A DECOMPOSITION THEOREM FOR BV FUNCTIONS

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Abstract. The Jordan decomposition states that a function $f : \mathbb{R} \to \mathbb{R}$ is of bounded variation if and only if it can be written as the difference of two monotone increasing functions.

In this paper we generalize this property to real valued BV functions of many variables, extending naturally the concept of monotone functions. Our result is an extension of a result obtained by Alberti, Bianchini and Crippa.

A counterexample is given which prevents further extensions.

1. Introduction

One of the necessary and sufficient properties, which characterizes real valued BV functions of one variable, is the well-known Jordan decomposition: it states that a function $f : \mathbb{R} \to \mathbb{R}$ is of bounded variation if and only if it can be written as the difference of two monotone increasing functions.

The aim of this work is to generalize this property to real valued BV functions of many variables.

The starting point is a recent result presented in Section 7 of [1], which shows that a real Lipschitz function of many variables with compact support can be decomposed in sum of monotone functions. Precisely the authors give the following definition of monotone function

Definition 1. A function $f : \mathbb{R}^N \to \mathbb{R}^m$, which belongs to $[\mathrm{Lip}(\mathbb{R}^N)]^m$, is said to be *monotone* if the level sets $\{f = t\} := \{x \in \mathbb{R}^N |\ f(x) = t\}$ are connected for every $t \in \mathbb{R}^m$.

and state the theorem below.

Theorem 1. *Let f be a function in $\mathrm{Lip_c}(\mathbb{R}^N)$ with compact support. Then there exists a countable family $\{f_i\}_{i\in\mathbb{N}}$ of functions in $\mathrm{Lip_c}(\mathbb{R}^N)$ such that $f = \sum_i f_i$ and each f_i is monotone. Moreover there is a pairwise disjoint partition $\{A_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^N$ such that ∇f_i is concentrated on A_i.*

In the case of BV functions, which are defined $\mathcal{L}^N$-a.e., an appropriate generalization of the concept of monotone function has to involve super-level sets and the concept of indecomposable set, as given in [2].

Definition 2. A set $E \subseteq \mathbb{R}^N$ with finite perimeter is said to be *decomposable* if there exists a partition (A, B) of E such that $P(E) = P(A) + P(B)$ and both $|A|$ and $|B|$ are strictly positive. A set E is said to be *indecomposable* if it is not decomposable.

Here and in the following $|E|$ means the Lebesgue measure of E, for E measurable.

Definition 3. A function $f : \mathbb{R}^N \to \mathbb{R}$, which belongs to $L^1_{loc}(\mathbb{R}^N)$, is said to be *monotone* if the super-level sets $\{f > t\} := \{x \in \mathbb{R}^N |\ f(x) > t\}$ are of finite perimeter and indecomposable for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$.

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Differences and analogies from the case of functions of one variables arise.

On the one hand, it can be found an L^1 monotone function, which is not of bounded variation, that is a counterexample to the fact that monotonicity is a sufficient condition for being of bounded variation (Example 3.1).

On the other hand, it can be stated that a BV function is decomposable in a countable sum of monotone functions, similarly to the case of BV functions of one real variable.

The main result of the paper is the following.

Theorem 2 (Decomposition Theorem for BV functions). *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a $BV(\mathbb{R}^N)$ function. Then there exists a finite or countable family of monotone $BV(\mathbb{R}^N)$ functions $\{f_i\}_{i\in I}$, such that*

$$f = \sum_{i\in I} f_i \quad \textit{and} \quad |Df| = \sum_{i\in I} |Df_i|.$$

This decomposition is in general not unique, see Remark 2.2.

The main tool for proving this theorem is a decomposition theorem for sets of finite perimeter, presented here in the form given in [2].

Theorem 3 (Decomposition Theorem for sets). *Let E be a set with finite perimeter in $\mathbb{R}^N$. Then there exists a unique finite or countable family of pairwise disjoint indecomposable sets $\{E_i\}_{i\in I}$ such that*

$$|E_i| > 0 \quad \textit{and} \quad P(E) = \sum_{i\in I} P(E_i).$$

Moreover, denoting with

$$\mathring{E}^M := \{x \in \mathbb{R}^N | \lim_{r\to 0^+} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1\}$$

the essential interior of the set E, it holds

$$\mathcal{H}^{N-1}\big(\mathring{E}^M \setminus \bigcup_{i\in I} \mathring{E}_i^M\big) = 0$$

and the E_i's are maximal indecomposable sets, i.e. any indecomposable set $F \subseteq E$ is contained, up to $\mathcal{L}^N$-negligible sets, in some set E_i.

The property stated in Theorem 1, for which there is a disjoint partition $\{A_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^N$ such that every derivative ∇f_i of the decomposition is concentrated on A_i, is no longer preserved in the case of BV functions. Example 2.1 shows that, in general, this decomposition can generate monotone BV functions without mutually singular distributional derivatives.

Finally, we conclude the paper showing that there is no hope for a further generalization of this decomposition to vector valued BV functions, apart from the case of a function $f : \mathbb{R} \to \mathbb{R}^m$ where the analysis is straightforward. We consider Lipschitz functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ and the relative definition of monotone function. In this particular case, we can construct a counterexample showing that the decomposition property could not be true, see Example 3.2.
In fact, we show that a necessary condition for the decomposability of a Lipschitz function, from $\mathbb{R}^2$ to $\mathbb{R}^2$, is that some of its level sets must be of positive $\mathcal{H}^1$-measure. This is an additional property, which is clearly not shared by all the Lipschitz functions.

The paper is organized as follows.

In Section 2 we prove the Main Theorem and show that this decomposition can generate monotone BV functions without mutually singular distributional derivatives.

In Section 3 we give two counterexamples: the first to the fact that a monotone function is always a BV function, the second to a further extension of the Main Theorem to vector valued functions.

