2×2 Systems of Conservation Laws with $\mathbb{L}^\infty$ Data

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Abstract

Consider a hyperbolic system of conservation laws with genuinely nonlinear characteristic fields. We extend the classical Glimm-Lax result [11, Theorem 5.1] proving the existsnce of solution for $\mathbb{L}^\infty$ initial datum, relaxing the assumptions taken therein on the geometry of the shock–rarefaction curves.

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1 Introduction

Consider the following non-linear 2×2 system of conservation laws

$$\partial_t u + \partial_x \left[f(u)\right] = 0 \tag{1.1}$$

and the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x \left[f(u)\right] = 0 \\ u(0,x) = \bar{u}(x) \end{cases} \tag{1.2}$$

Our aim is to extend the classical result [11, Theorem 5.1] relaxing the assumptions taken therein on the geometry of the shock–rarefaction curves. More precisely, as is well known, the assumptions in [11] ensure that the interaction of two shocks of the same family yields a shock of that family and a *rarefaction* of the other family. Here, no assumption whatsoever of this kind is assumed. Nevertheless, the result of Theorem 1.1 is the same of that in [11, Theorem 5.1], namely the existence of a weak entropy solution to (1.2) for all initial data with sufficiently small $\mathbb{L}^\infty$ norm.

On the flow f in (1.1) we assume the following Glimm-Lax condition, analogously to [11, formula (1.4)]:

(GL) $f\colon B(0,r) \to \mathbb{R}^2$, for a suitable $r > 0$, is smooth with $Df(0)$ strictly hyperbolic and with both characteristic fields genuinely non linear.

The main result of this paper is the following:

Theorem 1.1 *Under the assumption* ***(GL)****, there exists a sufficiently small $\eta > 0$ such that for every initial condition $\bar{v} \in \mathbb{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^2)$ with:*

$$\left\|\bar{v}\right\|_\infty \le \eta \tag{1.3}$$

the Cauchy problem (1.2) admits a weak entropy solution for all $t \ge 0$.

The solution is constructed as limit of the ε–approximate solutions v^ε constructed through the front tracking algorithm as in [6]. First, as in [11], careful decay estimates on a trapezoid (see Figure 1) allow to bound the positive variation and the $\mathbb{L}^\infty$ norm of v^ε on the upper side of the trapezoid. Under the further assumption that a suitable $\mathbb{L}^\infty$ estimate on v^ε holds, see condition **(A)** below, a technique based on the hyperbolic rescaling allows to extend the previous bound to any positive time. The approximate solutions can hence be defined globally in time.

A key point is now to provide estimates that allow to abandon condition **(A)**. This is achieved through $\mathbb{L}^\infty$ estimates essentially based on the conservation form of (1.1) and on the previous results on the trapezoids. It is here that the integral estimates in Section 6 allow us to extend the result in [11].

As a byproduct, we also obtain Theorem 3.12, under the standard Lax condition

(L) $f\colon B(0,r) \to \mathbb{R}^2$, for a suitable $r > 0$, is smooth with $Df(0)$ strictly hyperbolic and each characteristic field is either genuinely non linear or linearly degenerate.

Indeed, Theorem 3.12 is an existence result valid for all initial data having small $\mathbb{L}^\infty$ norm and bounded, not necessarily small, total variation.

In this connection, we recall that in the case of systems with coinciding shock and rarefaction waves, the well posedness of (1.2) in $\mathbb{L}^\infty$ was proved in [4] under condition **(GL)**, extending the previous results [2, 3].

This paper is organized as follows. Section 2 is devoted to introduce the notation. Then, ε–approximate solutions are defined in Section 3 and suitable bounds are proved, in the case of bounded total variation. Section 4 uses the previous results to construct the ε–approximate solutions globally in time under the further assumption **(A)**. This latter assumption is abandoned in Section 5, which relies on the integral estimates in Section 6. The more technical details are collected in the final Section 7.

2 Notations

As a general reference on the theory of conservation laws, we refer to [5, 9]. Throughout, we let $B(u,r)$ be the open sphere in $\mathbb{R}^2$ centered at u with radius r.

Denote by $A(u)$ the 2×2 hyperbolic matrix $Df(u)$, by λ_1, λ_2 its eigenvalues and by l_1, l_2 (resp. r_1, r_2) its left (resp. right) eigenvectors, normalized so that

$$\left\|r_i(u)\right\| = 1, \qquad \langle l_j(u), r_i(u)\rangle = \left\{ \begin{array}{ll} 1 & j = i \\ 0 & j \neq i \end{array} \right. \qquad i,j = 1,2\,.$$

By **(GL)**, a suitable choice of the direction of the eigenvectors yields

$$\nabla\lambda_i(u)\cdot r_i(u) \geq c > 0 \quad \text{for} \quad i = 1,2 \quad \text{and} \quad u \in B(0,r) \tag{2.1}$$

for a suitable c. By **(GL)**, $\sup_{B(0,r)} \lambda_1 < \inf_{B(0,r)} \lambda_2$.

By a linear change of coordinates, we can always assume that $f(0) = 0$, $A(0) = \operatorname{diag}\big(\lambda_1(0), \lambda_2(0)\big)$ and that $\lambda_1(0) = -1$, $\lambda_2(0) = 1$. We are thus led to assume that f can be written as follows:

$$\begin{array}{rcl} f_1(u) & = & -u_1 + \dfrac{1}{2}\alpha_{11}\, u_1^2 + \alpha_{12}\, u_1\, u_2 + \dfrac{1}{2}\alpha_{22}\, u_2^2 + \mathcal{O}(1)\,\|u\|^3 \\ f_2(u) & = & u_2 + \dfrac{1}{2}\beta_{11}\, u_1^2 + \beta_{12}\, u_1\, u_2 + \dfrac{1}{2}\beta_{22}\, u_2^2 + \mathcal{O}(1)\,\|u\|^3 \end{array} \tag{2.2}$$

with $\alpha_{ij} := \dfrac{\partial^2 f_1}{\partial u_i\, \partial u_j}(0)$ and $\beta_{ij} := \dfrac{\partial^2 f_2}{\partial u_i\, \partial u_j}(0)$.

Following [5, formula (5.38)], introduce the Lax curves as the gluing of the shock and rarefaction curves:

$$L_i(u,\sigma) := \left\{ \begin{array}{ll} S_i(u,\sigma) & \sigma < 0\,, \\ R_i(u,\sigma) & \sigma \geq 0\,. \end{array} \right. \tag{2.3}$$

As in [5, formula (7.36)], call $E = E(u^-, u^+)$ the map giving the sizes of the waves in the solution to the Riemann problem for (1.1) with data u^- and u^+:

$$(\sigma_1, \sigma_2) = E(u^-, u^+) \quad \text{if and only if} \quad u^+ = L_2\left(L_1(u^-, \sigma_1), \sigma_2\right)\,.$$

Recall now the continuous version of the Glimm potentials, see [7, (1.14) and (1.15)] or [8, (4.2)–(4.4)]. For $u \in \mathbb{BV}\left(\mathbb{R}; B(0,r)\right)$ and for a measurable $\Omega \subseteq \mathbb{R}$, define the wave measures μ_i for $i = 1, 2$, as

$$\mu_i(\Omega) := \int_\Omega l_i(u)\, d\mu_c + \sum_{x\in\Omega} E_i\left(u(x-), u(x+)\right)$$

where μ_c is the continuous part of the distributional derivative of u. Then, let

$$\rho := |\mu_2| \otimes |\mu_1| + \sum_{i=1}^{2} \left(\mu_i^- \otimes \mu_i^- + \mu_i^+ \otimes \mu_i^- + \mu_i^- \otimes \mu_i^+\right) \tag{2.4}$$

and, as in [1, 5, 7, 8], set

$$\begin{array}{rcl} Q(u) & := & \rho\left(\left\{(x,y) \in \mathbb{R}^2 \colon x < y\right\}\right) \\ V(u,I) & := & |\mu_1|(I) + |\mu_2|(I) \qquad I \subseteq \mathbb{R} \text{ interval} \\ \Upsilon(u) & := & V(u,\mathbb{R}) + Q(u) \end{array}$$

where $|\mu_i|$ is the total variation of measure μ, $V(u,\mathbb{R})$ is the *total strength of waves* in u and $Q(u)$ is the *interaction potential* of u. For a $u \in \mathbb{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^2)$, define its total variation by:

$$\mathrm{TV}(u) := \sup \left\{ \sum_{i=1}^{2} \sum_{l} \left|u_i(x_{l+1}) - u_i(x_l)\right| \colon \begin{array}{l} x_1, \ldots, x_N \in \mathbb{R} \text{ with} \\ x_1 < \cdots < x_N \end{array} \right\}. \tag{2.5}$$

Obviously, the total variation and the functional $V(\cdot, \mathbb{R})$ are equivalent. In the following, for $L > 0$, it will be useful also the notation:

$$\mathrm{TV}(u; L) := \sup_{x\in\mathbb{R}} \mathrm{TV}\left(u_{|[x,x+L]}\right)$$

where $u_{|[x,x+L]}$ is the restriction of u to the interval $[x, x+L]$.

Below, we consider also the positive part of the signed measure μ_i, denoted by μ_i^+, and the positive total variation of the i–th component of u, denoted by $\mathrm{TV}^+(u_i)$.

For a function $u \colon \mathbb{R} \to B(0,r)$, we use below the $\mathbb{L}^\infty$ norm

$$\|u\|_\infty := \operatorname*{supess}_{x\in\mathbb{R}} \left|u_1(x)\right| + \operatorname*{supess}_{x\in\mathbb{R}} \left|u_2(x)\right|\,.$$

Let $\hat\lambda$ be an upper bound for the moduli of the characteristic speeds in $B(0,r)$, i.e.

$$\hat\lambda > \sup_{i=1,2;\, \|u\|\leq r} \left|\lambda_i(u)\right|\,. \tag{2.6}$$

3 Construction of Solutions with Bounded Total Variation and Small $\mathbb{L}^\infty$ Norm

In this section, we modify the wave front tracking algorithm in [6, Section 2] to construct a solution to (1.2) under the assumption that the initial datum has bounded total variation and small $\mathbb{L}^\infty$ norm. More precisely, let $\bar{u}$ belong to

$$\mathcal{D}(\eta, \bar{K}) := \left\{ u \in \mathbb{L}^1_{\text{loc}} \left(\mathbb{R}; B(0,\eta)\right) : \text{TV}(u) \leq \bar{K} \right\}, \tag{3.1}$$

where $\bar{K}, \eta$ are positive constants.

