Hilbert Schemes, Quiver Varieties, and the Boson-Fermion Correspondence

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3. Geometric description

Definition (Heisenberg algebra)

Complex Lie algebra $\mathfrak{s} = \bigoplus_{k \in \mathbb{Z} - \{0\}} \mathbb{C}\alpha(k) \oplus \mathbb{C}c$,

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Action on $\mathbb{C}[x_1, x_2, \dots]$ (bosonic Fock space) via

$$\alpha(k) \mapsto \frac{\partial}{\partial x_k}, \quad \alpha(-k) \mapsto kx_k, \quad k > 0,$$

$$c \mapsto \mathrm{id}.$$

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Realizations of $V_{\text{basic}}(\widehat{\mathfrak{g}})$

- Given a HSA, as $\widehat{\mathfrak{g}}$-modules,

$$V_{\text{basic}}(\widehat{\mathfrak{g}}) \cong \Omega \otimes \mathbb{C}[x_1, x_2, \dots],$$

where $\Omega = \{v \in V_{\text{basic}}(\widehat{\mathfrak{g}}) \mid \alpha(k) \cdot v = 0 \text{ for all } k > 0\}$.

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- Principal HSA:

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- Homogeneous HSA $\longleftrightarrow \underline{r} = (1, 1, \ldots, 1)$.

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- This defines an irreducible representation of the Clifford algebra.

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- "Tweak" slightly to get realizations of $V_{\text{basic}}(\widehat{\mathfrak{sl}}_r)$.

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- $\mathcal{H}_T^*(X) \cong \mathcal{H}_T^*(X^T)$.

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Operator

$\alpha \in \mathcal{H}_T^{2(n_1+n_2)}(X_1 \times X_2)$ gives a map $\alpha : \mathcal{H}_T^{2n_1}(X_1) \to \mathcal{H}_T^{2n_2}(X_2)$ with structure constants

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Other cases

Some "blend" of the previous two.

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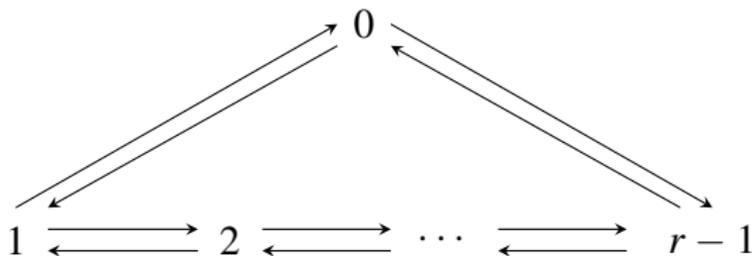
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Fix Q: $Q_0 = \mathbb{Z}_r$, $Q_1 = \{k \to k+1\}_{k \in \mathbb{Z}_r} \cup \{k \to k-1\}_{k \in \mathbb{Z}_r}$

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- $\mathbb{Z}_r$-graded vector space V, $\mathbf{v} = (\dim V_k)_{k \in \mathbb{Z}_r}$, $|\mathbf{v}| = \sum_k \mathbf{v}_k$.
- Let $G_{\mathbf{v}} := \prod_{k \in \mathbb{Z}_r} \mathrm{GL}(V_k)$.

Define $M :=$ variety whose points consist of

- $i \in V_0$,
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  1. $V_{k-1} \; \underset{C_k^-}{\overset{C_{k-1}^+}{\rightleftarrows}} \; V_k \; \underset{C_{k+1}^-}{\overset{C_k^+}{\rightleftarrows}} \; V_{k+1}$ commutes,

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The Nakajima quiver variety is

$$\mathfrak{M}(\mathbf{v}) = M /\!/ G_{\mathbf{v}}.$$

Theorem (Barth, 1977)

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Fact

$$\mathcal{M}(1, n)^{\mathbb{Z}_r} \cong \coprod_{|\mathbf{v}|=n} \mathfrak{M}(\mathbf{v}).$$

$$\mathcal{H}_T^*(-) \quad \leftarrow \text{ Define operators}$$

$$\text{Heisenberg} \quad \curvearrowright \mathcal{M}(1,n) \quad \supseteq \quad \mathcal{M}(1,n)^{\mathbb{Z}_r} \quad \supseteq \quad \mathcal{M}(1,n)^T \quad \curvearrowleft \quad \text{Clifford}$$
$$\text{algebra} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{algebra}$$

$$\cong$$

$$\coprod_{\mathbf{v}} \mathfrak{M}(\mathbf{v}) \quad \curvearrowleft \quad \widehat{\mathfrak{sl}}_r$$

Torus action

Fix $T = \mathbb{C}^*$.

