)$. We call *optimal ray* the segment $[(t,x),(0,y)]$ defined in $[0,t]$.

Proposition 2.15. *Let $[(t,x),(0,y)]$ and $[(t,x)',(0,y')]$ be two optimal rays in $[0,t]$, for $x, x' \in \Omega_t$ $y, y' \in \Omega_0$ then they cannot intersect except at time 0 or t.*

Proof. It follows from Theorem 2.13-(iii). □

Proposition 2.16. *Let u_0 be a semiconcave function. Then the unique viscosity solution $u(t,x)$ of (2.2) is semiconcave in x, for all $t \in [0,T]$.*

Theorem 2.17 (Semiconcavity Theorem)**.** *Suppose H is locally uniformly convex. Then for any t in $(0,T]$, $u(t,\cdot)$ is locally semiconcave with semiconcavity constant $C(t) = \frac{C}{t}$. Thus for any fixed $\tau > 0$ there exists a constant $C = C(\tau)$ such that $u(t,\cdot)$ is semiconcave with constant less than C for any $t \geq \tau$.*

Moreover u is also locally semiconcave in both the variables (t,x) in $(0,T] \times \mathbb{R}^n$.

2.7. **Duality solutions.** We consider a fixed interval of time $[0,1]$, and we define duality solutions in this time interval.

Definition 2.18. Setting $u^+(1,z) := u(1,z)$, we define *duality solutions* for $s \in [0,1]$ and $z \in \Omega_s$, the *backward solution*

$$u^-(s,z) := \max_{x\in\Omega_1} \left\{ u^+(1,x) - (1-s)L\left(\frac{x-z}{1-s}\right) \right\}, \tag{2.5}$$

and the *forward solution*

$$u^+(s,z) := \min_{y\in\Omega_0} \left\{ u^-(0,y) + sL\left(\frac{z-y}{s}\right) \right\}. \tag{2.6}$$

Remark 2.19. Note that the function $v(\tau,y) := u^-(1-\tau,y)$ is a viscosity solution of

$$\begin{cases} \partial_\tau v - H(D_y v) = 0 & \text{in } \Omega \subset [0,T] \times \mathbb{R}^n \\ v(0,y) = u(1,y) & \text{for all } y \in \Omega_1. \end{cases} \tag{2.7}$$

Moreover the forward solution is the viscosity solution of

$$\begin{cases} \partial_t u + H(D_x u) = 0 & \text{in } \Omega \subset [0,T] \times \mathbb{R}^n, \\ u(0,x) = u^-(0,x) & \text{for all } x \in \Omega_0. \end{cases} \tag{2.8}$$

Thanks to the previous remark Theorems 2.13, 2.17 and Propositions 2.15, 2.16 hold for v and the forward solution u^+.

Proposition 2.20. *From the definitions above, u^+ and u^- satisfy the following properties for $x \in \Omega_1$, $y \in \Omega_0$ and $z \in \Omega_s$ for $s \in (0,1)$*

$$u^-(1,x) = u^+(1,x) = u(1,x), \quad u^+(0,y) = u^-(0,y) \le u(0,y), \quad u^-(s,z) \le u^+(s,z) \le u(s,z).$$

Proof. The first two equalities are a consequence of the fact that u^0 and u^1, defined as follows, are $L(x-y)$ conjugate functions. First, for $x \in \Omega_1$, set

$$u^1(x) := \min_{y\in\Omega_0} \{u(0,y) + L(x-y)\},$$

i.e. $u^1(x) = u(1,x)$.
Then, for $y \in \Omega_0$, set

$$u^0(y) := \max_{x\in\Omega_1} \{u^1(x) - L(x-y)\},$$

i.e. $u^0(y) = u^-(0,y)$.

From these definitions it follows $u^0(y) \le u(0,y)$ and

$$u^1(x) = \min_{y\in\Omega_0} \{u^0(y) + L(x-y)\}.$$

Indeed, let $\tilde{x} \in \Omega_1$ a maximizer for $u^0(y)$ then

$$u^0(y) = u^1(\tilde{x}) - L(\tilde{x}-y) \le u(0,y) + L(\tilde{x}-y) - L(\tilde{x}-y) = u(0,y).$$

Nevertheless, from $u^0(y) \le u(0,y)$, it follows

$$\min_{y\in\Omega_0} \{u^0(y) + L(x-y)\} \le \min_{y\in\Omega_0} \{u(0,y) + L(x-y)\} = u^1(x).$$

On the other hand, let $\tilde{y}$ be a minimizer for $\min_{y\in\Omega_0}\{u^0(y) + L(x-y)\}$, then we have

$$\begin{aligned} \min_{y\in\Omega_0} \{u^0(y) + L(x-y)\} &= u^0(\tilde{y}) + L(x-\tilde{y}) \\ &\ge u^1(x) - L(x-\tilde{y}) + L(x-\tilde{y}) \\ &= u^1(x). \end{aligned}$$

Note that the definition of $u^-(s,z)$ and $u^+(s,z)$ implies that $u^-(1,x) = u^1(x)$ and $u^+(0,y) = u^0(y)$.

For s in $(0,1)$,the last inequality follows by

$$\begin{aligned} u^-(s,z) &= \max_{x\in\Omega_1}\left\{u^1(x)-(1-s)L\left(\frac{x-z}{1-s}\right)\right\} \\ &= \max_{x\in\Omega_1}\left\{\min_{y\in\Omega_0}\left\{u^0(y)+L(x-y)-(1-s)L\left(\frac{x-z}{1-s}\right)\right\}\right\} \\ &\le \min_{y\in\Omega_0}\left\{u^0(y)+sL\left(\frac{z-y}{s}\right)\right\} = u^+(s,z), \end{aligned}$$

where the inequality is given by the convexity of L

$$L(x-y)\le sL\left(\frac{z-y}{s}\right)+(1-s)L\left(\frac{x-z}{1-s}\right).$$

Note that, from the strict convexity of L, the equality holds if and only if $\frac{x-z}{1-s}=\frac{z-y}{s}$, i.e. $z=sx+(1-s)y$. That is z belongs to the segment joining the maximizer x to the minimizer y.

Furthermore, due to the fact that $u^-(0,y)\le u(0,y)$, we have $u^+(s,z)\le u(s,z)$. □

Proposition 2.21. *Suppose H is a smooth uniformly convex Hamiltonian. Then on a smaller set $\tilde\Omega\subset\Omega$ there exists a $C^{1,1}(\tilde\Omega)$ function w such that $u^-(s,z)\le w(s,x)\le u^+(s,z)$.*

Proof. Thanks to the fact that H is uniformly convex u^- is semiconvex and u^+ is semiconcave, thus the proposition is a consequence of the Ilmanen's Lemma, see Lemma 4G in [17]. □

The definition of backward and forward solutions can be easily generalized for every time interval $[\tau,t]\subset[0,T]$. Propositions 2.20 and 2.21 hold even in this case.

Definition 2.22. Setting $u^+_{t,\tau}(t,z):=u(t,z)$, we define *duality solutions* for $s\in[\tau,t]$ and $z\in\Omega_s$, the *backward solution*

$$u^-_{t,\tau}(s,z):=\max_{x\in\Omega_t}\left\{u^+_{t,\tau}(t,x)-(t-s)L\left(\frac{x-z}{t-s}\right)\right\},$$

and the *forward solution*

$$u^+_{t,\tau}(s,z):=\min_{y\in\Omega_\tau}\left\{u^-_{t,\tau}(\tau,y)+(s-\tau)L\left(\frac{z-y}{s-\tau}\right)\right\}.$$

3. Extension and preliminary properties of the vector field d

We consider a viscosity solution u of the Hamilton-Jacobi equation

$$\partial_t u+H(D_x u)=0 \quad \text{in an open set } \Omega\subset\mathbb{R}^+\times\mathbb{R}^n, \tag{3.1}$$

where H is $C^2(\mathbb{R}^n)$ convex and

$$\lim_{|p|\to\infty}\frac{H(p)}{|p|}=+\infty.$$

As already noticed, thanks to the time invariance of the equation and to Proposition 2.11, it is enough to consider the unique viscosity solution of the following Cauchy problem

$$\begin{cases} \partial_t u+H(D_x u)=0 & \text{in } \Omega\subset[0,T]\times\mathbb{R}^n, \\ u(0,x)=u_0(x) & \text{for all } x\in\Omega_0, \end{cases} \tag{3.2}$$

where $u_0(x)$ is a bounded Lipschitz function on Ω_0. Here we restrict to the set $\Omega=\Omega^{\varepsilon,M}$ as in Theorem 2.12.