Moreover, in the first two paragraphs below, it is not necessary to assume that both characteristic fields be genuinely nonlinear. The standard Lax [13, Section 9] condition **(L)** is sufficient.

3.1 The Algorithm

Fix $\varepsilon > 0$. Denote by v the Riemann coordinates of (1.1), see [9, Definition 7.3.2], and call $\mathcal{L}_i$, $\mathcal{R}_i$ and $\mathcal{S}_i$ the Lax, the rarefaction and the shock curves in the Riemann coordinates:

$$\mathcal{L}_i(v,\sigma) := \left\{ \begin{array}{ll} \mathcal{S}_i(v,\sigma) & \sigma < 0\,, \\ \mathcal{R}_i(v,\sigma) & \sigma \geq 0\,. \end{array} \right. \tag{3.2}$$

In these variables, as in [6], we parametrize the rarefaction and the shock curves as follows:

$$\begin{array}{ll} \mathcal{R}_1(v,\sigma) = (v_1+\sigma, v_2), & \mathcal{S}_1(v,\sigma) = \left(v_1+\sigma, v_2 + \hat{\psi}_2(v,\sigma)\cdot\sigma^3\right) \\ \mathcal{R}_2(v,\sigma) = (v_1, v_2+\sigma), & \mathcal{S}_2(v,\sigma) = \left(v_1 + \hat{\psi}_1(v,\sigma)\cdot\sigma^3, v_2+\sigma\right) \end{array} \tag{3.3}$$

where $\hat{\psi}_1$ and $\hat{\psi}_2$ are suitable smooth functions of their arguments. First, the initial datum $\bar{v}$ is substituted by a piecewise constant $\bar{v}^\varepsilon$ such that:

$$\lim_{\varepsilon\to 0+} \|\bar{v}^\varepsilon - \bar{v}\|_{\mathbb{L}^1} = 0\,, \quad \text{TV}(\bar{v}^\varepsilon) \leq \text{TV}(\bar{v}) \leq \bar{K}\,, \quad \|\bar{v}^\varepsilon\|_\infty \leq \eta\,.$$

For the proof, see [6, Lemma 2 and Lemma 3]. At each point of jump in $\bar{v}^\varepsilon$, the resulting Riemann problem is solved as in [6, Section 2]. Let $\varphi \in \mathbf{C}^\infty(\mathbb{R};\mathbb{R})$ be such that

$$\begin{array}{rclcrcl} \varphi(\sigma) & = & 1 & \text{for} & \sigma & \leq & -2 \\ \varphi(\sigma) & = & 0 & \text{for} & \sigma & \geq & -1 \\ \varphi'(\sigma) & \in & [-2,0] & \text{for} & \sigma & \in & [-2,-1] \end{array}$$

and introduce the ε-approximate Lax curves

$$\mathcal{L}_i^\varepsilon(v,\sigma) = \varphi(\sigma/\sqrt{\varepsilon})\,\mathcal{S}_i(v,\sigma) + \left(1 - \varphi(\sigma/\sqrt{\varepsilon})\right)\mathcal{R}_i(v,\sigma) \quad \text{for } i = 1,2.$$

An ε–solution to the Riemann problem for (1.1) is obtained gluing ε–rarefactions and ε–shocks. ε–rarefactions of the first, respectively second, family are subsituted by rarefaction fans attaining values in $\varepsilon\mathbb{Z}\times\mathbb{R}$, respectively $\mathbb{R}\times\varepsilon\mathbb{Z}$, travelling with the characteristic speed of the state on the right of each wave. Shocks with size larger than $2\sqrt{\varepsilon}$ travel with the exact Rankine–Hugoniot speed, otherwise we assigne to these jumps an interpoleted speed λ_i^φ defined as in [6, formulæ (2.18) and (2.19)]. For every $\sigma_i < 0$, it holds

$$\lambda_i(v^+) < \lambda_i^\varphi(v^-, \sigma_i) < \lambda_i(v^-). \tag{3.4}$$

where v^- and v^+ are respectively the left and the right states and $i = 1, 2$. We refer to [6, Section 2] for further details. Gluing the solutions to the Riemann problems at the points of jump in $\bar{v}^\varepsilon$ we obtain an ε–solution defined on a non trivial time interval $[0, t_1]$, t_1 being the first time at which two or more waves interact. Any interaction yields a new Riemann problem, so that a piecewise constant ε–solution of the form

$$v^\varepsilon = \sum_\alpha v^\alpha \chi_{]x_\alpha, x_{\alpha+1}]} \qquad \text{with} \qquad v^{\alpha+1} = \mathcal{L}_2^\varepsilon\left(\mathcal{L}_1^\varepsilon(v^\alpha, \sigma_{1,\alpha}), \sigma_{2,\alpha}\right) \tag{3.5}$$

is recursively extended to any time $t > 0$. Hence, we obtain a sequence of ε–approximate solutions. Here, the meaning of by ε–approximate solutions is slightly different from that in [6, Definition 1], namely:

Definition 3.1 *A piecewise constant function $v^\varepsilon = v^\varepsilon(t, x)$ is an ε–approximate solution if all its lines of discontinuities are ε–admissible wave fronts.*

By an ε–admissible wavefront of the first family we mean a line $x = x(t)$ across which a function v^ε has a jump, say with $v^- = (v_1^-, v_2^-)$, $v^+ = (v_1^+, v_2^+)$, satisfying the following conditions:

- *If $v_1^+ \geq v_1^-$, then $v_2^+ = v_2^-$ and*

$$v_1^+ \leq v_1^- + \varepsilon, \quad \dot{x} = \lambda_1(v^+) \tag{3.6}$$

- *If $v_1^+ \leq v_1^-$, then $v^+ = \mathcal{L}_1^\varepsilon(v^-, \sigma_1)$ for some $\sigma_1 < 0$ and $\dot{x}$ coincide with the speed λ_1^φ defined in [6, formula (2.19)] and satisfies*

$$\lambda_1(v^+) < \dot{x} < \lambda_1(v^-). \tag{3.7}$$

The ε–admissible wave fronts of the second family are defined in an entirely similar way.

It may happen that three or more fronts interact at the same point. But, observing that two rarefactions of the same family never interact with each other, it is sufficiently a little perturbation of the shocks speed in order to have interactions only between two waves. If this perturbation is small enough, the bound (3.7) is still true.

3.2 Existence and Properties of the Approximate Solutions

In this paragraph we show that the ε–approximate solutions constructed by the previous algorithm are well defined, see Theorem 3.10.

Throughout, by C we denote a positive constant dependent only on f and r as in **(L)**.

The following Lemma provides the standard interaction estimates.

Lemma 3.2 *There exists a positive C such that for any interaction resulting in the waves σ_1^+ and σ_2^+, the following estimates hold.*

1. *If the interacting waves are σ_1^- of the first family and σ_2^- of the second family,*

$$\left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| = C\left|\sigma_1^- \sigma_2^-\right| \left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right).$$

2. If the interacting waves σ' and σ'' both belong to the first family, we have

$$\left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| = C|\sigma'\sigma''| \left(|\sigma'| + |\sigma''|\right) .$$

3. If the interacting waves σ' and σ'' both belong to the second family, we have

$$\left|\sigma_1^+\right| + \left|\sigma_2^+ - (\sigma' + \sigma'')\right| = C|\sigma'\sigma''| \left(|\sigma'| + |\sigma''|\right) .$$

The proof is in [6, Lemma 2. and LemmaR 3.].

Assume now that the ε–approximate solution v^ε is defined up to time $T > 0$. For $i = 1, 2$, $t \in [0, T]$ and $x \in \mathbb{R}$, introduce the quantities

$$\begin{aligned} \check{\lambda}_i(t,x) &:= \min\left\{\lambda_i\left(v^\varepsilon(t,x-)\right), \lambda_i\left(v^\varepsilon(t,x+)\right)\right\} \\ \hat{\lambda}_i(t,x) &:= \max\left\{\lambda_i\left(v^\varepsilon(t,x-)\right), \lambda_i\left(v^\varepsilon(t,x+)\right)\right\} . \end{aligned}$$

For any $X \in \mathbb{R}$, the generalized i-th characteristic through (T, X) is an absolutely continuous solution $x(t)$ to the differential inclusion

$$\begin{cases} \dot{x} \in \left[\check{\lambda}_i(t,x),\ \hat{\lambda}_i(t,x)\right] \\ x(T) = X . \end{cases}$$

The *minimal* backward i-th characteristic through (T, X) is the generalized i-th characteristic such that, for $t \in [0, T]$,

$$y_i(t) := \min\left\{x(t) \colon x \text{ is a generalized } i\text{-th characteristic through } (T, X)\right\} .$$

We omit the dependence of $y_i(t)$ from (T, X). It is clear that $y_i(t)$ is well defined, for v^ε piecewise constant.

As a reference about minimal backward characteristics on exact solutions, see [9, Paragraph 10.3]. Backward characteristics on wave front tracking solutions were used, for instance, in [7, Section 4].

To estimate the norm $\left\|v^\varepsilon(T)\right\|_\infty$, for $T > 0$, we follow backward the i–coordinate v_i^ε along the minimal characteristic $y_i(t)$ through (T, X), for all $X \in \mathbb{R}$. Using the Lax inequality (3.4) and the choice adopted for the speed of rarefaction waves, we can conclude that y_i does not interact with any i–shock with size $\sigma < -\sqrt{\varepsilon}$, it can coincide on a non-trivial time interval with an i–wave coming from the left with size $\sigma \geq -\sqrt{\varepsilon}$, it can cross a wave of the other family or pass through an interaction point where a rarefaction of its family arises

Lemma 3.3 *Let $t > 0$ be such that $v_1\left(t^+, y_1(t^+)\right) \neq v_1\left(t^-, y_1(t^-)\right)$. Then, either y_1 crosses a 2–wave σ_2, and*

$$\left|v_1^\varepsilon\left(t^+, y_1(t^+)\right)\right| - \left|v_1^\varepsilon\left(t^-, y_1(t^-)\right)\right| \leq C\left|\sigma_2\right|^3 , \tag{3.8}$$

or y_1 passes through an interaction point between waves σ', σ'' of the second family and

$$\left|v_1^\varepsilon\left(t^+, y_1(t^+)\right)\right| - \left|v_1^\varepsilon\left(t^-, y_1(t^-)\right)\right| \leq C\left(|\sigma'| + |\sigma''|\right)^3 . \tag{3.9}$$

The proof directly follows from (3.3) and 3. in Lemma 3.2. An entirely analogous result holds along 2-characteristics.