2. The Decomposition Theorem for BV functions from $\mathbb{R}^N$ to $\mathbb{R}$

To generalize the Jordan decomposition property, let us concentrate on functions $f : \mathbb{R}^N \to \mathbb{R}$, which belong to $BV(\mathbb{R}^N)$. From now on $N > 1$.

Since we will consider functions of bounded variation, the Definition 3 of monotone function becomes the following:

Definition 4. A BV function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be *monotone* if the super-level sets $\{f > t\} = \{x \in \mathbb{R}^N |\ f(x) > t\}$ are indecomposable, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$.

We recall that, for BV functions, the super-level set $\{f > t\}$ is of finite perimeter for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$.

We now prove the main theorem of this paper.

Proof of Theorem 2. The proof will be given in several steps.

Before entering into details, let us consider the following simple case.
Let $f = \chi_E$ with $E \subseteq \mathbb{R}^N$ a set of finite perimeter. Thanks to the Decomposition Theorem for sets, there exists a unique finite or countable family of pairwise disjoint indecomposable sets $\{E_i\}_{i \in I}$ such that

$$|E_i| > 0 \quad \text{and} \quad P(E) = \sum_{i \in I} P(E_i).$$

From the properties of these sets, it follows that the functions χ_{E_i} are $BV(\mathbb{R}^N)$ and monotone, so that the decomposition of χ_E,

$$\chi_E = \sum_{i \in I} \chi_{E_i},$$

gives $|D\chi_E| = \sum_{i \in I} |D\chi_{E_i}|$ as required.

Step 0. We can assume without loss of generality that $f \geq 0$: in the general case one can decompose f^+ and f^- separately.

Step 1. The sets $E^t := \{f > t\}$ are of finite perimeter for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}^+$, thanks to the hypothesis that f is $BV(\mathbb{R}^N)$ and coarea formula. Therefore, the Decomposition Theorem for sets gives, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}^+$, pairwise disjoint indecomposable sets $\{E_i^t\}_{i \in I_t}$ such that

$$\left| E^t \setminus \bigcup_{i \in I_t} E_i^t \right| = 0.$$

In particular, the property of maximal indecomposability yields a natural partial order relation between these sets: since $t_1 \geq t_2$ gives $E^{t_1} \subseteq E^{t_2}$, it follows that, for $\mathcal{L}^1$-a.e. $t_1 \geq t_2 \in \mathbb{R}^+$,

$$\forall i \in I_{t_1}\ \exists!\ i' \in I_{t_2} \quad \text{s.t.} \quad E_i^{t_1} \subseteq E_{i'}^{t_2} \pmod{\mathcal{L}^N}.$$

Taken a countable dense subset $\{t_j\}_{j \in J}$ of $\mathbb{R}^+$, such that, for all $j \in J$, the sets $E^j := E^{t_j}$ are of finite perimeter, the countable family $\{E_i^j\}_{j \in J, i \in I_{t_j}}$ can be equipped with the partial order relation

$$E_i^j \leq E_{i'}^{j'} \iff t_j \leq t_{j'},\ E_i^j \supseteq E_{i'}^{j'} \pmod{\mathcal{L}^N}.$$

Therefore there exists at least one maximal countable ordered sequence (here we do not need the Axiom of Choice).

Let $\{E_{i(j)}^j\}_{j \in J}$ one of these maximal countable ordered sequences.

Notice that, once one of these sequences is fixed, the index i is a function of j, by the uniqueness of the decomposition $\{E_i^j\}_{i\in I_{t_j}}$.

Step 2. Define

$$\tilde{f}(x) := \begin{cases} 0 & x \notin \bigcup_{j\in J} E^j_{i(j)} \\ \sup\{t_j |\ j\in J, x\in E^j_{i(j)}\} & \text{otherwise} \end{cases}$$

Clearly $0 \le \tilde{f}(x) \le f(x)$ for all $x \in \mathbb{R}^N$. Indeed, the set

$$\{t_j |\ j \in J, x \in E^j_{i(j)}\} \subseteq \{t_j |\ j \in J, x \in E^j\} \quad \forall x \in \mathbb{R}^N,$$

passing to the supremum one has $\tilde{f}(x) \le f(x)$ for all $x \in \mathbb{R}^N$. Moreover $f \in \mathrm{L}^1_{loc}(\mathbb{R}^N)$ and $0 \le \tilde{f} \le f$ give $\tilde{f} \in \mathrm{L}^1_{loc}(\mathbb{R}^N)$.

Step 3. Fix $t \in \mathbb{R}^+$ such that E^t is a set of finite perimeter. Define $\tilde{E}^t := \{\tilde{f} > t\}$ and let $E^t_{i(t)}$ the indecomposable component of E^t which is contained in a set $E^j_{i(j)}$ of the maximal countable ordered sequence and contains another $E^{j'}_{i(j')}$, for certain $j, j' \in J$, up to $\mathcal{L}^N$-negligible sets. This is possible for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}^+$.

Due to the maximal indecomposability property, one has that

$$E^{j'}_{i(j')} \subseteq E^t_{i(t)} \subseteq E^j_{i(j)} \ (\mathrm{mod}\ \mathcal{L}^N) \quad \forall t_{j'}, t_j,$$

where $t_{j'} > t > t_j$.
Notice that, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}^+$, there exists only one of such an $E^t_{i(t)}$ among all the indecomposable sets $E^t_i, i \in I_t$.

We show that $\tilde{E}^t = E^t_{i(t)}$ (mod $\mathcal{L}^N$), for $\mathcal{L}^1$-a.e t in $\mathbb{R}^+$, in two steps.