The vector field $d(t,x):=H_p(D_x u(t,x))$ is well defined where $u(t,x)$ is differentiable in x, i.e. $\mathcal{H}^n$-a.e. on Ω_t, for every $t\in[0,T]$.

Thanks to the Lipschitz regularity of $u(t,\cdot)$ and the fact that H is smooth, the vector field $d(t,\cdot)$ belongs to $[L^\infty(\Omega_t)]^n$.

Moreover d is constant along optimal rays. Indeed, thanks to Theorem 2.13-(iii), we have

$$d(t,x)=d(s,x-(t-s)d(t,x))$$

for all $0\le s\le t$.

A natural extension of d to Ω is $\mathcal{D}(\cdot):\Omega\to\mathbb{R}^n$

$$\mathcal{D}(t,x) := \left\{ \frac{x-y}{t} \;|\; y \text{ is a minimum for } u^+_{t,0}(t,z) \right\}.$$

$\mathcal{D}(t,x)$ is a multi-valued function which coincides with $d(t,x)$ in the points (t,x) where $u(t,x)$ is differentiable in x. Indeed, where $u(t,\cdot)$ is differentiable, $u(t,x)=u^+_{t,0}(t,x)$ and they both admit as unique minimizer $y=x-tH_p(D_xu(t,x))$ in Ω_0.

Following the results of Bianchini and Gloyer in [6], we can prove that $\mathcal{D}(t,x)$ has closed graph and thanks to the fact that $\mathcal{D}(t,x)$ is closed. For all $x'\in\Omega_t$, for all $\varepsilon>0$, there exists $\delta>0$ such that

$$\mathcal{D}(t,x)\subset\mathcal{D}(t,x')+B(0,\varepsilon)$$

for $x\in B(x',\delta)\subset\Omega_t$. Moreover $\mathcal{D}(t,x)$ is a Borel measurable function and $\mathrm{div}d(t,\cdot)$ a locally finite Radon measure. We repeat the proof for the reader's convenience.

Theorem 3.1. *For every $t\in(0,T]$, the divergence $\mathrm{div}d(t,\cdot)$ is a locally finite Radon measure with negative singular part.*

Proof. Consider an approximation of our vector field done by taking a dense sequence of points $\{y_i\}_{i=1}^\infty$ in Ω_0. Fix an integer $I>0$, call $\Omega_0^I := \{y_i|\; i=1,\dots,I\}$ and define for any $x\in\Omega_t$

$$u^+_I(t,x) := \min_{i\in I}\left\{ u^-_{t,0}(0,y_i) + tL\left(\frac{x-y_i}{t}\right)\right\}.$$

Through this approximation the set Ω_t is split into at most I open regions Ω^i_t, $i=1,\dots,I$, defined by

$$\Omega^i_t := \text{interior of } \{x\in\Omega_t|\; \exists y_i \text{ minimizer for } u^+_I(t,x)\},$$

together with the set

$$J^I_t := \bigcup_{i\neq j}\left(\overline{\Omega}^i_t\cap\overline{\Omega}^j_t\right)$$

of negligible $\mathcal{H}^n$-measure. Indeed, even for $u^+_I(t,\cdot)$ the set of points with more than one minimum is the set of points of non differentiability of $u^+_I(t,\cdot)$ and this set has $\mathcal{H}^n$-measure zero. We define the vector field d^I on Ω so that on each open set Ω^i_t

$$d^I(t,x) := \frac{x-y_i}{t}.$$

Using explicitly the definition of d^I and the fact that $\mathcal{H}^n(J^I_t)=0$,

$$\mathrm{div}d^I(t,x)\le\frac{n}{t}.$$

Thanks to the pointwise convergence of d^I to d

$$\mathrm{div}d(t,\cdot)-\frac{n}{t}\mathcal{H}^n\le 0,$$

i.e. $\mathrm{div}d(t,\cdot)-\frac{n}{t}\mathcal{H}^n$ is a negative definite distribution, hence it is a locally finite Radon measure. Thus $\mathrm{div}d(t,\cdot)$ is itself a locally finite Radon measure.

Moreover

$$\mathrm{div}d(t,\cdot)\le\frac{n}{t}\mathcal{H}^n,$$

implies that the singular part of this measure can be only negative. □

From now on we will denote $\mu(t,\cdot):=\mathrm{div}d(t,\cdot)$.

Since we have proven that $\mu(t,\cdot)$ is a locally finite Radon measure, it makes sense to ask whether is possible or not that $\mu(t,\cdot)$ has Cantor part for all t in $[0,T]$. Note that if a Cantor part is different from zero then it must be negative for Theorem 3.1.

3.1. **General strategy.** In order to prove that $\mu(t,\cdot)$ has Cantor part only for a countable number of t's, the general idea is now standard, see [2], [5].

We reduce to a smaller interval $[\tau, T]$, for a fixed $\tau > 0$, and we construct, on this interval, a monotone bounded functional $F(t)$. Then, we relate the presence of a Cantor part for the measure $\mu(t,\cdot)$, for a certain t in $[\tau, T]$, with a jump of the functional F in t. Since this functional is bounded monotone it can have only a countable number of jumps. Thus, the Cantor part of $\mu(t,\cdot)$ can be different from zero only for a countable number of t's.

To define F we consider the following maps: $X_{t,\tau}(x) : \Omega_t \to \Omega_\tau$

$$X_{t,\tau}(x) := x - (t-\tau)\mathcal{D}(t,x),$$

and its restriction to the set U_t of points where $\mathcal{D}(t,x)$ is single-valued, $\chi_{t,\tau}(x) : U_t \to U_\tau$

$$\chi_{t,\tau}(x) := x - (t-\tau)d(t,x).$$

We will sometimes write $\chi_{t,\tau}(\Omega_t)$ for $\chi_{t,\tau}(U_t)$.

We define the functional $F : (\tau, T] \to \mathbb{R}$

$$F(t) := \mathcal{H}^n(\chi_{t,\tau}(U_t)).$$

The functional F is bounded, and, due to the fact that optimal rays do not intersect except at time t or 0, F is a monotone decreasing functional.

In order to apply the strategy above we need two estimates of the following type:

i) For any Borel set $A \subset U_t$ for t in $(\tau, T]$

$$\mathcal{H}^n(X_{t,\tau}(A)) \geq C_1\mathcal{H}^n(A) - (t-\tau)C_2\mu(t,A), \tag{3.3}$$

where C_1, C_2 are fixed positive constants.

ii) For any Borel set $A \subset \Omega_t$, for t in $(\tau, T]$ and for every $0 \leq \delta \leq t-\tau$

$$\mathcal{H}^n(X_{t,\tau+\delta}(A)) \geq \left(\frac{t-(\tau+\delta)}{t-\tau}\right)^m \mathcal{H}^n(X_{t,\tau}(A)), \tag{3.4}$$

where $m \in \mathbb{N}, m > 0$ is fixed.

Indeed with the estimates above we can prove the following lemma.

Lemma 3.2. *For any t in $(\tau, T]$ such that $\mu_c(t, \Omega_t) < 0$ there exists a Borel set $A \subset U_t$ such that*

i) *$\mathcal{H}^n(A) = 0$, $\mu_c(t,A) < 0$ and $\mu_c(t, \Omega_t \setminus A) = 0$;*

ii) *$X_{t,\tau}$ is single-valued on A;*

iii) *and for any δ in $(0, T-t]$*

$$\chi_{t,\tau}(A) \cap \chi_{t+\delta,\tau}(\Omega_{t+\delta}) = \emptyset.$$

Proof. The set of points where $d(t,\cdot)$ is not single-valued, which coincides with the set of points where $u(t,\cdot)$ is not differentiable, is a $\mathcal{H}^{n-1}$-rectifiable set, due to the Lipschitz regularity of $u(t,\cdot)$. Hence, the Radon measure $\mu(t,\cdot)$ has null Cantor part on it. This and the definition of Cantor part of a measure imply the existence of a Borel set A such that

- $d(t,x)$ is single-valued for every $x \in A$,
- $\mathcal{H}^n(A) = 0$,
- $\mu_c(\Omega_t \setminus A) = 0$ and $\mu_c(A) < 0$.