The total size of the j-waves, with $j \neq i$, which may potentially interact with $y_i(t)$ after time t is given by the functionals

$$\tilde{Q}_1(t) := \sum_{\alpha:\, x_\alpha < y_1(t)} \left|\sigma_{2,\alpha}\right| \quad \text{and} \quad \tilde{Q}_2(t) := \sum_{\alpha:\, x_\alpha > y_2(t)} \left|\sigma_{1,\alpha}\right| \tag{3.10}$$

where we referred to the form (3.5) of v^ε. To estimate $\Delta\tilde{Q}_i(t)$, we analize all the cases:

Lemma 3.4 *Let $i, j = 1, 2$ and $i \neq j$. Fix $t > 0$. If at time t there is*

1. *no interaction and $y_i(t)$ does not cross any wave, then $\Delta\tilde{Q}_i(t) = 0$;*
2. *no interaction and $y_i(t)$ crosses a j–wave σ_j, then $\Delta\tilde{Q}_i(t) = -\left|\sigma_j\right|$;*
3. *an interaction between σ' and σ'', and $y_i(t)$ does not cross any wave, then $\Delta\tilde{Q}_i(t) \leq C\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right)$;*
4. *an interaction between σ' and σ'', and $y_i(t)$ crosses a j–wave σ_j, then $\Delta\tilde{Q}_i(t) \leq C\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right) - \left|\sigma_j\right|$;*
5. *an interaction between the j-waves σ' and σ'', and $y_i(t)$ crosses the interaction point, then $\Delta\tilde{Q}_i(t) \leq -\left|\sigma'\right| - \left|\sigma''\right|$.*

Proof. Points 1., 2. and 5. directly follow from the definition (3.10). Points 3. and 4. follow from Lemma 3.2 and (3.10). $\square$

Now we also define, as usual, the *total strength of waves* and the *interaction potential*:

$$V(v^\varepsilon) := \sum_{i,\alpha} \left|\sigma_{i,\alpha}\right|, \qquad Q(v^\varepsilon) := \sum_{(\sigma_{i,\alpha}, \sigma_{j,\beta}) \in \mathcal{A}} \left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|, \tag{3.11}$$

where $\mathcal{A}$ is the set of all couples of approaching wave-fronts, see [5, Paragraph 3, Section 7.3].

Proposition 3.5 *Fix a positive M'. Let the ε–approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ be defined up to time $t > 0$. At time t an interaction between two waves σ' and σ'' takes place. If $\mathrm{TV}\left(v^\varepsilon(t^-)\right) < M'$ and $\left\|v^\varepsilon(t^-)\right\|_\infty$ is sufficiently small, then v^ε can be defined beyond time t and*

$$\Delta Q(v^\varepsilon(t)) \leq -\frac{\left|\sigma'\sigma''\right|}{2}.$$

Proof. Using Lemma 3.2 and (3.11), we have

$$\begin{aligned} \Delta Q(v^\varepsilon(t)) &\leq -\left|\sigma'\sigma''\right| + C\,\mathrm{TV}\left(v^\varepsilon(t^-)\right)\left|\sigma'\sigma''\right|\left(\left|\sigma'\right| + \left|\sigma''\right|\right) \\ &\leq \left|\sigma'\sigma''\right|\left(-1 + CM'\left\|v^\varepsilon(t^-)\right\|_\infty\right) \end{aligned}$$

Choosing $\left\|v^\varepsilon(t^-)\right\|_\infty < 1/(2CM')$, we obtain

$$\Delta Q(v^\varepsilon(t)) \leq -\frac{\left|\sigma'\sigma''\right|}{2}.$$

$\square$

We introduce now the following two functionals:

$$\begin{aligned}\Upsilon^\varepsilon(t) &:= V(v^\varepsilon(t)) + K\,Q(v^\varepsilon(t)) &(3.12)\\ \Theta_i^\varepsilon(t) &:= \left(\left|v_i^\varepsilon\left(t, y_i(t)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right)\, e^{\tilde{H}\,\tilde{Q}_i(t)+H\,Q(v^\varepsilon(t))} &(3.13)\end{aligned}$$

where $i = 1, 2$, $\tilde{H}, H$ and K are positive constants to be precisely defined below.

Proposition 3.6 *Fix positive M, M'. Choose an initial datum $\bar{v}^\varepsilon$ such that $\|\bar{v}^\varepsilon\|_\infty < \eta$. Assume that the ε-approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ is defined up to time $t > 0$. If η is sufficiently small,* TV $\left(v^\varepsilon(t^-)\right) < M'$ *and $\left\|v^\varepsilon(t-)\right\|_\infty < M\|\bar{v}^\varepsilon\|$, then, there exist positive $\tilde{H}, H$ and K such that*

$$\begin{aligned}\Delta\Upsilon^\varepsilon(t) &\le 0 &(3.14)\\ \Delta\Theta_i^\varepsilon(t) &\le 0 \ \textit{for } i = 1, 2. &(3.15)\end{aligned}$$

Proof. First, we suppose that at time t there is no interaction and y_i crosses the wave σ_j. Obviously, $\Delta\Upsilon^\varepsilon = 0$ and $\left\|v^\varepsilon(t^+)\right\|_\infty = \left\|v^\varepsilon(t^-)\right\|_\infty$. Moreover:

$$\begin{aligned}&\Delta\Theta_i^\varepsilon(t)\\ \le\ & \left(\left|v_i^\varepsilon\left(t^+, y_i(t^+)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right) e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))}\\ & - \left(\left|v_i^\varepsilon\left(t^-, y_i(t)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right) e^{\tilde{H}\tilde{Q}_i(t^-)+H\,Q(v^\varepsilon(t^-))}\\ \le\ & \left(\left|v_i^\varepsilon\left(t^+, y_i(t^+)\right)\right| - \left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right|\right) e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))}\\ & + \left(\left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right)\left(e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} - e^{\tilde{H}\tilde{Q}_i(t^-)+H\,Q(v^\varepsilon(t^-))}\right)\\ \le\ & C\left|\sigma_j\right|^3 e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} - \tilde{H}\,\|\bar{v^\varepsilon}\|_\infty\left|\sigma_j\right| e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))}\\ \le\ & 0\,,\end{aligned}$$

provided $\tilde{H} \ge CM^2\,\|\bar{v^\varepsilon}\|_\infty$.

Suppose now that at time t the waves σ' and σ'' interact and y_i does not pass through the interaction point. Hence, using Lemma 3.2 and the estimate of Proposition 3.5,

$$\Delta\Upsilon^\varepsilon(t) \le C\,\left(\left|\sigma'\right| + \left|\sigma''\right|\right)\left|\sigma'\sigma''\right| - \frac{K}{2}\left|\sigma'\sigma''\right| \le 0 \qquad (3.16)$$

if $K \ge 2C\,\left(\left|\sigma'\right| + \left|\sigma''\right|\right)$. For the functional Θ_i^ε, we consider separately two cases. If $y_i(t)$ does not cross any wave at time t, we get:

$$\begin{aligned}&\Delta\Theta_i^\varepsilon(t)\\ \le\ & \left(\left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right)\\ & \left(e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} - e^{\tilde{H}\tilde{Q}_i(t^-)+H\,Q(v^\varepsilon(t^-))}\right)\\ \le\ & \|\bar{v^\varepsilon}\|_\infty\left(\tilde{H}\,\left(\left|\sigma'\right| + \left|\sigma''\right|\right) - \frac{H}{2}\right)\left|\sigma'\sigma''\right| e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))}\\ \le\ & 0\,,\end{aligned}$$

provided $H \geq 2\tilde{H}\,\left(\left|\sigma'\right| + \left|\sigma''\right|\right)$. If $y_i(t)$ crosses a j–wave:

$$
\begin{aligned}
& \Delta\Theta_i^\varepsilon(t) \\
\leq\ & \left(\left|v_i^\varepsilon\left(t^+, y_i(t^+)\right)\right| - \left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right|\right) e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} \\
& + \left(\left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right) \\
& \left(e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} - e^{\tilde{H}\tilde{Q}_i(t^-)+H\,Q(v^\varepsilon(t^-))}\right) \\
\leq\ & C\left|\sigma_j\right|^3 e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} \\
& + \|\bar{v^\varepsilon}\|_\infty \left(-\tilde{H}\left|\sigma_j\right| + \tilde{H}\,\left(\left|\sigma'\right| + \left|\sigma''\right|\right)\left|\sigma'\sigma''\right| - \frac{H}{2}\left|\sigma'\sigma''\right|\right) \\
& e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} \\
\leq\ & 0
\end{aligned}
$$

provided $\tilde{H} > CM^2\,\|\bar{v^\varepsilon}\|_\infty$ and $H \geq 2\tilde{H}\,\left(\left|\sigma'\right| + \left|\sigma''\right|\right)$.