- First we show that $\tilde{E}^t \subseteq E^t_{i(t)}$ (mod $\mathcal{L}^N$) for $\mathcal{L}^1$-a.e t in $\mathbb{R}^+$.
  For $x \in \tilde{E}^t = \{\tilde{f} > t\}$, there exist $j_1 = j_1(x), j_2 = j_2(x)$ such that
  $$\tilde{f}(x) > t_{j_1} > t > t_{j_2} \quad \text{and} \quad x \in E^{j_1}_{i(j_1)} \cap E^{j_2}_{i(j_2)}.$$
  Since it holds for all $t_{j_1} > t > t_{j_2}$
  $$E^{j_1}_{i(j_1)} \subseteq E^t_{i(t)} \subseteq E^{j_2}_{i(j_2)} \ (\mathrm{mod}\ \mathcal{L}^N),$$
  it follows that for $\mathcal{L}^N$-a.e $x \in \tilde{E}^t$ it holds $x \in E^t_{i(t)}$, hence $\tilde{E}^t \subseteq E^t_{i(t)}$ (mod $\mathcal{L}^N$).
- Next we show the other inclusion up to countably many values of t.
  Observe that set $E^t_{i(t)}$ is contained in $\tilde{E}^{t'}$ for all $t' < t$. In fact $x \in E^t_{i(t)}$ implies $f(x) > t > t_j > t'$ for some $j \in J$, hence $\tilde{f}(x) \ge t_j > t'$. Thus for every $t'_n \nearrow t$ one has $\bigcap_{t'_n<t} \tilde{E}^{t'_n} \supseteq E^t_{i(t)}$.
  Suppose $|E^t_{i(t)} \setminus \tilde{E}^t| > 0$: from $\tilde{E}^t \subseteq E^t_{i(t)}$ it follows
  $$0 < \left| \bigcap_{t'_n<t} \tilde{E}^{t'_n} \setminus \tilde{E}^t \right| = \left|\{\tilde{f} \ge t\} \setminus \tilde{E}^t\right|$$
  and this implies $|\{\tilde{f} = t\}| > 0$. This last condition can be satisfied only for a countable number of $t \in \mathbb{R}^+$.

Therefore the set of t such that $E^t_{i(t)}$ does not coincide with $\tilde{E}^t$, has zero Lebesgue measure, i.e. for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}^+$ the sets $\tilde{E}^t$ coincide with $E^t_{i(t)}$ up to $\mathcal{L}^N$-negligible sets. Since the property of being indecomposable is invariant up to $\mathcal{L}^N$-negligible sets, they are indecomposable.

In the following we will denote with $\tilde{t}_k,\ k \in K$, the countable family of values such that

$$H_k := \{\tilde{f} = \tilde{t}_k\}, \quad |H_k| > 0.$$

Step 4. The function $\tilde{f}$ is $BV(\mathbb{R}^N)$ and monotone.

The indecomposability of the super-level sets of $\tilde{f}$, proved in the previous step, gives immediately that $\tilde{f}$ is monotone.

Using coarea formula, see for example Theorem 2.93 of [3], we get

$$\begin{aligned} |D\tilde{f}| &= \int_{-\infty}^{+\infty} P(\{\tilde{f} > t\})dt \\ &= \int_{-\infty}^{+\infty} P(E^t_{i(t)})dt \\ &\le \int_{-\infty}^{+\infty} P(E^t)dt = |Df| < +\infty. \end{aligned}$$

Thus the function $\tilde{f}$ is $BV(\mathbb{R}^N)$.

Step 5. Define the function $\hat{f} := f - \tilde{f}$. Clearly $\hat{f}$ is $BV(\mathbb{R}^N)$. The aim of the following steps is to show that its total variation satisfies

$$|D\hat{f}| = |Df| - |D\tilde{f}|.$$

Denote with E^t_1 the super-level sets used to generate the function $\tilde{f}$: this can be done setting $i(t) = 1$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}^+$.

It has been proved that, $\mathcal{L}^1$-a.e. $t \in \mathbb{R}^+$, one has $\{\tilde{f} > t\} = E^t_1$, up to $\mathcal{L}^N$-negligible sets, therefore for such t's

$$\begin{aligned} P(\{f > t\}) &= \sum_{i \in I_t} P(E^t_i) \\ &= \sum_{i \in I_t,\ i > 1} P(E^t_i) + P(\{\tilde{f} > t\}). \end{aligned}$$

We would like to show that, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}^+$, for every $i \in I_t,\ i > 1$, E^t_i is equal, up to $\mathcal{L}^N$-negligible sets, to one of the indecomposable components $\hat{E}^{\hat{t}}_i$ of $\{\hat{f} > \hat{t}\}$, where $\hat{t} = t - \tilde{t}_i$ for a certain $\tilde{t}_i$.

The index i in $\tilde{t}_i$ refers to the fact that its value varies with the indecomposable component E^t_i, $i \in I_t, i > 1$.

We prove it in the following three steps.

Step 6. Let t such that the set E^t is of finite perimeter and $\{E^t_i\}_{i \in I_t}$ are its indecomposable components.

Let us prove that there exists a unique $k \in K$ such that the set E^t_i, $i \in I_t, i > 1$, is contained in H_k, up to $\mathcal{L}^N$-negligible sets.

The set E^t_i is indecomposable and $E^t_i \cap E^t_1 = \varnothing$. Being, up to $\mathcal{L}^N$-negligible sets, $E^j_1 \subseteq E^t_1$ for all $t_j \ge t$, it follows

$$\left|E^t_i \cap E^j_1\right| = 0 \ \forall t_j \ge t.$$

Therefore, from the definition of $\tilde{f}$, for $\mathcal{L}^N$-a.e. $x \in E^t_i$ one has $\tilde{f}(x) \le t$.

Again from the indecomposability of E^t_i and from the fact that E^t_i is contained in $\{f > t_j\}$ for all $t_j \le t$, it follows that there exists a unique $l \in I_{t_j}$ such that,

$$E^t_i \subseteq E^j_l \ (\mathrm{mod}\ \mathcal{L}^N) \quad \text{and} \quad \left|E^t_i \cap E^j_m\right| = 0 \ \forall m \ne l,\ m \in I_{t_j},$$

for all $t_j \le t$.

