Fractional Brownian Motion, Levy Flights, and Fractional Diffusion

@ Transport and long-range interaction workshop SISSA

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Three Questions

- What does the fractional in fractional Brownian Motion or in fractional derivative mean?
- What is the connection between Levy Flights and fBM?
- Which role does long-range correlation play in fractional processes?

What does fractional even mean?

Long range correlation

Part I: Limit theorems

Gaussian Processes

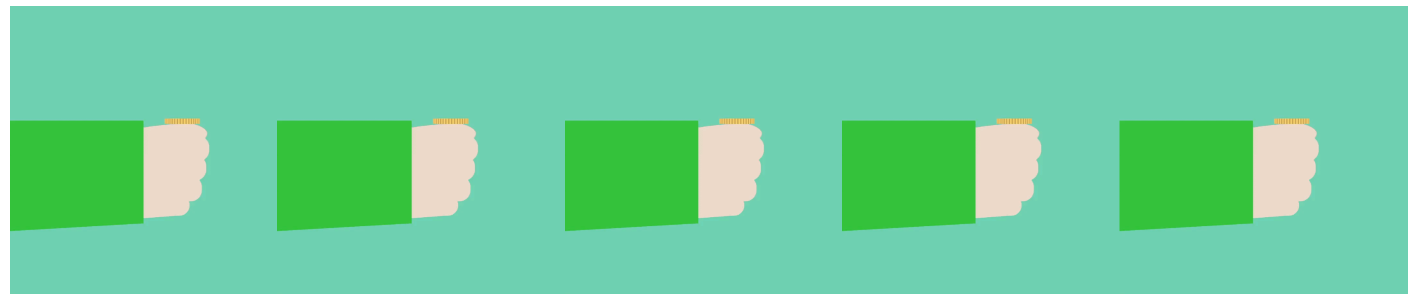
Fractional Brownian Motion

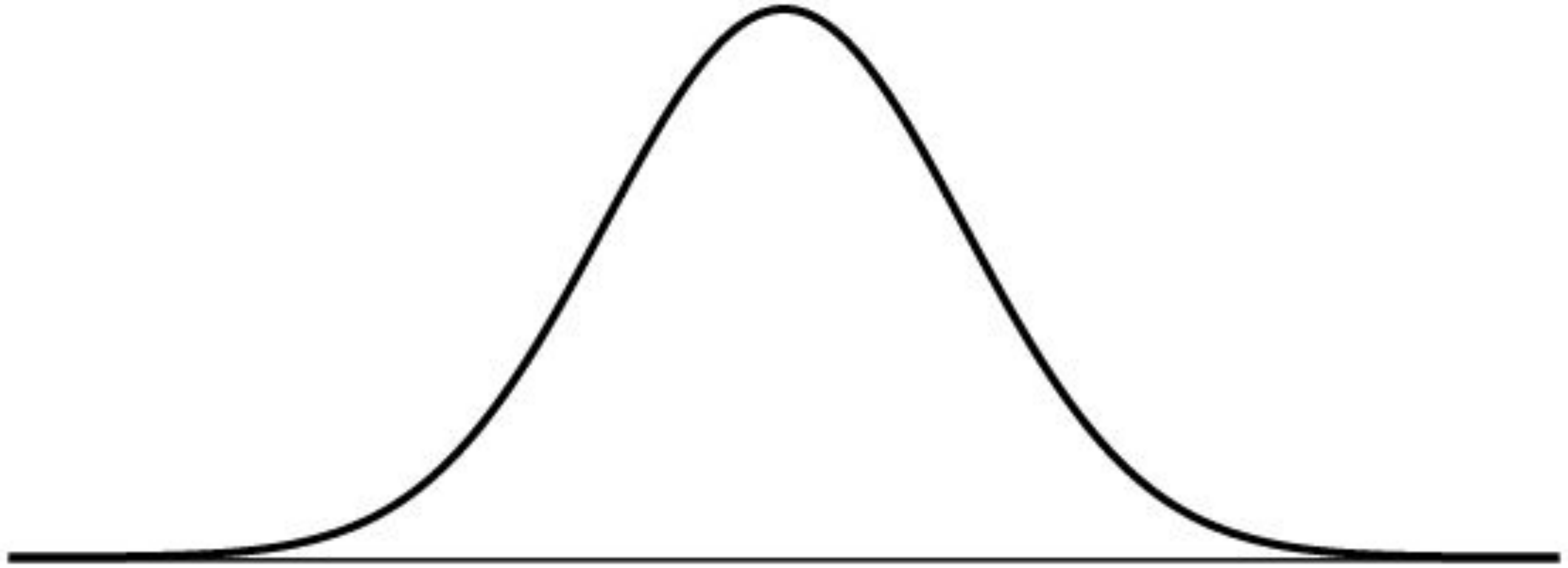
Scaling limits

Levy processes

Fractional Calculus

Part II: Fractional diffusion





Part I: Limit Theorems

Scaling limits

Central Limit Theorem

- Microscopic hopping on a lattice -> Continuous stochastic process
- Example: **Discrete time random walk**
- $X_1, X_2, \ldots, X_N \sim p_X$ iid, and $S_n = \sum_{k=1}^{n} X_n$ with $\mu = \langle X \rangle, \sigma^2 = \left\langle (X-\mu)^2 \right\rangle$ finite
