On a Conjecture of Donagi-Morrison

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Preliminaries on Brill-Noether Theory

Let C be a smooth irreducible curve of genus g.

Having fixed two integers r, d, look at the Brill-Noether variety $W^r_d(C)$:

$$\operatorname{supp} W^r_d(C) := \{A \in \operatorname{Pic}^d(C) \mid h^0(C, A) \geq r+1\}.$$

$$\operatorname{expdim} W^r_d(C) \setminus W^{r+1}_d(C) = g - (r+1)(g-d+r) =: \rho(g, r, d).$$

An element $A \in W^r_d(C) \setminus W^{r+1}_d(C)$ is called a complete g^r_d on C.

Def: If $A \in \operatorname{Pic}(C)$ satisfies $h^i(C, A) \geq 2$ for $i = 0, 1$, we say that A contributes to the Clifford index and set

$$\operatorname{Cliff}(A) := \deg(A) - 2h^0(C, A) + 2.$$

$$\operatorname{Cliff}(C) := \min\{\operatorname{Cliff}(A) \,:\, A \in \operatorname{Pic}(C),\, h^i(C, A) \geq 2 \text{ for } i = 0, 1\}.$$

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Brill-Noether theory of $K3$-sections

From now on, assume that C lies on a smooth, projective $K3$ surface S. Let $L := \mathcal{O}_S(C)$ be ample.

Theorem (Lazarsfeld 1986)

Assume that $\mathrm{Pic}(S) = \mathbb{Z} \cdot L$ and let $C \in |L|$.
If $\rho(g, r, d) < 0$, then $W^r_d(C) = \emptyset$.
If instead $\rho(g, r, d) \geq 0$ and $C \in |L|$ is general, then $W^r_d(C)$ is smooth of the expected dimension.

Remark: This implies that the same holds true for a general curve in M_g (Gieseker-Petri Theorem).

Question: What happens if S and L are arbitrary? We will analyze the cases where $\rho(g, r, d) < 0$.

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Example 1: $\pi : S \xrightarrow{2:1} \mathbb{P}^2$ branched along a smooth sextic, $H := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$.
A curve $C \in |3H|$ has genus 10 and can be either

- a $2:1$ cover of an elliptic curve $\rightsquigarrow$ g^1_4, or
- isomorphic to a smooth plane sextic $\rightsquigarrow$ g^1_5.

For $k = 4, 5$, one has $\rho(10, 1, k) < 0$.

Example 2: $S \subset \mathbb{P}^3$ a general quartic hypersurface, $H := \mathcal{O}_S(1)$.
A curve $C \in |2H|$ has genus 9 and is the complete intersection $C := S \cap Q$.
$\rightsquigarrow$ a ruling of Q gives a g^1_4 on C.
One has $\rho(9, 1, 4) = -3$.

Example 3: $S \subset \mathbb{P}^3$ quartic hypersurface containing a single line E.
$\mathrm{Pic}(S) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E$, with $H^2 = 4$, $E^2 = -2$, $H \cdot E = 1$.
A curve $C \in |2H + E|$ has genus 6 and $H \otimes \mathcal{O}_C$ is a g^2_5.
One has $\rho(6, 2, 5) = -3$.

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Def: A line bundle $M \in \mathrm{Pic}(S)$ is adapted to $|L|$ whenever:

- $h^0(S, M) \geq 2$, $h^0(S, L \otimes M^\vee) \geq 2$,
- $\mathrm{Cliff}(M \otimes \mathcal{O}_C)$ is independent of the curve $C \in |L|$.

Remark: The first condition assures that $M \otimes \mathcal{O}_C$ contributes to the Clifford index. The second condition is satisfied if either $h^1(S, M) = 0$, or $h^1(S, L \otimes M^\vee) = 0$.

Conjecture (Donagi-Morrison 1989)

Let A be a complete, base point free g^r_d on $C \subset S$ such that $d \leq g - 1$ and $\rho(g, r, d) < 0$
$\Longrightarrow$ $\exists$ $M \in \mathrm{Pic}(S)$ *adapted to $|L|$ such that:*

(i) $|A|$ *is contained in* $|M \otimes \mathcal{O}_C|$ *(i.e.,* $h^0(C, A^\vee \otimes M \otimes \mathcal{O}_C) > 0$*);*

(ii) $\mathrm{Cliff}(M \otimes \mathcal{O}_C) \leq \mathrm{Cliff}(A)$.