By contradiction suppose there exists a compact set $K \subset A$ such that

$$\mu_c(t,K) < 0$$

and

$$X_{t,\tau}(K) = \chi_{t,\tau}(K) \subset \chi_{t+\delta,\tau}(\Omega_{t+\delta}).$$

Then there exists a Borel set $\tilde{K} \subset \Omega_{t+\delta}$ such that $\chi_{t,\tau}(K) = \chi_{t+\delta,\tau}(\tilde{K})$. Moreover, thanks to the fact that we are considering optimal rays starting from $\tilde{K}$, we have

$$\chi_{t+\delta,t}(\tilde{K}) = K \quad \text{and} \quad \chi_{t+\delta,\tau}(\tilde{K}) = \chi_{t,\tau}(K).$$

Using the estimate (3.4),

$$\mathcal{H}^n(K) = \mathcal{H}^n(X_{t+\delta,t}(\tilde{K})) \geq \left(\frac{\delta}{t+\delta-\tau}\right)^m \mathcal{H}^n(X_{t+\delta,\tau}(\tilde{K})) = \left(\frac{\delta}{t+\delta-\tau}\right)^m \mathcal{H}^n(X_{t,\tau}(K)).$$

Hence

$$\mathcal{H}^n(K) \geq \left(\frac{\delta}{t+\delta-\tau}\right)^m \mathcal{H}^n(X_{t,\tau}(K)).$$

Moreover applying estimate (3.3)

$$\mathcal{H}^n(K) \geq \left(\frac{\delta}{t+\delta-\tau}\right)^m \left(C_1\mathcal{H}^n(K) - (t-\tau)C_2\mu(t,A)\right).$$

Since $\mathcal{H}^n(K)=0$ we obtain $\mu(t,A) \geq 0$ in contrast with the fact that $\mu_c(t,A)<0$. □

The estimate (3.3) and Lemma 3.2 lead us to the expected conclusion.
Suppose there exists a t in (τ,T) such that

$$\mu_c(t,\Omega_t) < 0,$$

then, for any $\delta>0$, let A be the set of Lemma 3.2. According to Lemma 3.2-(iii) we have

$$F(t+\delta) \leq F(t) - \mathcal{H}^n(X_{t,\tau}(A)).$$

Moreover, the estimate (3.3) gives

$$F(t+\delta) \leq F(t) + (t-\tau)C_2\mu_c(t,A).$$

Hence, letting $\delta \to 0$, we obtain

$$\limsup_{\delta\to 0} F(t+\delta) < F(t).$$

Therefore t is a point of discontinuity for F, as we wanted to prove.

4. One-dimensional case

We first consider the one-dimensional case. In this case we don't need any further assumption on d or L to prove the following theorem.

Theorem 4.1. *The vector field $d(t,\cdot)$ belongs to $SBV(\Omega_t)$, out of a countable number of $t \in [0,T]$.*

In the uniformly convex case, Theorem 4.1 is a corollary of Ambrosio and De Lellis's result in [2].

Proof. Since we are in the one-dimensional case, $\mathrm{div} d(t,x) = \frac{\partial}{\partial x} d(t,x)$. Hence, Theorem 3.1 implies that $d(t,x)$ belongs to $BV(\Omega_t)$, for every $t \in (0,T]$.

Moreover, $\mathcal{D}(t,\cdot)$ is semimonotone. Indeed, since we are following optimal rays for $u^+_{t,0}$, they do not intersect except at time 0 or t. Thus for $x_1,x_2 \in \Omega_t$, $x_1 < x_2$ and $d_1 \in \mathcal{D}(t,x_1), d_2 \in \mathcal{D}(t,x_2)$, it must hold

$$x_1 - td_1 \leq x_2 - td_2,$$

otherwise the rays cross each other at a time $s \in (0,t)$. Hence the function $\frac{1}{t}x - \mathcal{D}(t,x)$ is monotone increasing and $\mathcal{D}(t,x)$ is semimonotone with constant $C=\frac{1}{t}$.

Let us consider the map $X_{t,\tau}$ for any $t \in (\tau,T]$, $\tau>0$ fixed. The fact that we are in the one-dimensional case implies that for t,x, such that $\mathcal{D}(t,x)$ is multi-valued,

$$\mathcal{D}(t,x) = [d_1,d_2],$$

where $d_1,d_2 \in \mathbb{R}$ are the speeds of the optimal rays for $u(t,x)$. Indeed, for every $\bar{d},\tilde{d} \in [d_1,d_2]$, the ray $[(t,x),(\tau,x-(t-\tau)\bar{d})]$ cannot cross $[(t,x),(\tau,x-(t-\tau)\tilde{d})]$, since they are straight lines starting in the same point. So they fill the triangle delimited by $[(t,x),(\tau,x-(t-\tau)d_1)]$, $[(t,x),(\tau,x-(t-\tau)d_2]$. Moreover, optimal rays starting in other points cannot cross $[(t,x),(\tau,x-(t-\tau)d_1)]$ and $[(t,x),(\tau,x-(t-\tau)d_2)]$, at intermediate time, since they are optimal. Thus they cannot cross any other ray $[(t,x),(\tau,x-(t-\tau)d)]$, where $d \in [d_1,d_2]$. For this reason these rays are optimal for $u^+_{t,0}(t,x)$. Thus optimal rays for the forward solution completely fill the set $\{\Omega_s \mid s \in [0,t]\}$.

Remark 4.2. This argument holds also in the multi-dimensional case but only for a set of points of non differentiability of zero-dimension. The argument is not true in general when the points of non differentiability lie on a surface of dimension greater than zero, since rays starting in two different points of this surface can intersect even at intermediate times.

The above consideration ensures that the map $X_{t,\tau}$ is injective for $\tau > 0$, however this map is multi-valued. To recover the Lipschitzianity we use the Hille-Yosida transformation as seen in [1] and [7].

For any Borel set $A \subset \Omega_t$, let $z \in B := A + T(A)$, $T(x) := (Cx - \mathcal{D}(t,x))$ and $w(z) := (Id_1 + (T)^{-1})^{-1}(z)$. Then the following 1-Lipschitz transformations

$$\begin{cases} x(z) = z - w(z) \\ p(z) = Cz - (C+1)w(z), \end{cases} \tag{4.1}$$

transform our graph

$$\{(x,p)|\ x \in A,\ p \in \mathcal{D}(t,x)\}$$

into the equivalent graph of a maximal monotone function

$$\{(z - w(z), Cz - (C+1)w(z))|\ z \in B\}.$$

Recall that C is the semimonotonicity constant of $\mathcal{D}(t,\cdot)$.

Following optimal rays starting in A with speed in $\mathcal{D}(t,A)$, we can now pass from $X_{t,\tau}(x)$ to a Lipschitz map defined on B

$$\xi(\tau,z) := z - w(z) - (t-\tau)(Cz - (C+1)w(z)).$$

Note that

$$\{(Cz - (C+1)w(z))|\ z \in x + T(x)\} = \mathcal{D}(t,x)$$

so that $X_{t,\tau}(x) = \{\xi(\tau,z)|\ z \in x + T(x)\}$ and $X_{t,\tau}(A) = \xi(\tau,B)$.

We can now apply the Area Formula to $\xi(\tau,\cdot)$

$$\int_{\xi(\tau,B)} \mathcal{H}^0(\xi(\tau,\cdot)^{-1}(w))dw = \int_B |\xi_z(\tau,z)|dz. \tag{4.2}$$

Thanks to the injectivity of the map $X_{t,\tau}$, which is preserved when passing to the Lipschitz parametrization, the left term of (4.2) is precisely the measure of the set $\xi(\tau,B)$. Hence, we have

$$\int_{\xi(\tau,B)} \mathcal{H}^0(\xi(\tau,\cdot)^{-1}(w))dw = \mathcal{H}^1(\xi(\tau,B)) = \mathcal{H}^1(X_{t,\tau}(A)).$$

Moreover, differentiating ξ we respect to z we denote

$$\xi_z(\tau,z) = \xi_z(t,z) - (t-\tau)\dot{\xi}_z(t,z),$$

where $\xi_z(t,z) := \frac{\partial}{\partial z}(z - w(z))$ and $\dot{\xi}_z(t,z) := \frac{\partial}{\partial z}(Cz - (C+1)w(z))$.

