Higgs bundles over elliptic curves

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Introduction

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Higgs bundles over elliptic curves

http://www.icmat.es/Thesis/EFrancoGomez.pdf

Study Higgs bundles over a compact Riemann surface of genus 1 for different structure groups

- Classical complex reductive Lie groups (*arXiv 1302.2881*, 12 Feb 2012)

$$\mathrm{GL}(n,\mathbb{C}),\ \mathrm{SL}(n,\mathbb{C}),\ \mathrm{PGL}(n,\mathbb{C}),\ \mathrm{Sp}(2m,\mathbb{C}),\ \mathrm{O}(n,\mathbb{C}) \text{ and } \mathrm{SO}(n,\mathbb{C}).$$

- Real forms of $\mathrm{GL}(n,\mathbb{C})$ (*to appear*)

$$\mathrm{U}(p,q),\ \mathrm{GL}(n,\mathbb{R}) \text{ and } \mathrm{U}^*(2m).$$

- G arbitrary complex reductive Lie group (*to appear*).

In this talk we focus on $G = \mathrm{GL}(n,\mathbb{C})$.

Elliptic curves and Higgs bundles

X elliptic curve (compact Riemann surface of genus 1)

- the Abel-Jacobi of degree 1 is an isomorphism $\mathrm{aj}_1 : X \xrightarrow{\cong} \mathrm{Pic}^1(X)$;
- $X \cong \mathrm{Pic}^0(X)$, group structure on X;
- the canonical bundle is trivial $K_X \cong \mathcal{O}$;
- so is the cotangent bundle $T^*X \cong X \times \mathbb{C}$.

(E, Φ) Higgs bundle $\qquad$ (E vector bundle, $\Phi \in H^0(X, \mathrm{End}\, E)$ endomorphism)

- (E, Φ) is (semi)stable if $\mu(F)(\leq) < \mu(E)$ for all $F \subset E$ Φ-invariant (i.e. $\Phi(F) \subset F$)
- (E, Φ) is semistable has a Jordan-Hölder filtration of Φ-invariant subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E, \qquad \text{s.t. } \left({}^{E_i}\!/_{E_{i-1}}, \overline{\Phi}_i \right) \text{ stable Higgs bundle}$$

- $\mathrm{gr}(E, \Phi) \cong \bigoplus_i \left({}^{E_i}\!/_{E_{i-1}}, \overline{\Phi}_i \right)$ associated graded object,
- (E, Φ) polystable if $(E, \Phi) \cong \mathrm{gr}(E, \Phi)$ (direct sum of stable).

Atiyah: Vector bundles over an elliptic curve

Atiyah 1957 before GIT, see Tu 1993 and LePotier's book for moduli interpretation.

- Rank and degree coprime, $\gcd(n,d) = 1$,
  - $\det : M(\mathrm{GL}(n,\mathbb{C}))_d \xrightarrow{\cong} \mathrm{Pic}^d(X)$, then
  $$M(\mathrm{GL}(n,\mathbb{C}))_d \cong X.$$
  - $\Phi \in H^0(X, \mathrm{End}\, E) \Longrightarrow \Phi = \lambda \cdot \mathrm{id}_E$.
  - Family $\mathcal{V}_{(n,d)} \to X \times X$ parametrizing all stable vector bundles $\mathrm{rk} = n$ and $\deg = d$

- $\gcd(n,d) = h > 1$, set $n' = \frac{n}{h}$ and $d' = \frac{d}{h}$
  - There are no stable vector bundles, $M^{st}(\mathrm{GL}(n,\mathbb{C}))_d = \emptyset$.
  - E polystable $\Longrightarrow E \cong E'_1 \oplus \ldots \oplus E'_h$, each E'_i stable, rank n' and degree d';
  - $M(\mathrm{GL}(n,\mathbb{C}))_d \cong \mathrm{Sym}^h(M(\mathrm{GL}(n',\mathbb{C}))_{d'})$ and then
  $$M(\mathrm{GL}(n,\mathbb{C}))_d \cong \mathrm{Sym}^h(X).$$

Stability of Higgs bundles in terms of the underlying vector bundle

Proposition

(E, Φ) *semistable Higgs bundle* $\Longleftrightarrow$ *E semistable vector bundle.*

Take the Harder-Narasimhan filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m = E$. Since $\Omega^1_X \cong \mathcal{O}$, the the Higgs field $\Phi \in H^0(X, \operatorname{End} E)$ is an endomorphism and preserves the first term F_1.