Finally, we consider the case in which $y_i(t)$ is an interaction point where an i-rarefaction arises. Then, $\Delta\Upsilon(t) \leq 0$, as in (3.16). Concerning $\Delta\Theta_i^\varepsilon(t)$, call σ', σ'' the sizes of the interacting j-waves.

$$
\begin{aligned}
& \Delta\Theta_i^\varepsilon(t) \\
\leq\ & \left(\left|v_i^\varepsilon\left(t^+, y_i(t^+)\right)\right| - \left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right|\right) e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} \\
& + \left(\left|v_i^\varepsilon\left(t^-, y_i(t^-)\right)\right| + \|\bar{v^\varepsilon}\|_\infty\right) \\
& \left(e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} - e^{\tilde{H}\tilde{Q}_i(t^-)+H\,Q(v^\varepsilon(t^-))}\right) \\
\leq\ & C\left(\left|\sigma'\right| + \left|\sigma''\right|\right)^3\ e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} \\
& + \|\bar{v^\varepsilon}\|_\infty \left(-\tilde{H}\left(\left|\sigma'\right| + \left|\sigma''\right|\right) + \tilde{H}\,\left(\left|\sigma'\right| + \left|\sigma''\right|\right)\left|\sigma'\sigma''\right| - \frac{H}{2}\left|\sigma'\sigma''\right|\right) \\
& e^{\tilde{H}\tilde{Q}_i(t^+)+H\,Q(v^\varepsilon(t^+))} \\
\leq\ & 0
\end{aligned}
$$

provided $\tilde{H} > 4CM^2\,\|\bar{v^\varepsilon}\|_\infty$ and $H \geq 2\tilde{H}\,\left(\left|\sigma'\right| + \left|\sigma''\right|\right)$. $\square$

Proposition 3.7 *There exist positive M and C_2 such that, for all η, ε sufficiently small, if the ε-approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ corresponding to the initial datum $\bar{v}^\varepsilon \in \mathcal{D}(\eta, \bar{K})$ is defined up to time T, then, for all $t \in [0,T]$,*

$$
\mathrm{TV}\left(v^\varepsilon(t)\right) \leq C_2\bar{K} \qquad \textit{and} \qquad \left\|v^\varepsilon(t)\right\|_\infty \leq M\|\bar{v}^\varepsilon\|_\infty .
$$

Proof. Let $t \in [0,T]$. To bound the total variation, apply recursively the previous results:

$$\begin{array}{rcll} \mathrm{TV}(v^\varepsilon(t)) & \leq & C_1\, \Upsilon^\varepsilon(t) & \text{by (3.12)} \\ & \leq & C_1 \Upsilon^\varepsilon(0) & \text{by Proposition 3.6} \\ & \leq & C_2\, \mathrm{TV}(\bar{v}^\varepsilon) & \text{by (3.12)} \\ & \leq & C_2 \bar{K} & \text{by (3.1).} \end{array}$$

Similarly, to bound the $\mathbb{L}^\infty$ norm, for any $x \in \mathbb{R}$,

$$\begin{array}{rcll} \left|v_i^\varepsilon(t,x)\right| & \leq & \Theta_i^\varepsilon(t) & \text{by (3.13)} \\ & \leq & \Theta_i^\varepsilon(0) & \text{by Proposition 3.6} \\ & \leq & M \|\bar{v}^\varepsilon\|_\infty & \text{by (3.13).} \end{array}$$

for $i = 1,2$. Taking the supremum with respect to x, we obtain the desired bound. □

Hence, by the Proposition 3.7, if $\bar{v}^\varepsilon \in \mathcal{D}(\eta, \bar{K})$ and if the approximate solution v^ε can be constructed on some initial interval $[0,T]$, then $v^\varepsilon(t,\cdot) \in \mathcal{D}(M\eta, C_2\bar{K})$ for all $t \in [0,T]$. In order to prove that v^ε can actually be defined for all $t > 0$, it remains to show that the total number of wave fronts and of points of interaction remains finite. For this aim, we use the next two propositions.

Proposition 3.8 *[6, Proposition 2] Let $v^\varepsilon = v^\varepsilon(t,x)$ be an ε–approximate solution constructed by the previous algorithm, with $v^\varepsilon(t,\cdot) \in \mathcal{D}(M\eta, C_2\bar{K})$ for all $t > 0$. Then, all of the shocks with size $\sigma < -\sqrt{\varepsilon}$ are located along a finite number of polygonal lines.*

Proposition 3.9 *[6, Proposition 3] Let $v^\varepsilon = v^\varepsilon(t,x)$ be an ε–approximate solution constructed by the previous algorithm, with $v^\varepsilon(t,\cdot) \in \mathcal{D}(M\eta, \bar{K})$ for all $t > 0$. Then, the set of all points where two fronts interact has no limit point in the (t,x)–plane.*

These two propositions are proved exactly as in [6]. The above results complete the proof of the following Theorem.

Theorem 3.10 *Let **(L)** hold. Fix a positive $\bar{K}$. Then, there exist positive η and M such that for every initial condition $\bar{v} \in \mathcal{D}(\eta, \bar{K})$ and for every sufficiently small $\varepsilon > 0$, the Cauchy problem (1.2) admits an ε–approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ such that*

$$\left\|v^\varepsilon(t)\right\|_\infty \leq M \left\|\bar{v}\right\|_\infty . \tag{3.17}$$

Under condition **(GL)**, we also have the following decay estimate.

Theorem 3.11 *Let **(GL)** hold. Fix a positive $\bar{K}$. Then, there exist positive η and $\mathcal{M}$ such that for every initial condition $\bar{v} \in \mathcal{D}(\eta, \bar{K})$ and for every sufficiently small $\varepsilon > 0$, the ε–approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ to the Cauchy problem (1.2) constructed in Theorem 3.10 satisfies for all $t > 0$, for all $a, b \in \mathbb{R}$ and for $i = 1,2$:*

$$\mathrm{TV}^+\left(v_i^\varepsilon(t);[a,b]\right) \leq \frac{b-a}{c\,t} + \mathcal{M}\left(\left\|\bar{v}\right\|_\infty \mathrm{TV}\left(\bar{v};[a-\hat{\lambda}t, b+\hat{\lambda}t]\right) + \varepsilon\right) \tag{3.18}$$

with c as in (2.1) and $\hat{\lambda}$ as in (2.6).

Proof. Under the present hypotheses, we use the usual decay estimate, see [5, Theorem 10.3] or [7, Theorem 1]:

$$\begin{aligned}
\mathrm{TV}^+\left(v_i^\varepsilon(t);[a,b]\right) &\le \frac{b-a}{ct} + C\left[Q\left(\bar v_{|[a-\hat\lambda t, b+\hat\lambda t]}\right) - Q\left(v^\varepsilon(t)_{|[a,b]}\right) + \varepsilon\right]\\
&\le \frac{b-a}{ct} + C\,Q\left(\bar v_{|[a-\hat\lambda t, b+\hat\lambda t]}\right) + C\varepsilon\\
&\le \frac{b-a}{ct} + \mathcal{M}\left(\left\|\bar v\right\|_\infty \mathrm{TV}\left(\bar v;[a-\hat\lambda t, b+\hat\lambda t]\right) + \varepsilon\right)
\end{aligned}$$

completing the proof. □

3.3 Existence of Solutions

For the sake of completeness, we pass the ε–approximate solutions to the limit $\varepsilon \to 0$. This standard application of Helly compactness Theorem yields a slight extension of the wave front tracking construction exhibited in [6]. Indeed, the mere existence of solutions to (1.2) is here obtained under the assumptions that the total variation of the initial datum be bounded.

Theorem 3.12 *Let **(L)** hold. Fix a positive $\bar K$. Then, there exist positive η, M such that for all $\bar u \in \mathcal{D}(\eta, \bar K)$, the Cauchy problem (1.2) admits a weak entropy solution, which is the limit of the wave front tracking approximate solutions constructed above and satisfying*

$$\left\|v(t)\right\|_\infty \le M\,\left\|\bar v\right\|_\infty .$$

*Moreover, if also **(GL)** holds, then there exists a positive $\mathcal{M}$ such that for all $t > 0$, for all $a, b \in \mathbb{R}$ and for $i = 1, 2$,*

$$\mathrm{TV}^+\left(v_i(t);[a,b]\right) \le \frac{b-a}{c\,t} + \mathcal{M}\left\|\bar v\right\|_\infty \mathrm{TV}\left(\bar v;[a-\hat\lambda t, b+\hat\lambda t]\right)$$

with c as in (2.1) and $\hat\lambda$ as in (2.6).

Thanks to the estimates proved above, the proof is standard and, hence, omitted.

4 Construction of a Solution with small $\mathbb{L}^\infty$ norm

We now prove Theorem 1.1 in the case of initial data satisfying the stronger conditions

$$\bar v \in \mathbf{C}^1\left(\mathbb{R}; B(0,\eta)\right) \qquad \text{with} \qquad \left\|\frac{d\bar v}{dx}\right\|_\infty \le \mathcal{L} \tag{4.1}$$

see [11, i), ii) and iii) in Section 5].

We are going to use an inductive method. Define, for $m = 0, 1, 2, \ldots$ and for every $L > 0$, the m-trapezoid by

$$\triangle_m := \left\{(t,x) \in [0,+\infty[\,\times\mathbb{R}\colon \begin{array}{rcl} t & \in & [t_m, t_m+\Delta t_m] \text{ and} \\ x & \in & [-2^m L + \hat\lambda t, 2^m L - \hat\lambda t] \end{array}\right\} \tag{4.2}$$

Figure 1: Construction of the trapezoids.

see Figure 1, where:

$$t_m = (2^m - 1)L/2\hat{\lambda} \quad \text{and} \quad \Delta t_m = 2^{m-1}L/\hat{\lambda}\,. \tag{4.3}$$

The upper side of $\triangle_m$ measures $2^m L$ and the lower one $2^{m+1}L$. The upper bases of 4 trapezoids $\triangle_{m-1}$ cover the lower basis of $\triangle_m$. We denote by $\triangle_m(x)$ the translation of the m–trapezoid: $\triangle_m(x) := (0,x) + \triangle_m$. Correspondingly, we introduce the domains

$$\mathcal{D}_m(\delta, 20\frac{\hat{\lambda}}{c}) := \left\{ v \in \mathbb{L}^1_{\text{loc}}\left(\mathbb{R}; B(0,\delta)\right) : \mathrm{TV}(v; 2^{m+1}L) \le 20\frac{\hat{\lambda}}{c} \right\}. \tag{4.4}$$

4.1 Construction in the 0–Trapezoid

In this paragraph we show that we are able to construct a solution in $\triangle_0(x)$, for all $x \in \mathbb{R}$. In fact, since the initial datum satisfies (4.1), we can always choose $L > 0$ such that

$$\mathrm{TV}(\bar{v}, 2L) \le 20\hat{\lambda}/c \tag{4.5}$$

Then, with reference to (4.4), we prove the following result.