If there exists a j' such that $\left|E^t_i \cap E^{j'}_1\right| = 0$ then

$$\forall t_j,\ 0 \le t_{j'} \le t_j \le t \quad \left|E^t_i \cap E^j_1\right| = 0,$$

on the other hand if there exists a j'' such that $E_i^t \subseteq E_1^{j''}$, up to $\mathcal{L}^N$-negligible sets, then

$$\forall t_j,\ 0 \le t_j \le t_{j''} \quad E_i^t \subseteq E_1^j \ (\mathrm{mod}\ \mathcal{L}^N).$$

Thus, being the definition

$$\tilde{f}(x) := \begin{cases} 0 & x \notin \bigcup_{j\in J} E_1^j \\ \sup\{t_j |\ j \in J, x \in E_1^j\} & \text{otherwise} \end{cases}$$

equivalent to

$$\tilde{f}(x) := \inf\{t_j |\ j \in J, x \notin E_1^j\},$$

it follows that, up to $\mathcal{L}^N$-negligible subsets of E_i^t, $\tilde{f}\ |_{E_t^i} = constant$, which belongs to $\{\tilde{t}_k\}_{k\in K}$.

In particular, we can order the sets $E_i^t, i \in I_t, i > 1$, as $E_{(k,i)}^t$ where

$$\{E_{(k,i)}^t |\ i \in B_k^t\} = \Big\{E_i^t |\ i \in I_t,\ i > 1,\ \ E_i^t \subseteq H_k \ (\mathrm{mod}\ \mathcal{L}^N)\Big\}.$$

Note that B_k^t could be empty for some $t \in \mathbb{R}^+,\ k \in K$.

Step 7. Let $\hat{t} > 0$ such that the set $\hat{E}^{\hat{t}}$ is of finite perimeter and $\{\hat{E}_i^{\hat{t}}\}_{i\in \hat{I}_{\hat{t}}}$ are its indecomposable components.

Let us prove that there exists a unique $k \in K$, such that the set $\hat{E}_i^{\hat{t}}$ is contained in H_k, up to $\mathcal{L}^N$-negligible sets.

Define

$$\bar{t} := \sup\{0,\ t_j |\ j \in J,\ \hat{E}_i^{\hat{t}} \subseteq E_1^j \ (\mathrm{mod}\ \mathcal{L}^N)\}.$$

It follows that

$$f|_{\hat{E}_i^{\hat{t}}} = \hat{f}|_{\hat{E}_i^{\hat{t}}} + \tilde{f}|_{\hat{E}_i^{\hat{t}}} > \hat{t} + \bar{t} > \bar{t}.$$

For every t_j, in the countable dense sequence, such that $\bar{t} < t_j < \bar{t} + \hat{t}$ there exists a unique $\bar{i} \in I_{t_j}$ such that

$$\hat{E}_i^{\hat{t}} \subseteq E_{\bar{i}}^j \ (\mathrm{mod}\ \mathcal{L}^N),$$

due to the indecomposability of $\hat{E}_i^{\hat{t}}$, and, for the definition of $\bar{t}$, the index $\bar{i}$ must be greater than 1.

Therefore $\tilde{f}|_{\hat{E}_i^{\hat{t}}} = \bar{t}$ and $\bar{t}$ belongs to $\{\tilde{t}_k\}_{k\in K}$.

In particular, we can order the sets $\hat{E}_i^{\hat{t}},\ i \in \hat{I}_{\hat{t}}$, as $\hat{E}_{(k,i)}^{\hat{t}}$ where

$$\{\hat{E}_{(k,i)}^{\hat{t}} |\ i \in \hat{B}_k^{\hat{t}}\} = \{\hat{E}_i^{\hat{t}} |\ i \in \hat{I}_{\hat{t}},\ \hat{E}_i^{\hat{t}} \subseteq H_k \ (\mathrm{mod}\ \mathcal{L}^N)\}.$$

Note that $\hat{B}_k^{\hat{t}}$ could be empty for some $\hat{t} \in \mathbb{R}^+,\ k \in K$.

Step 8. In this step we prove that, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}^+$, $k \in K$ fixed,

$$\{E_{(k,i)}^t |\ i \in B_k^t\} = \{\hat{E}_{(k,i)}^{t-\tilde{t}_k} |\ i \in \hat{B}_k^{t-\tilde{t}_k}\}.$$

Indeed, fix $i \in B_k^t$

$$\hat{f}|_{E_{(k,i)}^t} = f|_{E_{(k,i)}^t} - \tilde{f}|_{E_{(k,i)}^t} > t - \tilde{t}_k.$$

Let us consider only the t's such that the set $\{\hat{f} > t - \tilde{t}_k\}$ is of finite perimeter. For its indecomposabiltity, $E_{(k,i)}^t$ must be contained, up to $\mathcal{L}^N$-negligible sets, in $\hat{E}_{(k,i')}^{t-\tilde{t}_k}$ for a unique $i' \in \hat{I}_{t-\tilde{t}_k}$.

Take then the set $\hat{E}_{(k,i')}^{t-\tilde{t}_k}$:

$$f|_{\hat{E}_{(k,i')}^{t-\tilde{t}_k}} = \tilde{f}|_{\hat{E}_{(k,i')}^{t-\tilde{t}_k}} + \hat{f}|_{\hat{E}_{(k,i')}^{t-\tilde{t}_k}} > \tilde{t}_k + t - \tilde{t}_k = t.$$

For its indecomposability, $\hat{E}^{t-\tilde{t}_k}_{(k,i')}$ must be contained, up to $\mathcal{L}^N$-negligible sets, in $E^t_{(k,i'')}$ for a unique $i'' \in I_t, i'' > 1$. Thus $i'' = i$ and $E^t_{(k,i)} = \hat{E}^{t-\tilde{t}_k}_{(k,i')}$, up to $\mathcal{L}^N$-negligible sets.
Hence

$$\{E^t_{(k,i)}|\ i \in B^t_k\} \subseteq \{\hat{E}^{t-\tilde{t}_k}_{(k,i)}|\ i \in \hat{B}^{t-\tilde{t}_k}_k\}.$$

