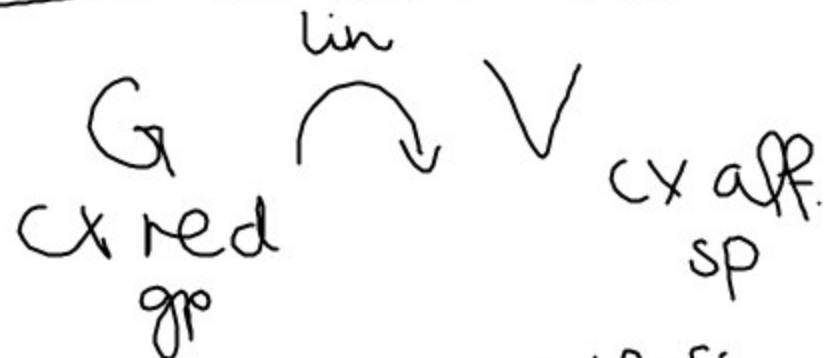


Three Stratifications for representation spaces of quivers

I Geometric Invariant Theory



Use $\rho: G \rightarrow \mathbb{C}^\times$ to linearise

ie. $G \curvearrowright L = V \times \mathbb{C}$

$$g \cdot (v, c) = (gv, \rho(g)c)$$

Mumford's GIT

$$V^{\rho\text{-ss}} \longrightarrow V //_{\rho} G$$

Hilbert-Mumford (king) v is ρ -ss $\iff (\rho, \lambda) \geq 0 \forall 1\text{-PS } \lambda: \mathbb{C}^\times \rightarrow G$

s.t. $\lim_{t \rightarrow \infty} \lambda(t) \cdot v$ exists

$$\rho \circ \lambda(t) = t^{(\rho, \lambda)}$$

$$\underline{\text{Ex}} : Q = (V, A, h, t) \quad \underline{d}$$

$$V \cong \text{Rep}(Q, \underline{d}) = \bigoplus_{a \in A} \text{Hom}(\mathbb{C}^{d_{t(a)}}, \mathbb{C}^{d_{h(a)}})$$

$$G = \prod_{v \in V} \text{GL}(\mathbb{C}^{d_v})$$

$$\rho : G \rightarrow \mathbb{C}^\times \quad \rho_v \in \mathbb{Z} \quad \rho(g_v) = \prod (\det g_v)^{\rho_v}$$

assume: $\sum \rho_v d_v = 0$

Thm(king) A rep^W of dim \underline{d} is ρ -ss $\Leftrightarrow \forall W' \subseteq W$

$$\rho(W') = \sum \rho_v \dim W'_v \geq 0$$

Instability

Choose G -inv norm $\|\cdot\|$ on $\lambda_*(G)_\mathbb{R}$

$$\text{Let } M^p(v) = \inf_{\substack{\text{l-PS } \lambda \\ \text{s.t. } \lim_{t \rightarrow \infty} \lambda(t) \cdot v \\ \text{exists}}} \frac{(\rho, \lambda)}{\|\lambda\|}$$

$$\text{HM: } v \text{ is } p\text{-ss} \\ \Leftrightarrow M^p(v) \geq 0$$

Defn: A l-PS λ is adapted to v (a p -unstable pt)

$$\text{if } M^p(v) = \frac{(\rho, \lambda)}{\|\lambda\|} \quad \& \quad \lim_{t \rightarrow \infty} \lambda(t) \cdot v \text{ exists}$$

$$\Lambda^p(v) = \{ \text{non-div adapted l-PS to } v \} \neq \emptyset \quad \text{ Kempf}$$

$[\lambda]$ = cony class of λ

Hesselink's Stratification $V - V^{\rho-ss} = \bigsqcup_{[\lambda]} S_{[\lambda]}^H$

where $S_{[\lambda]}^H = \{v : \Lambda^{\rho}(v) \cap [\lambda] \neq \emptyset\}$

Structure of strata?