Proposition 4.1 *Let **(GL)** and (4.1) hold. Then, there exist a sufficiently small $\eta > 0$ and positive $M, \mathcal{M}$ such that for every initial condition $\bar{v} \in \mathcal{D}_0(\eta, 20\hat{\lambda}/c)$, the Cauchy problem (1.2) admits a weak entropy solution $v = v(t,x)$ defined for all $t \in [0, L/2\hat{\lambda}]$ and*

$$\begin{aligned} &\left\|v(t)\right\|_\infty \le M \left\|\bar{v}\right\|_\infty \\ &\mathrm{TV}^+\left(v_i(t); 2(L - \hat{\lambda}t)\right) \le \frac{2}{c}\,\frac{L - \hat{\lambda}t}{t} + \mathcal{M}\left\|\bar{v}\right\|_\infty \mathrm{TV}(\bar{v}; 2L)\,. \end{aligned}$$

Proof. Construct the sequence v^ε of approximate solutions to (1.2) as defined in Paragraph 3.1. For any $x \in \mathbb{R}$, apply Theorem 3.10 and Theorem 3.11 in $\triangle_0(x)$ with $\bar{K} = 20\hat{\lambda}/c$ to prove that v^ε is well defined for all $t \in [0, L/(2\hat{\lambda})]$ and satisfies the bounds (3.17) and (3.18) on any $\triangle_0(x)$.

Then, apply the Helly type theorem [9, Theorem 1.7.3] to obtain the convergence as $\varepsilon \to 0$ of a subsequence of the v^ε on all the strip $[0, L/(2\hat{\lambda})] \times \mathbb{R}$. The limit clearly satisfies the integral entropy inequality, the bounds on the $\mathbb{L}^\infty$ norm and on the positive variation. □

4.2 Construction in the m–Trapezoid

Now we prove that, if a solution v to (1.2) satisfies suitable conditions at time $t = t_m$, then this solution can be extended on all the interval $[t_m, t_{m+1}]$. We also provide suitable estimates for later use.

Proposition 4.2 *Let **(GL)** hold. Then, there exists a sufficiently small $\eta > 0$ and positive $M, \mathcal{M}$ such that if $v(t_m) \in \mathcal{D}_m(K\sqrt{\eta}, 20\hat{\lambda}/c)$, then the problem (1.1) with datum $v(t_m)$ admits a weak entropy solution $v = v(t,x)$ defined for $t \in [t_m, t_{m+1}]$ satisfying*

$$\left\|v(t)\right\|_\infty \le M \left\|\bar{v}\right\|_\infty \tag{4.6}$$

$$\mathrm{TV}^+\left(v_i(t); 2(2^m L - \hat{\lambda}t)\right) \le \frac{2}{c}\,\frac{2^m L - \hat{\lambda}t}{t - t_m} + \mathcal{M}\left\|\bar{v}\right\|_\infty \mathrm{TV}(\bar{v}; 2^{m+1}L). \tag{4.7}$$

Above, $\mathcal{D}_m(K\sqrt{\eta}, 20\hat{\lambda}/c)$ is defined in (4.4). The proof is entirely similar to that of Proposition 4.1.

4.3 Existence of a Global Solution

In this paragraph we assume the following a priory bound:

(A) Whenever it is possible to define up to time t_m a solution v to (1.2) with an initial datum satisfying (4.1), then there exists $K > 0$ such that, for all $m \in \mathbb{N}$, $\|v(t_m)\|_\infty \leq K\sqrt{\eta}$, where η is an upper bound for $\|\bar{v}\|_\infty$.

It is motivated by the recursive proof of Theorem 1.1 and by the following Proposition.

Proposition 4.3 *Suppose there exists up to time t_m a weak entropy solution $v = v(t,x)$ to (1.2) with an initial datum satisfying (4.1). Let **(GL)**, (4.5) and **(A)** hold. Then, for all sufficiently small $\eta > 0$, if $\|\bar{v}\|_\infty \leq \eta$, for all $m \in \mathbb{N}$ we have the estimate*

$$\mathrm{TV}\left(v(t_m); 2^{m+1}L\right) \leq 20\frac{\hat{\lambda}}{c}\,.$$

Proof. Condition (4.5) immediately implies the desired bound for $m = 0$.

Let $m \geq 1$ and proceed by induction. Using the definition (4.2) of $\triangle_m(x)$ and the estimate (4.7), we get:

$$\begin{aligned}
& \mathrm{TV}^+\left(v_i(t_m); 2^{m+1}L\right) \\
\leq\ & 4\,\mathrm{TV}^+\left(v_i(t_m); 2^{m-1}L\right) \\
\leq\ & \frac{2^{m+1}}{c(t_m - t_{m-1})} + 4\mathcal{M}\left\|v(t_{m-1})\right\|_\infty \mathrm{TV}\left(v(t_{m-1}); 2^m L\right) \\
\leq\ & 8\frac{\hat{\lambda}}{c} + 4\mathcal{M}\left\|v(t_{m-1})\right\|_\infty \mathrm{TV}\left(v(t_{m-1}); 2^m L\right)\,.
\end{aligned}$$

Since $\mathrm{TV}(v) \leq \left(\mathrm{TV}^+(v_1) + \mathrm{TV}^+(v_2)\right) + 2\|v\|_\infty$, we obtain:

$$\mathrm{TV}\left(v(t_m); 2^{m+1}L\right) \leq 16\frac{\hat{\lambda}}{c} + 8\mathcal{M}\left\|v(t_{m-1})\right\|_\infty \mathrm{TV}\left(v(t_{m-1}); 2^m L\right) + 2\left\|v(t_m)\right\|_\infty.$$

By **(A)** and choosing η small enough we get the thesis. □

Proof of Theorem 1.1 under condition (A).

Assume first that the initial data satisfies (4.1). By an application of Proposition 4.1, we are able to construct a solution for all $t \in [0, L/2\hat{\lambda}]$. Now, assume that a solution exists up to time t_m, with $m \geq 1$. Then, by **(A)**, we may apply Proposition 4.3 to obtain the TV bound at time t_m. Therefore, again thanks to **(A)**, we apply Proposition 4.2 to extend the solution up to time t_{m+1}. The proof is thus obtained inductively.

Consider now a general initial datum satisfying only (1.3). As in [11, Section 5], we approximate the initial datum $\bar{v}$ by a sequence of mollified data $\bar{v}_n$ such that each $\bar{v}_n$ satisfies (4.1). So, we are able to construct a sequence of solutions v_n to (1.1) releted to the initial data $\bar{v}_n$. Then by Helly's theorem we can select a subsequence that converges to a limit v, which is a weak entropy solution to (1.2). □

5 The $\mathbb{L}^\infty$ Estimate

The next step consists in proving that the a priori bound **(A)** is in fact a consequence of the other assumptions in Theorem 1.1 when the initial datum satisfies (4.1).

Proposition 5.1 *There exists a positive K such that for all initial datum $\bar{v}$ in (1.2), satisfying (1.3)–(4.1) and for all $m \in \mathbb{N}$, on the solution $v = v(t,x)$ to (1.2) the following estimate holds:*

$$\left\| v(t_m) \right\|_\infty \leq K\sqrt{\eta}\,,$$

where t_m is defined in (4.3).

Proof. For $m = 0$ the thesis holds, provided $K > \sqrt{\eta}$. Now, by induction, suppose that the theorem holds true up to $m-1$.

The lower basis of $\triangle_m$ is covered exactly by the upper basis of 4 $(m-1)$–trapezoids. Denote by T_{m-1} the union of these trapezoids. Then, divide T_{m-1} by horizontal segments $b^0_{m-1}, \ldots, b^N_{m-1}$ into N subtrapezoids, say $T^1_{m-1}, \ldots, T^N_{m-1}$. Each subtrapezoid T^j_{m-1} has height $h_N = 2^{m-2}L/(N\hat{\lambda})$, upper basis b^j_{m-1} and lower basis b^{j-1}_{m-1}, for $j = 1, \ldots, N$. Obviously, b^0_{m-1} and b^N_{m-1} are the lower and upper basis of T_{m-1}.

At least one of these trapezoids, call it T^n_{m-1}, is such that

$$\begin{aligned}
& Q\left(v\left(t_{m-1} + (n-1)h_N\right)_{|b^{n-1}_{m-1}} \right) - Q\left(v\left(t_{m-1} + nh_N\right)_{|b^n_{m-1}} \right) \\
\leq\ & \frac{1}{N}\left[Q\left(v(t_{m-1})_{|b^0_{m-1}} \right) - Q\left(v(t_m)_{|b^N_{m-1}} \right) \right] \\
\leq\ & \frac{1}{N} Q\left(v(t_{m-1})_{|b^0_{m-1}} \right) \\
\leq\ & \frac{1}{N} \left\| v(t_{m-1}) \right\|_\infty \mathrm{TV}(v(t_{m-1})) \\
\leq\ & \frac{1}{N} \left\| v(t_{m-1}) \right\|_\infty \frac{20\hat{\lambda}}{c}
\end{aligned} \tag{5.1}$$

by Proposition 4.3. Now, fix (t,x) and (t,y) on b^n_{m-1} with $x < y$. Then, using together the usual decay estimate [5, Theorem 10.3] or [7, Theorem 1] on the region T^n_{m-1}, together with (5.1), we have:

$$v_i(t,y) \quad \leq \quad v_i(t,x) + \frac{N}{L}\frac{y-x}{2^{m-2}}\frac{\hat{\lambda}}{c} + \frac{\mathcal{M}}{N}\frac{20\hat{\lambda}}{c}\left\| v(t_{m-1}) \right\|_\infty .$$

Assume $y - x < l$ and integrate in y to obtain

$$\frac{1}{l}\int_x^{x+l} v_i(t,y)\, dy \leq v_i(t,x) + \frac{N}{L}\frac{l}{2^{m-1}}\frac{\hat{\lambda}}{c} + \frac{\mathcal{M}}{N}\frac{20\hat{\lambda}}{c}\left\| v(t_{m-1}) \right\|_\infty . \tag{5.2}$$

Similarly, integrating in x, we get

$$v_i(t,y) \leq \frac{1}{l}\int_{y-l}^{y} v_i(t,x)\, dx + \frac{N}{L}\frac{l}{2^{m-1}}\frac{\hat{\lambda}}{c} + \frac{\mathcal{M}}{N}\frac{20\hat{\lambda}}{c}\left\| v(t_{m-1}) \right\|_\infty . \tag{5.3}$$

Using together (5.2) and (5.3), we obtain

$$\left|v_i(t,x)\right| \leq \frac{1}{l}\left|\int_{y-l}^{y} v_i(t,x)\,dx\right| + \frac{N}{L}\,\frac{l}{2^{m-1}}\,\frac{\hat{\lambda}}{c} + \frac{\mathcal{M}}{N}\,\frac{20\hat{\lambda}}{c}\left\|v(t_{m-1})\right\|_\infty . \tag{5.4}$$

At this point we consider three different cases, depending on which coefficients in (2.2) vanish. We defer the proofs of the corresponding integral estimates to Section 6.