Thus we have

$$\mathcal{H}^1(X_{t,\tau}(A)) = \int_B |\xi_z(t,z) - (t-\tau)\dot{\xi}_z(t,z)|dz \geq \int_B \xi_z(t,z)dz - (t-\tau)\int_B \dot{\xi}_z(t,z)dz.$$

Observing that

$$\int_B \dot{\xi}_z(t,z)dz = \int_B \frac{\partial}{\partial z}(Cz - (C+1)w(z))dz = \mu(t,A),$$

we have proven the following estimate: given a Borel set $A \subset \Omega_t$ for t in $(\tau,T]$, we have

$$\mathcal{H}^1(X_{t,\tau}(A)) \geq \mathcal{H}^1(A) - (t-\tau)\mu(t,A). \tag{4.3}$$

Moreover, since for every $0 \leq \delta \leq t-\tau$

$$\xi_z(t,z) - (t-(\tau+\delta))\dot{\xi}_z(t,z) = \frac{\delta}{t-\tau}\xi_z(t,z) + \frac{t-(\tau+\delta)}{t-\tau}(\xi_z(t,z) - (t-\tau)\dot{\xi}_z(t,z)),$$

and $\xi_z(t,z) > 0$, we have

$$\xi_z(t,z) - (t-(\tau+\delta))\dot{\xi}_z(t,z) \geq \frac{t-(\tau+\delta)}{t-\tau}(\xi_z(t,z) - (t-\tau)\dot{\xi}_z(t,z)).$$

Thus, integrating the last equation over B, we obtain the following estimate: given a Borel set $A \subset \Omega_t$ for t in $(\tau,T]$, then for every $0 \leq \delta \leq t-\tau$ we have

$$\mathcal{H}^1(X_{t,\tau+\delta}(A)) \geq \frac{t-(\tau+\delta)}{t-\tau}\mathcal{H}^1(X_{t,\tau}(A)). \tag{4.4}$$

The estimates (4.3) and (4.4) are of type (3.3) and (3.4) respectively, thus they are enough to prove the SBV regularity of d, as seen in Subsection 3.1.

□

Remark 4.3. The SBV regularity of $d(t,\cdot)$ does not necessarily implies the one of $\frac{\partial}{\partial x}u(t,\cdot)$ as well as the BV regularity of $d(t,\cdot)$ does not necessarily implies the one of $\frac{\partial}{\partial x}u(t,\cdot)$. However, in the one-dimensional case when H is strictly convex, the divergence of $d(t,\cdot)$ controls $\frac{\partial^2}{\partial x^2}u(t,\cdot)$ when $\frac{\partial}{\partial x}u(t,\cdot)$ is BV. Therefore, this result can be seen as an extension of the one in [19].

5. The multi-dimensional case

In [5] Bianchini, De Lellis and Robyr proved that the estimates (3.3) and (3.4) hold for the uniformly convex Hamiltonian $H_\epsilon(p) := H(p) + \frac{\epsilon}{2}|p|^2$ for every $\varepsilon > 0$ in a small interval of time and with constants strictly depending on ϵ. Thus, the two estimates cannot pass to the limit.

Nevertheless, we can prove that the divergence $\mathrm{div}d(t,\cdot)$ has Cantor part only for a countable number of t's, adding some hypothesis on the regularity of d and on the structure of the the set of points where L is not twice differentiable.

As already noticed, the Lagrangian corresponding to a smooth convex Hamiltonian is strictly convex but non smooth in general. Particular conditions on the set of points where L is not twice differentiable will allow us to reduce iteratively our problem to a problem of lower dimension, down to the one-dimensional case, where, as we have seen, SBV regularity can be proven without additional assumptions.

Before going on with the proof we set some notations. We will denote with $(x_1, x_2, \ldots, x_n)$ the components of the vector $x \in \mathbb{R}^n$ and, to contract the notation, for a fixed $j = 1, \ldots, n-1$ we call $\hat{x} \in \mathbb{R}^{n-j}$ the vector defined so that

$$(x_1, \ldots, x_j, \hat{x}) = (x_1, x_2, \ldots, x_n).$$

Given a set $E \subset [0,T] \times \mathbb{R}^n$ we will denote with

$$E_t := \{x \in \mathbb{R}^n |\ (t,x) \in E\}$$

and for $j = 1, \ldots, n-1$

$$E_{x_1,\ldots,x_j} := \Big\{(t, x_{j+1}, \ldots, x_n) |\ (t, x_1, \ldots, x_j, x_{j+1}, \ldots x_n) \in E\Big\}.$$

As before we will sometimes denote with $\mu(t,\cdot)$ the Radon measure $\mathrm{div}d(t,\cdot)$ defined on Ω_t.

(HYP(0)) Suppose that the vector field $d(t,\cdot)$ belongs to $[BV(\Omega_t)]^n$ for any $t \in [0,T]$.

Remark 5.1. This hypothesis is certainly satisfied when the initial datum in semiconcave, as a consequence of Proposition 2.16. However it is not true in general, see Remark 3.7. in [4] for an example of an Hamilton-Jacobi equation with convex Hamiltonian whose vector field d is not BV.

The measure $\mathrm{div}d$ can have Cantor part only on a subset of the points of differentiability in x of $u(t,x)$, i.e. the points where $\mathcal{D}(t,x)$ is single-valued. Thus we can reduce to the study of our measure on the set

$$U := \Omega \setminus \{(t,x) |\ \mathcal{D}(t,x) \text{ is multi-valued}\}.$$

Call V the set of points where L is not twice differentiable:

$$V := \{v \in \mathbb{R}^n |\ L(\cdot) \text{ is not twice differentiable in } v\}.$$

Then the set U can be split into two subsets:

$$\Sigma := \{(t,x) \in U |\ d(t,x) \in V\} \quad \text{and} \quad \Sigma^c := U \setminus \Sigma.$$

(HYP(n)) Suppose V is contained in a finite union of hyperplanes.

Remark 5.2. The set $V^c := U \setminus V$, of points where L is twice differentiable, is clearly open because of (HYP(n)). Moreover, since d is continuous, Σ^c, the pre-image of V^c through d, is relatively open in U.

Claim 1.(n) The vector field $d(t,\cdot)$ belongs to $[SBV(\Sigma^c_t)]^n$ out of a countable number of t's in $[0,T]$.

Claim 2.(n) The Radon measure $\mathrm{div}d(t,\cdot)$, restricted to Σ_t, can have Cantor part only for a countable number of t's in $[0,T]$.

The regularity of $\mathrm{div}d$ will follow from the previous claims and the fact that $U = \Sigma \cup \Sigma^c$.

Proof of Claim 1.(n). For a fixed $(\bar{t}, \bar{x}) \in \Sigma^c$, the Hessian of L exists and is continuous in $\bar{v} := d(\bar{t}, \bar{x})$. Thus there exist $r > 0$ and a $(n+1)$-dimensional ball $B_r^{n+1}(\bar{t}, \bar{x}) \subset \Omega \setminus \Sigma$ where L and H are uniformly convex.

We can also find an open cone $C_{n+1}(\bar{t}, \bar{x}) \subset B_r^{n+1}(\bar{t}, \bar{x})$, properly containing $(\bar{t}, \bar{x})$, over which an Hamilton-Jacobi equation can be solved. Indeed, we take an n-dimensional ball as base,

$$B^n \subset (B_r^{n+1}(\bar{t}, \bar{x}))_{\bar{t}-\sigma} \subset (\Omega \setminus \Sigma)_{\bar{t}-\sigma},$$

for a certain $0 < \sigma < r$, and we fix the height of length $l \in \mathbb{R}$, $0 < l < 2r$. The height must be chosen according to the speed of propagation of the solution and such that $\bar{t} < \bar{t} - \sigma + l$.

Consider now the viscosity solution $\bar{u}$ of the Cauchy problem

$$\begin{cases} \partial_t \bar{u} + H(D_x \bar{u}) = 0 & \text{in } C_{n+1}(\bar{t}, \bar{x}), \\ \bar{u}(t-\sigma, x) = u(t-\sigma, x)\mathbb{1}_{B^n}(x), \end{cases}$$

where $\mathbb{1}_E(x)$ is the indicator function of the set E. Note that $u(t,x) = \bar{u}(t,x)$ on $C_{n+1}(\bar{t}, \bar{x})$.

Thanks to the uniform convexity of H over $C_{n+1}(\bar{t}, \bar{x})$, the main theorem of [5] ensures that the vector field

$$\bar{d}(t, \cdot) := H_p(D_x \bar{u}(t, \cdot))$$

is SBV out of a countable number of t's in $[\bar{t} - \sigma, \bar{t} - \sigma + l]$.

The vector fields $d(t, \cdot)$ and $\bar{d}(t, \cdot)$ are both BV and coincide on $(C_{n+1}(\bar{t}, \bar{x}))_t$, thus, for Proposition 2.3,

$$D_x d(t, \cdot) = D_x \bar{d}(t, \cdot).$$

Therefore $d(t, \cdot)$ belongs to $SBV((C_{n+1}(\bar{t}, \bar{x}))_t)$ out of a countable number of t's in $[\bar{t} - \sigma, \bar{t} - \sigma + l]$.