- What are statistical properties of $S_n, n \to \infty$? Need to **rescale** appropriately
- $\tilde{S}_n = \dfrac{N^{-1} S_n - \mu}{N^{-1/2}\sqrt{\sigma^2}} \to \mathcal{N}(0,1)$ with Gaussian $p_G(y) = \dfrac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$

Scaling limits

Discrete time Random Walk to Brownian Motion

- Define linear interpolation $W_{LI}(t) = S_{[t]} + (t - [t])(S_{[t+1]} - S_{[t]})$
- $W_N(t) = N^{-1/2} W_{LI}(N^1 t) \longrightarrow W(t) \sim \mathcal{N}(0,t) \qquad N \to \infty$
- Limit is simple Brownian Motion, Gaussian, independent increments with correlator $\langle W(t)W(s) \rangle = \min(t, s)$
- $\left\langle (W(t) - W(s))^2 \right\rangle = |t - s|^1$
- $W_{ct} \sim c^{1/2} W_t$

Simple Brownian Motion natural scaling limit of discrete time random walks

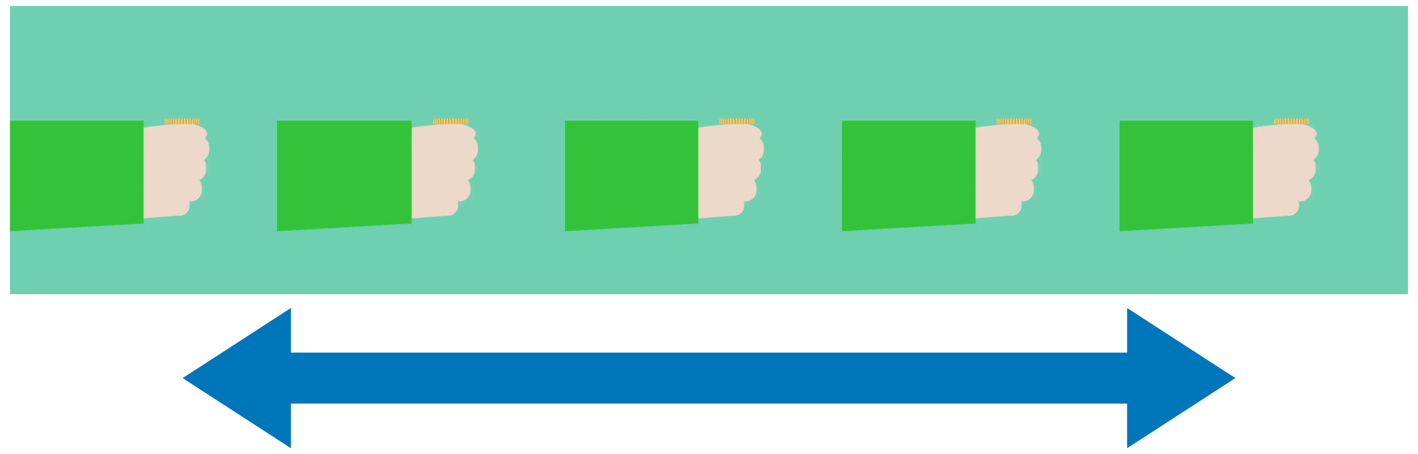
Scaling limits of correlated series

How robust is the CLT?

