Geometrical Structures in Supersymmetric (Q)FT

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PARTS I and II

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Lectures Notes
SISSA 2008–2009

Draft not for public circulation

Contents

Introduction

This course replaces a conventional course about Supergravity (SUGRA). Instead of focusing on constructing locally supersymmetric Lagrangians — which is a specialized and somehow technical activity — I will discuss the aspects of supersymmetry (global and local) which are most general and ubiquous in theoretical physics, the aspects that (in my view) no theoretical physicist could ignore given their far reaching implications, in particular for the quantum theory at the non–perturbative level.

Stated differently, I will (try to) explain SUGRA/SUSY not in the historical order — which was rather casual — but rather using the deeper insights we have today, after 30 years of developments. Physical insights, not just technical refinements.

In doing so, I follow Steve Weinberg who once wrote [**1**]:

> In George Orwell's novel, *1984*, the hero is employed by the Ministry of Truth in the task of re–writing history. Any lecturer or author who attempts to summarize progress in an area of theoretical physics is faced with a similar task. The actual historical development of a physical theory is always confused by false starts, theoretical misapprehensions, experimental errors, and the play of personalities. To make sense out of all this, one has to go back and re–write history according to one's best understanding of the underlying logic of the subject. The result is not good history, but it sometimes makes sense, in a way that history rarely does.

Well, here is my personal best understanding of the underlying logic of (local) supersymmetry. It is a geometrical logic. I am referring to the web of *'special geometries'* which appear everywhere in supersymmetric theories (SUSY, SUGRA, superstrings, M–theory, ...). The resulting geometrical wisdom has many crucial applications. To mention just a few:

(1) construct ($\mathcal{N}$–extended) supersymmetric Lagrangians in D space–time dimensions;
(2) construct ($\mathcal{N}$–extended) supergravity in D space–time dimensions;
(3) prove general theorems in General Relativity (*e.g.* positivity of mass);
(4) solve non–perturbatively the ($\mathcal{N}$–extended) susy QFT (*e.g.* Seiberg–Witten);
(5) compute non–perturbatively BPS mass–spectra (domain–wall formulae, . . .);
(6) understand the low-energy physics of *confining* $4D$ supersymmetric gauge theories;

(7) construct gravitational/gauge instantons, their moduli space, and associated dynamics;
(8) construct solutions to Einstein equations, and in particular find solutions to SUGRA of the form $\mathcal{M}_4 \times K_{d-4}$, which correspond to compactifications to four dimensions of a higher dimensional theory, preserving $\mathcal{N}$ supersymmetries;
(9) construct 'vacua' configuration of superstring and M–theory;
(10) find the dualities between the various theories arising as above;
(11) understand the integrable models *via* (holomorphic) symplectic geometry;
(12) prove a bunch of fantastic new theorems in pure geometry[1];
(13) ....;
(14) only the *devil* knows what else;
(15) the Langlands program?

Not worry! I will not talk about all this stuff. I have no time. I want to give you just the big frame, not the complete picture. And, sometimes, I will allow myself to cheat a little bit, to save time and print, and to make long stories reasonably short.

Plan of the course

WARNING!

Following Cassels [**2**], from time to time I will insert in the text some comments (even entire sections) which are meant for the *cognoscenti*, that is readers already introduced to the Higher Mysteries. My best advice is: *NEVER READ THEM!*

Section marked with an asterisk $*$ are meant as side material, typically pedantic and tedious, which is not (too) essential.

Prerequisites

[1] E. Witten got the Fields Medal for this.

Part 1

Supersymmetric Field Theory: How Geometric Patterns Arise

CHAPTER 1

Geometrical Structures in (Q)FT

In this introductory chapter we show how geometrical structures[1] arise naturally from classical and quantum field theories on quite general grounds. Later we shall specialize to the supersymmetric case (both rigid and local SUSY).

This chapter serves as a first *motivation* for our geometric approach. It is somehow elementary, but it shows how dualities, modularity and other 'stringy' patterns are in fact universal structures in field theory.

If the motivations presented here are not enough to convince you of the merits of the subject, think of the huge quantity of boring algebra (heavy Fierzing ect.) the geometrical approach will spare you.

1. (Gauged) σ–models

For pedagogical reasons, we begin by considering a theory with only scalar fields. In the next section we shall add fields of arbitrary spin.

Most QFT's have scalar fields. Usually we can understand a lot about the dynamics of a QFT just by studying its scalar sector. This is *a fortiori* true if the theory is (enough) supersymmetric, since then all other sectors are related to the scalar one by a symmetry. The understanding of the scalars' geometry is relevant even for theories, like QCD, which do not have fundamental scalar fields in their microscopic formulation. At low energy, QDC is well described by an effective scalar model whose fields represent pions (the lightest particles in the hadronic spectrum). This effective theory is the original σ–model. It encodes all of the current algebra of QCD, and its phenomenological predictions are quite a success [**3, 4, 5**]. So, let us begin by generalizing it.

1.1. The target space $\mathcal{M}$. We consider a general QFT in D space–time dimensions having scalar fields.

Let ϕ^i, $i = 1, 2, \ldots, n$, be the scalar fields of the given theory. Limiting ourselves to Lagrangians having (at most) two derivatives, we have

$$\mathcal{L} = -\frac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i\, \partial^\mu \phi^j + \text{terms at most linear in } \partial_\mu \phi^i \tag{1.1}$$

for some symmetric ϕ–dependent matrix $g_{ij}(\phi)$. Unitarity requires positivity of the kinetic terms, so $g_{ij}(\phi)$ should be positive definite. Physical quantities (S–matrix elements, correlations of observables, ect.) should be independent

[1]By 'geometric structures' I mean properties which are best understood in terms of differential geometry of smooth manifolds and bundles.

of which fields we use in the Lagrangian, that is they should be invariant under field reparameterizations

$$\phi^i \to \varphi^i = \varphi^i(\phi). \tag{1.2}$$

In the new fields φ^i the Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{2}\widetilde{g}_{ij}(\varphi)\,\partial_\mu\varphi^i\,\partial^\mu\varphi^j + \cdots \tag{1.3}$$

where

$$\widetilde{g}_{ij}(\varphi) \equiv \frac{\partial\phi^k}{\partial\varphi^i}\,g_{kl}\,\frac{\partial\phi^l}{\partial\varphi^j}. \tag{1.4}$$

The above equations have a simple geometric interpretation: the fields ϕ^i are local coordinates on a (smooth) manifold $\mathcal{M}$ and the tensor g_{ij} is a Riemannian metric for $\mathcal{M}$ with the correct transformation under diffeomorphisms, eqn.(1.4). This allows us to describe the situation in more geometric terms: We have two manifolds, the target one $\mathcal{M}$, which can have a non–trivial topology[2], and the space–time manifold Σ. A classical field configuration is just a (smooth) map

$$\Phi\colon \Sigma \to \mathcal{M} \tag{1.5}$$

which is given, in local coordinates, by the functions $\phi^i(x^\mu)$. The Lagrangian is simply the trace (with respect to the space–time metric) of the pull–back (induced) metric[3] $\Phi^* g$.

(Q)FT's defined by maps $\Sigma \to \mathcal{M}$ and the above Lagrangian (without other couplings) are called *σ–models.* We stress again that in these models *all* physical quantities are differential–geometric invariants of the Riemannian manifold $\mathcal{M}$. This observation is very powerful. We give an example.

1.1.1. *Example: the RG β–functions.* To show the power of the geometrical viewpoint, we discuss the one–loop beta–functions of the σ–model. We take $D = 2$, the number of space–time dimensions in which the model is power–counting renormalizable. We introduce $\hbar$ as a loop–counting device. The action is

$$\mathcal{L} = -\frac{1}{2\hbar}\int_\Sigma g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j. \tag{1.6}$$

We see that a rescaling $\hbar \to \lambda\hbar$ is equivalent to $g_{ij} \to \lambda^{-1} g_{ij}$, so the weak-coupling limit $\hbar \to 0$ is just the large volume limit for $\mathcal{M}$. The k–loop contribution to the β–functions scales like $\hbar^{k-1}$.

A general σ–model has an *infinite* number of coupling constants, $g_{i_1 i_2\cdots i_l}$,

$$S = -\frac{1}{2\hbar}\int \sum_{l=1}^{\infty} g_{i_1 i_2\cdots i_l}\,\phi^{i_3}\phi^{i_4}\cdots\phi^{i_l}\,\partial_\mu\phi^{i_1}\partial^\mu\phi^{i_2},$$

namely the Taylor coefficients[4] of the metric $g_{ij}(\phi)$. We can conveniently combine the infinite system of beta–functions into a symmetric tensor, $\beta_{ij}(\phi)$,

[2]Hence the fields ϕ^i are, in general, only locally defined on $\mathcal{M}$.

[3] A mathematician would said that the action is 'the energy in the sense of harmonic maps'.

[4]Assuming the metric is of class C^ω.

whose Taylor coefficients are the β–functions of the couplings $g_{i_1 i_2 \cdots i_l}$. The RG flow then reads

$$\mu \frac{\partial}{\partial \mu} \frac{g_{ij}(\phi)}{\hbar} = \beta_{ij}(\phi). \tag{1.7}$$

The geometric principle implies that $\beta_{ij}(\phi)$ is a *covariant* symmetric tensor made out of the metric g_{ij} and its derivatives. Moreover, $\beta_{ij}(\phi)$ should vanish for a flat metric, since in that case the QFT is free. Therefore $\beta_{ij}(\phi)$ is a symmetric tensor which has an expansion as a sum of products of Riemann tensor derivatives, $\nabla_{i_1} \cdots \nabla_{i_s} R_{jkl}{}^m$, with the indices contracted in a suitable way using the inverse metric g^{ij}. A term in this expansions scales with the volume as

$$\hbar^{r+s-1}$$

where r is the number of Riemann tensors and $2s$ the total number of covariant derivatives. At one–loop this leaves only one possibility: one Riemann tensor and no derivative. Thus

$$\beta_{ij}|_{\text{one loop}} = c_1\, R_{ij} + c_2\, g_{ij}\, R.$$

Next we show that $c_2 = 0$ and that c_1 is a universal coefficient which does not depend on $\mathcal{M}$, not even on its dimension. Indeed, take $\mathcal{M} = \mathbb{R}^n \times \mathcal{N}$. The fields of the flat factor are free, and then $\beta_{ij}|_{\mathbb{R}^n} = 0$, whereas the above formula gives $c_2\, R\, \delta_{ij}$. So $c_2 = 0$. On the other hand, $\beta_{ij}|_{\mathcal{N}} = c_1\, R_{ij}$ cannot depend on $n \equiv$ the number of flat directions (they correspond to decoupled free fields). Hence c_1 should be independent of $\dim \mathcal{M}$.

These arguments are even more powerful when the theory is invariant under (extended) supersymmetry. In that case there are severe geometric restrictions on the polynomials in the $\nabla_{i_1} \cdots \nabla_{i_s} R_{jklm}$'s which may appear in the loop expansion of β_{ij}.

$\ast$ Remark. As we shall see in chapt.2, the σ–model admits a SUSY completion. The above discussion goes trough, word–for–word in the supersymmetric case too. However, in that case, one can compute by purely geometrical means even the numerical value[5] of c_1, without making any reference to Feymann diagrams. A possible strategy is: since c_1 is independent of the manifold $\mathcal{M}$, choose it to be compact an two–dimensional. Then the correct value of c_1 is the unique one which makes the Kallan–Symanzik equation of RG identical to the Riemann–Roch formula[6]. Riemann–Roch is — of course — an exact statement, not just a 'one–loop' result. Thus one realizes that the geometric approach allows to compute *exactly* (even non–perturbatively) some β–functions in the supersymmetric theory. In fact geometry fully determines 'most' of the β–functions of a SUSY model. Results of this kind are known as *supersymmetric non–renormalization theorems.* Although they are usually proven using other techniques, they are most easily (and deeply!) understood *via* geometry.

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[5] Obviously $c_1|_{\text{susy}}$ is different from of $c_1|_{\text{bosonic}}$, since the first gets contributions also from the fermionic loop.

[6] For the mathematics see, *e.g.* ref. [**6**], for its meaning in terms of $2D$ field theory, see ref. [**7**].