Finally, using the fact that $\mathbb{R}^n$ is a countable union of bounded sets, we can apply Besicovitch covering Theorem, see [3], to prove that the set Σ^c can be fully covered by a countable number of cones C_{n+1}^i, for $i \in \mathbb{N}$, with the property stated above. Thus $d(t, \cdot)$ belongs to $[SBV(\Sigma_t^c)]^n$ out of a countable number of t's in $[0, T]$. □

We consider now the behavior of div$d$ on the set Σ. In order to prove Claim 2.(n), in the n-dimensional case, $n > 2$, we need some other hypothesis on L and its restriction to the set of points where L is not twice differentiable. No additional hypotheses are needed in the case $n = 2$.

Proof of Claim 2.(n). **2-dimensional case.** First, suppose V is a single straight line. Without loss of generality we can fix $V = \{v \in \mathbb{R}^2 | \; v_1 = 0\}$.

Call $L_V : \mathbb{R} \to \mathbb{R}$ the restriction of the Lagrangian L to V,

$$L_V(v_2) := L(0, v_2)$$

for any $v_2 \in \mathbb{R}$. Call $I \subset \mathbb{R}$ the set of every x_1 in $\mathbb{R}$ such that Σ_{x_1} is non empty. Note that if $(t, x_2) \in \Sigma_{x_1}$ then $(0, x_2 - td_2(t, (x_1, x_2)))$ belongs to Σ_{x_1} because $d(t, (x_1, x_2)) = (0, d_2(t, (x_1, x_2)))$.

For every $x_1 \in I$, we consider the one-dimensional Hamilton-Jacobi equation for the function $u_{x_1}(t, x_2)$.

$$\begin{cases} \partial_t u_{x_1} + H_V(D_{x_2} u_{x_1}) = 0 & \text{in } \Sigma_{x_1}, \\ u_{x_1}(0, x_2) = u(0, (x_1, x_2)) & \forall x_2 \in (\Sigma_{x_1})_0, \end{cases}$$

where $H_V(p)$ is the Hamiltonian associated to $L_V(v)$.

The viscosity solution $u_{x_1}(t, x_2)$ is equal to $u(t, (x_1, x_2))$ for every $(t, (x_1, x_2)) \in \Sigma$. Indeed

$$u_{x_1}(t, x_2) = \min_{y_2 \in \mathbb{R}} \left\{ u(0, x_1, y_2) + tL_V\left(\frac{x_2 - y_2}{t}\right) \right\} = \min_{y_2 \in \mathbb{R}} \left\{ u(0, x_1, y_2) + tL\left(0, \frac{x_2 - y_2}{t}\right) \right\} = u(t, (x_1, x_2)),$$

where the last equality follows from the fact that, for (t, x) in Σ, the unique minimizer in the representation formula (2.3) is $y = (x_1 - td_1(t, x), x_2 - td_2(t, x))$ and $d(t, x) = (0, d_2(t, x))$.

Let us define as usual

$$d_{x_1}(t, x_2) := (H_V)_{p_2}(D_{x_2} u_{x_1}(t, x_2))$$

and

$$\mu_{x_1}(t, \cdot) := \frac{\partial}{\partial x_2} d_{x_1}(t, \cdot).$$

The vector field $d_{x_1}(t, \cdot)$ is one-dimensional. Hence, for Theorem 3.1, $d_{x_1}(t, \cdot)$ belongs to $BV((\Sigma_{x_1})_t)$ for any $x_1 \in I$, for any $t \in [0, T]$.

On the set $\Sigma \subset U$, the matrix of Radon measures $D_x d$ has no jump part. Moreover, since Σ_t is contained on the set $\{x|\ d_1(t,x)=0\}$ and $d(t,\cdot)$ is BV, Proposition 2.4 implies

$$\frac{\partial}{\partial x_1} d_1(t,\Sigma_t)=0 \quad \text{and} \quad \frac{\partial}{\partial x_2} d_1(t,\Sigma_t)=0.$$

Therefore

$$\mathrm{div} d(t,\cdot)=\frac{\partial}{\partial x_2} d_2(t,\cdot) \quad \text{on } \Sigma_t.$$

For every $(t,x)\in\Sigma$, $u_{x_1}(t,x_2)=u(t,(x_1,x_2))$ implies

$$d_2(t,x)=d_{x_1}(t,x_2).$$

The vector field $d_2(t,(x_1,\cdot))$ is a one-dimensional restriction of $d_2(t,\cdot)$ thus, for Proposition 2.5, belongs to $BV((\Sigma_{x_1})_t)$ for $\mathcal{H}^1$-a.e. $x_1\in I$. Since even $d_{x_1}(t,\cdot)$ is BV on $(\Sigma_{x_1})_t$, Proposition 2.3 implies

$$\frac{\partial}{\partial x_2} d_2(t,(x_1,\cdot))=\frac{\partial}{\partial x_2} d_{x_1}(t,\cdot)$$

for $\mathcal{H}^1$-a.e. $x_1\in I$. Therefore taken a Borel set $A\subset\Sigma_t$ and any $\phi\in C_c^\infty(\Sigma_t)$,

$$\int_A \phi(x)d\mu(t,x)=\int_I\int_{A_{x_1}} \phi(x)d\mu_{x_1}(t,x_2)dx_1.$$

Thanks to the convexity of L_V, we can apply Theorem 4.1 to $\mu_{x_1}(t,\cdot)$ and obtain the following estimates.

For any $\tau>0$, let A be a Borel set in Σ_t, for $t\in(\tau,T]$. Then for any $0\le\delta\le t-\tau$, and every section A_{x_1}, for $x_1\in I$, we have

$$\mathcal{H}^1(X^{x_1}_{t,\tau}(A_{x_1}))\ge\mathcal{H}^1(A_{x_1})-(t-\tau)\mu_{x_1}(t,A_{x_1}),$$

$$\mathcal{H}^1(X^{x_1}_{t,\tau+\delta}(A_{x_1}))\ge\frac{t-(\tau+\delta)}{t-\tau}\mathcal{H}^1(X^{x_1}_{t,\tau}(A_{x_1})).$$

Here we denote with $X^{x_1}_{t,\tau}(x_2)$ the one-dimensional map defined on $(\Sigma_{x_1})_t$

$$X^{x_1}_{t,\tau}(x_2):=x_2-(t-\tau)d_{x_1}(t,x_2).$$

The corresponding 2-dimensional map

$$X_{t,\tau}(x):=x-(t-\tau)d(t,x),$$

reduces to

$$X_{t,\tau}(x)=(x_1,X^{x_1}_{t,\tau}(x_2))$$

for every $x\in\Sigma_t$.

We can integrate the previous estimates with respect to $\mathcal{H}^1$ on $I\subset\mathbb{R}$ to recover estimates of type (3.3) and (3.4).

For any $\tau>0$, given a Borel set $A\subset\Sigma_t$, for t in $(\tau,T]$, we have

$$\mathcal{H}^2(X_{t,\tau}(A))\ge\mathcal{H}^2(A)-(t-\tau)\mu(t,A). \tag{5.1}$$

For any $\tau>0$, given a Borel set $A\subset\Sigma_t$, for t in $[\tau,T]$ and $0\le\delta\le t-\tau$ we have

$$\mathcal{H}^2(X_{t,\tau+\delta}(A))\ge\frac{t-(\tau+\delta)}{t-\tau}\mathcal{H}^2(X_{t,\tau}(A)). \tag{5.2}$$

Thus the strategy seen in the Subsection 3.1 can be easily applied to prove that $\mu(t,\cdot)$, restricted to Σ_t, can have Cantor part only for a countable number of t's in $[0,T]$.

Remark 5.3. Note that in this case nothing can be said about the Cantor part of $\frac{\partial}{\partial x_1}d_2(t,\cdot)$. Thus we cannot say that $d(t,\cdot)$ belongs to $[SBV(\Omega_t)]^2$.

Consider now the case in which V consists of a finite number of straight lines. When we consider $\mu(\cdot,\cdot)$ restricted to the points of Σ such that $d(t,x)$ belongs only to a part of one of the straight lines, we can apply the considerations done in the case where V consists only of a single straight line. On the other hand, when we consider $\mu(\cdot,\cdot)$ restricted to the points of Σ such that $d(t,x)$ belongs to an intersection point (v_1,v_2) of two, or more, straight lines, the divergence div$d(t,\cdot)$ must be zero on every Borel subset of $\{x|\ d_1(t,x)=v_1, d_2(t,x)=v_2\}$, for Proposition 2.4. Thus the measure $\mu(t,\cdot)$, restricted to Σ_t, can have Cantor part only for a countable number of t's in $[0,T]$ even when V consists of a finite number of straight lines. The case in which V is contained in a finite number of straight lines is analogous.

n-dimensional case. We prove the claim iterating a subdivision of Σ down to the dimension one.