- **Infinite second moment for jump sizes?**
  - $p_X \simeq |x|^{-1-\alpha}, \alpha \in (0{,}2)$ Levy-stable distribution
  - $p_X \simeq \dfrac{1}{1+x^2}$, Cauchy-distribution

Levy-stable distributions

- **Correlated random variables?**
  - Weak dependency ("short range") — CLT wins
  - 2D Ising model at critical temperature with LxL=N sites, $S_N = \dfrac{1}{N}\sum_i s_i$, $N^{-1/4} S_N \to S$
  - Many open problems (eg in Hamiltonian Mean Field Model)



H Hilhorst, arXiv:0901.1249

Various universal attractor distributions

Levy-stable distributions*

- Four-parameter family of sum-invariant distributions
- $X_1, X_2 \sim L(\alpha, \beta, \mu, \sigma) \Rightarrow aX_1 + bX_2 \sim cX + d \qquad X \sim L(\alpha, \beta, \mu, \sigma)$
- Special cases: Gaussian, α-Levy, Cauchy…

Invariant under sum (modulo rescaling) = fixpoints of distribution

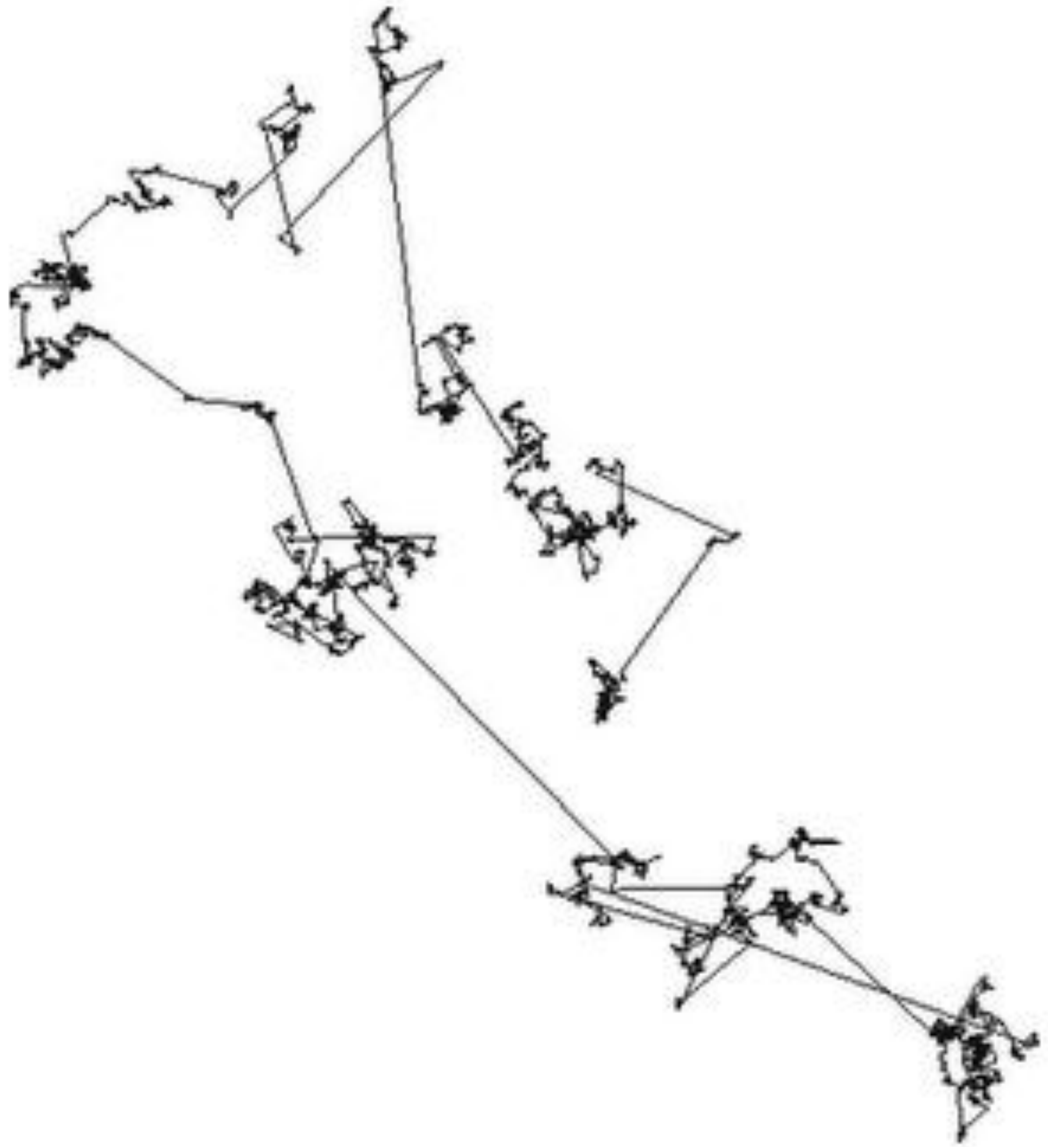
* Different to Levy distribution!