1.2. Symmetries, gaugings and Killing vectors. The geometry says a lot more. Assume our Lagrangian has an internal symmetry group G acting non trivially on the scalars. G should be, in particular, a symmetry of the two–derivative terms, hence it should leave invariant g_{ij}, that is *it should be an isometry of the Riemannian manifold* $\mathcal{M}$. The infinitesimal symmetries, $\phi^i \to \phi^i + \epsilon\, K^i(\phi)$, are generated by vectors $K^i_A \partial_i$ ($A = 1, 2 \dots, \dim G$) on $\mathcal{M}$ satisfying the Killing condition

$$\pounds_{K_A}\, g_{ij} = \nabla_i K_{Aj} + \nabla_j K_{Ai} = 0 \tag{1.8}$$

and

$$\pounds_{K_A}\, K_B = [K_A, K_B] = f_{AB}{}^C K_C \tag{1.9}$$

where $\pounds_v$ denotes the Lie derivative with respect to the vector field v and $f_{AB}{}^C$ are the structure constants of $\mathfrak{g}$ (the Lie algebra of G).

The existence of a group of isometries — in particular a non–Abelian group — is a strong requirement on the geometry of $\mathcal{M}$. For instance, if $\mathcal{M}$ is compact and negatively curved, it definitely has no Killing vector[7].

1.2.1. *Gauging a subgroup of* Iso($\mathcal{M}$). We may wish to gauge a subgroup G of the isometry group Iso($\mathcal{M}$). The (minimal) coupling of the vector fields A^A_μ to the scalars is then dictated by the geometry of $\mathcal{M}$ trough the Killing vectors. One replaces the ordinary derivative by the covariant one according to the dictionary

$$\partial_\mu \phi^i \to D_\mu \phi^i := \partial_\mu \phi^i - A^A_\mu K^i_A. \tag{1.13}$$

The gauge transformation then reads

$$\delta\phi^i = \Lambda^A\, K^i_A(\phi) \tag{1.14}$$

$$\delta A^A_\mu = \partial_\mu \Lambda^A + f^A{}_{BC} A^B_\mu \Lambda^C, \tag{1.15}$$

so

$$\begin{aligned} \delta D_\mu \phi^i &= \Lambda^A(\partial_j K^i_A)\, D_\mu \phi^j + \\ &+ A^B_\mu \Lambda^C \big[K^j_B \partial_j K^i_C - K^j_C \partial_j K^i_B\big] - f_{BC}{}^A A^B_\mu \Lambda^C\, K^i_A = \\ &= \Lambda^A(\partial_j K^i)\, D_\mu \phi^j \equiv \pounds_{(\Lambda^A K_A)} D_\mu \phi^j, \end{aligned} \tag{1.16}$$

where Λ^A are arbitrary functions in space–time. The covariance of $D_\mu \phi^i$ follows from the closure of the gauge algebra, eqn.(1.9), while the invariance of the kinetic term $g_{ij} D_\mu \phi^i D^\mu \phi^j$ also requires $\pounds_{(\Lambda^A K_A)}\, g_{ij} = 0$ that is the Killing condition, eqn.(1.8).

We stress the following crucial

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[7] *Proof.* Let K^i be Killing, *i.e.* $\nabla_i K_j + \nabla_j K_i = 0$. Contracting this equation with g^{ij}, we get $\nabla_i K^i = 0$, while taking the derivative

$$\nabla_k \nabla_i K_j = -\nabla_k \nabla_j K_i = -[\nabla_k, \nabla_j] K_i - \nabla_j \nabla_k K_i = \tag{1.10}$$

$$= -R_{kji}{}^l K_l - \nabla_j \nabla_k K_i. \tag{1.11}$$

Contracting with g^{ki} and using $\nabla_i K^i = 0$, we get

$$-\nabla^2 K_i = R_{ij} K^j. \tag{1.12}$$

Multiply by K^i and integrate on $\mathcal{M}$. On one side we get $\int_{\mathcal{M}} |\nabla_k K_i|^2 \geq 0$, while on the other we have $\int R_{ij} K^i K^j$ which cannot be non–negative if R_{ij} is negative definite.

GENERAL LESSON 1.1. *The physics of the* (gauged) *σ–model is controlled by the differential geometry of the target manifold $\mathcal{M}$.*

This principle as a useful

COROLLARY 1.1 (target space equivalence principle). *Any quantity that depends only on the metric and its first derivative can be safely computed using a flat target space.*

2. Adding fields with arbitrary spin

2.1. Couplings and geometric structures on $\mathcal{M}$. We have seen that the scalars' couplings to gauge vectors are specified by a set of vector fields on the manifold $\mathcal{M}$ which satisfy certain differential–geometric constraints, namely the Killing equation and the Lie algebra condition.

This is a first example of a general phenomenon: *all* the couplings in a Lagrangian can be identified with suitable (differential–)geometric structures on the scalars' manifold $\mathcal{M}$.

The geometric viewpoint is particularly useful in the supersymmetric case. To make our point, we shall work out the details of a specific example, in which only spin–1/2 fermions are present. However the reader can easily realize that the ideas and arguments are pretty general, and work — *mutatis mutandis* — for fields of any spin.

2.1.1. *General* 2*d model with fermions.* To give a relevant example, let us consider the general theory with scalars ϕ^i, and fermions ψ^a, $i = 1, 2, \dots, n$, $a = 1, 2, \dots, m$. We choose $D = 2$ which is the number of dimensions in which these models have the more interesting applications [as world-sheet theories of some (super)string [**9**]] and in which they make sense quantum mechanically. The arguments, however, are manifestly dimension–independent (apart for questions of existence as quantum field theories) and we shall use them in *diverse* dimensions.

Limiting ourselves to power–counting renormalizable theories, the most general Lagrangian, compatible with locality and Poincaré invariance, is

$$\begin{aligned} L = -\frac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j + b_{ij}(\phi)\, \epsilon^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j + V(\phi) + \\ + i h_{ab}(\phi)\, \bar{\psi}^a \gamma^\mu \partial_\mu \psi^b + i \tilde{h}_{ab}(\phi)\, \bar{\psi}^a \gamma_3 \gamma^\mu \partial_\mu \psi^b + \\ + k_{abi}(\phi)\, \bar{\psi}^a \gamma^\mu \psi^b \partial_\mu \phi^i + \tilde{k}_{abi}(\phi)\, \bar{\psi}^a \gamma^\mu \gamma_3 \psi^b \partial_\mu \phi^i + \\ + y_{ab}(\phi)\, \bar{\psi}^a \psi^b + \tilde{y}_{ab}(\phi)\, \bar{\psi}^a \gamma_3 \psi^b + \\ + s_{abcd}(\phi)\, \bar{\psi}^a \psi^c \bar{\psi}^b \psi^d + \text{other 4–fermi terms} \end{aligned} \tag{2.1}$$

All couplings are arbitrary functions of the scalars field ϕ^i.

Each term in the Lagrangian can be interpreted as a geometric structure on $\mathcal{M}$. Let us look at each of them. The coupling $b_{ij}(\phi)$ is antisymmetric in the indices i, j and hence can be seen as a differential 2–form $b = b_{ij}(\phi)\, d\phi^i \wedge d\phi^j$ on $\mathcal{M}$. The value of this contribution to the action S from a given field configuration (map) $\Phi\colon \Sigma \to \mathcal{M}$ is given by the very geometrical (in fact functorial) formula

$$\int_\Sigma \Phi^* b. \tag{2.2}$$

This, in particular, implies that — up to space–time boundary phenomena — the physics should be invariant under the *target space* gauge invariance

$$b \to b + d\xi \tag{2.3}$$

for all 1–forms ξ on $\mathcal{M}$. Indeed, under this variation of the $b_{ij}(\phi)$ coupling, the action changes as

$$S \to S + \int_{\Sigma} \Phi^* d\xi = S + \int_{\Sigma} d\Phi^* \xi = S + \int_{\partial\Sigma} \Phi^* \xi \tag{2.4}$$

so $b \to b + d\xi$ is a symmetry whenever we are allowed to ignore boundary terms in the action, *e.g.* if space–time Σ is closed. In such a situation, the physics should depend only on the gauge–invariant field–strength 3–form of b, $H = db$, as well as on the harmonic projection of b. In particular, when $H = 0$ — *i.e.* b is closed — the physics depends only on the cohomology class of b and the coupling $\int \Phi^* b$ is purely topological: it measures the class $[X^* b]$ as a multiple of the fundamental class of Σ. Thus it does not change under continuous deformations of the map Φ.

The next coupling, $V(\phi)$, is easy. It is just a scalar field on $\mathcal{M}$. In many situations $V(\phi)$ also carries interesting structure. For instance, if we gauge a bosonic symmetry, we must have

$$\pounds_{K_A} V = 0, \tag{2.5}$$

which is the geometrical statement of gauge invariance. Other geometrical structures related to the scalars' potential will be presented at the due time.

To discuss the other couplings, we change notation and use Majorana–Weyl (one–component) fermions $\psi^a_\pm = \pm\gamma_3 \psi^a_\pm$. The natural interpretation of the second line of eqn. (2.1) is that the chiral fermions $\psi^a_\pm$ live in vector bundles[8] over Σ which are the pull-backs $\Phi^* \mathcal{V}_\pm$ of bundles $\mathcal{V}_\pm \to \mathcal{M}$ of rank m (in fact, we may as well choose the two bundles $\mathcal{V}_+$ and $\mathcal{V}_-$ to have different ranks[9] m_+ and m_-) and having *fiber metrics*

$$h^\pm_{ab} = h_{ab} \pm \tilde{h}_{ab}. \tag{2.6}$$

Again, $h^\pm_{ab}$ should be positive definite and hence invertible. We write $h^{ab}_\pm$ for their inverse metrics. The couplings in the third line of (2.1) take the form

$$ik^+_{ab\,i}(\phi)\, \psi^a_+ \psi^b_+\, \partial_- \phi^i + ik^-_{ab\,i}(\phi)\, \psi^a_- \psi^b_-\, \partial_+ \phi^i \tag{2.7}$$

where $k^\pm_{ab\,i} = -k^\pm_{ba\,i}$. Defining $\omega^{\pm\,a}_{i\ \ b} := h^{ac}_\pm k^\pm_{cb\,i}$, and writing

$$D_\mu \psi^a_\pm := \partial_\mu \psi^a_\pm + \partial_\mu \phi^i\, \omega^{\pm\,a}_{i\ \ b} \psi^b_\pm, \tag{2.8}$$

we can combine the second and third line of eqn.(2.1) into the very geometric form

$$ih^+_{ab}\, \psi^a_+ D_- \psi^b_+ + (+ \leftrightarrow -) \tag{2.9}$$

where now the covariant derivative contains a connection $\partial_\mu \phi^i\, \omega^{\pm\,a}_{i\ \ b}$ which is just the pull–back, trough the scalars' map Φ, of the connection on the vector bundles $\mathcal{V}_\pm \to \mathcal{M}$ given by $\omega^\pm_{i\,ab}$. In particular, this means that the

[8]For the moment we take Σ to be flat Minkowski/Euclidean space. For the general case, see below.

[9]Paying attention to cancel axial anomalies, if we wish to have a sensible quantum theory.

couplings in the third line of (2.1) *do* transform as a connection under arbitrary field redefinitions. Moreover, Hermiticity of the Lagrangian requires $D_\mu(h^{\pm}_{ab}\,\psi^a_{\pm}) = h^{\pm}_{ab}\,D_\mu\psi^b_{\pm}$, that is $\omega^{\pm}_{i\,ab}$ *is a metric connection with respect to the fiber metric* h_{ab}.

Thus the couplings in the second and third lines of eqn.(2.1), although *a priori* totally generic and structureless, *do* correspond to very nice geometric objects on $\mathcal{M}$ namely *vector bundles with fiber metrics and compatible connections.*

As for the remaining couplings, we rewrite the last two lines of eqn.(2.1) in the general form

$$Y_{ab}(\phi)\,\psi^a_+\psi^b_- + S_{abcd}(\phi)\,\psi^a_+\psi^b_+\psi^c_-\psi^d_- \tag{2.10}$$

($S_{abcd} = -S_{bacd} = -S_{abdc}$). It is evident that the Yukawa coupling Y_{ab} is a section of the (pull-back of the) bundle[10] $\mathcal{V}^\vee_+ \otimes \mathcal{V}^\vee_-$, while the 4–Fermi coupling is a section of $\wedge^2\mathcal{V}^\vee_+ \otimes \wedge^2\mathcal{V}^\vee_-$. Both these bundles have natural fiber metrics and connections inherited from $\mathcal{V}_\pm$.