The same argument, reversed, shows that, once $i' \in \hat{B}^{t-\tilde{t}_k}_k$ is fixed, $\hat{E}^{t-\tilde{t}_k}_{(k,i')} = E^t_{(k,i)}$, up to $\mathcal{L}^N$-negligible sets, for a certain $i \in B^t_k$. Hence

$$\{E^t_{(k,i)}|\ i \in B^t_k\} \supseteq \{\hat{E}^{t-\tilde{t}_k}_{(k,i)}|\ i \in \hat{B}^{t-\tilde{t}_k}_k\}.$$

In an equivalent way, we can also say that, for $\mathcal{L}^1$-a.e. $\hat{t} \in \mathbb{R}^+$, $k \in K$ fixed,

$$\{\hat{E}^{\hat{t}}_{(k,i)}|\ i \in \hat{B}^{\hat{t}}_k\} = \{E^{\hat{t}+\tilde{t}_k}_{(k,i)}|\ i \in B^{\hat{t}+\tilde{t}_k}_k\}.$$

In the following $i = i'$.

Step 9. Coarea formula gives

$$\begin{aligned} |Df| &= \int_{-\infty}^{+\infty} P(\{f > t\})dt \\ &= \int_{-\infty}^{+\infty} \sum_{i \in I_t, i>1} P(E^t_i)dt + \int_{-\infty}^{+\infty} P(\{\tilde{f} > t\})dt. \end{aligned}$$

The final steps consist in showing that

$$\int_{-\infty}^{+\infty} \sum_{i \in I_t, i>1} P(E^t_i)dt = |D\hat{f}|.$$

Step 10. Let $\{\tilde{t}_k|\ k \in K\}$ the countable set of values such that $\left|\tilde{f}^{-1}(\tilde{t}_k)\right| > 0$.

Step 6 shows that, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}^+$ and for all $i \in I_t,\ i > 1$, there exists a unique $k \in K$ such that $\tilde{f}\ |_{E^t_i} = \tilde{t}_k$.

For every $k \in K$, let $\{E^t_{(k,i)}|\ i \in B^t_k\}$ be the set of indecomposable components of E^t such that $\tilde{f}\ |_{E^t_{(k,i)}} = \tilde{t}_k$, $i > 1$.
Observe that $\sum_{i \in B^t_k} P(E^t_{(k,i)})$ are measurable functions of t, for all $k \in K$: indeed we have

$$\begin{aligned} \left|D(f - \tilde{t}_k)\chi_{H_k}\right| &= \int_{-\infty}^{+\infty} \sum_{i \in I_t, i>1 \text{ s.t. } \{f>t\}_i \subseteq \{\tilde{f}=\tilde{t}_k\}} P(\{f > t\}_i)dt \\ &= \int_{-\infty}^{+\infty} \sum_{i \in B^t_k} P(\{f > t\}_i)dt \leq |Df|(\mathbb{R}^N) < +\infty. \end{aligned}$$

Therefore the function $t \mapsto \sum_{i \in B^t_k} P(E^t_i)$ is integrable for all $k \in K$.

Using this notation, we can write

$$\begin{aligned}\int_{-\infty}^{+\infty}\sum_{i\in I_t, i>1} P(E_i^t)dt &= \int_{-\infty}^{+\infty}\sum_{k\in K}\sum_{i\in B_k^t} P(E_{(k,i)}^t)dt\\ &= \sum_{k\in K}\int_{-\infty}^{+\infty}\sum_{i\in B_k^t} P(E_{(k,i)}^t)dt\\ &= \sum_{k\in K}\int_{-\infty}^{+\infty}\sum_{i\in \hat{B}_k^{t-\tilde{t}_k}} P(\{\hat{f} > t-\tilde{t}_k\ \}_{(k,i)})dt\\ &= \sum_{k\in K}\int_{-\infty}^{+\infty}\sum_{i\in \hat{B}_k^{\hat{t}}} P(\{\hat{f} > \hat{t}\ \}_{(k,i)})d\hat{t}\\ &= \int_{-\infty}^{+\infty}\sum_{k\in K}\sum_{i\in \hat{B}_k^{\hat{t}}} P(\{\hat{f} > \hat{t}\ \}_{(k,i)})d\hat{t}.\end{aligned}$$

Since it holds from Step 7

$$\begin{aligned}\hat{E}^{\hat{t}} &= \bigcup_i \{\hat{E}_i^{\hat{t}} |\ i \in \hat{I}_{\hat{t}}\}\\ &= \bigcup_i \bigcup_{k\in K} \{\hat{E}_{(k,i)}^{\hat{t}} |\ \tilde{f}\ |_{\hat{E}_i^{\hat{t}}} = \tilde{t}_k, i \in \hat{I}_{\hat{t}}\}\\ &= \bigcup_{k\in K}\bigcup_i \{\hat{E}_{(k,i)}^{\hat{t}} |\ i \in \hat{B}_k^{\hat{t}}\},\end{aligned}$$

we can write

$$\begin{aligned}\int_{-\infty}^{+\infty}\sum_{k\in K}\sum_{i\in \hat{B}_k^{\hat{t}}} P(\{\hat{f} > \hat{t}\ \}_{(k,i)})d\hat{t} &= \int_{-\infty}^{+\infty}\sum_{i\in \hat{I}_{\hat{t}}} P(\{\hat{f} > \hat{t}\}_i)d\hat{t}\\ &= \int_{-\infty}^{+\infty} P(\{\hat{f} > \hat{t}\})d\hat{t} = |D\hat{f}|.\end{aligned}$$

Step 11. Finally we have

$$\begin{aligned}|Df| &= \int_{-\infty}^{+\infty} P(\{f > t\})dt\\ &= \int_{-\infty}^{+\infty} P(\{\hat{f} > t\})dt + \int_{-\infty}^{+\infty} P(\{\tilde{f} > t\})dt\\ &= |D\hat{f}| + |D\tilde{f}|.\end{aligned}$$

Since f has bounded variation we can iterate this process at most a countable number of times generating the family of monotone functions $f_i \in BV(\mathbb{R}^N)$ which satisfies the theorem.