Part I: Limit Theorems

“Normal” variables tend to to normal distributions

Uncorrelated or short-range correlated microscopics leads to Gaussian macroscopics

Breaking jump size variance leads to Levy distributions, breaking independence leads to completely different physics



Part II: Fractional Diffusion

Fractional Brownian Motion (fBM)

- Gaussian Process with correlator $\langle X_s X_t \rangle = |t|^{2H} + |s|^{2H} - 2|t-s|^{2H}$ $\quad H \in (0,1)$
- $H = \frac{1}{2}$ is simple Brownian Motion
- Otherwise non-Markovian
- Self-similar $X_{ct} \sim c^H X_t$
- $\langle X_t^2 \rangle \sim 2t^{2H}$
- Increments are correlated via $\langle \dot{X}_s \dot{X}_t \rangle = 2H(2H-1)|t-s|^{2H-2} + 2H|t-s|^{2H-1}\delta(t-s)$

fBM models self-similar sub- and superdiffusive processes

Phase transition long-range short-range

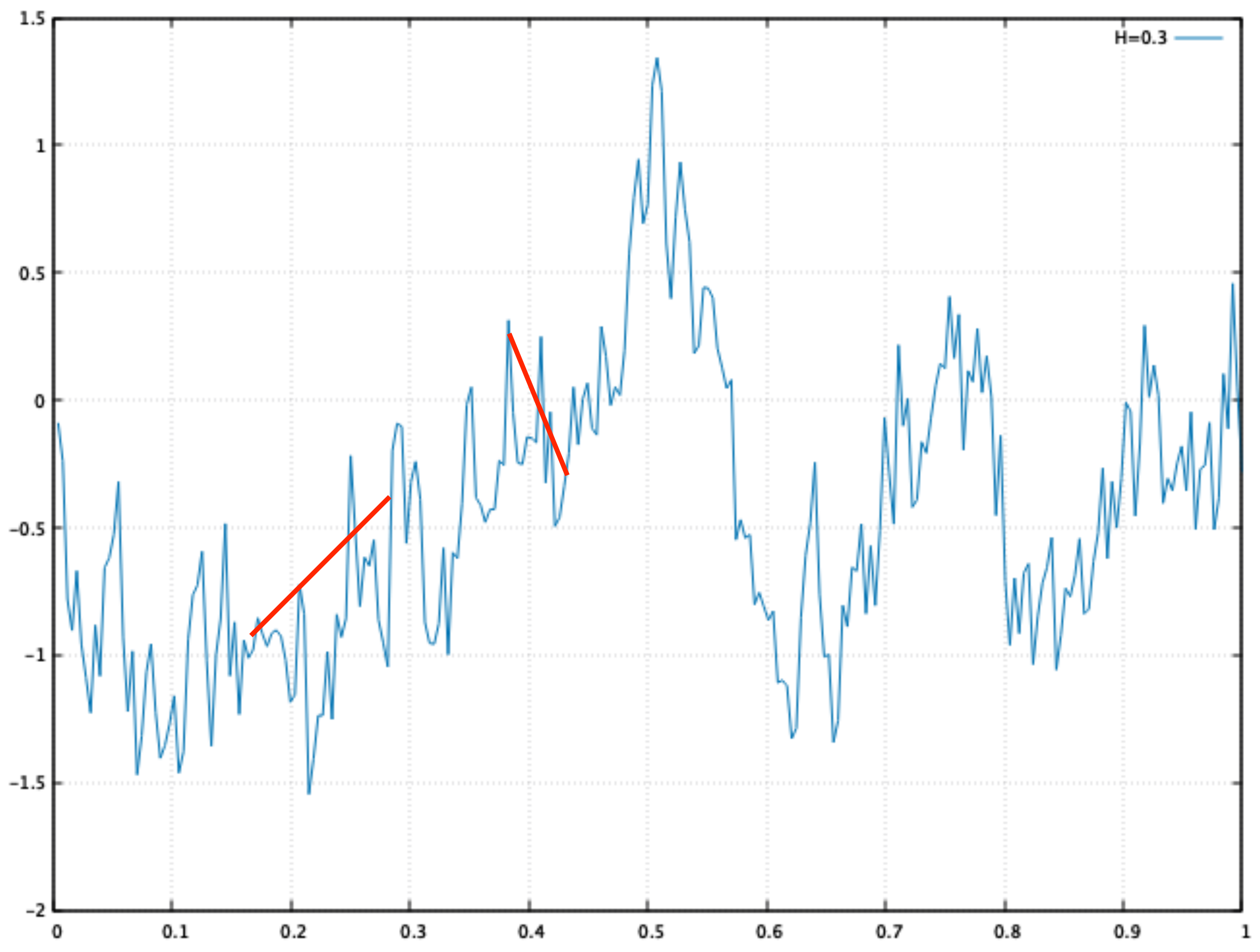
$$\langle \dot{X}_s \dot{X}_t \rangle = 2H(2H-1)\,|t-s|^{2H-2} + 2H\,|t-s|^{2H-1}\,\delta(t-s)$$