The above analysis is correct for a flat space–time Σ. For general Σ's, the spinors themselves are sections of some spin–bundle $\mathcal{S}_\pm \to \Sigma$, with a spin–connection related to the Christoffel connection by the usual formulae of Riemannian geometry. When both Σ and $\mathcal{M}$ are non–trivial, the spinor bundles get 'twisted' by the (pull-back of) the above target–space bundles $\mathcal{V}_\pm \to \mathcal{M}$.

In conclusion,

General lesson 2.1. *Everything in (Q)FT is differential–geometric!! Fields with (any) spin take values (i.e. are sections) of vector bundles of the form*

$$\mathcal{S}_R \otimes \Phi^*\mathcal{V} \to \Sigma \tag{2.11}$$

where $\mathcal{S}_R \to \Sigma$ *is the vector bundle associated to the spin representation* R *of the given field*[11] *and* $\mathcal{V}$ *is target–space bundle describing the interactions of the field with the scalars. The derivative couplings are determined by covariant derivatives whose connection is the natural ('functorial') one on the bundle* (2.11)*. [For higher spin fields there is typically some additional structure*[12]*].*

Remark. Now (I hope) it is clear why we focus on the scalar fields. They are the only fields which can enter in the Lagrangian in a non polynomial way, and hence have general reparameterization symmetries. Everything else — having a natural affine structure — should live on vector bundles over the scalars' manifold $\mathcal{M}$ and have metrics and connections (possibly

[10] *Notation:* if V is a vector space or a vector bundle, we write $V^\vee$ for the dual vector space or bundle, that is for the space (bundle) of $\mathbb{K}$–linear maps $V \to \mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}, \mathbb{C}$ or the sheaf $\mathcal{A}$ of smooth functions, according to the case at hand.

[11] $\mathcal{S}_R$ is defined as follows. If the space–time Σ has Euclidean signature, the Riemannian geometry defines a metric connection on the tangent bundle $T\Sigma$ taking values in $\mathfrak{so}(\dim\Sigma)$, and so defines an $SO(\dim\Sigma)$ principal bundle. If Σ is a *spin* manifold, this principal bundle can (by definition) be uplifted to a $Spin(\dim\Sigma)$ principal bundle $\mathcal{P}$. $\mathcal{S}_R$ is the vector bundle associated to $\mathcal{P}$ corresponding to the representation R of $Spin(\dim\Sigma)$. In the Minkowski signature case, one replaces $SO(\dim\mathcal{M})$ with $SO(\dim\mathcal{M}-1,1)$.

[12] See the sections below.

trivial) over it. Hence all couplings get an interpretation in term of (rather standard) differential geometry of the target space $\mathcal{M}$.

Remark. In the above no supersymmetry is implied. Supersymmetry, when present, selects specific bundles $\mathcal{V}_\pm \to \mathcal{M}$. Unifying bosons and fermions, SUSY requires that the couplings of the spin–1/2 fields are related to those of the scalars. Geometrically, this means that $\mathcal{V}_\pm$ should be natural bundles whose properties are uniquely fixed by the Riemannian geometry of $\mathcal{M}$. Typically $\mathcal{V}_\pm = T\mathcal{M}$, and the fiber metric and connection coincide with the Riemmannian ones. We will see more of this below. Thus, while *all* theories have couplings which are described by nice geometrical objects, SUSY theories have couplings which are described by *canonical* geometric objects. This explains why the geometric approach is the most convenient one in the supersymmetric case. Besides, it is also the approach which is more directly related to the *physical* meaning of the theory, as we discuss in the section. 3.

Remark. In GENERAL LESSON 2.1 we made an implicit assumption, namely that the (Q)FT has no 'defects', that is degrees of freedom which live on proper submanifolds $S \subset \Sigma$, called $(\dim S - 1)$–branes. If such 'defects' are present, you have to replace the vector bundles by the more general notion of *sheaves* (typically *coherent* sheaves).

2.1.2. *Language.* Sometimes it is more convenient to rephrase the above concept in a somewhat fancier language. Consider, for instance, the Yukawa coupling $Y_{ab}\psi_+^a\psi_-^b$, or the 4–Fermi one, $S_{ab\,cd}\,\psi_+^a\psi_+^b\,\psi_-^c\psi_-^d$. We saw before that they are, respectively, sections of $\mathcal{V}_+^\vee \otimes \mathcal{V}_-^\vee$ and $\wedge^2\mathcal{V}_+^\vee \otimes \wedge^2\mathcal{V}_-^\vee$. By the very definition of duality, we can see them as *bundle morphisms* (see ref.[**8**])

$$\mathcal{V}_- \xrightarrow{\;Y\;} \mathcal{V}_+^\vee \xrightarrow{\;h^+\;} \mathcal{V}_+, \tag{2.12}$$

$$\wedge^2\mathcal{V}_- \xrightarrow{\;S\;} \wedge^2\mathcal{V}_+^\vee \;\simeq\; \wedge^2\mathcal{V}_+, \tag{2.13}$$

where each coupling Y_{ab}, S_{abcd}, and h_{ab}^+ is written as a bundle morphism. It follows from the above GENERAL LESSON 2.1 that *all* couplings in a (Q)FT can be written in this 'arrow' form. The 'arrow' point of view is useful when the given coupling, viewed as an arrow, fits into a larger arrows' diagram (exact sequences and similar stuff) and we can work out the geometry of the given couplings in a purely *arrow–theoretical* manner (*i.e.* using what Steenrod used to call *'abstract nonsense'*). This saves tremendous quantities of time and print, expecially in relation with dualities.

As a matter of nomeklature, we will say that a bundle morphism is a *monomorphism* if it is fiberwise injective, a *epimorphism* if it is fiberwise surjective, and an *isomorphism* if it is both. A bundle morphism of the form $\mathcal{E} \to \mathcal{E}$ is called an *endomorphism*, and an *automorphim* if it is also an isomorpshims. The bundle morphisms $\mathcal{A} \to \mathcal{B}$ can be (equivalently) seen as sections of $\mathcal{B} \otimes \mathcal{A}^\vee$. We shall write $\mathrm{End}(\mathcal{E})$ for $\mathcal{E} \otimes \mathcal{E}^\vee$, and $\mathrm{Aut}(\mathcal{E})$ for the corresponding group of automorphisms.

For instance, the kinetic term coupling h_{ab}^+ should be an *isomorphism* $\mathcal{V}_+ \to \mathcal{V}_+^\vee$, if the kinetic terms are to be non–degenerate.

2.1.3. *Philosophy.* One could push the above viewpoint even further. Grothendieck revolutionized the way of thinking of geometry emphasizing that what really matters, and should be studied, is not the properties of the single manifold $\mathcal{M}$ (or variety, or scheme) but rather all the possible maps (morphisms) of this manifold with all the others. The properties of $\mathcal{M}$ are best defined, studied and understood as *properties of the maps,* that is in relation with all other objects in the category of $\mathcal{M}$. Most of the analysis we shall present in the present course is evidence for this point of view of Grothendieck also in Field Theory and, particularly, supergravity. The theory is best formulated in terms of relations between the different models, rather than in terms of the properties of the single one.

This point of view has, however, the major disadvantage that it would lead us to a notation and language very distant from the one commonly used in physics. We prefer to adhere to the tradition, and speak of Lagrangians, not arrows. Anyhow, it should be kept in the back of our mind that what matters is not the single Lagrangian but the 'category of Lagrangian Field Theories' and that the morphism are more relevant than objects.

2.2.∗ (Forced) gaugings and parallel structures. [13]

Vector bundles make us think of gauge theories, and all that. So, maybe this is the right place to make a general, almost philosophical, comment about gauged symmetries in the present general context. As an illustration, we consider a model of fermions coupled to scalars in D dimensions, having (say) the general structure in eqn.(2.1). To keep formulae simpler, we set to zero all parity–odd couplings.

Geometrically speaking, the fiber metric $h_{ab}(\phi)$ reduces the structure group of the fermion bundle to a subgroup $\mathcal{H}$ of the group $\mathcal{G} \subset \mathrm{Aut}(\mathcal{V}_+)$ of fiber isometries (which is isomorphic, fiberwise, to $U(n)$ or $SO(n)$ in the real case).

However, not all elements of $\mathcal{G}$ are symmetries of the theory: only a transformation commuting with the connection term in the covariant derivative of eqn.(2.8) is potentially a *(global)* symmetry. The connection takes value in the Lie algebra of $\mathcal{H}$. Hence we are lead to transformations of the form

$$\psi_+^a \;\mapsto\; m^a{}_b(\phi)\,\psi_+^b \tag{2.14}$$

where $m^a{}_b$ now belongs to the normalizer $\mathcal{N}$ of $\mathcal{H}$ in $\mathcal{G}$.

The (obvious) point I wish to stress is that such a transformation is seldom a *global* symmetry of the Lagrangian, since the matrices $m \in \mathcal{N}$, in general, depend non–trivially on the scalar fields ϕ^i and hence, *via* the pull–back to Σ, from the space–time coordinates x

$$m^a{}_b\big(\phi(x)\big).$$

Although $\mathcal{N}$ cannot be realized as a group of global symmetries, it can be realized as a local (gauged) symmetry. The space–dependent nature of a gauge transformation, makes the non–trivial dependence of the $m^a{}_b(x)$'s to be not a problem in this case. We take a basis of $\mathfrak{Lie}(\mathcal{N})$, given by field

[13] ∗ means that this subsection may be skipped.

dependent matrices $\tau^{A\,a}{}_b(\phi)$, satisfying $[\tau^A, \tau^B] = f^{AB}{}_C\, \tau^C$, and all we have to do is to add to the covariant derivative acting on fermions a term

$$-A^A_\mu\, \tau^{A\,a}{}_b(\phi)$$

This works (at least as long as the structure constants $f^{AB}{}_C$ can be taken to be constants). Geometrically, one describes the situation as a further twisting of the bundle $\Phi^*\mathcal{V}$ by some space–time bundle associated to the gauge principal bundle.

The situation is a bit more intricate when the symmetry acts non–trivially on *both* the fermions and the scalars (trough an isometry of $\mathcal{M}$). We defer the discussion of this case to the analysis of specific models in future chapters.

To summarize:

General lesson 2.2. *In presence of non–trivial scalar dependence of the fermions' kinetic terms, a transformation of the ψ's, leaving the scalars inert, may be a symmetry (generically) only if* it is gauged.

Remark. The statement refers to fermions but it holds — *mutatis mutandis* — for the other fields as well.

Thus, in presence of non–trivial spaces, we are often *forced* to gauge a symmetry if we want to have it. This is what we call *a forced gauging.* There are two possible sources of forced gauging: either from the non–triviality of $\mathcal{M}$ or the non–triviality of Σ. sugra itself may be seen as an example of forced gauging: in presence of gravity, we cannot have supersymmetry *unless it is gauged.*

2.2.1. *Parallel structures.* The statement above contains the disclaimer *(generically).* This means that exceptions to the rule *are possible* and also very interesting. Under certain circumstances, the matrix $m^a{}_b(\phi)$ (representing the given automorphism of $\mathcal{V}$) happens to be a constant (numerical) matrix in some (preferred) trivialization of $\mathcal{V}$.

$m^a{}_b$ is a section of the bundle $\mathcal{V}\otimes\mathcal{V}^\vee$, which has a connection induced by that of $\mathcal{V}$, $\omega_i{}^a{}_b$. Thus the existence of a preferred trivialization, in which $m^a{}_b$ is constant, requires[14]

$$D_i m \equiv \partial_i m + [\Omega_i, m] = 0. \tag{2.15}$$

A non–zero section m satisfying the above condition is called *parallel.* A manifold $\mathcal{M}$ together with a vector bundle $\mathcal{E}$ with a parallel section s is called *a parallel structure.* Then

General lesson 2.3. *A linear transformation $\psi^a \mapsto m^a{}_b\, \psi^b$ can be a* global *symmetry only if $(\mathcal{M}, \mathrm{Aut}(\mathcal{V}), m)$ is a parallel structure.*

The relevance of this observation is that often we *know,* from physical considerations, that some global symmetry should be present. Then the above general lesson puts a strong constraint on the possible scalars' geometries, which helps to determine the Lagrangian.