Proposition

(E, Φ) *stable Higgs bundle* $\Longleftrightarrow$ *E stable vector bundle.*

Using Atiyah's description of vector bundles over an elliptic curve.

Corollary

Set $h = \gcd(n, d)$, $n' = \frac{n}{h}$ *and* $d' = \frac{d}{h}$. *Let* (E, Φ) *polystable of rank n and degree d, then*

$$(E, \Phi) \cong \bigoplus_{i=1}^{h} (E_i, \lambda_i \operatorname{id}_{E_i}) \qquad \lambda_i \in \mathbb{C}, \quad \operatorname{rk}(E_i) = n' \text{ and } \deg(E_i) = d'.$$

Independently by Biswas-Fiorentino for Abelian varieties.

A family of polystable Higgs bundle

- $\gcd(n', d') = 1$,
  - Recall $\mathcal{V}_{(n',d')}$ family of stable Higgs bundles parametrized by X
  - $\mathcal{E}_{(n',d')}$ family of stable Higgs bundles parametrized by $T^*X = X \times \mathbb{C}$

$$(\mathcal{E}_{(n',d')})_z = \Big(\quad (\mathcal{V}_{(n',d')})_x \quad , \quad \lambda \cdot \mathrm{id}_{(\mathcal{V}_{(n',d')})_x} \quad \Big), \qquad z = (x, \lambda) \in T^*X.$$

  - $\mathcal{E}_{(n',d')}$ parametrizes all stable Higgs bundles

- $\gcd(n, d) = h > 1$
  - $\mathcal{E}_{(n,d)}$ = fibre product over X of h copies of $\mathcal{E}_{(n',d')}$, parametrized by $T^*X \times \overset{h}{\cdots} \times T^*X$

$$(\mathcal{E}_{(n,d)})_{\overline{z}} = (\mathcal{E}_{(n',d')})_{z_1} \oplus \cdots \oplus (\mathcal{E}_{(n',d')})_{z_h}, \qquad \overline{z} = (z_1, \ldots, z_h) \in T^*X \times \overset{h}{\cdots} \times T^*X.$$

  - $\mathcal{E}_{(n,d)}$ parametrizes all polystable Higgs bundles
  - $(\mathcal{E}_{(n,d)})_{\overline{z}} \cong (\mathcal{E}_{(n,d)})_{\overline{z}'} \iff \overline{z}'$ permutation of $\overline{z}$.

The moduli of Higgs bundles

By moduli theory $\mathcal{E}_{(n,d)}$ induces a morphism

$$T^*X \times \overset{h}{\cdots} \times T^*X \longrightarrow \mathcal{M}(n,d)$$

that factors through a bijective morphism

$$\chi : \mathrm{Sym}^h(T^*X) \xrightarrow{1:1} \mathcal{M}(n,d)$$

Theorem

Let $n \leq 4$, then

$$\mathcal{M}(n,0) \cong \mathrm{Sym}^n(T^*X).$$

If $n \leq 4$ we know that $\mathcal{M}(n,0)$ is normal and therefore, in this case Zarisky's Main Theorem implies that χ is an isomorphism.

We don't know if $\mathcal{M}(n,d)$ is normal in general $\Longrightarrow$ change of strategy.

Families with the local universal property

A family $\mathcal{E}$ parametrized by Z has the *local universal property* if for any other family $\mathcal{F}$ parametrized by Y and any $y \in Y$, there exists $y \in U \subset Y$ open and $f : U \to Z$ such that

$$f^*\mathcal{E} \sim \mathcal{F}|_U.$$

Proposition (Newstead)

Suppose $\mathcal{E}$ parametrized by Z has the local universal property, Γ group acting on Z such that $\mathcal{E}_{z_1} \sim \mathcal{E}_{z_2} \iff z_2 = \gamma \cdot z_1$ for some $\gamma \in \Gamma$. Then, a categorical quotient of Z by Γ is a coarse moduli space if and only if it is an orbit space.