Towards white noise (H=0) ←

$$\frac{1}{|t-s|^{2-2H}}$$

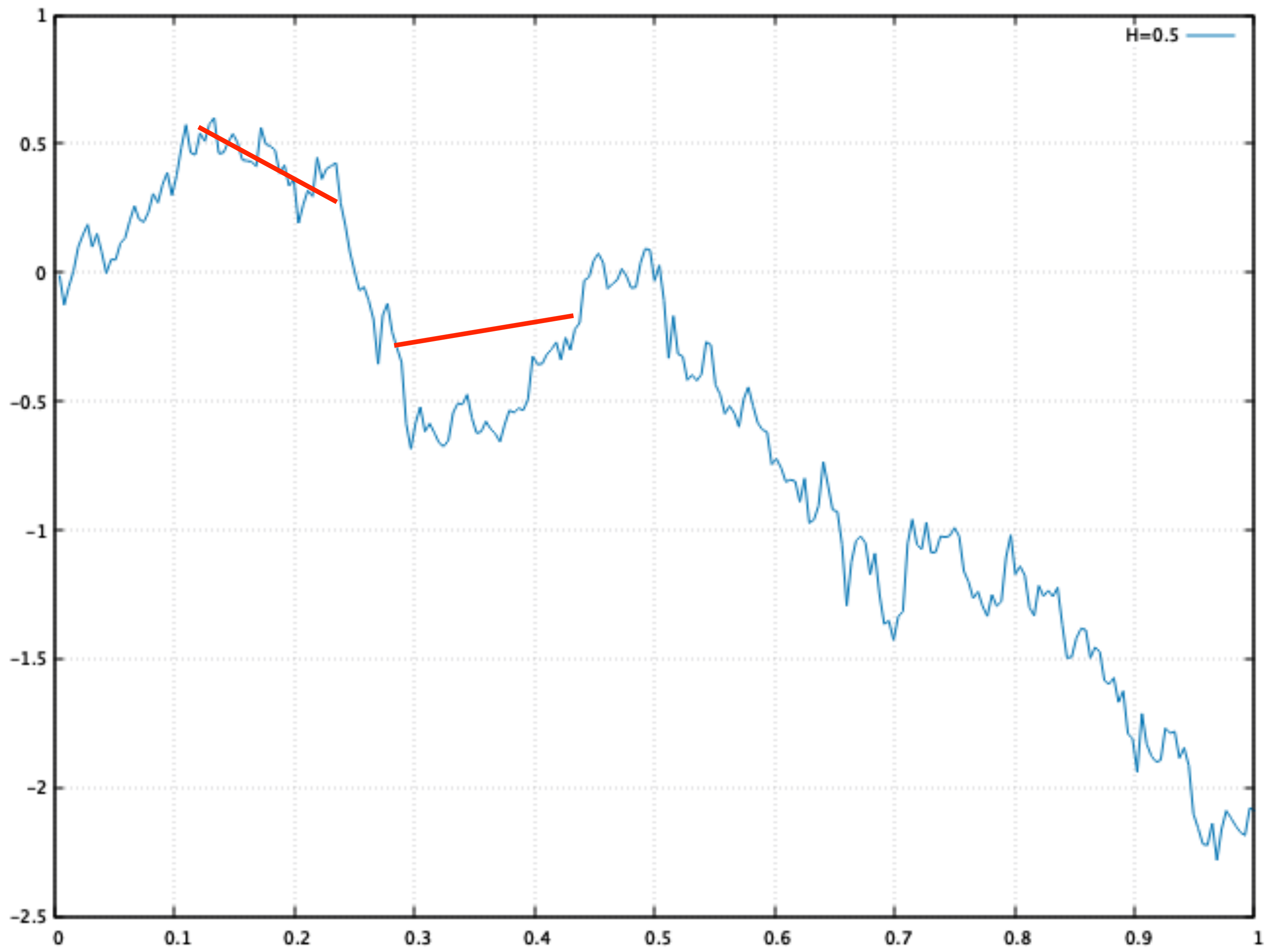
→ **Towards straight line (H=1)**

H = 0.3



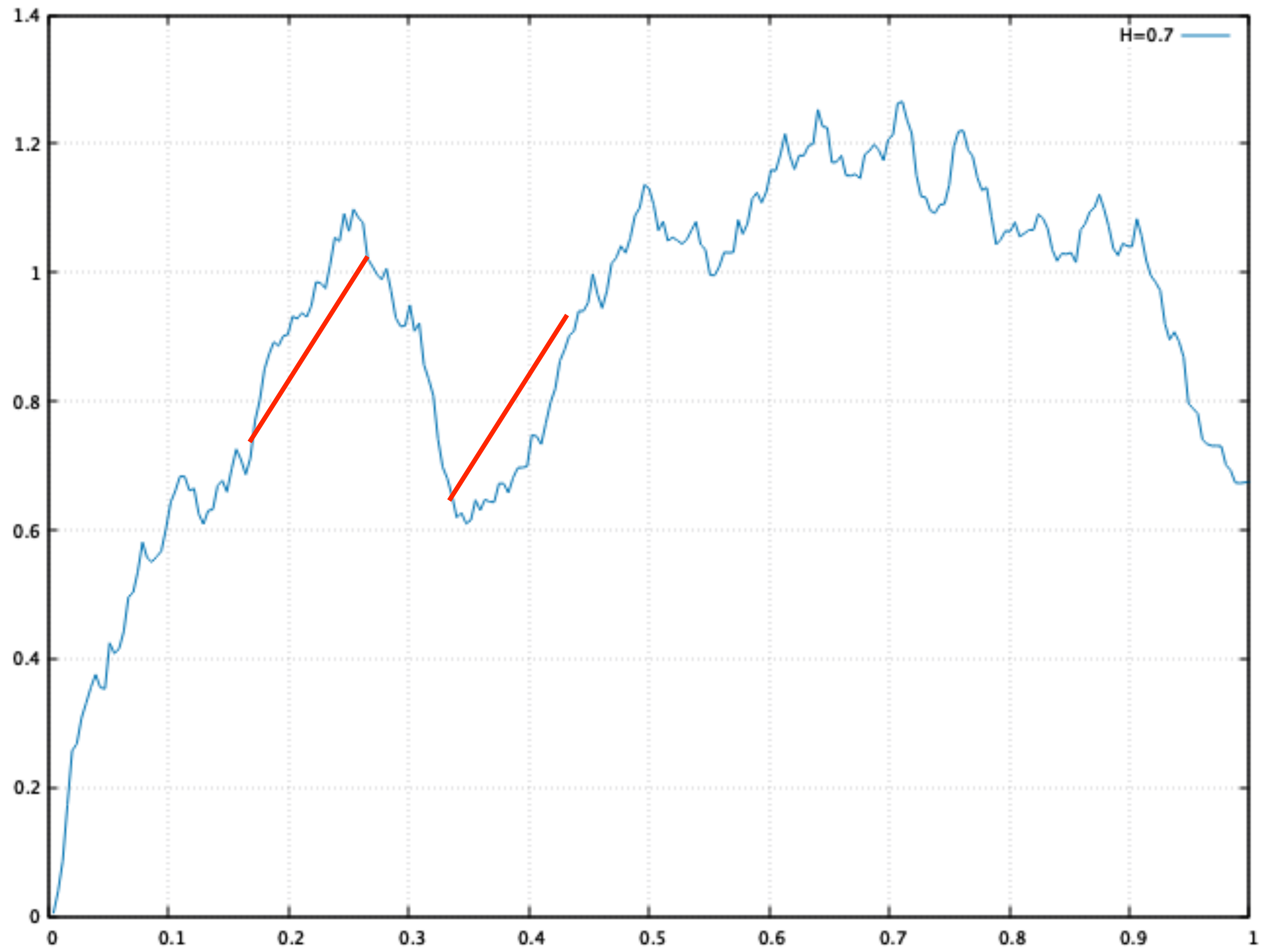
Short range anticorrelated for H<1/2

H = 0.5



Simple Brownian Motion = Critical Point

H = 0.7



Long range correlated for H>1/2

But what *is* fractional Brownian Motion?

- $$B_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\int_0^t \frac{1}{|t-s|^{\frac{1}{2}-H}} dW_s$$

White noise

fractional kernel

Mandelbrot & Van Ness, 1968

- "Fractional integral"

- $$D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)$$

B_t^H is $H+\frac{1}{2}$ -fractional integral over white noise

Fractional integral/derivative is "long-range" for non-integer α

$$\partial_t^{H+\frac{1}{2}} x_t = \xi_t \leftrightarrow (-i\omega)^{H+\frac{1}{2}}\tilde{x}_\omega = \tilde{\xi}_\omega$$

fractional Langevin equation

But what *is* fractional Brownian Motion?