Eqn.(2.15) has an integrabily condition

$$0 = [D_i, D_j]m = [F_{ij}, m], \tag{2.16}$$

[14] Assuming $\mathcal{M}$ to be simply–connected.

where F_{ij} is the curvature of the connection ω_i. Thus, the presence of a parallel section gives an algebraic constraint on the curvature.

Since SUGRA is just an example of *forced gauging,* the above remark would fix the geometry of $\mathcal{M}$, and hence all canonical geometric structures on it, which means — as we observed above — *all* the couplings. At this point you have only to apply the *target space equivalence principle* to get the explicit form of the Lagrangian.

3. How strings come about

This section contains only a remark — albeit fundamental — about sec. 2.

The fact that the couplings can be seen as nice tensor fields on the target space $\mathcal{M}$, with the correct properties under general diffeomorphism as well bundle morphism, is not only a useful mathematical nicety. It is a tremendously deep physical issue. The couplings of any theory — in particular of any two–dimensional model — can be interpreted as local fields in some target space $\mathcal{M}$, in which we have general covariance (that is general relativity) gauge–transformations, local supersymmetry, ... much as it was a physical space–time. In fact the $2d$ σ–model approach to (super)string theory just takes seriously this interpretation: it is the target space $\mathcal{M}$ which is the 'physical' space, whereas the source bidimensional space Σ is 'just' a mathematical space of (perturbative) parameters.

For this interpretation to hold, any theory should have this geometrical feature, without extra requirements on the couplings, like SUSY or conformal invariance. Such additional conditions on the couplings, on the contrary, may be seen as differential equations for the target–space fields, which are naturally reinterpreted as space–time equations of motion. *E.g.* the SUSY requirement becomes part of the on–shell condition for the target–space fermions. The equation of motion for the bosons can be seen as the vanishing of the β–function, which — at 1–loop and in absence of other target–space fields — is just (§.1.1.1) $R_{ij}|_{\text{target space}} = 0$, *i.e.* the vacuum Eistein equations.

This target–space interpretation is even more general than Lagrangian field theory. This follows from the Zamolodchikov analysis of the Renormalization Group (RG) [**10**].

The point I wish to stress is that the geometric structures that make the current physical interpretation of the (super)string possible, are in fact much more general that the mere application to that specific problem. These structures are quite useful — both at the fundamental and the technical levels — for the construction and analysis of general (Q)FT as well fancier theories.

4. Gauge dualities

We have seen that all non–scalar fields live in some bundle over $\mathcal{M}$. We wish to study in more detail the particular geometry of such bundles in some very relevant cases, related to a class of dualities generalizing the celebrated *eletric–magnetic* duality in four dimensions. Again we do not assume supersymmetry. However, historically, the following results where first obtained in the context of SUGRA, as they are needed in order to construct the Lagrangian of the maximal extended $\mathcal{N}=8$ supergravity in $D=4$ (see Cremmer and Julia ref.[**11**] and also [**12**][**13**]).

We take space–time to have dimension $D=2m$, with Minkowski signature $(-,+,+,\cdots,+)$.

We consider models with scalars ϕ^i living on some target manifold $\mathcal{M}$ together with a set B^a, $a=1,2,\cdots,n$ of $(m-1)$–form fields with Abelian gauge invariance

$$B^a \to B^a + d\Lambda^a \tag{4.1}$$

for arbitrary $(m-2)$–forms Λ^a. For $m>1$, the most general two–derivative Lagrangian is

$$\mathcal{L} = -\frac{1}{2} f_{ab}(\phi)\, F^a \wedge *F^b + \frac{1}{2}\widetilde{f}_{ab}(\phi)\, F^a \wedge F^b - \frac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j + \cdots\,, \tag{4.2}$$

where $F^a = dB^a$ is the m–form field strength of B^a, and $*$ is the Hodge dual. $f_{ab}(\phi)$ is some positive–definite symmetric matrix depending, in general, on the point in $\mathcal{M}$, while $\widetilde{f}_{ab}$ is antisymmetric for m odd and symmetric for m even. The case $D=2$ is special in the sense that zero–forms are 'almost' the same thing as scalars. For the present purpose, a zero–form field is a scalar σ endowed with a Peccei–Quinn invariance, $\sigma \to \sigma + \text{const.}$. With this specification, our arguments apply also to two dimensions.

We are interested in the possible symmetries of this system. The tricky point is that not all interesting symmetries are symmetries of the Lagrangian $\mathcal{L}$. Some symmetries — called *dualities* — hold only at the level of the equations of motion. They are, however, also symmetries of the energy–momentum tensor $T_{\mu\nu}$, and hence are physical symmetries (even at the quantum level, when an appropriate quantization exists).

4.1. Duality transformations (morphisms). Before discussing the symmetries of the Lagrangian (4.2), we have to discuss the morphisms of the formalism, namely the changes of field variables which produce Lagrangians of the same general form. These morphisms are the analogue for this class of theories of the diffeomorphisms for the σ–models, operations which leave invariant the structure of the Lagrangian $-g_{ij}\partial_\mu\phi^i\partial\phi^j$, but are not symmetries of the particular field theory unless the diffeomorphism is actually an isometry of the given g_{ij}.

Since it will cost us no more work, we consider an even more general Lagragian,

$$\mathcal{L} = L(F^a, \phi^i, \partial_\mu\phi^j, \chi^p, \partial_\mu\chi^q, \psi^I_\mu, \partial_\mu\psi^I_\nu, A^\lambda, dA^\kappa),$$

allowing for actions which are non quadratic in the field strengths F^a's, but are still 'algebraic' in the sense that do not contain any derivative of the F^a's. Thus the arguments of this section apply, say, to Dirac–Born–Infeld (DBI) Lagrangians in $4D$ as well. For future applications to supergravity we have added generic couplings to spin–$\frac{1}{2}$ fermions χ^m and spin-$\frac{3}{2}$ gravitinos ψ^I_μ allowing for Pauli–like couplings

$$H_{apq}(\phi)\, F^a_{\mu_1\mu_2\cdots\mu_m}\, \bar{\chi}^p \gamma^{\mu_1\mu_2\cdots\mu_m} \chi^q$$

and generalizations thereoff, as well as other form–fields A^λ of degree $\neq m-1$, thus allowing, *e.g.*, Chern–Simons couplings of the generic form

$$\int A^{\lambda_1}\wedge dA^{\lambda_2}\wedge\cdots\wedge dA^{\lambda_r}\wedge F^a.$$

Since $d^2=0$, the m-form field strengths satisfy the Bianchi identities

$$dF^a=0. \tag{4.3}$$

Define m–forms G_a as follows

$$G_a=*\frac{\partial L}{\partial F^a}. \tag{4.4}$$

The equation of motion of the B^a fields read

$$dG_a=0 \tag{4.5}$$

which has exactly the same form as the Bianchi identity (4.3). Therefore the combined system of equations, (4.3)(4.5), is invariant under real linear transformations mixing the m-forms (F^a, G_b).

It is convenient to work with eigenforms of $*$. In a $D=2m$ space of Minkowski signature, $*^2=(-1)^{m-1}$ when acting on m–forms. Hence $*$ has eigenvalues ± 1 in $D=4k+2$, and $\pm i$ in $D=4k$. We write $F^a_\pm$ for the component of F^a on which $*$ acts as multiplication by ± 1 or, respectively, $\pm i$.

Remark. Notice that, for $D=4k$, $(F^a_\pm)^*=F^a_\mp$ while for $D=4k+2$, $F^a_\pm$ are *real* and hence we can impose the (anti)self–dual condition $F^a_-=0$ as a physical constraint. Field strengths having only the self–dual or the anti–self–dual part are called *chiral.* They are tricky both at the classical level (typically their equations of motion do not follow from an action) and at the quantum level (they lead to anomalies, in particular gravitational anomalies [**14**], and other deep subtleties) but they *do* exist in some of the more remarkable physical theories as, for instance,

- the world–wheet theory of the heterotic string
- the space–time theory of Type IIB superstrings
- the exotic 'gauge' theories in six dimensions.

In the $*$–diagonal basis eqn.(4.4) becomes

$$G_{a\pm}=\pm i\frac{\partial L}{\partial F^a_\pm} \qquad \text{for } D=4k \tag{4.6}$$

$$G_{a\pm}=\ \pm\frac{\partial L}{\partial F^a_\mp} \qquad \text{for } D=4k+2. \tag{4.7}$$

We look for transformations which act on the scalars as diffeomorphism $f\colon \mathcal{M}\to\mathcal{M}$, and on the field strengths in the form

$$\begin{pmatrix}F^a\\ G_b\end{pmatrix}_\pm \to \begin{pmatrix}\hat F^a\\ \hat G_b\end{pmatrix}_\pm=\begin{pmatrix}A^a{}_c & B^{ad}\\ C_{bc} & D_b{}^d\end{pmatrix}\begin{pmatrix}F^c\\ G_d\end{pmatrix}_\pm \tag{4.8}$$

where A, B, C, D are *real* constant matrices. They rotate the equations of motion into the Bianchi identity and *viceversa.* However not every transformation of this form is a morphism of the formalism. In fact, to recover the original formulation in the new basis, we also *need* a new Lagrangian $\hat{L}(\hat{F}^a, \cdots)$ such that

$$\hat{G}_a = * \frac{\partial \hat{L}}{\partial \hat{F}^a}. \tag{4.9}$$

This is an integrability condition which is most conveniently formulated in terms of differential forms. Let $\boldsymbol{\delta}$ be the de Rham differential in field strength space. Then (4.9) is integrable for $\hat{L}$ iff the 1–form in field–strength space

$$\hat{\boldsymbol{\eta}} := \begin{cases} \hat{G}_{a+} \boldsymbol{\delta} \hat{F}^a_+ - \hat{G}_{a-} \boldsymbol{\delta} \hat{F}^a_- & \text{for } D = 4k \\ \hat{G}_{a+} \boldsymbol{\delta} \hat{F}^a_- - \hat{G}_{a-} \boldsymbol{\delta} \hat{F}^a_+ & \text{for } D = 4k+2 \end{cases} \tag{4.10}$$

is $\boldsymbol{\delta}$–closed, $\boldsymbol{\delta}\hat{\boldsymbol{\eta}} = 0$. Equivalently, one requires

$$\hat{\boldsymbol{\eta}} - \boldsymbol{\eta} = \boldsymbol{\delta} \boldsymbol{\Phi} \tag{4.11}$$

where $\boldsymbol{\eta}$ is the same expression as in eqn.(4.10) but with the hatted quantities replaced by the unhatted ones, and $\boldsymbol{\Phi}(F, \hat{F}, \cdots)$ is some function of the F^a's and the other fields. This is precisely the equation which defines the *symplectomorphisms* (*a.k.a.* canonical transformations in classical mechanics) with respect to the symplectic pairing

$$\boldsymbol{\Omega} = \begin{cases} \Omega \otimes \sigma_3 \quad \equiv \big(i\sigma_2 \otimes \mathbf{1}_{n\times n}\big) \otimes \sigma_3 & D = 4k \\ \Sigma \otimes (i\sigma_2) \equiv \big(\sigma_1 \otimes \mathbf{1}_{n\times n}\big) \otimes i\sigma_2 & D = 4k+2 \end{cases} \tag{4.12}$$

where the last factor in the tensor product refers to the $(+,-)$ index. In the context of classical mechanics, $\boldsymbol{\Phi}(F, \hat{F}, \dots)$ is known as *the Hamilton–Jacobi function.*

Then our transformation in eqn.(4.8) should be a real linear symplectomorphism, with respect to $\boldsymbol{\Omega}$, having the special form[15]

$$\boldsymbol{S} \otimes \mathbf{1}_{2\times 2} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \mathbf{1}_{2\times 2}$$

The $2n \times 2n$ matrix $\mathbf{S}$ satisfies

$$\mathbf{S}^t \, \Omega \, \mathbf{S} = \Omega \qquad \text{for } D = 4k \tag{4.13}$$

$$\mathbf{S}^t \, \Sigma \, \mathbf{S} = \Sigma \qquad \text{for } D = 4k+2. \tag{4.14}$$

By definition, this means that

$$\mathbf{S} \in Sp(2n, \mathbb{R}) \qquad \text{for } D = 4k \tag{4.15}$$

$$\mathbf{S} \in SO(n, n) \qquad \text{for } D = 4k+2. \tag{4.16}$$

since they preserve, respectively, a non–degenerate antisymmetric pairing, and a symmetric pairing of signature (n, n).