Call $V_n := V$. At the step $n-j$, for $j=n,\dots,3$, we first suppose that V_j consists of a single $(j-1)$-dimensional plane, without loss of generality we can fix

$$V_j = \{v\in\mathbb{R}^n|\ v_1=0,\dots,v_{n+1-j}=0\}.$$

Call $L_{V_j}:\mathbb{R}^{j-1}\to\mathbb{R}$ the restriction of $L_{V_{j+1}}$ to V_j,

$$L_{V_j}(\hat v) := L_{V_{j+1}}(0,\hat v) = L(0,\dots,0,\hat v)$$

for any $\hat v\in\mathbb{R}^{j-1}$.

(HYP(j-1)) We require that the restriction L_{V_j} is twice $(j-1)$-differentiable out of the set V_{j-1},

$$V_{j-1} := \{\hat v\in\mathbb{R}^{j-1}|\ L_{V_j}(\cdot) \text{ is not twice differentiable in } \hat v\},$$

and V_{j-1} is contained in a finite number of $(j-2)$-dimensional planes.

Then we can subdivide Σ_j into two set:

$$\Sigma_{j-1} := \{(t,x)\in\Sigma_j|\ d(t,x)\in V_{j-1}\} \quad \text{and} \quad \Sigma^c_{j-1} := \Sigma_j\setminus\Sigma_{j-1}.$$

Thus, at every step, we have to prove the following claims.

Claim 1.(j-1) The Radon measure div$d(t,\cdot)$, restricted to $(\Sigma^c_{j-1})_t$, can have Cantor part only for a countable number of t's in $[0,T]$.

Claim 2.(j-1) The Radon measure div$d(t,\cdot)$, restricted to $(\Sigma_{j-1})_t$, can have Cantor part only for a countable number of t's in $[0,T]$.

Proof of Claim 1.(j-1) . We will prove it for $j=n$, in the other cases the proof is similar.

For a fixed $(\bar t,\bar x)\in\Sigma^c_{n-1}$, the Hessian of L_V exists and is continuous in $\hat v := (d_2(\bar t,\bar x),\dots,d_n(\bar t,\bar x))\in\mathbb{R}^{n-1}$. Thus there exist $r>0$ and a $(n+1)$-dimensional ball $B^{n+1}_r(\bar t,\bar x)\subset\Omega\setminus\Sigma_{n-1}$ where L_V and H_V are uniformly convex.

We can also find, as we did in the proof of Claim 1.(n), an open cone $C_{n+1}(\bar t,\bar x)\subset B^{n+1}_r(\bar t,\bar x)$ of height $[\bar t-\sigma,\bar t-\sigma+l]$, for a certain $0<\sigma<r$, $\bar t<\bar t-\sigma+l$ and base B^n, which contains properly $(\bar t,\bar x)$. On every section $(C_{n+1}(\bar t,\bar x))_{x_1}$, for every $x_1\in I := \{z\in\mathbb{R}|\ (C_{n+1}(\bar t,\bar x))_z\neq\emptyset\}$, we can consider the viscosity solution $\bar u_{x_1}$ of the $(n-1)$-dimensional Hamilton-Jacobi equation

$$\begin{cases} \partial_t\bar u_{x_1} + H_V(D_{\hat x}\bar u_{x_1}) = 0 & \text{in } (C_{n+1}(\bar t,\bar x))_{x_1}, \\ \bar u_{x_1}(\bar t-\sigma,\hat x) = u(\bar t-\sigma,x)\mathbb{1}_{B^n}(x). \end{cases}$$

As usual we define

$$\bar d_{x_1}(t,\hat x) := (H_V)_{\hat p}(D_{\hat x}\bar u_{x_1}(t,\hat x)).$$

and

$$\bar\mu_{x_1}(t,\cdot) := \mathrm{div}_{n-1}\bar d_{x_1}(t,\cdot).$$

The vector field $\bar d_{x_1}(t,\cdot)$ belongs to $[BV(((C_{n+1}(\bar t,\bar x))_{x_1})_t)]^{n-1}$ for any $x_1\in I$, for any $t\in[\bar t-\sigma,\bar t-\sigma+l]$. Indeed in every $(C_{n+1}(\bar t,\bar x))_{x_1}$ H_V is uniformly convex.

Since we have a uniform convexity constant for H_V, which holds on every $(C_{n+1}(\bar t,\bar x))_{x_1}$, for $x_1\in I$, we can arrange l small enough, eventually subdividing the cone, so that the following two estimates hold with uniform constants $C_1, C_2>0$, which do not depend on x_1.

Let $\bar t-\sigma<\tau<\bar t-\sigma+l$, let A be a Borel set in $(C_{n+1}(\bar t,\bar x))_t$, for t in $[\tau,\bar t-\sigma+l]$. Then, for any $0\le\delta\le t-\tau$ and every set A_{x_1}, for $x_1\in I$, we have

$$\mathcal{H}^{n-1}(\bar X^{x_1}_{t,\tau}(A_{x_1})) \ge C_1\mathcal{H}^{n-1}(A_{x_1}) - (t-\tau)C_2\bar\mu_{x_1}(t,A_{x_1}),$$

$$\mathcal{H}^{n-1}(\bar X^{x_1}_{t,\tau+\delta}(A_{x_1})) \ge \left(\frac{t-(\tau+\delta)}{t-\tau}\right)^{n-1}\mathcal{H}^{n-1}(\bar X^{x_1}_{t,\tau}(A_{x_1})).$$

Here the $(n-1)$-dimensional map $\bar{X}^{x_1}_{t,\tau}(\hat{x})$ is defined

$$\bar{X}^{x_1}_{t,\tau}(\hat{x}) := \hat{x} - (t-\tau)\bar{d}_{x_1}(t,\hat{x}).$$

Consider now the vector field d.

On the set $C_{n+1}(\bar{t},\bar{x}) \subset U$, the matrix of Radon measures $D_x d$ has no jump part. Moreover, since $(C_{n+1}(\bar{t},\bar{x}))_t$ is contained on the set $\{x|\ d_1(t,x)=0\}$ and $d(t,\cdot)$ is BV, Proposition 2.4 implies

$$\frac{\partial}{\partial x_j} d_1(t,(C_{n+1}(\bar{t},\bar{x}))_t) = 0 \quad \text{ for } j=1,\dots,n.$$

Therefore

$$\mathrm{div} d(t,\cdot) = \mathrm{div}_{n-1}\hat{d}(t,\cdot) \quad \text{on } (C_{n+1}(\bar{t},\bar{x}))_t,$$

$\hat{d}(t,x) := (d_2(t,x),\dots,d_n(t,x))$.

For every $(t,x) \in C_{n+1}(\bar{t},\bar{x})$, $u_{x_1}(t,x_2) = u(t,(x_1,x_2))$ implies

$$\hat{d}(t,x) = d_{x_1}(t,\hat{x}).$$

The vector field $\hat{d}(t,x_1,\cdot)$, being a $(n-1)$-dimensional section of the BV vector field $d(t,\cdot)$, belongs, for Proposition 2.5, to $[BV((\Sigma_{n-1})_{x_1})]^{n-1}$ for $\mathcal{H}^1$-a.e. x_1 such that $(\Sigma_{n-1})_{x_1}$ is non empty.

Since even $\bar{d}_{x_1}(t,\cdot)$ is BV on $(C_{n+1}(\bar{t},\bar{x}))_t$, Proposition 2.3 implies

$$\mathrm{div}_{n-1}\hat{d}(t,(x_1,\cdot)) = \mathrm{div}_{n-1}\bar{d}_{x_1}(t,\cdot)$$

for almost every x_1 such that $(\Sigma_{n-1})_{x_1}$ is non empty. Therefore taken a Borel set $A \subset (C_{n+1}(\bar{t},\bar{x}))_t$ and any $\phi \in C^\infty_c((C_{n+1}(\bar{t},\bar{x}))_t)$,

$$\int_A \phi(x) d\mu(t,x) = \int_I \int_{A_{x_1}} \phi(x) d\bar{\mu}_{x_1}(t,\hat{x}) dx_1.$$

Moreover, for every $x \in (C_{n+1}(\bar{t},\bar{x}))_t$

$$X_{t,\tau}(x) = x - (t-\tau)d(t,x) = (x_1, \bar{X}^{x_1}_{t,\tau}(\hat{x})).$$

The uniformity on every A_{x_1} allow us to integrate with respect to $\mathcal{H}^1$, over the set I, to obtain the following estimates.