(cont)

Gaussian PDF

Propagator

$$\rho(x_0, x, t_0, t) = \sqrt{\frac{H}{\pi}} \frac{\Gamma(H + \frac{1}{2})}{t^H} \exp\left(-H\Gamma^2 \left(\frac{1}{2} + H \right) \frac{(x - x_0)^2}{2(t - t_0)^{2H}} \right)$$

Calvo, Sanchez. arXiv:0805.1170

Solves

$$D_H = \frac{1}{2\Gamma^2(\frac{1}{2} + H)}$$

$$\frac{\partial}{\partial t}\rho(x,t) = D_H t^{2H-1} \partial_x^2 \rho(x,t) \Leftrightarrow \frac{\partial}{\partial (t^{2H})}\rho(x,t) = 2H \cdot D_H \partial_x^2 \rho(x,t)$$

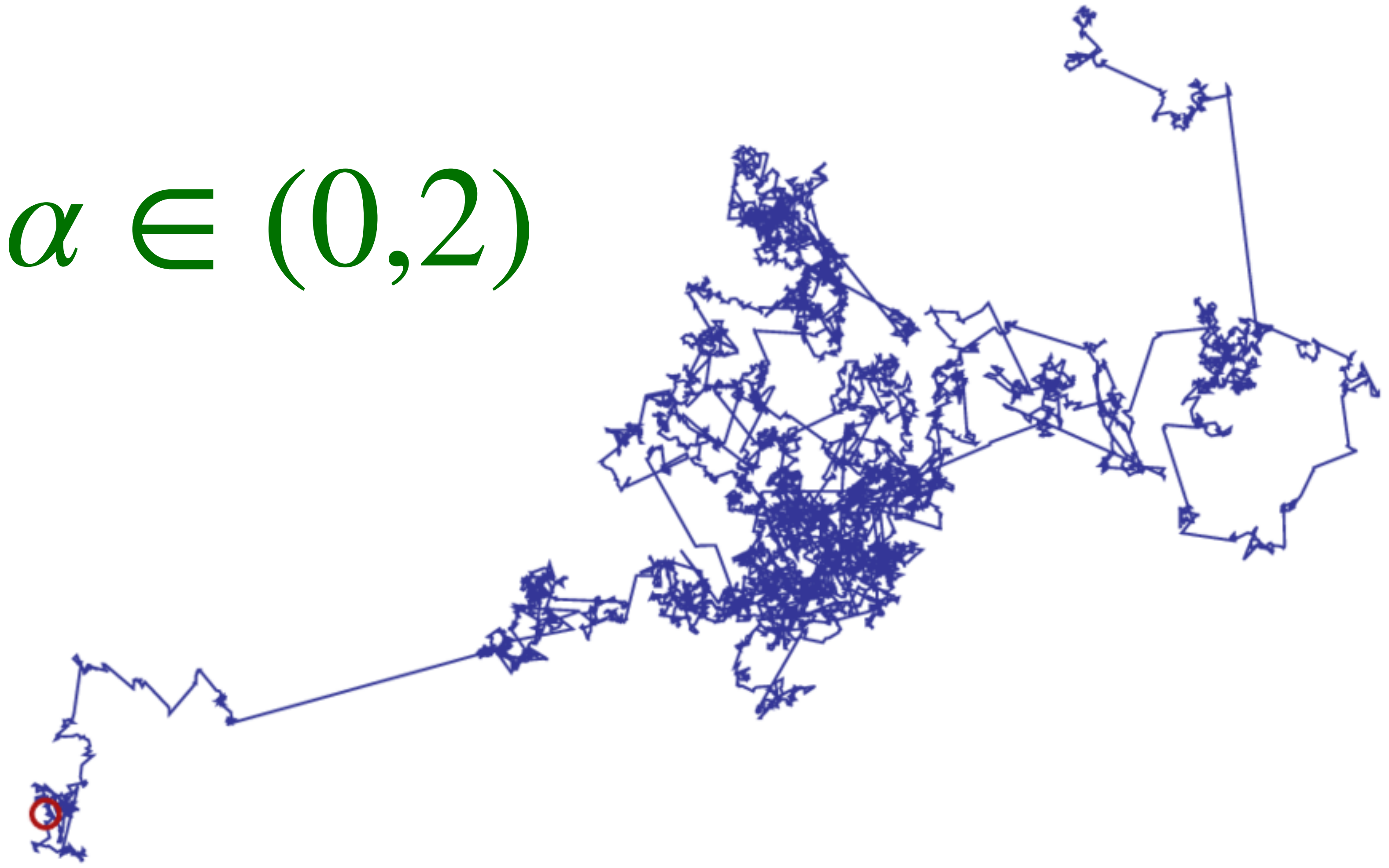
Time-stretched diffusion equation

Bovet. arXiv:1508.01879
Mainardi, Mura, Pagnini. arXiv:1004.2950

fBM solves time-stretched diffusion equation

Space-fractional diffusion equation

$$\partial_t \rho(x,t) = D_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha} \rho(x,t) \qquad \alpha \in (0,2)$$