Alternatively, eqns.(4.15)(4.16) can be obtained by requiring the invariance of the physical energy–momentum tensor (which, being measurable, should not change under any field redefinition, see [**12**] §.2.3 for the details.)

REMARK. *(Chiral Field–Strengths)* As we noted above, in dimension $D = 4k+2$ we may have chiral $2k+1$ field–strengths hence, in general, the number of F^a_+'s and F^i_-'s need not to be equal. Assume we have n self–dual

[15] Since (4.8) does not affect the ± indices.

and m anti–self–dual forms. Then the duality morphism group $SO(n,n)$ gets replaced, quite naturally, by

$$SO(n,m). \tag{4.17}$$

In the particular case of the world–sheet theory of the heterotic string, this group arises from the Narain lattices [**15**]. For a discussion along the present lines see [**16**].

The main point of the present section is that the bundles over $\mathcal{M}$ associated to $(m-1)$–form fields have a natural (possibly twisted) symplectic structure. This is a very useful observation, since symplectic geometry is a well–known subject. When — in extended supersymmetry — scalars and $(m-1)$–forms get related by a symmetry, the symplectic nature of the form–fields induces a symplectic structure also on $\mathcal{M}$. So the scalars' manifold is a symplectic manifold, and *all* the couplings in the Lagrangian are defined by symplectic geometry (*a.k.a.* classical mechanics). Remarkably, this structure is powerful enough to allow, for instance, to actually solve non–perturbatively non–Abelian $\mathcal{N}=2$ gauge theories in four dimensions. We shall see more of this.

4.1.1. *Transformation of the Lagrangian.* From eqn.(4.11), we see that, under duality, the Lagrangian is not invariant in value, instead the new and old Lagrangians are related by

$$\hat{L} = L + \begin{cases} \mathbf{\Phi} & D = 4k+2 \\ -i\mathbf{\Phi} & D = 4k. \end{cases} \tag{4.18}$$

For a linear symplectomorphism, the Hamilton–Jacobi function[16] $\mathbf{\Phi}$ can be chosen to be a *quadratic form* in the field–strengths $F^a_\pm$ and $\hat{F}^a_\pm$. To save print we write down explicitly the $D=4k$ case, leaving the other as an exercise to the reader. For the same reason we write down only the $+$ component of the field–strengths, and use matrix notation. Then

$$\mathbf{\Phi} = \frac{1}{2} F^t_+\, M\, F_+ + \hat{F}^t_+\, N\, F_+ + \frac{1}{2}\hat{F}^t_+\, P\, \hat{F}_+ + (\text{terms in } F_-, \hat{F}_-) \tag{4.19}$$

where M, N, P are real $n\times n$ matrices and $M^t = M$, $P^t = P$. From eqn.(4.10),

$$\hat{G}_+ = NF_+ + P\hat{F}_+ \tag{4.20}$$

$$G_+ = -MF_+ - N^t\,\hat{F}_+ \tag{4.21}$$

or, inverting,

$$\begin{pmatrix} \hat{F}_+ \\ \hat{G}_+ \end{pmatrix} = \begin{pmatrix} -(N^t)^{-1}M & -(N^t)^{-1} \\ N - P(N^t)^{-1}M & -P(N^t)^{-1} \end{pmatrix} \begin{pmatrix} F_+ \\ G_+ \end{pmatrix} \tag{4.22}$$

it is easy to check that the $2n\times 2n$ matrix in the RHS satisfies the condition (4.13). Comparing with eqn.(4.8), we get the identifications

$$A = -(N^t)^{-1}M \qquad B = -(N^t)^{-1} \tag{4.23}$$

$$C = N - P(N^t)^{-1}M \quad D = -P(N^t)^{-1} \tag{4.24}$$

[16]Recall that symplectomorphism ≡ canonical transformation of classical Hamilton–Jacobi mechanics. The statement in the text is well–known in that context.

or

$$M = B^{-1}A, \quad P = DB^{-1}, \quad N = -(B^t)^{-1}. \tag{4.25}$$

Finally, we get

$$\hat{L} = L - i\left(\frac{1}{2}F_+^t\, B^{-1}A\, F_+ - \hat{F}_+^t\, (B^t)^{-1}\, F_+ + \frac{1}{2}\hat{F}_+^t\, DB^{-1}\, \hat{F}_+ - \text{H.c.}\right) \tag{4.26}$$

(this holds in value; to get the effective functional form of $\hat{L}$ you need to re–express everything in terms of the $\hat{F}$'s, by inverting the transformation).

EXERCISE 4.1. (1) Check that the matrix in eqn.(4.22) belongs to $Sp(2n,\mathbb{R})$. (2) Work out the details for $D = 4k+2$.

NOTE ADDED. In the class it was pointed out by somebody in the audience that eqns.(4.19)–(4.26) hold only if the original Lagrangian L is quadratic in the field–strenghts F^a and not in full generality as claimed here.

This is not true. Eqns.(4.19)–(4.26) are valid even for *non–quadratic* L's. Simply, in that case, the $\hat{F}^a$'s are *not* linear expressions in the F^a's, and hence G^a, as given by eqn.(4.21), is not linear in the F^a's. Then also the Lagrangian, eqn.(4.26), is not quadratic in either the F^a's or the $\hat{F}^a$'s. The only thing which is linear, is the duality transformation, eqn.(4.22), as claimed.

REMARK. With respect to others approaches, we had to pain a little more, since we constructed *finite* duality transformations not just *infinitesimal* ones. The point is that there exist pairs of models which look quite different, but which in fact are equivalent under a *big* duality transformation. In the past this was not recognized, and the SUGRA literature is full of papers contrasting the properties of models which — with the help of the superstring insight — turned out to be the same theory!!

Moreover, the finite transformations are the relevant ones for the non–perturbative applications. Consider the basic case, four–dimensional electric–magnetic duality. To get something interesting, we have to rotate a given (e,m) dyon into another (e',m'). Since the charges are quantized, such a transformation belongs to a *discrete* subgroup of finite $Sp(2n,\mathbb{R})$ transformations, under which the Lagrangian changes in a very dramatic way (the ability of understanding such changes in geometrical terms is at the core of the possibility of understanding the non–perturbative dynamics).

4.2. Duality *symmetries*. Now we ask under which condition a morphism of the form discussed in §.4.1, is actually a symmetry of the theory, that is, $\hat{L}$ and L are equal in their functional form, although they may differ (in general) in value. To make things easy, and in view of the applications to the physically relevant models, we assume L to be quadratic in the field–strengths F^a, but for the rest we remain totally general, and in particular, we still allow the possibility of fermions, gravitini, and generic form potentials/field strengths, as in §.4.1. To make the formulae simpler, in this §. we adopt the following conventions: **(1)** Ω stands for the symplectic matrix $(i\sigma_2)\otimes \mathbf{1}_{n\times n}$ if $D=4k$, while it is the orthogonal matrix $\sigma_1 \otimes \mathbf{1}_{n\times n}$ if $D = 4k+2$; **(2)** consequently $O_\Omega(2n)$ denotes, respectively, $Sp(2n,\mathbb{R})$ or $SO(n,n)$; **(3)** j is a complex number with value i for $D=4k$ and 1 for $D = 4k+2$, so that $j\Omega$ is a Hermitean involution in all dimensions[17]; **(4)** the short–hand 'other fields' stands for any (unspecified) composite m–form which is a polynomial in χ^i, ψ^I_μ, A^λ and their derivatives, which does not contain the F^a's; **(5)** we combine the field strengths F^a and G_b into a column vector

$$\mathcal{F} = \begin{pmatrix} F^a \\ G_b \end{pmatrix}. \tag{4.27}$$

4.2.1. *The 'Vielbein' $\mathcal{E}$.* If the Lagrangian is quadratic in the F^a's we have

$$G_{a+} = -j\mathcal{N}_{ab}(\phi)\, F^b_+ + \text{'other fields'}, \tag{4.28}$$

where $\mathcal{N}_{ab}(\phi)$ is some $n\times n$ matrix whose entries are functions of the scalars. Unitarity (positivity of the kinetic energy) implies that $j\mathcal{N}$ is

(1) a symmetric complex matrix whose imaginary part is positive–definite for $D = 4k$;
(2) a real matrix whose symmetric part is negative–definite for $D = 4k+2$;

We rewrite the last equation in a more $O_\Omega(2n)$ covariant fashion

$$\big(\mathbf{1} - j\Omega\big)\mathcal{E}(\phi)\, \mathcal{F}_+ = \text{'other fields'}. \tag{4.29}$$

Here $\mathcal{E}(\phi)$ — called in the SUGRA literature the 'Vielbein' — is an element of the group $O_\Omega(2n)$ whose matrix elements depend on the scalars. $\mathcal{E}(\phi)$ encodes all the couplings of the F^a with the scalars; indeed writing

$$\mathcal{E} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \tag{4.30}$$

(where $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are $n\times n$ matrices) eqn.(4.29) becomes

$$G_+ = (j\mathcal{D} - \mathcal{B})^{-1}(\mathcal{A} - j\mathcal{C})F_+ \tag{4.31}$$

so eqns.(4.28) and (4.29) are equivalent provided

$$j\mathcal{N} = (\mathcal{B} - j\mathcal{D})^{-1}(\mathcal{A} - j\mathcal{C}). \tag{4.32}$$

We have still to show that the Vielbein $\mathcal{E}$ is an element of $O_\Omega(2n)$. It follows from the following very well–known lemma.

LEMMA 4.1. ***(1)*** *All complex symmetric matrices $i\mathcal{N}$ whose* imaginary part *is positive definite can be written in the form* (4.32) *for some $\mathcal{E} \in Sp(2n,\mathbb{R})$;* ***(2)*** *all real matrices $\mathcal{N}$ whose* symmetric part *is negative definite are of the form* (4.32) *for some $\mathcal{E} \in O(n,n)$.*

PROOF. Choose $\mathcal{E}$ in the special form

$$\mathcal{E} = \begin{pmatrix} Q^{-1}M & Q^{-1} \\ \mp Q^t & 0 \end{pmatrix} \tag{4.33}$$

where $M^t = \pm M$ (upper sign for $Sp(2n,\mathbb{R})$, lower one for $O(n,n)$). Then eqn.(4.32) gives

$$\mathcal{N} = \begin{cases} M + iQQ^t \\ M - QQ^t. \end{cases} \tag{4.34}$$

□

[17] In particular $P_\pm = \frac{1\mp j\Omega}{2}$ are projectors on subspaces of half the dimension.

Next we ask when two 'Vielbeins' $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ do correspond to the same coupling matrix $\mathcal{N}_{ab}$. Let $\mathcal{S} \in O_\Omega(2n)$ be an element commuting with $j\Omega$. From eqn.(4.29) it is obvious that $\mathcal{E}$ and $\mathcal{S}\mathcal{E}$ give equivalent constraints on the field strengths $\mathcal{F}$ (provided we also multiply by $\mathcal{S}$ the 'other fields' in the RHS). Hence multiplication on the left of the Vielbein by an element of the normalizer of Ω leaves $\mathcal{N}_{ab}(\phi)$ invariant. One has

LEMMA 4.2. *The above normalizer is $U(n)$ in the $Sp(2n, \mathbb{R})$ case, and $O(n) \times O(n)$ in the $O(n,n)$ one. In both cases it is the maximal compact subgroup of $O_\Omega(2n)$.*

PROOF. A simple computation shows that an element of the normalizer should have the form

$$\begin{pmatrix} A & B \\ \mp B & A \end{pmatrix} \qquad A^t A + B^t B = 1, \quad (B^t A)^t = \pm B^t A, \tag{4.35}$$

(upper sign for $Sp(2n, \mathbb{R})$, lower one for $SO(n,n)$). So

$$(A \mp iB)^t (A \pm iB) = 1 \qquad Sp(2n, \mathbb{R}) \text{ case} \tag{4.36}$$

$$(A \pm B)^t (A \pm B) = 1 \qquad O(n,n) \text{ case} \tag{4.37}$$

thus $U = A + iB \in U(n)$ while $O_\pm = A \pm B$ belong to distinct copies of $O(n)$. □

The crucial point we wish to stress is that the couplings matrices $\mathcal{N}_{ab}(\phi)$ allowed by unitarity are in one–to–one correspondence with the points of the coset spaces

$$\frac{Sp(2n, \mathbb{R})}{U(n)} \qquad D = 4k \tag{4.38}$$

$$\frac{SO_0(n,n)}{SO(n) \times SO(n)} \qquad D = 4k + 2 \tag{4.39}$$

which are well–known rank n Riemannian *symmetric spaces*, whose differential geometry is well–understood. Let us recall what they represent geometrically. Consider $\mathbb{R}^{2n}$ equipped with the symplectic structure given by our (constant) matrix Ω. A Lagrangian subspace $L \subset \mathbb{R}^{2n}$ is a n–dimensional subspace L such that $\Omega|_L = 0$. $Sp(2n, \mathbb{R})/U(n)$ is the space parameterizing the Lagrangian subspaces of $\mathbb{R}^{2n}$. Indeed, this is precisely what we proved above in our two lemmas. A point in $Sp(2n, \mathbb{R})/U(n)$ is a class of Vielbeins [up to the equivalence $\mathcal{E} \sim \mathcal{S}\mathcal{E}$, with $\mathcal{S} \in U(n)$] that determines which n–dimensional subspace of the $2n$–dimensional field–strengths' space we interpret as curvatures, $F^a = dA^a$. On $Sp(2n, \mathbb{R})/U(n)$ we have a *tautological* rank n bundle $\mathfrak{T}$

$$\mathfrak{T} := \{(L, x) \,|\, L \in Sp(2n, \mathbb{R})/U(n),\ x \text{ a point in } L\} \tag{4.40}$$

$$\pi \colon \mathfrak{T} \to Sp(2n, \mathbb{R})/U(n) \qquad \text{given by } \ (L, x) \mapsto L. \tag{4.41}$$

The tautological bundle $\mathfrak{T}$ is a first example of *homogeneous* bundle on a symmetric space (the bundles which we can construct using the group structure of the coset; we shall say more in Part 2 of these lecture notes).