1. $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$. Hence by Proposition 6.2,

$$\left|\int_l v_i(t,x)dx\right| \leq C'\eta(l + C''t) \qquad \text{for } i = 1,2 .$$

   (Note that it is this case that covers the situation considered in [11]).

2. $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = 0$. hen, using Proposition 6.3

$$\left|\int_l v_i(t,x)dx\right| \leq C'\eta(l + C''t) + C\left\|v(t)\right\|_\infty^3 t \qquad \text{for } i = 1,2 .$$

3. $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = 0$ (or $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$). Hence, by an application of Proposition 6.4:

$$\left|\int_l v_i(t,x)dx\right| \leq C'\eta(l + C''t) + C\left\|v(t)\right\|_\infty^3 t \qquad \text{for } i = 1,2 .$$

So that in the first case by (5.4) we obtain:

$$\left|v_i(t,x)\right| \;\leq\; C'\eta\left(1 + C''\frac{t}{l}\right) + \frac{N}{L}\,\frac{l}{2^{m-1}}\,\frac{\hat{\lambda}}{c} + \frac{\mathcal{M}}{N}\,\frac{20\hat{\lambda}}{c}\left\|v(t_{m-1})\right\|_\infty .$$

Setting $l/t = \sqrt{\eta}$ and using the fact that $t \leq t_m$, we have

$$\begin{aligned}
\left\|v(t)\right\|_\infty &\leq C\left(\eta + \sqrt{\eta}\right) + \frac{N}{c}\,\frac{\sqrt{\eta}t}{2^{m-1}}\,\frac{\hat{\lambda}}{L} + \frac{\mathcal{M}}{N}\,\frac{20\hat{\lambda}}{c}\left\|v(t_{m-1})\right\|_\infty \\
&\leq C\sqrt{\eta} + \frac{N}{c}\sqrt{\eta} + \frac{\mathcal{M}}{N}\,\frac{20\hat{\lambda}}{c}\left\|v(t_{m-1})\right\|_\infty \\
&\leq CN\sqrt{\eta} + \frac{C}{N}\left\|v(t_{m-1})\right\|_\infty .
\end{aligned}$$

Choosing $N = 4CM$ and $K = 4MNC$, by the inductive hypothesis, we get $\left\|v(t)\right\|_\infty \leq \frac{K}{2M}\sqrt{\eta}$. So, we can conclude:

$$\left\|v(t_m)\right\|_\infty \leq 2M\left\|v(t)\right\|_\infty \leq K\sqrt{\eta} .$$

Similarly, in the second and in the third cases,

$$\left|v_i(t,x)\right| \leq C'\eta\left(1 + C''\frac{t}{l}\right) + C\left\|v(t)\right\|_\infty^3\frac{t}{l} + \frac{N}{L}\,\frac{l}{2^{m-1}}\,\frac{\hat{\lambda}}{c} + \frac{\mathcal{M}}{N}\,\frac{20\hat{\lambda}}{c}\left\|v(t_{m-1})\right\|_\infty .$$

If $l/t = \sqrt{\eta + \|v(t)\|_\infty^3}$, we have again

$$\|v(t)\|_\infty \;\; \le \;\; CN\sqrt{\eta} + \frac{CK}{N}\sqrt{\eta}.$$

Then, choosing N and K as above, we obtain also in this case:

$$\|v(t_m)\|_\infty \le 2M\|v(t)\|_\infty \le K\sqrt{\eta}.$$

□

Obviuosly, the proof is exactly the same if, instead of $\triangle_m$, we consider a generic trapezoid $\triangle_m(x)$ for some $x \in \mathbb{R}$.

6 The Integral Estimate

Lemma 6.1 *Let $u = u(t,x)$ be the solution to (1.2) constructed in the previous sections, with an initial data satisfying (1.3) and (4.1). If $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ (respectively $\frac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$), then there exists an invariant region for the variable u_1 (respectively u_2). More precisely, there exists a positive constant $\mathcal{K}$ such that, for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, it holds:*

$$u_1(t,x) \ge -\mathcal{K}\eta\,, \qquad \textit{respectively} \quad u_2(t,x) \ge -\mathcal{K}\eta\,.$$

Proof. At first we consider the ε–approximate solutions constructed above. Let v_1 and v_2 be the corresponding Riemann coordinates. The map $\mathcal{T}\colon v = (v_1, v_2) \mapsto u = (u_1, u_2)$ is smooth and maps the origin into the origin. So, using the hypothesis $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$, lemma 7.2 implies that

$$\left[\dddot{\mathcal{S}}_2(v,\sigma) - \dddot{\mathcal{R}}_2(v,\sigma)\right]_1 = \left[\dddot{\mathcal{S}}_2(v,\sigma)\right]_1 \neq 0 \tag{6.1}$$

for v sufficiently small.
Let u^- and u^+ denote the left and the right states in a Riemann initial value problem, and let u^* denote the intermediate state, connected to u^- across a left wave and to which u^+ is connected across a right wave.
If $\left[\dddot{\mathcal{S}}_2(v,\sigma)\right]_1 \ge 0$ then we have that the Riemann invariant v_1^ε doesn't change along a right rarefaction and increases along a right shock, i.e.

$$v_1^\varepsilon(u^*) \le v_1^\varepsilon(u^+). \tag{6.2}$$

Obviously, this inequality holds also whenever the right shock has strenght less then $2\sqrt{\varepsilon}$, in fact in this case we interpolate a rarefaction and an entropic shock Using (6.2) and tha fact that $v_1^\varepsilon(0,x) = \bar{v}_1^\varepsilon(x) \le \eta$, we obtain $v_1^\varepsilon(t,x) \le \eta$ for any $t > 0$. Now, we can always assume that the map $\mathcal{T}$ is such that:

$$u_1^\varepsilon(t,x) \ge -\mathcal{K}\eta.$$

Similarly, if $\left[\dddot{\mathcal{S}}_2(v,\sigma)\right]_1 \le 0$, v_1^ε doesn't change along a right rarefaction and decreases along a right shock, i.e.

$$v_1^\varepsilon(u^*) \ge v_1^\varepsilon(u^+). \tag{6.3}$$

Now, using the fact that $v_1^\varepsilon(0,x) = \bar{v}_1^\varepsilon(x) \geq -\eta$ and (6.3), we get: $v_1^\varepsilon(t,x) \geq -\eta$ for any $t > 0$. As above, we can suppose that the map $\mathcal{T}$ is such that:

$$u_1^\varepsilon(t,x) \geq -\mathcal{K}\eta.$$

Clearly, the result still holds when we pass to the limit.
Similarly, if $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$, it holds $u_2(t,x) \geq -\mathcal{K}\eta$. □

Proposition 6.2 *Let $v = v(t,x)$ be the solution to (1.2) constructed in the previous sections, with an initial data satisfying (1.3) and (4.1). If $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$, then, for all segment l and for all $\bar{t} \geq 0$:*

$$\left| \int_l v_i(\bar{t},x)dx \right| \leq C'\eta\Big(l + C''\bar{t}\Big). \tag{6.4}$$

Proof. By an application of Lemma 6.1, we get:

$$|u_1| \leq u_1 + 2\mathcal{K}\eta \;\;,\;\; |u_2| \leq u_2 + 2\mathcal{K}\eta \tag{6.5}$$

Then, let us consider in the t,x plain the trapezoid with the lower basis l_0 equals to $[(0,x^l),(0,x^r)]$ and the upper basis l equals to $[(\bar{t},x^l+\vartheta\bar{t}),(\bar{t},x^r-\vartheta\bar{t})]$, where ϑ is positive. Then, using the Divergence Theorem

$$\begin{aligned}
&\int_l [u_1(\bar{t},x)+u_2(\bar{t},x)]dx = \int_{l_0} [u_1(0,x)+u_2(0,x)]dx \\
&- \int_{x^l}^{x^l+\vartheta\bar{t}} \left\{ [u_1(\frac{x-x^l}{\vartheta},x) + u_2(\frac{x-x^l}{\vartheta},x)] - \frac{1}{\vartheta}\,[f_1(u(\frac{x-x^l}{\vartheta},x)) + f_2(u(\frac{x-x^l}{\vartheta},x))] \right\} dx \\
&- \int_{x^r-\vartheta\bar{t}}^{x^r} \left\{ [u_1(\frac{x^r-x}{\vartheta},x) + u_2(\frac{x^r-x}{\vartheta},x)] + \frac{1}{\vartheta}\,[f_1((\frac{x^r-x}{\vartheta},x)) + f_2((\frac{x^r-x}{\vartheta},x))] \right\} dx\,.
\end{aligned} \tag{6.6}$$

Since f_1 and f_2 depend smoothly on u_1 and u_2 it holds that $|f_1|+|f_2| \leq C\,(|u_1|+|u_2|)$. Then, using this last estimate and (6.5) we get

$$\begin{aligned}
&[u_1(\frac{x-x^l}{\vartheta},x) + u_2(\frac{x-x^l}{\vartheta},x)] - \frac{1}{\vartheta}\,[f_1(u(\frac{x-x^l}{\vartheta},x)) + f_2(u((\frac{x-x^l}{\vartheta},x)] \\
&\geq \left(\left|u_1(\frac{x-x^l}{\vartheta},x)\right| + \left|u_2(\frac{x-x^l}{\vartheta},x)\right| \right)(1-\frac{C}{\vartheta}) - 2\mathcal{K}\eta
\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}
&[u_1(\frac{x^r-x}{\vartheta},x) + u_2(\frac{x^r-x}{\vartheta},x)] + \frac{1}{\vartheta}\,[f_1(u(\frac{x^r-x}{\vartheta},x)) + f_2(u(\frac{x^r-x}{\vartheta},x))] \\
&\geq [u_1(\frac{x^r-x}{\vartheta},x) + u_2(\frac{x^r-x}{\vartheta},x)] - \frac{1}{\vartheta}\,\left[\left|f_1(u(\frac{x^r-x}{\vartheta},x))\right| + \left|f_2(u((\frac{x^r-x}{\vartheta},x))\right| \right] \\
&\geq \left(\left|u_1(\frac{x^r-x}{\vartheta},x)\right| + \left|u_2(\frac{x^r-x}{\vartheta},x)\right| \right)(1-\frac{C}{\vartheta}) - 2\mathcal{K}\eta
\end{aligned} \tag{6.8}$$