$$S_{[0]}^H = V^{\rho-ss}$$

$$V_{\mu}^{\lambda} = \{v : \lim_{t \rightarrow 0} \lambda(t) \cdot v \text{ exists}\} \supseteq S_{\lambda} = \{v : \lambda \in \Lambda^{\rho}(v)\}$$

$$\begin{array}{c} \downarrow P_{\lambda} \\ V^{\lambda} \end{array}$$

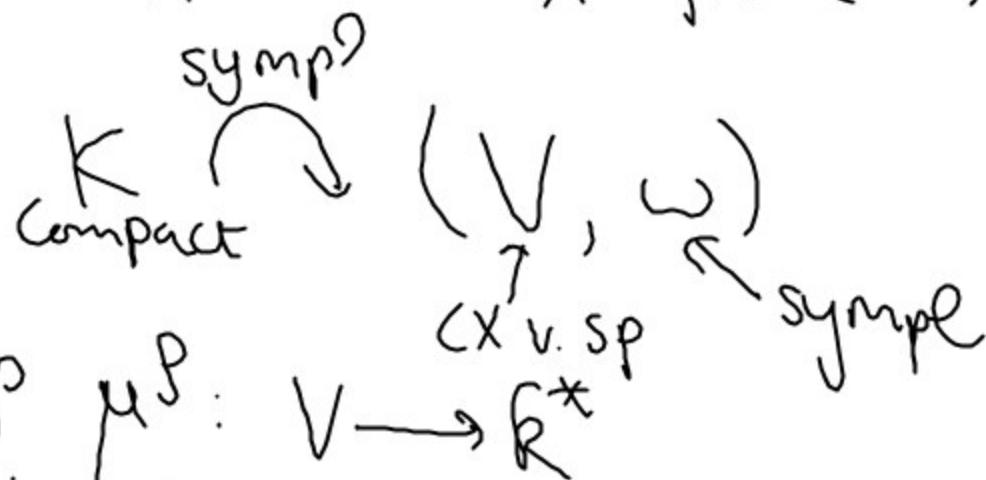
$$\downarrow$$

$$Z_{\lambda} = \{v : \lambda \in \Lambda^{\rho}(v) \cap \Omega_v\}$$

Prop 1) $Z_\lambda = G \cdot T$ ss for smaller red gp $\curvearrowright V^\lambda$

$$2) S_{[\lambda]}^H = G S_\lambda \quad S_\lambda = p_\lambda^{-1}(Z_\lambda)$$

II Moment maps



for $p: K \rightarrow S^1$, m. map $\mu^p: V \rightarrow \mathbb{R}^*$

given by $\mu(v) \cdot A = \frac{1}{2} \omega(Av, v) - \frac{1}{2\pi i} dp \cdot A$

Fix p & $\mu = \mu^p$.

Choose k -inv norm on \mathbb{R}

Idea. $\|p\|^2: V \rightarrow \mathbb{R} \rightsquigarrow$ Morse strat of V

As $\|p\|^2$ is K -inv, so is crit $\|p\|^2$

Morse strat:

$$V = \coprod_{K \cdot \beta} S_{K \cdot \beta}^M$$

$$\cong \coprod_{K \cdot \beta} C_{K \cdot \beta}$$

+ Structure results.

pts flowing
to $C_{K \cdot \beta}$

III Comparison

Aff case: $G \xrightarrow{\omega} V$ $\omega = \text{Im} H$
 $\mathbb{C}^x \text{ red}$ $\mathbb{C}^x \text{ v-space}$

assume: $K \subseteq G$
 max compact
 acts symple

K -inv norm $\|-\|$ on $\mathbb{R} \rightsquigarrow \|-\|$ on $\mathcal{X}_*(G)/\mathbb{R}$

$$\lambda: \mathbb{C}^x \rightarrow G \quad \lambda' = g \lambda g^{-1} \quad K \rightarrow S'$$

$$\|\lambda\| = \|d\lambda'(2\pi i)\|$$

Thm (H). In this affine set up, for any $\rho \in \mathfrak{h}$,
 Hesselink's strat = Morse strat.

Idea of pf:

• Affine Kempf-Ness w.r.t g : lowest strata agree

• Structure results of strata $V^{\beta-ss} = \sum_{\sigma \in \Sigma} \mu^{-1}(0)$

$$S_{k \cdot \beta}^M = G \cdot S_{\beta}$$

$$Z_{\beta} = P_{\beta}^{-1}(\overline{Z_{\beta}}) \text{ lowest morse strata}$$

$$Z_{\beta} = Z_{\lambda}$$

where $\lambda = 1 - \rho_S$ assoc to a +ve multiple of $\beta \in \mathbb{R}^*$

IV Quiver reps

$$G(\underline{Q}, \underline{d}) \curvearrowright V = \text{Rep}(\underline{Q}, \underline{d})$$

$$g \leftrightarrow p_v \in \mathbb{Z} \text{ s.t. } \sum p_v d_v = 0$$

$$\|\cdot\|_\alpha \leftrightarrow \alpha_v \in \mathbb{N}$$

Every rep W has a! HN filtr wrt $(g \& \alpha)$

Thm(H): The stratification of V by HN types agrees with the Morse = Hess strat.