The same holds for $SO(n,n)/SO(n)\times SO(n)$. This coset parameterize the *null* subspaces in $\mathbb{R}^{2n}$ with respect the indefinite inner product Ω. Again we have a *tautological* bundle $\mathfrak{T}$.

We summarize the results of this section in the following:

THEOREM 4.1. *Let F be m–form field–strengths of a $D=2m$ dimensional FT. The* (scalars)*FF couplings define a map of Riemannian manifolds*[18]

$$\mu\colon \mathcal{M}\to\begin{cases} Sp(2n,\mathbb{R})/U(n)\\ SO(n,n)/SO(n)\times SO(n)\\ SO(n,m)/SO(n)\times SO(m) \quad \textit{(chiral case)}\end{cases} \tag{4.42}$$

given by[19]

$$\phi^i\mapsto \mathcal{E}^t H,\qquad \text{where } H=\begin{cases} U(n)\\ SO(n)\times SO(n)\\ SO(n)\times SO(m).\end{cases} \tag{4.43}$$

All the bundles describing couplings of the F's to other fields $(\chi,\psi_\mu,\cdots)$ are the pull–back via μ of the homogeneous bundles over the coset space in the RHS. *In particular, the Abelian curvatures dA^a take value in the pull–back $\mu^*\mathfrak{T}$ of the* tautological *bundle.*

CRUCIAL REMARK. The bundle (over $\mathcal{M}$) on which the F^a live should be flat (by the Bianchi identity). So we get for free a *flat non–metric* connection on the associated bundle, from which the metric and metric connection can be deduced by a standard procedure (sometimes called tt^*).

EXERCISE 4.2. Show that the energy–momentum tensor $T_{\mu\nu}$ is $O_\Omega(n)$ invariant. HINT: Try writing $T_{\mu\nu}$ as a term proportional to $\widetilde{\mathcal{F}}_\mu^{t\ \rho}\,\Omega\,\mathcal{F}_{\rho\nu}+$ terms not containg field–strenghts.

4.2.2. *When a duality transformation is a symmetry?* Armed with the previous theorem, we can easily answer this basic question. A duality transformation $\mathbf{S}\in O_\Omega(n)$ acts on the field–strengths as

$$\mathcal{F}_\pm\to\mathbf{S}\,\mathcal{F}_\pm \tag{4.44}$$

so, in view of eqn.(4.29), it can be compensated by a change in the Vielbein $\mathcal{E}$

$$\mathcal{E}\to\mathcal{E}\,\mathbf{S}^{-1}. \tag{4.45}$$

Suppose there exist an isometry of $s\colon\mathcal{M}\to\mathcal{M}$ such that

$$\mu\circ s=(\mathbf{S}^t)^{-1}\,\mu \tag{4.46}$$

that is, in coordinates

$$\mathcal{E}\big(s^i(\phi)\big)=\mathcal{E}(\phi^i)\,\mathbf{S}^{-1}. \tag{4.47}$$

[18]We call it μ since in the historical case, Maxwell theory, these couplings give rise (phenomenologically) to the magnetic susceptibility of the medium.

[19]We use $\mathcal{E}^t$ instead of $\mathcal{E}$ to convert a right action into a standard left action and *viceversa*.

Then the following is a symmetry of both the scalars' and the forms' kinetic terms

$$\mathcal{F}_\pm \to \mathbf{S}\,\mathcal{F}_\pm \tag{4.48}$$

$$\phi^i \to s^i(\phi). \tag{4.49}$$

This is a symmetry of the full Lagrangian provided the expression in the RHS of eqn.(4.29), *'other fields'*, transforms covariantly under $O_\Omega(2n)$ that is, if the other fields $\chi, \psi_m u, \cdots$ live in the (pull–back of) the right homogeneous bundles, as the theorem requires.

EXAMPLE (Prototypical). The ovious example of the above situation, which is the generic situation for $D = 4$ supergravity with $\mathcal{N} \geq 3$, is when the scalars' manifold $\mathcal{M}$ is itself a symmetric space G/H where G is a subgroup, respectively, of $Sp(2n, \mathbb{R})$ or $O(n, n)$ and $H = G \cap U(n)$ or, respectively, $H = G \cap (SO(n) \times SO(n))$. The metric on $\mathcal{M}$ is taken to be the symmetric one, which has a group G of isometries acting by right multiplication.

As the map μ one takes the one induced by the subgroup embedding $i\colon G \to Sp(2n, \mathbb{R})$ (say) given by the defining representation

$$\begin{array}{ccc} G & \xrightarrow{\quad i \quad} & Sp(2n,\mathbb{R}) \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{M} = G/\big(G \cap U(n)\big) & \xrightarrow{\quad \mu \quad} & Sp(2n,\mathbb{R})/U(n). \end{array}$$

In this case the property (4.46) holds by construction, and we have a full group G of symmetries. This duality business is a first reason of why coset spaces G/H are ubiquous in supergravity, super–string etc. Luckily it is not the only one, as we shall see.

REMARK. This is a trivial remark, but in the literature there is quite a fuss about it, so I mention it: the group G of duality symmetries need not to be compact. In fact seldom is. So the symmetry cannot be linearly realized on the spectrum.

5. The emergence of modularity

5.1. $D = 4k$**.** Without doubt the reader has noticed that eqn.(4.32) resembles the modular transformation of the period matrix Ω_{ab} of a Riemann surface. It is not a coincidence. It is a *very* (*very!*) deep fact.

We recall the basics.

5.1.1. *Period matrix of a Riemann surface.* On a genus g Riemann surface σ we have g linearly independent holomorphic differentials ω_α, $\alpha = 1, 2, \ldots, g$. On $H_1(\sigma, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ we have a non–degenerate symplectic pairing $\langle \cdot, \cdot \rangle$ dual to the Poincaré one $\int_\sigma \alpha \wedge \beta$. We take a canonical (symplectic) basis of cycles (A_α, B^β) $(\alpha, \beta = 1, 2, \cdots, g)$

$$\langle A_\alpha, A_\beta \rangle = \langle B^\alpha, B^\beta \rangle = 0, \qquad \langle A_\alpha, B^\beta \rangle = -\langle B^\beta, A_\alpha \rangle = \delta_a{}^\beta. \tag{5.1}$$

The holomorphic differentials are normalized by fixing their integrals on the A–cycles

$$\int_{A_\alpha} \omega_\beta = \delta_{\alpha\beta}. \tag{5.2}$$

The integrals over the B–cycles define the period matrix $\Omega_{\alpha\beta}$

$$\int_{B^\alpha} \omega_\beta = \Omega_{\alpha\beta}. \tag{5.3}$$

One has

THEOREM∗ (Riemann bilinear relations). *The period matrix is symmetric $\Omega_{ab} = \Omega_{ba}$ and the real quadratic form* $\mathrm{Im}\,\Omega_{ab}$ *is positive–definite.*

Notice that these are exactly the properties of the coupling matrix $j\mathcal{N}$ for $D = 4k$ (see discussion after eqn.(4.28)).

The space of complex $n \times n$ symmetric matrices with positive–definite imaginary part is called *Siegel's upper half–space,* $\mathfrak{H}_n$, For $n = 1$ it is the usual upper half–plane $\mathfrak{h} = \{z \in \mathbb{C} | \mathrm{IM}\, z > 0\}$. Lemmas 4.14.2 gives

$$\mathfrak{H}_n \simeq \frac{Sp(2n, \mathbb{R})}{U(n)}. \tag{5.4}$$

We do not prove this (well–known) theorem since we shall prove a much more general result when the time comes. However

EXERCISE 5.1. Prove the Riemann bilinear relations.

The symplectic basis (5.1) is not unique. We get all the other by a linear transformation of the form

$$\begin{pmatrix} B^\alpha \\ A_\beta \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{A}^\alpha{}_\gamma & \mathcal{B}^{\alpha\delta} \\ \mathcal{C}_\beta{}^\gamma & \mathcal{D}_\beta{}^\delta \end{pmatrix} \begin{pmatrix} B^\gamma \\ A_\delta \end{pmatrix}, \tag{5.5}$$

where the $2g \times 2g$ square matrix in the RHS should have integral entries (in order to transform *integral* cycles into integral cycles) and symplectic (to preserve the pairing in eqn.(5.1)). Thus it is an element of $Sp(2g, \mathbb{Z}) \subset Sp(2g, \mathbb{R})$.

How $Sp(2g, \mathbb{R})$ acts on the period matrix Ω? One has (we suppress indices and use block–matrix notation)

$$\begin{pmatrix} \int_B \omega \\ \int_A \omega \end{pmatrix} \equiv \begin{pmatrix} \Omega \\ 1 \end{pmatrix} \int_A \omega \longrightarrow \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \int_B \omega \\ \int_A \omega \end{pmatrix} \equiv \begin{pmatrix} \Omega' \\ 1 \end{pmatrix} \int_{A'} \omega$$

so, in block–notation

$$\Omega' = (\mathcal{A}\Omega + \mathcal{B})(\mathcal{C}\Omega + \mathcal{D})^{-1}. \tag{5.6}$$

Lemmas 4.1 and 4.2 state that all Ω's satisfying Riemann's bilinear relations can be obtained from any given one, say $i\delta_{\alpha\beta}$ by an $Sp(2g, \mathbb{R})$ transformation, and that two transformations differing by multiplication (on the left) by an element of the $U(g)$ subgroup. Two Ω's related by an *integral* $Sp(2g, \mathbb{Z})$ correspond to the *same* Riemann surface. This $Sp(2g, \mathbb{Z})$ invariance is called *modular symmetry.*

5.1.2. *Relation with eqn.*(4.32). Now it is obvious that the structure is just the same as we encountered in eqn.(4.32). We have only a change of conventions. We interchange the role of F_+ and G_+, taking the dual field–strengths G_+ as fundamental. Then write

$$\mathcal{F}_+ = \begin{pmatrix} \Omega G_+ \\ G_+ \end{pmatrix}$$

where the 'period matrix' $\Omega \equiv i\mathcal{N}^{-1}$. The coupling Ω transforms exactly as in eqn.(5.6) under $Sp(2n,\mathbb{R})$. To make contact with the original form of eqn.(4.32), notice that — since $\mathcal{N}$ is symmetric — it can be rewritten as

$$\Omega \equiv (-i\mathcal{N})^{-1} = (i\mathcal{D}^t - \mathcal{B}^t)(-i\mathcal{C}^t + \mathcal{A}^t)^{-1}$$

which is exactly the right expression if we recall that a Vielbein $\mathcal{E}$ should be interpreted as the duality transformation $\mathcal{E}^{-1}$ and

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{D}^t & -\mathcal{B}^t \\ -\mathcal{C}^t & \mathcal{A}^t \end{pmatrix}.$$

In summary,

General lesson 5.1. *The symmetry under (subgroups of) $Sp(2n,\mathbb{R})$ or $SO(n,m)$ arises quite universally in theories with form fields. These are symmetries of the kinetic terms, but not necessarily of the full theory, since other couplings (e.g. the gauge couplings) may spoil the invariance, typically leaving unbroken only a discrete subgroup, usually congruent to $Sp(2n,\mathbb{Z})$ or $SO(n,m,\mathbb{Z})$. Thus the theory gets, on quite general grounds, a kind of* 'modular invariance'.