Let $\bar{t}-\sigma < \tau < t$, let A be a Borel set in $(C_{n+1}(\bar{t},\bar{x}))_t$, for t in $[\bar{t}-\sigma, \bar{t}-\sigma+l]$. Then for any $0 \le \delta \le t-\tau$, it holds

$$\mathcal{H}^n(X_{t,\tau}(A)) \ge C_1\mathcal{H}^n(A) - (t-\tau)C_2\mu(t,A),$$

$$\mathcal{H}^n(X_{t,\tau+\delta}(A)) \ge \left(\frac{t-(\tau+\delta)}{t-\tau}\right)^{n-1} \mathcal{H}^n(X_{t,\tau}(A).$$

Therefore, repeating the standard procedure seen in Subsection 3.1, we can prove that $\mu(t,\cdot) := \mathrm{div} d(t,\cdot)$ has Cantor part only for a countable number of t's in $[\bar{t}-\sigma, \bar{t}-\sigma+l]$.

Finally, using again Besicovitch Theorem, the set Σ^c_{n-1} can be fully covered by a countable number of cones C^i_{n+1} for $i \in \mathbb{N}$ with the property stated above. Thus the Radon measure $\mathrm{div} d(t,\cdot)$ can have Cantor part on $(\Sigma^c_{n-1})_t$ only for a countable number of t's in $[0,T]$. □

We iterate the procedure subdividing Σ_{j-1} in Σ_{j-2} and Σ^c_{j-2}. Hence to prove Claim 2.(j-1) is enough to prove Claim 2.(2), i.e. for $j=3$.

Claim 2.(2) The Radon measure $\mathrm{div} d(t,\cdot)$, restricted to $(\Sigma_2)_t$, can have Cantor part only for a countable number of t's in $[0,T]$.

Proof. The proof is equal to the one done in the 2-dimensional case. We rewrite it with the notation which applies in this case.

First, suppose V_2 is a single straight line. Without loss of generality we can fix

$$V_2 = \{v \in \mathbb{R}^n|\ v_1 = 0,\dots,v_{n-1}=0\}.$$

Recall that V_2 is a straight line in $V_3 = \{v \in \mathbb{R}^n|\ v_1=0,\dots,v_{n-2}=0\}$.

Call $L_{V_2} : \mathbb{R} \to \mathbb{R}$ the restriction of the Lagrangian L_{V_3} to V_2,

$$L_{V_2}(v_n) := L_{V_3}(0,v_n) = L(0,\dots,0,v_n)$$

for any $v_n \in \mathbb{R}$. For $i = 1 \dots, n-1$, call $I_i \subset \mathbb{R}$ the set of every x_i in $\mathbb{R}$ such that $(\Sigma_2)_{x_i}$ is non empty and $I := I_1 \times \cdots \times I_{n-1} \subset \mathbb{R}^{n-1}$.

For every $(x_1, \dots, x_{n-1}) \in I$, we consider the one-dimensional Hamilton-Jacobi equation for the function $u_{x_1,\dots,x_{n-1}}(t, x_n)$.

$$\begin{cases} \partial_t u_{x_1,\dots,x_{n-1}} + H_{V_2}(D_{x_n} u_{x_1,\dots,x_{n-1}}) = 0 & \text{in } (\Sigma_2)_{x_1,\dots,x_{n-1}}, \\ u_{x_1,\dots,x_{n-1}}(0, x_n) = u(0,(x_1,\dots,x_n)) & \forall x_n \in ((\Sigma_2)_{x_1,\dots,x_{n-1}})_0, \end{cases}$$

where $H_{V_2}(p_n)$ is the Hamiltonian associated to $L_{V_2}(v_n)$.

The viscosity solution $u_{x_1,\dots,x_{n-1}}(t, x_n)$ is equal to $u(t,(x_1,\dots,x_n))$ for $(t,(x_1,\dots,x_n)) \in \Sigma_2$. Indeed

$$u_{x_1,\dots,x_{n-1}}(t, x_n) = \min_{y_n \in \mathbb{R}} \left\{ u(0,(x_1,\dots,x_{n-1},y_n)) + tL_{V_2}\left(\frac{x_n - y_n}{t}\right)\right\} = u(t,(x_1,\dots,x_n)),$$

where the last equality follows from the fact that, for (t,x) in Σ_2, the unique minimizer in (2.3) is $y = (x_1 - td_1(t,x),\dots,x_n - td_n(t,x))$ and $d(t,x) = (0,\dots,0,d_n(t,x))$ on Σ_2.

Let us define as usual

$$d_{x_1,\dots,x_{n-1}}(t,x_n) := (H_{V_2})_{p_n}(D_{x_n} u_{x_1,\dots,x_{n-1}}(t,x_n)),$$

and

$$\mu_{x_1,\dots,x_{n-1}}(t,\cdot) := \frac{\partial}{\partial x_n} d_{x_1,\dots,x_{n-1}}(t,\cdot).$$

The vector field $d_{x_1,\dots,x_{n-1}}(t,\cdot)$ is one-dimensional. Hence, for Theorem 3.1, $d_{x_1,\dots,x_{n-1}}(t,\cdot)$ belongs to $BV(((\Sigma_2)_{x_1,\dots,x_{n-1}})_t)$ for any $(x_1,\dots,x_{n-1}) \in I$, for any $t \in [0,T]$.

On the set $\Sigma_2 \subset U$, the matrix of Radon measures $D_x d$ has no jump part. Moreover, since $(\Sigma_2)_t$ is contained on the set $\{x \mid d_1(t,x) = 0,\dots,d_{n-1}(t,x) = 0\}$ and $d(t,\cdot)$ is BV, Proposition 2.4 implies

$$\frac{\partial}{\partial x_l} d_i(t,(\Sigma_2)_t) = 0 \quad \text{for } i = 1,\dots,n-1 \text{ and } l = 1,\dots,n.$$

Therefore

$$\mathrm{div} d(t,\cdot) = \frac{\partial}{\partial x_n} d_n(t,\cdot) \quad \text{on } (\Sigma_2)_t.$$

For every $(t,x) \in \Sigma_2$, $u_{x_1,\dots,x_{n-1}}(t,x_n) = u(t,(x_1,\dots,x_n))$ implies

$$d_n(t,x) = d_{x_1,\dots,x_{n-1}}(t,x_n).$$

The vector field $d_n(t,(x_1,\dots,x_{n-1},\cdot))$ is a one-dimensional restriction of $d_n(t,\cdot)$ thus, for Proposition 2.5, belongs to $BV(((\Sigma_2)_{x_1,\dots,x_{n-1}})_t)$ for almost every $(x_1,\dots,x_{n-1}) \in I$. Since even $d_{x_1,\dots,x_{n-1}}(t,\cdot)$ is BV on $((\Sigma_2)_{x_1,\dots,x_{n-1}})_t$, Proposition 2.3 implies

$$\frac{\partial}{\partial x_n} d_n(t,(x_1,\dots,x_{n-1},\cdot)) = \frac{\partial}{\partial x_n} d_{x_1,\dots,x_{n-1}}(t,\cdot)$$

for $\mathcal{H}^{n-1}$-a.e. $(x_1,\dots,x_{n-1}) \in I$. Therefore taken a Borel set $A \subset (\Sigma_2)_t$ and any $\phi \in C_c^\infty((\Sigma_2)_t)$,

$$\int_A \phi(x) d\mu(t,x) = \int_I \int_{A_{x_1,\dots,x_{n-1}}} \phi(x) d\mu_{x_1,\dots,x_{n-1}}(t,x_n) d(x_1,\dots,x_{n-1}).$$

Thanks to the convexity of L_{V_2}, we can apply Theorem 4.1 to $\mu_{x_1,\dots,x_{n-1}}(t,\cdot)$ and obtain the following estimates.