□

Remark 2.1. Notice that we have also proved that

$$\hat{f} = \sum_{k\in K} f|_{H_k} - \tilde{t}_k.$$

Remark 2.2. In general the decomposition of f in BV monotone functions is not unique as the following example shows.

The function f in Figure 1(c) can be decomposed either in the way shown in Figure 1(b) or in Figure 1(b).

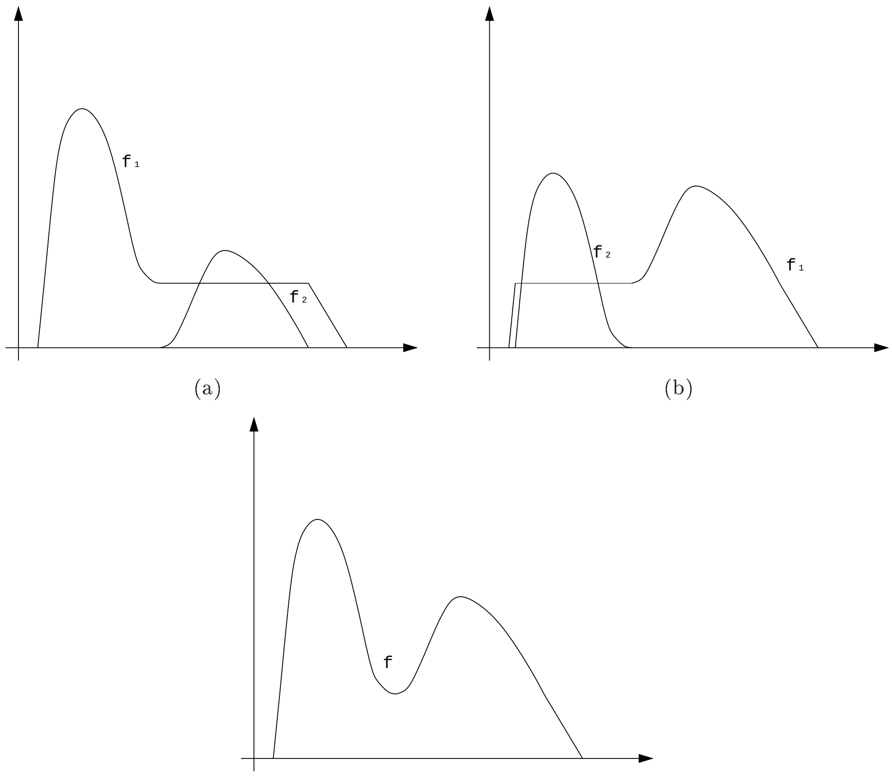

FIGURE 1.

In the simple case, where f is the characteristic function of a set of finite perimeter, there exists a unique subdivision of f as a countable sum of BV monotone functions. Moreover in that case, due to the fact that the sets E_i are pairwise disjoint, $D\chi_{E_i}$ are mutually singular for all $i \in I$.

This property, which has been proved also for the decomposition of Lipschitz functions in Theorem 1, can be false in the general case. As shown in the example below, one can have monotone BV functions, whose distributional derivatives are concentrated on sets with non empty intersection.

Example 2.1. Let us consider a BV function f as in the Figure 2. In this case the Decomposition Theorem gives two BV monotone functions f_1 and f_2 such that $f = f_1 + f_2$. Their distributional derivatives are

$$|Df_1| = 2\delta_0 + \delta_1 + \delta_3 \quad \text{and} \quad |Df_2| = 2\delta_2 + 2\delta_3,$$

where δ_x is the Dirac measure, $\delta_x(A) = 1$ if x belongs to the set A, $\delta_x(A) = 0$ otherwise. Clearly these distributional derivatives are not mutually singular, since both have an atom in $x = 3$.

One can show that for any possible decomposition it is impossible to find two disjoint sets on which the distributional derivatives are concentrated.

3. COUNTEREXAMPLES

As we said in the Introduction, the definition of monotone function could be given, in the same way, even for a function which is only $L^1_{loc}(\mathbb{R}^N)$. In that case

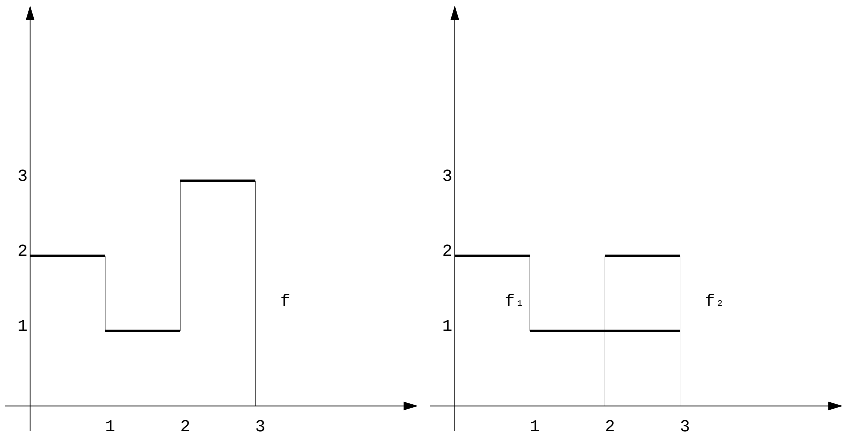

FIGURE 2.

one has to require that this function must have super-level sets with finite perimeter, which is true $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ for the super-level sets of a BV function.

The Jordan decomposition states that monotonicity is a sufficient condition for a function of one variable to be of bounded variation. However, we cannot say that every monotone function $f : \mathbb{R}^N \to \mathbb{R}$ defined as in Definition 3 is of bounded variation.

A counterexample is given below by a function, whose super-level sets are progressive configurations of the construction of a Koch snowflake.

Example 3.1. The Koch snowflake is a curve generated iteratively from a unitary triangle T adding each time, on each edge, a smaller centered triangle with edges one third of the previous edge, see Figure 3.