An important application of this observation will be the analysis of the moduli space of compactifications of the superstring (or, more generally of $D=2$ conformal field theories) where the geometry of supergravity, the Zamoldochikov geometry, will match together leading to the so called 'target space modular invariance' [**17**] which is a powerful tool to study string and M–theories dualities (and other problems as well).

Remark. The coupling matrix $\mathcal{N}$ can also be interpreted as a period matrix for the $SO(n,m)/SO(n)\times SO(m)$ case, albeit for a complex manifold of larger dimension (in fact even dimension). We shall not work out the details at this point, the interested reader may read, *e.g.*, §. of ref.[**47**] and references therein.

6. More dualities

The dualities described in the previous two sections are morphisms of the formalism, in the sense that they transform a Lagrangian into another one of the same 'geometric' form. Under suitable circumstances, a (sub)group of the duality group may be actually a physical symmetry (often with deep non–perturbative implications). There are other dualities which map a Lagrangian with a given field content to another one with a different field content. These are also useful in sugra. Symmetries which are hidden in one formulation may be explicit in a dual one. Structures that do not look 'geometric' in one framework may appear so in dual variables. More importantly, these dualities allow to put the Lagrangian in a 'canonical' form, namely the one in which the physical/geometrical structures are easier to analyze.

We discuss two examples we shall need below to simplify things.

6.1. Abelian dualities. These are the obvious generalization of the ones discussed above. We take the space–time, Σ, to have dimension n and Minkowski signature. The field content consists of a set p–form fields A^a, with gauge invariance $A^a \sim A^a + d\Lambda^a$, Λ^a $(p-1)$–forms, which enter in the Lagrangian L only through their field–strengths $F^a \equiv dA^a$, scalars ϕ^i, and other fields Ψ in non–trivial representations of the Lorentz group which are taken to be inert under the gauge transformation $A^a \to A^a + d\Lambda^a$ (they may even be p'–forms to which the same argument will apply).

The Bianchi identities and equations of motion of the A^a's,

$$dF^a = 0 \quad \text{(Bianchi identity)} \tag{6.1}$$

$$d\left(*\frac{\partial L}{\partial F^a}\right) = 0 \quad \text{(eqns. of motion)} \tag{6.2}$$

are symmetric under the interchange of the $(p+1)$–form F^a with the $(n-p-1)$–form $G_a \equiv *\partial L/\partial F^a$. Thus the theory can be formulated replacing the p–form 'electric' potentials A^a with the dual 'magnetic' $(n-p-2)$–form potentials B_a, which automatically solve the equations of motion

$$G_a = dB_a. \tag{6.3}$$

One passes from a formulation to the dual one by performing a Legendre transform of the Lagrangian L. We recall the well–known procedure: we add to L Lagrange multipliers $(n-p-2)$–forms B_a enforcing the Bianchi identity, eqn.(6.1).

$$S = \int d^n x\, L(F,\phi,\Psi) + \int_\Sigma B_a \wedge dF^a \tag{6.4}$$

This allow us to take the field–strengths F^a as integration variables in the path integral instead than the potentials A^a. Next we perform the integral over the F^a's. At the leading semi–classical order we get

$$S' = \int d^n x\; L\Big(F(G,\phi,\Psi),\phi,\Psi\Big) + (-1)^{n-p-1} \int_\Sigma G_a \wedge F^a(G,\phi,\Psi) \tag{6.5}$$

$$F^a(G,\phi,\Psi) \text{ is obtained by inverting } \frac{\partial L}{\partial F^a} = (-1)^{n-p-1} * G_a. \tag{6.6}$$

[CHECK SIGNS!!] This result is *exact* quantum mechanically for L quadratic in the F^a's since in that case the path integral is Gaussian.

Exercise 6.1. We was sketchy in the above manipulations. Fill in the details:(1) functional measure; (2) proper treatment of the zero modes.

For instance, the above Legendre transform allows to replace two–form gauge fields with Abelian gauge vectors in $D = 5$, with Peccei–Quinn scalars[20] in $D = 4$. The relevance of this duality for the (Q)FT geometric structures is better illustrated by an example.

[20] By a Peccei–Quinn scalar we mean scalar φ which enters in the Lagrangian only trough its derivatives $\partial_\mu \varphi$. Hence it has a "gauge" symmetry $\varphi \to \varphi + \text{const.}$ which is called a Peccei–Quinn (PQ) symmetry.

Example (Tensor theories in $D=4$). Consider a model in $D=4$ which has both scalars (living on some manifold $\mathcal{M}$) and gauge two–forms, A^a ($a=1,\dots,m$). In the original 'mixed' formulation, the geometric structure of the theory is essentially the differential geometry of the manifold $\mathcal{M}$. However, by performing a duality, the m two–forms are replaced by m PQ scalars, and we get a bigger scalars' manifold $\widetilde{\mathcal{M}}$ of dimension $\dim\mathcal{M}+m$. We have a much 'bigger' geometry to play with. Thus writing all degree of freedom in terms of scalar fields, we put the Lagrangian in its *canonical* form, in which the geometry is completely manifest, while in the mixed one is partly hidden. However, there are deep and useful interplays between the geometry of $\mathcal{M}$, $\widetilde{\mathcal{M}}$ and the PQ symmetries (which correspond to m Killing vectors on $\widetilde{\mathcal{M}}$). On the other hand, the Riemannian geometry of a manifold like $\widetilde{\mathcal{M}}$, which can be obtained through duality is rather peculiar, and often we can us the duality trick to construct Riemannian metrics which have some prescribed properties.

General lesson 6.1. *To make the underlying geometric structure manifest, one has to put the Lagrangian into canonical form by dualizing the higher–degree forms to lower degree ones.*

Example (Vectors in $D=3$). One–forms (= gauge vectors) in $D=3$ behave much in the same way as two–forms in four dimensions. Again we can replace the vector by PQ scalars (usually with a *compact* PQ symmetry) and study the geometry of the resulting bigger target manifold both locally and in the large (the global geometry is relevant for non–perturbative effects). However, here there is an important difference. Gauge vectors may be *non–Abelian* gauge vectors; the above procedure does not work in this case, so we cannot replace non–Abelian vectors by scalars (except at zero coupling) and we cannot make explicit all the underlying geometric structure. However, precisely in three dimensions, one can perform a non–Abelian duality, although less elementary than the one presented above. This will be our next topic.

6.2. Non–Abelian duality in $D=3$. The reader may wonder why we focus on such a peculiar case, vector–scalar duality in $D=3$. The reason is twofold. There is a technical motivation and a physical one. The physical one is easier to explain: generalized (supersymmetric) Chern–Simons–matter models are believed to correspond to the world–volume theory of a stack of coinciding M2 branes (the membranes of M–theories) [**55**] so, in some sense, they should be considered, fundamental theories, and the duality we are going to describe is one of their more deep properties. On the technical side, the non–Abelian duality is an essential ingredient for these lectures. The plane of the present course is to start the study of sugra in $D=3$, arguing that there *everything is differential–geometric,* and then go on to $D>3$. But, as discussed in the examples of §.6.1, everything is explicitly geometric only in a formulation in which all the bosonic propagating degrees of freedom are represented by scalar fields. Otherwise we have access only to the geometry of the submanifold $\mathcal{M}\subset\widetilde{\mathcal{M}}$. Luckily, thanks to de Wit, Herger and Samtleben, [**50**] (see also [**58, 59**]), this can be done. The non–Abelian duality replaces vector fields with canonical kinetic terms of the form

$$-\frac{1}{4}M_{AB}(\phi)\,F^A_{\mu\nu}F^{B\,\mu\nu}$$

by scalars ϕ_A and vectors A^A_μ, $B_{A\,\mu}$ whose derivatives enter in the Lagrangian only in the form of Chern–Simons (CS) couplings. In suitable gauges, we can think of the scalars as the ones describing the physical local degrees of freedom, while the vectors are merely topological fields with no local physics[21].

Following [**50**], we consider a general $3D$ Lagrangian quadratic in the Yang–Mills field–strengths of the form

$$\mathcal{L} = -\frac{1}{4}\sqrt{-g}\big(F^A_{\mu\nu}+O^A_{\mu\nu}\big)M_{AB}(\Phi)\big(F^{B\,\mu\nu}+O^{B\,\mu\nu}\big)+\mathcal{L}'(A,\Phi), \qquad (6.7)$$

where A^A_μ are non–Abelian gauge fields, $F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - f_{BC}{}^A A^B_\mu A^C_\nu$ are the corresponding field–strengths, Φ stands for all other fields, possibly transforming in non trivial representations of the gauge group G_{YM}, and $O^A_{\mu\nu} \equiv O^A_{\mu\nu}(A,\Phi)$ are gauge *covariant* operators formed in any way from the fields Φ, and A^A_μ. The field–dependent matrix M_{AB} also transforms covariantly under G_{YM}. $\mathcal{L}'$ is gauge invariant, and contains A^A_μ only trough the covariant derivatives of fields and, possibly, Chern–Simons terms.

The Bianchi identities and equations of motion read

$$D_\mu \tilde{F}^{A\,\mu} = 0 \qquad (6.8)$$

$$D_{[\mu}\Big(M_{AB}(\tilde{F}_{\nu]}+\tilde{O}^A_{\nu]}\Big) = J_{A\,\mu\nu} \qquad (6.9)$$

where

$$\tilde{F}^A_\mu = \frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\rho}\,F^{A\,\nu\rho}, \qquad (6.10)$$

$$\tilde{O}^A_\mu = \frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\rho}\,O^{A\,\nu\rho}, \qquad (6.11)$$

$$J_{A\,\mu\nu} = \frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\rho}\,\frac{\partial\mathcal{L}'}{\partial A^A_\rho}. \qquad (6.12)$$

To perform the duality, we introduce new vector fields $B_{A\,\mu}$ and compensating scalars ϕ_A, both transforming in the adjoint of G_{YM}. They are defined by the equation

$$B_{A\,\mu} - D_\mu\phi_A = M_{AB}\big(\tilde{F}^A_\mu + \tilde{O}^A_\mu\big), \qquad (6.13)$$

which is invariant under the additional gauge transformation

$$\delta\phi_A = \Lambda_A, \qquad \delta B_{A\,\mu} = D_\mu\Lambda_A, \qquad (6.14)$$

the generators of this new Abelian gauge group T transform in the adjoint of G_{YM} and hence the total gauge group is $G_{YM}\ltimes T$, with gauge covariant field-strenghts $F^A_{\mu\nu}$ and $G_{A\,\mu\nu} = 2D_{[\mu}B_{A\,\nu]}$, which transform under T as $\delta F^A = 0$ and $\delta G_A = -\Lambda^C f_{AB}{}^C F^B$. The full covariant derivative of ϕ_A is

$$\hat{D}_\mu\phi_A = D_\mu\phi_A - B_{A\,\mu} \qquad (6.15)$$

[21] Recall that the pure CS model, $S=\int \mathrm{Tr}(AdA+\frac{2}{3}A^3)$, is a topological FT wich do not propagate any local degree of freedom [**56**][**57**].