We can choose $\vartheta = C$; now using (6.7) and (6.8) in the two last integrals on the right in (6.6) and (6.5) on the left, we get

$$\int_l \Big[\big|u_1(\bar t, x)\big| + \big|u_2(\bar t, x)\big| - 2\mathcal{K}\eta \Big] \, dx = \int_{l_0} \Big[\big|u_1(0, x)\big| + \big|u_2(0, x)\big| \Big] \, dx + 4\mathcal{K}C\bar t\eta$$

then

$$\left| \int_l \left[u_1(\bar t, x) + u_2(\bar t, x) \right] dx \right| \leq \int_l \Big[\big|u_1(\bar t, x)\big| + \big|u_2(\bar t, x)\big| \Big] \, dx \leq C'\eta(l + C''\bar t)$$

Since v_1 and v_2 are smooth functions of u_1 and u_2 also the inequality (6.4) is proved. □

Proposition 6.3 *Let $v = v(t, x)$ be the solution to (1.2) constructed in the previous sections, with an initial data satisfying (1.3) and (4.1). If $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = 0$, then, for all segment l and for all $\bar t \geq 0$:*

$$\left| \int_l v_i(\bar t, x) dx \right| \leq C'\eta\Big(l + C''\bar t\Big) + C\big\|v(\bar t)\big\|_\infty^3 \bar t. \tag{6.9}$$

Proof. Let us call l^- and l^+ the initial and the terminal point of l. For any curves $x^-(t)$ and $x^+(t)$ such that $x^-(\bar t) = l^-$ and $x^+(\bar t) = l^+$, by the Divergence Theorem, we get:

$$\begin{aligned} \int_l u_i(\bar t, x) dx \quad &= \quad \int_{x^-(0)}^{x^+(0)} u_i(0, x) dx + \int_0^{\bar t} [f_i(u(t, x^-(t))) - \dot x^-(t)\, u_i(t, x^-(t))] dt \\ &\quad + \int_0^{\bar t} [-f_i(u(t, x^+(t))) + \dot x^+(t)\, u_i(t, x^+(t))] dt \end{aligned}$$

for $i = 1, 2$. Hence, to obtain

$$\left| \int_l u_i(\bar t, x) dx \right| \leq C'\eta(l + C''\bar t) + C\big\|u(\bar t)\big\|_\infty^3 \bar t \tag{6.10}$$

it is sufficiently to solve on $[0, \bar t]$, up to terms of the order of $\big\|u(t)\big\|_\infty^2$, the ordinary differential equations:

$$\dot x^-(t) = \frac{f_i(u(t, x^-(t)))}{u_i(t, x^-(t))}, \qquad \dot x^+(t) = \frac{f_i(u(t, x^+(t)))}{u_i(t, x^+(t))} \tag{6.11}$$

with the initial conditions $x^\pm(\bar t) = l^\pm$. By the hypothesis $\dfrac{\partial^2 f_i}{\partial u_j^2}(0) = 0$, (6.11) admits generalized solutions $x_i^-(t)$ and $x_i^+(t)$ in the sense of Filippov (see [10, Chaper 2, Section 4]). It may happen that their graph coincide with the support of shocks of the function u on sets of positive $\mathcal{H}^1$–measure. By Proposition 7.3, there exist two Lipschitz functions $\tilde x_i^\pm$ with $\tilde x_i^\pm(\bar t) = l^\pm$ and

$$\Big\| \dot x_i^- - \dot{\tilde x}_i^- \Big\|_\infty \leq \|u\|_\infty^2, \qquad \Big\| \dot x_i^+ - \dot{\tilde x}_i^+ \Big\|_\infty \leq \|u\|_\infty^2$$

such that their graphs coincide with the shock of u on sets of zero $\mathcal{H}^1$–measure. Then, we have that (6.10) holds and, by the smoothness of v_1 and v_2, also the inequality (6.9) is proved. □

Proposition 6.4 *Let $v = v(t,x)$ be the solution to (1.2) constructed in the previous sections, with an initial data satisfying (1.3) and (4.1). If $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = 0$ (or $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$), then, for all segment l and for all $\bar{t} \geq 0$:*

$$\left| \int_l v_i(\bar{t},x)dx \right| \leq C'\eta\Big(l + C''\bar{t}\Big) + C\big\|v(\bar{t})\big\|_\infty^3 \bar{t}. \tag{6.12}$$

Proof. Let us consider $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = 0$, in fact in the opposite case the proof is exactly the same. By an application of Lemma 6.1, we get:

$$|u_1| \leq u_1 + 2\mathcal{K}\eta \tag{6.13}$$

Proceeding as in Proposition 6.2, we get:

$$\begin{aligned} \int_l u_1(\bar{t},x)dx \;&=\; \int_{l_0} u_1(0,x)dx - \int_{x^l}^{x^l+\vartheta\bar{t}} \left\{ u_1(\frac{x-x^l}{\vartheta},x) - \frac{1}{\vartheta}\, f_1(u(\frac{x-x^l}{\vartheta},x)) \right\} dx \\ &\quad - \int_{x^r-\vartheta\bar{t}}^{x^r} \left\{ u_1(\frac{x^r-x}{\vartheta},x) + \frac{1}{\vartheta}\, f_1((\frac{x^r-x}{\vartheta},x)) \right\} dx \end{aligned} \tag{6.14}$$

Since f_1 depends smoothly on u_1, it holds that $|f_1| \leq C\,|u_1|$. Then:

$$u_1(\frac{x-x^l}{\vartheta},x) - \frac{1}{\vartheta}\, f_1(u(\frac{x-x^l}{\vartheta},x)) \geq \left| u_1(\frac{x-x^l}{\vartheta},x) \right| (1 - \frac{C}{\vartheta}) - 2\mathcal{K}\eta \tag{6.15}$$

and

$$\begin{aligned} & u_1(\frac{x^r-x}{\vartheta},x) + \frac{1}{\vartheta}\, f_1(u(\frac{x^r-x}{\vartheta},x)) \\ & \geq u_1(\frac{x^r-x}{\vartheta},x) - \frac{1}{\vartheta} \left| f_1(u(\frac{x^r-x}{\vartheta},x)) \right| \geq \left| u_1(\frac{x^r-x}{\vartheta},x) \right| (1 - \frac{C}{\vartheta}) - 2\mathcal{K}\eta \end{aligned} \tag{6.16}$$

Choosing, as in Proposition 6.2 $\vartheta = C$ and using (6.15) and (6.16) in the two last integrals on the right in (6.14) and (6.13) on the left, we get

$$\int_l \Big[\big|u_1(\bar{t},x)\big| - 2\mathcal{K}\eta \Big]\, dx = \int_{l_0} \big|u_1(0,x)\big| dx + 4\mathcal{K}C\bar{t}\eta$$

then:

$$\begin{aligned} & \left| \int_l u_1(\bar{t},x)dx \right| \leq \int_l \big|u_1(\bar{t},x)\big| dx \\ & \leq C'\eta(l + C''\bar{t}) \leq C'\eta(l + C''\bar{t}) + C\big\|u(\bar{t})\big\|_\infty^3 \bar{t}. \end{aligned} \tag{6.17}$$

□

For the variable u_2 we follow exactly the same strategy used in the Proposition 6.3, so that we obtain:

$$\int_l \left|u_2(\bar{t},x)\right|dx \leq C'\eta(l+C''\bar{t})+C\left\|u(\bar{t})\right\|_\infty^3\bar{t}. \tag{6.18}$$

Now, using together (6.17) and (6.18) and the fact that v_1 and v_2 are smooth functions of u_1 and u_2 also the inequality (6.12) is proved.

7 Technical Details

Lemma 7.1 *If f is as in (2.2), then*

$$(Dr_2\, r_2)(0) = [-\alpha_{22}, 0]^T \qquad \textit{and} \qquad (Dr_1\, r_1)(0) = [-\beta_{11}, 0]^T \tag{7.1}$$

Proof. Recall the definition of the resolvent: $R(\xi,u) := (Df(u)-\xi I)^{-1}$ (see [12]). We have:

$$\begin{aligned} R(\xi,u) &= \Big(Df(0)+\big(Df(u)-Df(0)\big)-\xi I\Big)^{-1} \\ &= \big(Df(0)-\xi I\big)^{-1}\Big(I+\big(Df(u)-Df(0)\big)\big(Df(0)-\xi I\big)^{-1}\Big)^{-1} \\ &= (Df(0)-\xi I)^{-1}-(Df(0)-\xi I)^{-1}\big(Df(u)-Df(0)\big)\ (Df(0)-\xi I)^{-1}. \end{aligned}$$

Choose a closed curve Γ such that $\lambda_2(u)$ is the unique eigenvalue inside it. The projection P_2 can then be computed as:

$$P_2(u) = -\frac{1}{2\pi i}\oint_\Gamma R(\xi,u)d\xi = -\frac{1}{2\pi i}\oint_\Gamma \begin{bmatrix} -\frac{1}{\xi+1} & 0 \\ 0 & \frac{1}{1-\xi} \end{bmatrix} d\xi$$

$$+\frac{1}{2\pi i}\oint_\Gamma \begin{bmatrix} -\frac{1}{\xi+1} & 0 \\ 0 & \frac{1}{1-\xi} \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(u)+1 & \frac{\partial f_1}{\partial u_2}(u) \\ \frac{\partial f_2}{\partial u_1}(u) & \frac{\partial f_2}{\partial u_2}(u)-1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\xi+1} & 0 \\ 0 & \frac{1}{1-\xi} \end{bmatrix} d\xi$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2\pi i}\oint_\Gamma \begin{bmatrix} 0 & -\frac{\frac{\partial f_1}{\partial u_2}(u)}{(\xi+1)(1-\xi)} \\ -\frac{\frac{\partial f_2}{\partial u_1}(u)}{(\xi+1)(1-\xi)} & 0 \end{bmatrix} + \mathcal{O}\left(\frac{1}{(1-\xi)^2}\right) + \mathcal{O}\left(\frac{1}{(\xi+1)^2}\right) d\xi$$

$$= \begin{bmatrix} -1 & -\alpha_{12}u_1-\alpha_{22}u_2 \\ -\beta_{11}u_1-\beta_{12}u_2 & 1 \end{bmatrix} + \mathcal{O}\left(u^2\right)$$

Since $P_2(u) = r_2(u)\otimes l_2(u)$,

$$r_2(u) = [-\alpha_{12}u_1-\alpha_{22}u_2, 1]^T + \mathcal{O}(1)\left\|u\right\|^2. \tag{7.2}$$

Finally $Dr_2(0) = \begin{bmatrix} -\alpha_{12} & -\alpha_{22} \\ 0 & 0 \end{bmatrix}$ and $(Dr_2\, r_2)(0) = [-\alpha_{22}, 0]^T$.