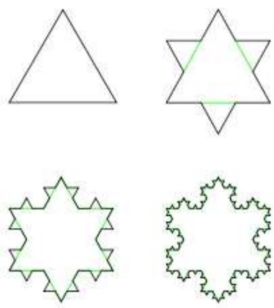

FIGURE 3. The first four iterations of the Koch snowflake

More precisely letting T_0 be the equilateral triangle T with unitary edge, and T_i the successive iterations of the curve, one has that at every stage

- the number of edges is $N_k = 3 \cdot 4^k$,
- the length of the edges is $L_k = \left(\frac{1}{3}\right)^k$,
- the perimeter of the iterated curve is $P(T_k) = 3 \cdot \left(\frac{4}{3}\right)^k$,

- the area of the iterated curve is

$$|T_k| = \left[1 + \frac{1}{3}\sum_{j=1}^{k}\left(\frac{4}{9}\right)^j\right] \cdot \frac{\sqrt{3}}{2}.$$

Denote with B the ball

$$B = \{x \in \mathbb{R}^2 | \ \|x\| < R\},$$

which contains the unitary triangle T centered in the origin: hence $T_i \subseteq B$ for all $i \in \mathbb{N}$.

Let $E_k := B \setminus T_k$ for $k \in \mathbb{N}$ and define $f : B \to \mathbb{R}$ in this way

$$f(x) := \sum_k \left(\frac{3}{4}\right)^k \chi_{E_k}(x).$$

Clearly $0 \leq f < 4$, therefore f belongs to $L^1(B)$ and coarea formula can be used to obtain its variation.

Let us note which are the super-level sets and their perimeter:

- for $t < 0$ the set $\{f > t\} = B$ and $P(B, B) = 0$,
- for $t = 0$ the set $\{f > t\} = E_0$ and $P(E_0, B) = 3$,
- for $0 < t < 4$ the set $\{f > t\} = E_{\bar{k}}$ for the first $\bar{k}$ such that $\sum_{k=0}^{\bar{k}} \left(\frac{3}{4}\right)^k > t$ and $P(E_{\bar{k}}, B) = 3 \cdot \left(\frac{4}{3}\right)^{\bar{k}}$,
- for $t \geq 4$ the set $\{f > t\} = \varnothing$ and $P(\varnothing, B) = 0$.

Thus this function is monotone and computing its variation one has

$$\begin{aligned}|Df|(B) &= \int_{-\infty}^{+\infty} P(\{f < t\}, B)dt \\ &= \int_0^4 P(\{f < t\}, B)dt \\ &= \sum_{k=0}^{+\infty} 3 \cdot \left(\frac{4}{3}\right)^k \cdot \left(\frac{3}{4}\right)^k = +\infty\end{aligned}$$

which implies that f does not belong to $BV(B)$.

The Decomposition Theorem for real valued BV functions of $\mathbb{R}^N$ is in some sense optimal. Considering BV functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ one can find counterexamples to this theorem, i.e. BV functions which cannot be decomposed in sum of BV monotone functions preserving total variation.

The crucial point is that we require to our decomposition, besides being the sum of BV monotone functions, to preserve the the total variation, i.e.

$$|Df| = \sum_{i \in I} |Df_i|.$$

Remark 3.1. For example, let us generalize as follows our definition of BV monotone function to functions with values in a space of a greater dimension.

Definition 5. A function $f : \mathbb{R}^N \to \mathbb{R}^m$, which belongs to $[BV(\mathbb{R}^N)]^m$, is said to be *monotone* if the super-level sets

$$\{f > t\} := \{x \in \mathbb{R}^N | \ f_i(x) > t_i \ i = 1, ..., m\},$$

are indecomposable, for $\mathcal{L}^m$-a.e. $t \in \mathbb{R}^m$.

Let $f : \mathbb{R}^N \to \mathbb{R}^m$ a BV function $f = \begin{pmatrix} f_1 \\ ... \\ f_m \end{pmatrix}$.

For $i=1,...,m$, every f_i is a BV function from $\mathbb{R}^N$ to $\mathbb{R}$ so that Theorem 2 applies. Therefore, for every $i=1,...,m$, one has the decomposition in BV monotone functions $f_i=\sum_{j\in J_i} f_i^j$.

Note that, if $g:\mathbb{R}^N\to\mathbb{R}$ is a BV monotone function, the function $\begin{pmatrix}0\\ \dots\\ g\\ \dots\\ 0\end{pmatrix}$ is a BV monotone function too, from $\mathbb{R}^N$ to $\mathbb{R}^m$, in the sense of Definition 5.

It follows that we can decompose f in that way

$$f=\sum_{j\in J_1}\begin{pmatrix}f_1^j\\ 0\\ \dots\\ 0\end{pmatrix}+\dots+\sum_{j\in J_m}\begin{pmatrix}0\\ \dots\\ 0\\ f_m^j\end{pmatrix}.$$

However, this decomposition does not preserve the total variation of f and one can only say that

$$|Df|\le\sum_{j\in J_1}\begin{pmatrix}|Df_1^j|\\ 0\\ \dots\\ 0\end{pmatrix}+\dots+\sum_{j\in J_m}\begin{pmatrix}0\\ \dots\\ 0\\ |Df_m^j|\end{pmatrix}.$$

We give now a counterexample in the case of Lipschitz function from $\mathbb{R}^2$ to $\mathbb{R}^2$. In this situation we recall the Definition 1.

Definition 6. A function $f:\mathbb{R}^2\to\mathbb{R}^2$, which belongs to $[\mathrm{Lip}(\mathbb{R}^2)]^2$, is said to be *monotone* if the level sets $\{f=t\}=\{x\in\mathbb{R}^2|\ f(x)=t\}$ are connected for every $t\in\mathbb{R}^2$.