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- $\mathbb{Z}_r \subseteq T$.
- $\coprod_n \mathcal{M}(1, n)^T \overset{\sim}{\longleftrightarrow}$ basis of $F_0 \subseteq F$ (by a result of Nakajima, Yoshioka 2005).

Vector bundles on $\mathcal{M}(1, n)$

Consider $\mathcal{M}(1, n) \cong \mathfrak{M}(\mathbf{v})$ with $r = 1$.

$$V \times_{G_{\mathbf{v}}} M \to \mathcal{M}(1, n) \quad \text{and} \quad \mathbb{C} \times \mathcal{M}(1, n) \to \mathcal{M}(1, n)$$

Denote by $\mathcal{V}$ and $\mathcal{W}$, respectively.

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On the product $\mathcal{M}(1, n_1) \times \mathcal{M}(1, n_2)$

Have Hom-bundles:

$$\mathrm{Hom}(\mathcal{V}^1, \mathcal{V}^2), \quad \mathrm{Hom}(\mathcal{W}^1, \mathcal{V}^2), \quad \mathrm{Hom}(\mathcal{V}^1, \mathcal{W}^2).$$

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On the product $\mathcal{M}(1, n_1) \times \mathcal{M}(1, n_2)$

$$\operatorname{Hom}(\mathcal{V}^1, \mathcal{V}^2) \overset{\sigma}{\hookrightarrow} \begin{array}{c} t\operatorname{Hom}(\mathcal{V}^1, \mathcal{V}^2) \oplus t^{-1}\operatorname{Hom}(\mathcal{V}^1, \mathcal{V}^2) \\ \oplus \\ \operatorname{Hom}(\mathcal{W}, \mathcal{V}^2) \oplus \operatorname{Hom}(\mathcal{V}^1, \mathcal{W}) \end{array} \overset{\tau}{\longrightarrow\!\!\!\!\!\rightarrow} \operatorname{Hom}(\mathcal{V}^1, \mathcal{V}^2)$$

(Similar to construction of Nakajima's Hecke correspondence)

Picking the Right Vector Bundle

Theorem (Licata, Savage 2009)

$\ker \tau / \operatorname{im} \sigma$ *is a vector bundle on* $\mathcal{M}(1, n_1) \times \mathcal{M}(1, n_2)$.

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$(\ker \tau / \operatorname{im} \sigma)^{\mathbb{Z}_r}$ is a vector bundle on $\mathcal{M}(1, n_1)^{\mathbb{Z}_r} \times \mathcal{M}(1, n_2)^{\mathbb{Z}_r}$.

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Theorem (L.)

$(\ker \tau / \operatorname{im} \sigma)^{\mathbb{Z}_r} \cong \bigoplus_{|\mathbf{v}^1|=n_1, |\mathbf{v}^2|=n_2} \mathcal{B}_{\mathbf{v}^1, \mathbf{v}^2}$, *where each* $\mathcal{B}_{\mathbf{v}^1, \mathbf{v}^2}$ *is the vector bundle on* $\mathfrak{M}(\mathbf{v}^1) \times \mathfrak{M}(\mathbf{v}^2)$ *that gives rise to Nakajima's geometric description of* $\widehat{\mathfrak{sl}}_r \curvearrowright H_*(\mathfrak{M})$.

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Observation

Gives a geometric interpretation of $\alpha(1) = E_0 + \cdots + E_{r-1}$ and $\alpha(-1) = F_0 + \cdots + F_{r-1}$.

Putting it all together

$$(\ker \tau / \operatorname{im} \sigma)^{\mathbb{Z}_r} \qquad \cong \qquad \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} \mathcal{B}_{\mathbf{v}^1, \mathbf{v}^2}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\mathcal{M}(1, n_1)^{\mathbb{Z}_r} \times \mathcal{M}(1, n_2)^{\mathbb{Z}_r} \cong \coprod_{\mathbf{v}^1, \mathbf{v}^2} \mathfrak{M}(\mathbf{v}^1) \times \mathfrak{M}(\mathbf{v}^2)$$

$$\downarrow$$

Define
Heisenberg algebra
operators on
$\prod_n \mathcal{H}_T^{\text{middle}} \left(\mathcal{M}(1, n)^{\mathbb{Z}_r} \right)$

$\longleftrightarrow$

Define $\widehat{\mathfrak{sl}}_r$
operators on
$\prod_{\mathbf{v}} \mathcal{H}_T^{\text{middle}} \left(\mathfrak{M}(\mathbf{v}) \right)$

The End

Future goals

Give similar geometric descriptions for the other realizations of $V_{\text{basic}}(\widehat{\mathfrak{sl}}_r)$.

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That's all!

Thank you :-)