$\mathcal{E}_{(n,d)}$ doesn't have the local universal property (too many families)

A familly $\mathcal{F}$ parametrized by Y is **locally graded** if for every $y \in Y$ there exists $y \in U \subset Y$ open and $\mathcal{F}_1, \ldots, \mathcal{F}_h$ families of stable Higgs bundles, such that

$$\mathcal{F}|_U \sim \bigoplus_{i=1}^{h} \mathcal{F}_i.$$

Proposition

$\mathcal{E}_{(n,d)}$ has the local universal property among locally graded families.

A new moduli problem

New moduli functor

$$\begin{array}{rcl} \mathrm{Mod}' : (\text{Algebraic varieties}) & \longrightarrow & (\text{Sets}) \\ Y & \longmapsto & \left\{ \begin{array}{c} \text{S-equivalence classes of} \\ \text{locally graded} \\ \text{families parametrized by } Y \end{array} \right\} \end{array}$$

Since $\mathfrak{S}_n$ is finite $\Longrightarrow \mathrm{Sym}^h(T^*X) = {}^{(T^*X \times \overset{h}{\cdots} \times T^*X)}/_{\mathfrak{S}_h}$ is an orbit space,

Theorem

There exists a coarse moduli space $\mathcal{N}(n,d)$ for the new moduli functor Mod'*, and*

$$\mathcal{N}(n,d) \cong \mathrm{Sym}^h(T^*X).$$

There exists a bijection $\mathcal{N}(n,d) \xrightarrow{1:1} \mathcal{M}(n,d)$ and $\mathcal{N}(n,d)$ is the normalization of $\mathcal{M}(n,d)$.

There exists a bijection $\mathcal{N}(n,d) \xrightarrow{1:1} \mathcal{M}(n,d) \Longrightarrow$ No extra geometric structure.

If $\mathcal{M}(n,d)$ is normal $\Longrightarrow \mathcal{M}(n,d) \cong \mathcal{N}(n,d)$

Hitchin map

Hitchin 1987, evaluating the invariant polynomials $q_{n,1}, \ldots, q_{n,n}$ on the Higgs field

$$b : \mathcal{M}(n,d) \longrightarrow B = \bigoplus_i H^0(X, K^{\deg(q_{n,i})})$$

$$(E, \Phi) \longmapsto (q_{n,1}(\Phi), \ldots, q_{n,n}(\Phi))$$

Note in our case

$$\begin{tikzcd}
\mathrm{Sym}^h T^*X \arrow[r, "p"] \arrow[d, "\cong"'] & \mathrm{Sym}^h \mathbb{C} \arrow[d, hook, "\text{using the } q_{n,i}"] \\
\mathcal{N}(n,d) \arrow[r, "b"] & B(n,d) \subset \bigoplus_i H^0(X, \mathcal{O}) (\cong \mathbb{C}^n)
\end{tikzcd}$$

The fibre $p^{-1}\left([\lambda_1, \overset{m_1}{\ldots}, \lambda_1, \ldots, \lambda_\ell, \overset{m_\ell}{\ldots}, \lambda_\ell]_{\mathfrak{S}_n}\right) \cong \mathrm{Sym}^{m_1}(X) \times \cdots \times \mathrm{Sym}^{m_\ell}(X)$.

Corollary

The generic Hitchin fibre (all $m_i = 1$) is the abelian variety $X \times \overset{n}{\ldots} \times X$. The non-generic fibre is a holomorphic fibration over the abelian variety $X \times \overset{\ell}{\ldots} \times X$ with fibre $\mathbb{P}^{m_1 - 1} \times \overset{\ell}{\ldots} \times \mathbb{P}^{m_\ell - 1}$.