To prove the second equation it's sufficient to use exactly the previous argouments. □

Lemma 7.2 *If* $\dfrac{\partial^2 f_1}{\partial u_2^2}(0) = \alpha_{22} \neq 0$, $\dfrac{\partial^2 f_2}{\partial u_1^2}(0) = \beta_{11} \neq 0$ *and condition* ***(GL)*** *holds, then*

$$\begin{aligned}
\left[\dddot{S}_2\,(0,0) - \dddot{R}_2\,(0,0)\right]_1 &= \frac{1}{2}\frac{(D\lambda_2 \cdot r_2)(Dr_2\; r_2)\cdot r_1}{\lambda_2 - \lambda_1} \neq 0\,,\\
\left[\dddot{S}_1\,(0,0) - \dddot{R}_1\,(0,0)\right]_2 &= \frac{1}{2}\frac{(D\lambda_1 \cdot r_1)(Dr_1\; r_1)\cdot r_2}{\lambda_1 - \lambda_2} \neq 0\,.
\end{aligned}$$

Proof. Let us denote by $S_2(\sigma)$ and $R_2(\sigma)$ the shock and the rarefaction curve of the second family with starting point 0, by $A(\sigma)$ the Jacobian matrix $Df(S_2(\sigma))$ and by $r_i(\sigma)$ ($l_i(\sigma)$) the right (left) eigenvector $r_i(S_2(\sigma))$ ($l_i(S_2(\sigma))$).

Differentiating three times the Rankine-Hugoniot conditions w.r.t. σ we obtain:

$$\ddot{A}\dot{S}_2 + 2\dot{A}\ddot{S}_2 + A\dddot{S}_2 = \dddot{\Lambda}S_2 + 3\dot{\Lambda}\ddot{S}_2 + 3\ddot{\Lambda}\dot{S}_2 + \Lambda\dddot{S}_2.$$

At $\sigma = 0$ it becomes

$$\ddot{A}r_2 + 2\dot{A}(Dr_2\; r_2) = +\frac{3}{2}(D\lambda_2 \cdot r_2)(Dr_2\; r_2) - A\dddot{S}_2 + 3\ddot{\Lambda}r_2 + \lambda_2\dddot{S}_2. \tag{7.3}$$

Differentiating twice w.r.t. σ the identity $Ar_2 = \lambda_2 r_2$ at $\sigma = 0$ we find

$$\begin{aligned}
&\ddot{A}r_2 + 2\dot{A}(Dr_2\; r_2) + A(D^2 r_2\; r_2)r_2 + ADr_2(Dr_2\; r_2)\\
&= (D^2\lambda_2\; r_2 \cdot r_2)r_2 + (D\lambda_2 \cdot Dr_2\; r_2)r_2 + 2(D\lambda_2 \cdot r_2)(Dr_2\; r_2) + \lambda_2(D^2 r_2\; r_2)r_2 + \lambda_2 Dr_2(Dr_2\; r_2).
\end{aligned}$$

Using (7.3) in the last equation:

$$\begin{aligned}
&(A - \lambda_2 Id)(D^2 r_2\; r_2)r_2 + (A - \lambda_2 Id)Dr_2(Dr_2\; r_2) - (A - \lambda_2 Id)\dddot{S}_2 + 3\ddot{\Lambda}r_2\\
&= (D^2\lambda_2\; r_2 \cdot r_2)r_2 + (D\lambda_2 \cdot Dr_2\; r_2)r_2 + \frac{1}{2}(D\lambda_2 \cdot r_2)(Dr_2\; r_2).
\end{aligned} \tag{7.4}$$

Then, multiplying on the left by $l_2(0)$, it holds:

$$\ddot{\Lambda} = \frac{1}{3}D(D\lambda_2 \cdot r_2)\; r_2. \tag{7.5}$$

We can now substitute (7.5) in (7.4) and obtain

$$(\lambda_2 Id - A)\dddot{S}_2 = \frac{1}{2}(D\lambda_2 \cdot r_2)(Dr_2\; r_2) + (\lambda_2 Id - A)(D^2 r_2\; r_2)r_2 + (\lambda_2 Id - A)Dr_2(Dr_2\; r_2).$$

Hence, multiplying on the left by $l_1(0) = [1, 0] = r_1^T(0)$, we have that

$$\dddot{S}_2 \cdot r_1 = \frac{1}{2}\frac{(D\lambda_2 \cdot r_2)(Dr_2\; r_2)\cdot r_1}{\lambda_2 - \lambda_1} + ((D^2 r_2\; r_2)r_2)\cdot r_1 + (Dr_2(Dr_2\; r_2))\cdot r_1.$$

Now, since $\dddot{R}_2 \cdot r^1 = ((D^2 r_2\; r_2)r_2)\cdot r_1 + (Dr_2(Dr_2\; r_2))\cdot r_1$, using (7.1) and the genuine non linearity, we can conclude that:

$$\dddot{S}_2 \cdot r_1 - \dddot{R}_2 \cdot r^1 = \frac{1}{2}\frac{(D\lambda_2 \cdot r_2)(Dr_2\; r_2)\cdot r_1}{\lambda_2 - \lambda_1} \neq 0.$$

The second part of the statement is proved using exactly the same argouments. □

Proposition 7.3 *Let $u = u(t,x)$ be a weak entropy solution to (1.2) and denote by $\{y_m(t)\}_{m\in\mathbb{N}}$ the countable family of its shocks (see [5, Section 10.3]). Setting $L(T,X) := \{\varphi \in W^{1,\infty}[0,T] : \varphi(T) = X\}$ and $J := \bigcup_m \mathrm{graph}(y_m)$, we have that the set*

$$\mathcal{F} := \{\varphi \in L : \mathcal{H}^1(\mathrm{graph}(\varphi) \cap J) = 0\}$$

is dense in $L(T,X)$ endowed with the usual norm of $W^{1,\infty}$ (i.e. $\|\varphi\|_{W^{1,\infty}} := \|\varphi\|_\infty + \|\varphi'\|_\infty$).

Proof. L is complete, being a closed subset of a complete metric space. Observe that $\mathcal{F} = \bigcap_{m,n} \mathcal{F}_{n,m}$, where:

$$\mathcal{F}_{n,m} := \left\{\varphi \in L(T;X) \colon \mathcal{H}^1(\mathrm{graph}(\varphi) \cap \mathrm{graph}(y_m)) < 1/n\right\}.$$

By Baire Theorem, see [14, Proposition 3.5.4], it is sufficient to prove that each $\mathcal{F}_{n,m}$ is an open and dense subset of $L(T,X)$.

$\mathcal{F}_{n,m}$ **is open:** Fix $\varphi \in \mathcal{F}_{n,m}$ and define

$$\begin{aligned} D_\varphi &:= \left\{(t, y_m(t)) \in [0,T] \times \mathbb{R} \colon \varphi(t) = y_m(t)\right\} \\ D_\varphi^d &:= \left\{(t, y_m(t)) \in [0,T] \times \mathbb{R} \colon \left|\varphi(t) - y_m(t)\right| \le d\right\}. \end{aligned}$$

For every $\varepsilon \in \left]0, 1/n - \mathcal{H}^1(D_\varphi)\right[$, there exists a positive δ such that $\mathcal{H}^1(D_\varphi^\delta) = 1/n - \varepsilon$. Now, consider the open ball $\mathcal{B}(\varphi,\delta)$ in the space $(L(T,X), \|\cdot\|_{W^{1,\infty}})$. For every $\psi \in \mathcal{B}(\varphi,\delta)$, we have that $\psi(t) \neq y_m(t)$ whenever $(t, y_m(t)) \in \mathbb{R}^2 \backslash D_\varphi^\delta$. In fact, if $\psi(t) = y_m(t)$ with $(t, y_m(t)) \in \mathbb{R}^2 \backslash D_\varphi^\delta$, then $\left|\varphi(t) - \psi(t)\right| > \delta$ which is impossible since $\psi \in \mathcal{B}(\varphi,\delta)$. Hence, we obtain that $D_\psi \subseteq D_\varphi^\delta$, for all $\psi \in \mathcal{B}(\varphi,\delta)$, i.e. $\mathcal{B}(\varphi,\delta) \subset \mathcal{F}_{n,m}$. By the arbitrariness of φ, we conclude that $\mathcal{F}_{n,m}$ is open.

$\mathcal{F}_{n,m}$ **is dense:** Choose a $\varphi \in L$. We show that φ can be arbitrarily approximated by functions in $\mathcal{F}_{n,m}$, hence we can assume that $\mathcal{H}^1(\mathrm{graph}(\varphi) \cap \mathrm{graph}(y_m)) \ge 1/n$. By [5, Theorem 10.4]), $\varphi - y_m$ is Lipschitz on $[0,T]$. Then, call $\mathcal{C} = \left\{t \in [0,T] \colon \varphi(t) = y_m(t)\right\}$. $\mathcal{C}$ is closed and can be represented as $\mathcal{C} = \bigcup_{k=1}^N [a_k, b_k]$, for a suitable $N \ge 1$. Define, for instance, ψ as

$$\psi(t) := \varphi(t) + \delta^2 \sum_{k=1}^N e^{-1/\left((t-a_k)^2 (b_k - t)^2\right)} \chi_{[a_k,b_k]}(t) \tag{7.6}$$

Clearly, $\psi \in \mathcal{F}_{n,m}$. Moreover $\|\varphi - \psi\|_{W^{1,\infty}} \le \delta$, for δ small. Hence, $\psi \in \mathcal{B}(\varphi,\delta)$, proving the density of $\mathcal{F}_{n,m}$ in $L(T,X)$. □

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