Remark: Condition *(i) is equivalent to the requirement that, for some divisors $A_0 \in |A|$ and $M_0 \in |M|$, one has $A_0 \subset M_o \cap C$.*

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The g^1_4 (or the g^1_5) is contained in $H \otimes \mathcal{O}_C$, which is a g^2_6.
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One has $\rho(9, 1, 4) = -3$.
The g^1_4 is contained in $H \otimes \mathcal{O}_C$, which is a g^3_8.
$\mathrm{Cliff}(H \otimes \mathcal{O}_C) = \mathrm{Cliff}(g^1_4) = 2$.

Lazarsfeld-Mukai bundles

Let A be a complete, base point free g^r_d on $C \subset S$.
The Lazarsfeld-Mukai bundle $E_{C,A}$ is a vector bundle of rank $r+1$ on S:

$$0 \longrightarrow E^\vee_{C,A} \longrightarrow H^0(C,A) \otimes \mathcal{O}_S \xrightarrow{\mathrm{ev}} A \longrightarrow 0.$$

By dualizing, one finds:

$$0 \longrightarrow H^0(C,A)^\vee \otimes \mathcal{O}_S \longrightarrow E_{C,A} \longrightarrow \omega_C \otimes A^\vee \longrightarrow 0.$$

Properties of $E := E_{C,A}$:

- E is globally generated off the base locus of $\omega_C \otimes A^\vee$;
- $\mathrm{rk}\, E = r+1$, $\det E = L$, $c_2(E) = d$;
- $h^1(S,E) = h^2(S,E) = 0$;
- $\chi(S, E \otimes E^\vee) = 2(1-\rho(g,r,d))$.

Key Fact: If $\rho(g,r,d) < 0 \Longrightarrow E_{C,A}$ is non-simple.

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How to approach the conjecture?

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$$h^0(S,N) \geq 2 \implies \mathrm{Hom}(N, \mathcal{O}_S) = 0 \implies \alpha \circ \gamma \neq 0$$
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Case r=1:

Theorem (Donagi-Morrison 1989)

The conjecture holds for $r = 1$.

Idea of the proof: $E := E_{C,A}$ is non-simple of rank 2 (assume also indecomposable)
$\Longrightarrow \exists\ \phi : E \to E$ nilpotent and E is given by an extension:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M \otimes I_\xi & \longrightarrow & 0, \\ & & \| & & & & \| & & \\ & & \ker\phi & & & & \mathrm{Im}\phi & & \end{array}$$

where $N, M \in \mathrm{Pic}(S)$ and $\xi \subset S$ is a 0-dimensional subscheme.
In order to show that M is adapted to $|L|$ and $\mathrm{Cliff}(M \otimes \mathcal{O}_C) \leq \mathrm{Cliff}(A)$ one uses the fact that:

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Theorem (LC)

The conjecture holds for $r = 2$.

New idea: Bring into play the fact that:

$$E \text{ non-simple} \implies E \text{ is not } \mu_L\text{-stable}$$

$\rightsquigarrow$ Use Harder-Narasimhan and Jordan-Hölder filtrations!

Key point in the proof: Let A be a g^2_d on C such that $\rho(g, 2, d) < 0$ and let $E := E_{C,A}$ ($\mathrm{rk} E = 3$).
$\implies$ The HN filtration (or the JH filtration) of E has the form $0 \subset N \subset E$, with $N \in \mathrm{Pic}(S)$ and E/N is μ_L-stable of rank 2.

Indeed, all the other types of filtrations are incompatible with the inequality $\rho(g, 2, d) < 0$.

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Possible strategy: Use coherent systems, i.e., pairs (E, V) such that $E \in \mathrm{Coh}(S)$ and $V \subset H^0(S, E)$.

Notion of stability: It depends on the choice of a polynomial $q \in \mathbb{Q}[t]$ with positive leading coefficient.

Fix $q = p_S$: A coherent system (E, V) with E of dimension 2 is semistable (resp. stable) iff

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Look at the maximal destabilizing sequence

$$0 \longrightarrow (E_1, H^0(E_1)) \longrightarrow (E_{C,A}, H^0(E_{C,A})) \longrightarrow (E_2, V_2) \longrightarrow 0,$$

where:

- E_1 is a vector bundle which is generically generated by its global sections;
- E_2 has no torsion and is globally generated by $V_2 \subset H^0(E_2)$.

Lemma

The 1-dimensional locus where E_1 is not generated by global sections is a (possibly empty) union of (-2)-curves.