Nice picture

By the result on semistability, there exists a projection $\mathcal{N}(n,d) \xrightarrow{a} M(n,d)$. Together with the Hitchin fibration we have two important maps

$$\begin{tikzcd} & \mathcal{N}(n,d) \arrow[dl, "a"'] \arrow[dr, "b"] & \\ M(n,d) & & B(n,d) \end{tikzcd}$$

In our case we have an explicit descrition

$$\begin{tikzcd} & \mathrm{Sym}^h(T^*X) \arrow[dl] \arrow[dr] & \\ \mathrm{Sym}^h(X) & & \mathrm{Sym}^h(\mathbb{C}) \end{tikzcd}$$

Let $\widetilde{M}(n,d)$ and $\widetilde{\mathcal{N}}(n,d)$ be the orbifolds given by the quotients $(X\times \overset{h}{\cdots} \times X)\,/\,\mathfrak{S}_h$ and $(T^*X\times \overset{h}{\cdots} \times X)\,/\,\mathfrak{S}_h$, we have that

$$\widetilde{\mathcal{N}}(n,d) \cong \mathcal{T}^*\widetilde{M}(n,d)$$

where $\mathcal{T}^*$ denotes the cotangent orbifold bundle.

$\mathcal{N}(\mathrm{SL}(n,\mathbb{C}))$ and $\mathcal{N}(\mathrm{PGL}(n,\mathbb{C}))$

$$\begin{tikzcd}
\mathcal{N}(n,d) \arrow[r, "(\det,\mathrm{tr})"] \arrow[d, "\cong"'] & \mathrm{Pic}^d(X) \times H^0(X,\mathcal{O}) \arrow[d, "\cong"] \\
\mathrm{Sym}^h(T^*X) \arrow[r] & T^*X
\end{tikzcd}$$

$$[(x_1,\lambda_1),\ldots,(x_h,\lambda_h)]_{\mathfrak{S}_h} \longmapsto \textstyle\sum_{i=1}^h (x_i,\lambda_i).$$

Therefore

$$\mathcal{N}(\mathrm{SL}(n,\mathbb{C})) \cong (\det,\mathrm{tr})^{-1}(\mathcal{O},0) \cong (T^*X \times \overset{n-1}{\cdots} \times T^*X) \,/\, \mathfrak{S}_n\,,$$

$$\mathcal{N}(\mathrm{PGL}(n,\mathbb{C}))_0 \cong (\det,\mathrm{tr})^{-1}(\mathcal{O},0) \,/\, \mathrm{Pic}^0(X)[n] \cong (T^*X \times \overset{n-1}{\cdots} \times T^*X) \,/\, \mathfrak{S}_n \ltimes X[n]\,,$$

where the action of $\mathfrak{S}_n$ on $(T^*X)^{\times(n-1)}$ is defined thanks to the projection π from $(T^*X)^n$ to $(T^*X)^{n-1}$ of the first $(n-1)$ factors

$$\sigma\cdot((x_1,\lambda_1),\ldots,(x_{n-1},\lambda_{n-1})) = \pi\left(\sigma\cdot\left((x_1,\lambda_1),\ldots,(x_{n-1},\lambda_{n-1}),-\sum_{i=1}^{n-1}(x_i,\lambda_i)\right)\right)$$

The Hitchin fibration for $\mathcal{N}(\mathrm{SL}(n,\mathbb{C}))$ and $\mathcal{N}(\mathrm{PGL}(n,\mathbb{C}))$

The two Hitchin maps

$$\begin{tikzcd} \mathcal{N}(\mathrm{SL}(n,\mathbb{C})) \arrow[dr, "\hat{b}"] & & \mathcal{N}(\mathrm{PGL}(n,\mathbb{C}))_0 \arrow[dl, "\check{b}"'] \\ & \hat{B}_n \cong \check{B}_n & \end{tikzcd}$$

are interpreted as follows

$$\begin{tikzcd} (T^*X \times \overset{n-1}{\cdots} \times T^*X)\,/\,\mathfrak{S}_n \arrow[dr, "\hat{p}"] & & (T^*X \times \overset{n-1}{\cdots} \times T^*X)\,/\,\mathfrak{S}_n \times X[n] \arrow[dl, "\check{p}"'] \\ & (\mathbb{C} \times \overset{n-1}{\cdots} \times \mathbb{C})\,/\,\mathfrak{S}_n & \end{tikzcd}$$