For any $\tau > 0$, let A be a Borel set in $(\Sigma_2)_t$, $t \in (\tau,T]$. Then for any $0 \le \delta \le t-\tau$ and every section $A_{x_1,\dots,x_{n-1}}$, for $(x_1,\dots,x_{n-1}) \in I$, we have

$$\mathcal{H}^1(X_{t,\tau}^{x_1,\dots,x_{n-1}}(A_{x_1,\dots,x_{n-1}})) \ge \mathcal{H}^1(A_{x_1,\dots,x_{n-1}}) - (t-\tau)\mu_{x_1,\dots,x_{n-1}}(t, A_{x_1,\dots,x_{n-1}}),$$

$$\mathcal{H}^1(X_{t,\tau+\delta}^{x_1,\dots,x_{n-1}}(A_{x_1,\dots,x_{n-1}})) \ge \frac{t-(\tau+\delta)}{t-\tau}\mathcal{H}^1(X_{t,\tau}^{x_1,\dots,x_{n-1}}(A_{x_1,\dots,x_{n-1}})).$$

Here we denote with $X_{t,\tau}^{x_1,\dots,x_{n-1}}(x_n)$ the one-dimensional map defined on $((\Sigma_2)_{x_1,\dots,x_{n-1}})_t$

$$X_{t,\tau}^{x_1,\dots,x_{n-1}}(x_n) := x_n - (t-\tau) d_{x_1,\dots,x_{n-1}}(t,x_n).$$

The corresponding n-dimensional map defined on $(\Sigma_2)_t$

$$X_{t,\tau}(x) := x - (t-\tau) d(t,x),$$

reduces to

$$X_{t,\tau}(x) = (x_1, \ldots, x_{n-1}, X_{t,\tau}^{x_1,\ldots,x_{n-1}}(x_n))$$

for every $x \in (\Sigma_2)_t$.

We can integrate the previous estimates with respect to $\mathcal{H}^{n-1}$ over I to recover estimates of type (3.3) and (3.4). For any $\tau > 0$, given a Borel set $A \subset (\Sigma_2)_t$, for t in $[\tau, T]$, we have

$$\mathcal{H}^n(X_{t,\tau}(A)) \geq \mathcal{H}^n(A) - (t-\tau)\mu(t, A). \tag{5.3}$$

For any $\tau > 0$, given a Borel set $A \subset \Sigma_t$, for t in $[\tau, T]$ and $0 \leq \delta \leq t - \tau$ we have

$$\mathcal{H}^n(X_{t,\tau+\delta}(A)) \geq \frac{t-(\tau+\delta)}{t-\tau}\mathcal{H}^n(X_{t,\tau}(A)). \tag{5.4}$$

Thus the strategy seen in the Subsection 3.1 can be easily applied to prove that $\mu(t,\cdot)$, restricted to $(\Sigma_2)_t$, can have Cantor part only for a countable number of t's in $[0, T]$.

Consider now the case in which V_2 consists of a finite number of straight lines. When we consider $\mu(\cdot,\cdot)$ restricted to the points of Σ_2 such that $d(t,x)$ belongs only to a part of one of the straight lines, we can apply the considerations done in the case where V_2 consists only of a single straight line. On the other hand, when we consider $\mu(\cdot,\cdot)$ restricted to the points of Σ_2 such that $d(t,x)$ belongs to an intersection point of two, or more, straight lines, the divergence $\mathrm{div} d(t,\cdot) = \mu(t,\cdot)$ must be zero on every Borel set, as seen in the 2-dimensional case. The case in which V_2 is contained in a finite number of straight lines is analogous.

Thus the measure $\mu(t,\cdot)$ can have Cantor part only for a countable number of t's in $[0, T]$ even when V_2 consists of a finite number of straight lines. $\square$

Once Claim 2.(2) is proved, we can iteratively prove all the others Claims 2.(j-1) for $j = 4, \cdots, n$ just by repeating the same considerations for the general case in which V_j consists of a finite union of $(j-1)$-dimensional planes. This case can be treated as usual distinguishing the two cases. When we consider $\mu(\cdot,\cdot)$ restricted to the points of Σ_j such that $d(t,x)$ belongs only to a part of one of the $(j-1)$-dimensional planes, we can apply the considerations done in the case where V_j consists only of a single $(j-1)$-dimensional plane. On the other hand, when we consider $\mu(\cdot,\cdot)$ restricted to the points of Σ_j such that $d(t,x)$ belongs to a $(j-2)$-dimensional plane intersection of two, or more, $(j-1)$-dimensional planes, we can reduce the problem to the $(j-2)$-dimensional case. Indeed in this case we can apply again the iterative proof. The case in which V_j is contained in a finite number of $(j-1)$-dimensional planes is analogous.

The considerations above done for $j = n+1$ concludes even the proof of Claim 2.(n). $\square$

Let us recall all the necessary assumptions.

Suppose H is $C^2(\mathbb{R}^n)$ convex and

$$\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty.$$

(HYP(0)) The vector field $d(t,\cdot)$ belongs to $[BV(\Omega_t)]^n$ for every $t \in [0, T]$.

Define V_{π_n} as

$$V_{\pi_n} := \{v \in \mathbb{R}^n |\ L(\cdot) \text{ is not twice differentiable in } v\},$$

and

$$\Sigma_{\pi_n} := \{(t,x) \in U |\ d(t,x) \in V_{\pi_n}\} \quad \text{and} \quad \Sigma^c_{\pi_n} := U \setminus \Sigma_{\pi_n}.$$

(HYP(n)) We suppose V_{π_n} to be contained in a finite union of hyperplanes Π_{π_n}.

For $j = n, \ldots, 3$ for any $(j-1)$-dimensional plane π_{j-1} in Π_{π_j}, let $L_{\pi_{j-1}} : \mathbb{R}^{j-1} \to \mathbb{R}$ be the $(j-1)$-dimensional restriction of L to π_{j-1} and

$$V_{\pi_{j-1}} := \{v \in \mathbb{R}^{j-1} |\ L_{\pi_{j-1}}(\cdot) \text{ is not twice differentiable in } v\}.$$

Define

$$\Sigma_{\pi_{j-1}} := \{(t,x) \in \Sigma_{\pi_j} |\ d(t,x) \in V_{\pi_j}\} \quad \text{and} \quad \Sigma^c_{\pi_{j-1}} := \Sigma_{\pi_j} \setminus \Sigma_{\pi_{j-1}}.$$

(HYP(j-1)) We suppose $V_{\pi_{j-1}}$ to be contained in a finite union of $(j-2)$-dimensional planes $\Pi_{\pi_{j-1}}$, for every $\pi_{j-1} \in \Pi_{\pi_j}$.

Remark 5.4. There is no need to ask any assumption on the one-dimensional restriction of L to a straight line in any of the V_{π_2} for a plane π_2, since in the one-dimensional case the SBV regularity is proven without any other assumptions on L.

Theorem 5.5. *With the above assumptions (HYP(0)),(HYP(n)),...,(HYP(2)), the Radon measure* $\operatorname{div}d(t,\cdot)$ *has Cantor part on* Ω_t *only for a countable number of* t*'s in* $[0,T]$.

The following corollaries are easily obtained from Theorem 5.5.

Example 5.6. The Hamiltonian

$$H(p) := \sum_{i=1}^{n-1} \frac{(p_i)^4}{12} + \frac{(p_n)^2}{2}$$

is such that the hypothesis (HYP(n)),..., (HYP(2)) are satisfied. Indeed the corresponding Lagrangian

$$L(v) = \sum_{i=1}^{n-1} \frac{11}{12} v_i (3v_i)^{\frac{1}{3}} + \frac{(v_n)^2}{2}$$

is not twice differentiable on the set $V = \{v \in \mathbb{R}^n |\ v_1 = 0\} \cup \cdots \cup \{v \in \mathbb{R}^n |\ v_{n-1} = 0\}$ which is a finite union of hyperplanes. Every restriction on one of these hyperplanes is not twice differentiable on a finite union of $(n-2)$-planes and so on.

Corollary 5.7. *Let* $D_x u(t,\cdot)$ *belongs to* $[BV(\Omega_t)]^n$ *for every* $t \in [0,T]$ *and let L satisfy (HYP(n)),..., (HYP(2)), then the Radon measure* $\operatorname{div}d(t,\cdot)$ *has Cantor part on* Ω_t *only for a countable number of* t*'s in* $[0,T]$.

Proof. If $D_x u(t,\cdot)$ belongs to $[BV(\Omega_t)]^n$ for every $t \in [0,T]$, then $d(t,\cdot) = H_p(D_x u(t,\cdot))$ belongs to $[BV(\Omega_t)]^n$ for every $t \in [0,T]$. □

Corollary 5.8. *Let* $u(0,\cdot)$ *be semiconcave and let L satisfy (HYP(n)),..., (HYP(2)), then the Radon measure* $\operatorname{div}d(t,\cdot)$ *has Cantor part on* Ω_t *only for a countable number of* t*'s in* $[0,T]$.

Proof. It follows from Proposition 2.16. □

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