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Generalized Lazarsfeld-Mukai Bundles

Def: A torsion free sheaf E on S is called a generalized LM bundle iff $h^2(S, E) = 0$ and either

1. E is locally free and generated by global sections off a finite set; or
2. E is globally generated.

Remarks:

- If both the conditions are satisfies and $h^1(S, E) = 0$, then E is a classical LM bundle, i.e., $E = E_{C,A}$ for some smooth $C \subset S$ and $A \in \mathrm{Pic}(C)$ such that both A and $\omega_C \otimes A^\vee$ are b.p.f..
- If E is a g.LM bundle and $\Lambda \in G(\mathrm{rk}E, H^0(E))$ is general, then:

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Generalized Lazarsfeld-Mukai Bundles

Def: A torsion free sheaf E on S is called a generalized LM bundle iff $h^2(S, E) = 0$ and either

1. E is locally free and generated by global sections off a finite set; or
2. E is globally generated.

Remarks:

- If both the conditions are satisfies and $h^1(S, E) = 0$, then E is a classical LM bundle, i.e., $E = E_{C,A}$ for some smooth $C \subset S$ and $A \in \mathrm{Pic}(C)$ such that both A and $\omega_C \otimes A^\vee$ are b.p.f..
- If E is a g.LM bundle and $\Lambda \in G(\mathrm{rk}E, H^0(E))$ is general, then:

$$0 \longrightarrow \Lambda \otimes \mathcal{O}_S \xrightarrow{ev} E \longrightarrow B \longrightarrow 0,$$

where B is a pure sheaf of dimension 1 on S supported on an integral (possibly singular) curve $X \subset S$, i.e., $B \in \overline{J}^d(X)$.

Def: Let E be a generalized LM bundle. The Clifford index of E is:

$$\mathrm{Cliff}(E) := c_2(E) - 2(\mathrm{rk}E - 1).$$

Note: If $E = E_{C,A}$ is a classical LM bundle $\Longrightarrow \mathrm{Cliff}(E) = \mathrm{Cliff}(A)$.

Proposition

If E is a generalized LM bundle, then $\mathrm{Cliff}(E) \geq 0$ and equality holds only in the following cases:

- *E is a line bundle;*
- *$E = E_{C,\omega_C}$ for a smooth curve $C \subset S$.*

Remark: Both the sheaves E_1 and E_2 in the maximal destabilizing sequence of $(E_{C,A}, H^0(E_{C,A}))$ are generalized LM bundles.

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Linear series computing the Clifford index

Theorem (LC)

Let A be a g^r_d on $C \subset S$ such that $d \leq g-1$, $\rho(g,r,d) < 0$ and $\mathrm{Cliff}(A) = \mathrm{Cliff}(C)$.
$\Longrightarrow$ *A is of one of the following types:*

- *$A = M \otimes \mathcal{O}_C$ for some $M \in \mathrm{Pic}(S)$ adapted to $|L|$;*
- *A is a g^1_d, and there exists an embedding $S \hookrightarrow \mathbb{P}^n$ such that A is cut out by an $(n-2)$-plane which is $(2n-2)$-secant to C.*

Idea of the proof: Use the fact that $\det E_1 \otimes \mathcal{O}_C$ contributes to the Clifford index and

$$\mathrm{Cliff}(A) = \mathrm{Cliff}(E_1) + \mathrm{Cliff}(E_2) + \mathrm{Cliff}(\det E_1 \otimes \mathcal{O}_C).$$

$\Longrightarrow \mathrm{Cliff}(E_1) = \mathrm{Cliff}(E_2) = 0.$

Remark: In particular, this provides a new proof of the constancy of the Clifford index for smooth curves in $|L|$ (Green-Lazarsfeld 1987).

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Secant varieties

Given $A \in W^r_d(C)$ and having fixed integers $0 \leq f < e$, consider the variety of secant divisors $V^{e-f}_e(A)$:

$$\operatorname{supp} V^{e-f}_e(A) := \{E \in C_e \mid h^0(A(-E)) \geq r+1-e+f\}.$$

$$\operatorname{expdim} V^{e-f}_e(A) = e - f(r+1-e+f).$$

If A is very ample, then $V^{e-f}_e(A)$ parametrizes e-secant $(e-f-1)$-planes to $C \subset \mathbb{P}^r$.

Def: We say that the pair (C, A) has some unexpected secant varieties up to deformation if it can be deformed to a pair (C', A') such that:

- $C' \in |L|$ and A' is a g^r_d;
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Theorem (LC)

Let A be a g^r_d on $C \subset S$ such that $\rho(g,r,d) < 0$, and assume that the pair (C,A) has no unexpected secant varieties up to deformation.
$\Longrightarrow$ *the Donagi-Morrison Conjecture holds for A.*

Idea of the proof: Show that (C,A) has some unexpected secant varieties up to deformation as soon as $\mathrm{rk}E_1 > 1$.

Conclusion: The use of coherent systems enables to reduce the Donagi-Morrison Conjecture to a question of secant varieties! The existence of some unexpected secant varieties might even cause the failure of the conjecture, but no such example is known.

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Thank you all

for your attention!!!