Setting

$$(x_1, \overset{\ell}{\cdots}, x_\ell) \overset{\alpha_\ell}{\longmapsto} m_1 x_1 + \cdots + m_\ell x_\ell, \qquad \text{with} \qquad \ker \alpha_\ell \cong X \times \overset{\ell-1}{\cdots} \times X,$$

we can describe the fibres as follows

$$\hat{p}^{-1}([\lambda_1, \overset{m_1}{\cdots}, \lambda_1, \ldots, \lambda_\ell \overset{m_\ell}{\cdots}, \lambda_\ell]_{\mathfrak{S}_n}) \cong (\mathrm{Sym}^{m_1}(X) \times \cdots \times \mathrm{Sym}^{m_\ell}(X))\,|_{\ker \alpha_\ell}$$

$$\check{p}^{-1}([\lambda_1, \overset{m_1}{\cdots}, \lambda_1, \ldots, \lambda_\ell \overset{m_\ell}{\cdots}, \lambda_\ell]_{\mathfrak{S}_n}) \cong (\mathrm{Sym}^{m_1}(X) \times \cdots \times \mathrm{Sym}^{m_\ell}(X))\,|_{\ker \alpha_\ell} \,/\, X[n]$$

Duality of the fibres

We define a $(m_1, \ldots, m_\ell)$-weighted action of $x' \in X[n]$ on $X \times \overset{\ell-1}{\cdots} \times X$ which is equivalent to the action of $X[n]$ on $\ker \alpha_\ell$

$$x' \cdot (x_1, \ldots, x_{\ell-1}) = \left(m_1 x' + x_1, \ldots, m_\ell x' + x_\ell\right).$$

Proposition

For any value of $(m_1, \ldots, m_\ell)$

$$X \times \overset{\ell}{\cdots} \times X \,/\, X[n] \cong X \times \overset{\ell}{\cdots} \times X$$

Corollary

The generic fibre of $\mathcal{N}(\mathrm{SL}(n, \mathbb{C})) \to \hat{B}_n$ *and the corresponding fibre of* $\mathcal{N}(\mathrm{PGL}(n, \mathbb{C}))_0 \to \check{B}_n$ *are isomorphic to* $X \times \overset{n-1}{\cdots} \times X$ *which is a self-dual abelian variety. The non-generic fibre of* $\mathcal{N}(\mathrm{SL}(n, \mathbb{C})) \to \hat{B}_n$ *is a holomorphic fibration over the self-dual abelian variety* $X \times \overset{\ell-1}{\cdots} \times X$ *with fibre* $\mathbb{P}^{m_1-1} \times \overset{\ell}{\cdots} \times \mathbb{P}^{m_\ell}$.
The corresponding non-generic fibre of $\mathcal{N}(\mathrm{PGL}(n, \mathbb{C}))_0 \to \check{B}_n$ *is a holomorphic fibration over the self-dual abelian variety* $X \times \overset{\ell-1}{\cdots} \times X$ *with fibre* $(\mathbb{P}^{m_1-1} \times \overset{\ell}{\cdots} \times \mathbb{P}^{m_\ell})/X[r]$, *where* $r = \gcd(n, m_1, \ldots, m_\ell)$.

The other results of the thesis

Description of moduli spaces of Higgs bundles for classical Lie groups: $\mathrm{Sp}(2m,\mathbb{C})$, $\mathrm{O}(n,\mathbb{C})$ and $\mathrm{SO}(n,\mathbb{C})$

$$\text{Ex.} \qquad \mathcal{N}(\mathrm{Sp}(2m,\mathbb{C})) \cong \mathrm{Sym}^m(T^*X/\mathbb{Z}_2)$$

Description of moduli spaces of Higgs bundles for real forms of $\mathrm{GL}(n,\mathbb{C})$: $\mathrm{U}(p,q)$, $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{U}^*(2m)$

$$\text{Ex.} \qquad \mathcal{N}(\mathrm{U}^*(2m)) \cong \mathrm{Sym}^m(\mathbb{P}^1 \times \mathbb{C})$$

Description of moduli spaces of Higgs bundles for an arbitrary complex reductive Lie group G

$$\text{Ex.} \qquad G \text{ simply connected Lie group} \qquad \mathcal{N}(G) \cong {}^{(T^*X \otimes_{\mathbb{Z}} \Lambda)}/_{W}$$

where Λ corroot lattice and W weyl group.