- Solved in Fourier-space by $\tilde{\rho}(\mathbf{k},t) \sim e^{-|\mathbf{k}|^\alpha t}$
- Levy-flight
  - Step size of $p(x) \sim x^{-1-\alpha}, x \gg 1$ + isotropic direction + Poisson waiting times
  - $\rho(k,\omega) = \dfrac{1}{-i\omega + |k|^\alpha}$

$$\rho(x,t) = t^{-1/\alpha} L_\alpha(x t^{-1/\alpha})$$

Propagator

Bovet. arXiv:1508.01879

Fractional space-derivative encapsulates long-range jumps

fBM + Levy flight?

Time-stretched space-fractional diffusion

Calvo, Sanchez, Carreras. arXiv: 0805.1838

- Fractional space diffusion with time-stretch

$$\frac{\partial}{\partial(t^{\gamma})}\rho(x,t) = D_{\alpha,\gamma}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}\rho(x,t) \qquad (\gamma < \max(1,\alpha))$$

Fractional Levy Motion (fLM)

$$\rho(x,t) \sim t^{-\gamma/\alpha} L_{\gamma,\alpha}(x t^{-\gamma/\alpha})$$

Non-Gaussian non-Markovian

Scaling Limit of Continuous Time Random Walk

fractional space-time diffusion equation

$$\frac{\partial^{\beta}}{\partial(t)^{\beta}}\rho(x,t) = D_{\alpha,\beta}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}\rho(x,t)$$

Bovet. arXiv:1508.01879

$$\rho(x,t) \sim t^{-\beta/\alpha}K_{\alpha,\beta}(xt^{-\beta/\alpha})$$

Scaling Limit of CTRW with

Jump size distribution $p(\Delta x) \sim (\Delta x)^{-1-\alpha}, \Delta x \to \infty$

Waiting time distribution $p(\Delta t) \sim (\Delta t)^{-1-\beta}, \Delta t \to \infty$

Generalisation

time-stretched fractional space-time diffusion equation

General scaling form of propagator well understood for any α, β, γ

$\gamma = 1 \Rightarrow \rho(x,t) \sim t^{-\beta/\alpha} K_{\alpha,\beta}(x t^{-\beta/\alpha})$

$$\frac{\partial^{\beta}}{\partial (t^{\gamma})^{\beta}} \rho(x,t) = D_{\alpha,\beta,\gamma} \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} \rho(x,t)$$

Mainardi, Luchko, Pagnini. arXiv:0702419

Plenty of special cases which are more or less known

Standard Brownian Motion $\alpha = 2, \beta = 1, \gamma = 1$	Fractional Brownian Motion $\alpha = 2, \beta = 1, \gamma = 2H$	Levy Flight $\alpha < 2, \beta = 1, \gamma = 1$
Fractional Levy Matter $\alpha < 2, \beta = 1, \gamma = H\alpha$	Cauchy diffusion $\alpha = \beta < 1, \gamma = 1$	Anomalous slow diffusion $\alpha = 2, \beta < 1, \gamma = 1$

What does fractional mean?

$$\frac{\partial^{\beta}}{\partial (t^{\gamma})^{\beta}} \rho(x,t) = D_{\alpha,\beta,\gamma} \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} \rho(x,t)$$

- α spatial transport exponent - heavy tailed jumps in space
- β temporal transport exponent - heavy tailed waiting times
- γ encapsulates stationary and self similar correlated increments
- Power law decay in time and scaling in $xt^{-\frac{\gamma\beta}{\alpha}}$

Part II: Fractional diffusion

Fractional Processes generalise the CLT to long-range interacting processes

The macroscopic asymptotes are well understood for a variety of microscopic rules

Fractional calculus able to express long-range interacting systems

Three Questions

- What does the fractional in fractional Brownian Motion or in fractional derivative mean?

Convolution with an algebraic kernel

- What is the connection between Levy Flights and fBM?

One solves fractional-space diffusion, the other time-stretched diffusion

- Which role does long-range correlation play in fractional processes?

LRC breaks CLT and instead of Gaussian distribution tends to fractional processes

Many thanks