Example 3.2. From the area formula

$$\int_{\mathbb{R}^2}\mathcal{H}^0(f^{-1}(t))d\mathcal{L}^2(t)=\int_{\mathbb{R}^2}\det(\nabla f(x))dx$$

one can say that $f^{-1}(t)$ is finite for a.e. $t\in\mathbb{R}^2$, i.e. $f^{-1}(t)=\{x_1(t),...,x_{q(t)}(t)\}$. Therefore there exists a measurable selection $h:\mathbb{R}^2\to\mathbb{R}^2$ such that $h(t)\in f^{-1}(t)$ for all $t\in\mathbb{R}^2$.

Note that the graph

$$G(f)=\{(x,f(x))|\ x\in\mathbb{R}^2\}$$

is closed, thus for Theorem 5.8.11 of [5],

$$G(f)=\bigcup_{i\in I}\{(t,h_i(t))|\ t\in\mathbb{R}^2\},$$

where every h_i is a Borel function and I a countable set.

Define, for every $x\in A_i:=h_i(\mathbb{R}^2)$, the function $f_i(x):=h_i^{-1}(x)$.
Being A_i the set where h_i is invertible, $f_i:A_i\to\mathbb{R}^2$ is well defined and, in its domain, it is a Lipschitz function with constant equal to the one of f. One also has $f=f_i$ in A_i.
Due to the injectivity of f_i, for all $t\in f_i(\mathbb{R}^2)$ there exists a unique $x\in A_i$ such that $\{f_i=t\}=\{x\}$ which is a connected set. Therefore, for every $i\in I$ f_i is a Lipschitz monotone function in A_i.

Thus, we can decompose $f=\sum_{i\in I} f_i$. This decomposition in sum of Lipschitz monotone functions f_i preserves total variation as desired $|Df|=\sum_{i\in I}|Df_i|$. However, these functions are not defined on the all $\mathbb{R}^2$ but only on the sets $A_i\subseteq\mathbb{R}^2$ for which we just know measurability.

The fact that it is possible to extend these functions to the all $\mathbb{R}^2$ requires an additional property of the function f. Clearly every f_i can be extended to $\overline{A_i}$ preserving its Lipschitzianity [1].

Fix an $i \in I$. We have $\mathbb{R}^2 \setminus \overline{A_i} = \bigcup_{j\in J} O_j$ where the O_j are connected open sets. The extension of f_i on the all $\mathbb{R}^2$ must preserve monotonicity and the total variation of f_i. For this reason and due to the fact that we already know that $|Df| = \sum_{i\in I} |Df_i|$, the function f_i must be constant on the O_j with positive measure.
Therefore, to preserve the Lipschitzianity, f_i must be constant on ∂O_j. Thus, for every $j \in J$ such that O_j has positive measure, there must be a t_j for which $\mathcal{H}^1(\{f_i = t_j\}) > 0$.
Note that, if for every $j \in J$ the sets O_j have zero measure, the function f_i is the only one in the decomposition and is already monotone, therefore the only interesting case is when there exists at least a $j \in J$ where the corresponding set O_j has positive measure.

Thus one must have

$$\mathcal{H}^1(\{f = \bar{t}\ \}) \geq \mathcal{H}^1(\{f_i = \bar{t}\ \}) > 0$$

for at least a $\bar{t} \in \mathbb{R}^2$. The condition $\mathcal{H}^1(\{f = \bar{t}\ \}) > 0$ for at least a $\bar{t} \in \mathbb{R}^2$ is a necessary condition to the decomposition of a function in that particular way.

However, not all Lipschitz functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ have this particular property. For example consider

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x) = \begin{pmatrix} 1 - \cos(\frac{\pi x_1}{2}) \\ 1 - \cos(\frac{\pi x_2}{2}) \end{pmatrix}.$$

Then it follows

$$\{f = t\} = \left\{(x_1, x_2) \in \mathbb{R}^2 |\ 1 - \cos(\frac{\pi x_1}{2}) = t_1,\ 1 - \cos(\frac{\pi x_2}{2}) = t_2\right\},$$

i.e. we have

$$\{f = t\} = \left\{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 = \alpha_1 + 4k,\ x_2 = \alpha_2 + 4k\ k = 1, 2, ...\right\}$$

if $0 \leq t_i \leq 2$, for $i = 1, 2$, where $\alpha_i \in \{\pm\frac{2}{\pi} \arccos(1 - t_i)\}$ for $i = 1, 2$, or

$$\{f = t\} = \varnothing$$

otherwise.
In both cases they have zero length.

Thus, for this particular function, there could not exist any decomposition with the properties desired.

[1] Thanks to Kirszbraun's theorem, see Theorem 2.10.43 in [4], every f_i can be extended to a Lipschitz function of the all $\mathbb{R}^2$. However, for our purpose, it is sufficient to consider the basic Lipschitz extension to the closure.

4. Notations

$\mathcal{H}^K$	K-dimensional Hausdorff measure
$\mathcal{L}^N$	N-dimensional Lebesgue measure
$\mathbb{R}^+$	set of all non negative real number
$[L^1(\mathbb{R}^N)]^m$	Lebesgue space of functions from $\mathbb{R}^N$ to $\mathbb{R}^m$
$L^1(\mathbb{R}^N)_{loc}$	space of functions from $\mathbb{R}^N$ to $\mathbb{R}$ which are locally $L^1(\mathbb{R}^N)$
$[\mathrm{Lip}_c(\mathbb{R}^N)]^m$	space of c-Lipschitz functions from $\mathbb{R}^N$ to $\mathbb{R}^m$
$[BV(\mathbb{R}^N)]^m$	space of bounded variation functions from $\mathbb{R}^N$ to $\mathbb{R}^m$
∇f	gradient of the Lipschitz function f
Df	distributional derivative of the BV function f
$\lvert Df\rvert$	total variation of the function f
$P(E)$	perimeter of the set E
$\lvert E\rvert$	Lebesgue measure of the set E
$\mathring{E}^M$	essential interior of the set E
$\overline{E}$	closure of the set E
χ_E	characteristic function of the set E
$(\mathrm{mod}\ \mathcal{L}^N)$	up to $\mathcal{L}^N$-negligible sets
δ_x	Dirac measure

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