Consider the new Lagrangian

$$\mathcal{L}_{\text{new}} = -\frac{1}{2}\sqrt{-g}\,\hat{D}_\mu\phi_A\,M^{AB}\,\hat{D}^\mu\phi_B + \frac{1}{2}\varepsilon^{\mu\nu\rho}\big(F^A_{\mu\nu}\,B_{A\,\rho} - O^A_{\mu\nu}\,\hat{D}_\rho\phi_A\big) + \mathcal{L}', \tag{6.16}$$

where M^{AB} is the inverse of the matrix M_{AB}. The equations of motion obtained from this Lagrangian are:

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta B^\mu_A} = 0 \;&\rightarrow\; M^{AB}\big(D_\mu\phi_B - B^\mu_B\big) + \tilde{F}^A_\mu + \tilde{O}^A_\mu = 0 &&\equiv \text{eqn.}(6.13)\\
\frac{\delta\mathcal{L}}{\delta\phi_A} = 0 \;&\rightarrow\; D^\mu\big(M^{AB}(D_\mu\phi_B - B^\mu_B) - \tilde{O}^A_\mu\big) = 0 &&\equiv \text{eqn.}(6.8)\\
*\frac{\delta\mathcal{L}}{\delta A^A_\rho} = 0 \;&\rightarrow\; D_{[\mu}(B_{A\,\nu]} - D_{\nu]}\phi_A) = J_{A\mu\nu} &&\equiv \text{eqn.}(6.9),
\end{aligned}$$

together with the same equations as before for the fields Φ. Therefore the two Lagrangians are physically *equivalent.*

$\mathcal{L}_{\text{new}}$ is gauge–invariant under $G_{YM}\ltimes T$ up to a total derivative. The original Yang–Mills–like Lagrangian, eqn.(6.7), is replaced by a Chern–Simons–matter Lagrangian but with a larger gauge group and a new scalar manifold $\widetilde{\mathcal{M}}$ of dimension $(\dim\mathcal{M}+\dim G_{YM})$, equal to the total number of (bosonic) propagating degrees of freedom.

To recover the original Lagrangian (6.7), one has just to fix the 'unitary' gauge $\phi_A = 0$ and perform the Gaussian interal in $B_{A\,\mu}$. Thus the equivalence holds also at the *quantum* level (as long as the quantum theory exists!).

General lesson 6.2. *In $D = 3$ we can always reduce to a scalars' manifold of dimension equal the effective (local) propagating degrees of freedom at the price, in presence of non trivial gauge interactions, of introducing suitable Chern–Simons couplings and gaugings.*

CHAPTER 2

Extended Supersymmetry

This chapter is still introductory/motivational.

After some warm–up in two (and one) space–time dimensions, we shall go to our preferred SUSY laboratory: $D = 3$. *Three* is a very nice number of dimensions. It is the first element in the magic sequence $\mathbb{R} \leftrightarrow \mathbb{C} \leftrightarrow \mathbb{H} \leftrightarrow \mathbb{O}$ whose entries correspond, respectively, to SUSY in 3, 4, 6 and 10 space–time dimensions. The relation between $D = 3$ supergravity and $D \geq 4$ SUGRA is like *diet Coke* versus the real drink: *no sugar, no caffeine, but all the flavour of Coca–Cola.* Our $D = 3$ *diet* SUGRA has no propagating graviton, no propagating gravitino, no propagating gauge vector (after the dHS duality, §. 6.2 of chapt. 1), but still has all the field–theoretic, algebraic, and geometric structures of the real thing. *Last, but not least* $D = 3$ SUSY/SUGRA is related to the world–volume theory of a stack of M2–branes. Hence — after all — it may be the dimension of *physical* interest.

In sections 8 and 9 we shall add plenty of sugar and uplift our results to $D \geq 4$ rigid SUSY. Finally, in sect.10 we add the caffeine and study the geometrical structures emerging from extended $D = 4$ *supergravity.* Here we get the payoff of the work we did in $D = 3$: once one has understood the basic structures in $D = 3$ (real case), $D = 4$ SUGRA requires only minor extensions (except for a few subtleties with propagating vectors).

1. Susy in *diverse* dimensions

Supersymmetry is the (only) symmetry which relates bosons to fermions. In fact, it is the only S–matrix quantum symmetry which may connect fields of different spins [**19, 20**]. A classical symmetry between bosons of different spins is usually not preserved at the quantum level, unless there is some supersymmetry to protect it. Hence, in most instances, (enough) supersymmetry is required for the classical geometric structures of FT to make sense at the quantum level.

I assume the reader has some familiarity with the SUSY algebra (the Haag–Łopuszanski–Sohnius theorem [**20**]), and its general implications (such as: the number of *bosonic* propagating degree of freedom is equal to the number of *fermionic* ones; SUSY unbroken ⇔ there exists a state of zero energy in the Hilbert space; SUSY is unbroken ⇒ the Fermi/Bose masses are equal), as well as a knowledge of the representations of the algebra, at least in $D = 4$. I will not insist on these topics, I just mention them when needed.

The detailed form of (Poincaré) supersymmetry depends on the zoology of spinors existing in the various dimensions (and space–time signatures): Weyl, Majorana, Majorana–Weyl, symplectic–Majorana, *etc.* However, the general structure is quite universal, so we shall state the properties of the algebra in the different dimensions without elaborating too much on their derivation.

A word of caution. The SUSY algebra makes sense as an algebra of operators acting of a Hilbert space $\mathcal{H}$, only in rigid, Poincaré invariant, supersymmetry. In the case of local supersymmetry, SUGRA, the meaning of the algebra as 'operators acting on some Hilbert space' is rather tricky, and we shall return to it when we have enough geometrical tools to study supersymmetry in curved (space–time) manifolds. However, since the variation of a field[1] Φ, under the SUSY transformation of (spinorial/Grasmannian) parameter ϵ_α, is given by

$$\delta\Phi = -i\,[\bar{\epsilon}Q, \Phi], \tag{1.1}$$

in the local case when we write the RHS we actually mean the LHS which is well–defined.

As we already mentioned, one basic ingredient is the zoology of spinors in D dimensions. This is our first topic.

1.1. Spinors in D dimensions. A spinor[2] is an element of the vector space S on which the (universal) Dirac matrices Γ_μ act. Its (complex) dimension is $2^{[D/2]}$. In even dimension, $D = 2m$, S is reducible as a representation of $Spin(D)$ into the direct sum of two irreducible spin representations $S = S_+ \oplus S_-$, corresponding to the two possible chiralities[3] of the spinor. The elements of the irreducible spaces $S_\pm$ are called *Weyl* spinors.

As for the reality properties, we have the standard three possibilities: the vector space S can either have a real, or complex, or quaternionic structure, depending on D and the space–time signature (Minkowski or Euclidean).

DEFINITION 1.1. $E_\mathbb{C}$ is a complex vector space. $V_\mathbb{R}$ a real one.

(1) A *real structure* on $E_\mathbb{C}$ is an *anti*–linear map $R\colon E_\mathbb{C} \to E_\mathbb{C}$ such that $R^2 = \text{id}$. We say that a vector $v \in E_\mathbb{C}$ is *real* if $Rv = v$, and *purely imaginary* if $Rv = -v$. Let E be the $\mathbb{R}$–subspace of the real vectors: one has $E_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} E$.
(2) A *quaternionic structure* on $E_\mathbb{C}$ is an *anti*–linear map $J\colon E_\mathbb{C} \to E_\mathbb{C}$ such that $J^2 = -\text{id}$. One defines the multiplication (on the left) of a vector in $v \in E_\mathbb{C}$ by a quaternion $a + bi + cj + dk$ as $\big(a + bi + (c + di)J\big)v$: this makes $E_\mathbb{C}$ into a $\mathbb{H}$–module.
(3) A *complex structure* in a real vector space $V_\mathbb{R}$ is a matrix I (that is an element of $\text{End}(V_\mathbb{R})$) with $I^2 = -1$. One defines multiplication of a vector $v \in V_\mathbb{R}$ by a complex number $(a+bi)$ as the action of the matrix $a + bI$. On the complexified space $V_\mathbb{C} \equiv \mathbb{C} \otimes V_\mathbb{R}$ (of double real dimension!) we can diagonalize I, and write $V_C = V_{(1,0)} \oplus V_{(0,1)}$ with I acting as multiplication by $+i$, resp. $-i$ on $V_{(1,0)}$, resp. on $V_C = V_{(0,1)}$.

[1] Capital Φ stands for a generic field appearing in the Lagrangian $\mathcal{L}$, bosonic *or* fermionic. Q_α is the super–charge (= the generator of SUSY).

[2] Clifford algebras and modules, spin groups, and their relation with the classical division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, as well with Cayley's octonions, $\mathbb{O}$ and the G_2 Lie group, are reviewed in APPENDIX B. There the reader may find these topics discussed in (some) detail.

[3] Chirality = eigenvalue with respect the generalized γ_5 matrix: $\Gamma_{[D]} := \Gamma_1\Gamma_2\cdots\Gamma_D$.

(4) A *quaternionic structure* on a real vector space $V_{\mathbb{R}}$ is a pair of matrices I^a $(a = 1, 2)$ satisfying the Clifford algebra

$$I^a I^b + I^b I^a = -2\delta^{ab}.$$

The (left) multiplication by a quaternion $a + bi + cj + dk$ on $V_{\mathbb{R}}$ is defined as the action of the matrix $a + bI^1 + cI^2 + dI^1I^2$. The matrices I^1, I^2 and $I^3 = I^1I^2$ are called *complex structures.*

As a matter of notation, if E is a complex vector space with a real structure R, we shall write $[[E]]$ for the corresponding real space.

Remark. Let E_1, E_2 two complex spaces with quaternionic structures J_1 and J_2, respectively. The space $E_1 \otimes E_2$ has a natural *real* structure given by $R = J_1 \otimes J_2$. We write $[[E_1 \otimes E_2]]$ for the associated real space.

Definition 1.2. We say that in D dimensions with signature (p, q) there exist *Majorana* spinors, resp. *symplectic–Majorana*[4] spinors, if the corresponding spinor space S admits a real, resp. quaternionic, structure invariant under the action of the $Spin(p, q)$ group. In this case the (pseudo)real elements are called Majorana spinors and, respectively, symplectic–Majorana. If the chiral subspaces $S_\pm$ have a real, resp. quaternionic, structure the corresponding spinors are called *Majorana–Weyl*, and, respectively, *symplectic–Majorana–Weyl* spinors.

In practice, the *anti*–linear maps of definition 1.1 are given by

$$\begin{gathered} \psi \mapsto B^* \psi^*, \\ \text{with} \quad B = C\,\Gamma^0, \quad C = \text{charge conjugation matrix}; \end{gathered} \tag{1.2}$$

the matrix B has the property

$$BB^* = \begin{cases} +1 & \text{Majorana} \\ -1 & \text{symplectic–Majorana.} \end{cases} \tag{1.3}$$

We present the classification of spinors for spacetimes of Minkowskian signature in table 2.1. See Appendix B for general signatures (p, q) (and for the theory). The rows repeat themselves with periodicity 8 in the dimension D (Bott's periodicity). The signs $(\pm)$ in the table refer to the signature of the Clifford algebra in which we have Γ–matrices with *real* (resp. *quaternionic*) entries. For instance, a ✓ sign in the 'Majorana $(+)$' row means that, in that dimension, there exist *real* matrices Γ^a satisfying $\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\eta^{ab}$, where η^{ab} is the Minkowski metric with the *'mostly + signature'* $(-, +, +, \dots, +)$; on the contrary, a ✓ sign in the 'Majorana $(-)$' row means that, in that dimension, there are *real* matrices Γ^a satisfying $\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = -2\eta^{ab}$.

1.2. Supercharges and superalgebras. It is convenient to choose a realization of the supercharges which makes manifest the largest automorphism group of the SUSY algebra. For instance, in $4D$ we may use either Majorana (= Hermitean) supercharges or Weyl (chiral) supercharges. The two formalisms are equivalent, of course, but in the second one the full automorphism group $U(\mathcal{N})_R$ is (more) explicit.

[4] Some author calls the symplectic–Majorana spinors *pseudoreal* spinors.

TABLE 2.1. Zoology of spinors in Minkowski signature

D	1	2	3	4	5	6	7	8	9	10	11
Majorana (+)	-	✓	✓	✓	-	-	-	-	-	✓	✓
Majorana (−)	✓	✓	-	-	-	-	-	✓	✓	✓	-
symplectic–Majorana (+)	-	-	-	-	-	✓	✓	✓	-	-	-
symplectic–Majorana (−)	-	-	-	✓	✓	✓	-	-	-	-	-
Majorana–Weyl	-	✓	-	-	-	-	-	-	-	✓	-
symplectic–Majorana–Weyl	-	-	-	-	-	✓	-	-	-	-	-

We emphasize that the automorphism group of the SUSY algebra may or may not be a symmetry of the physical theory; it depends on the actual model. When a subgroup of the SUSY algebra automorphism is a symmetry of the given theory, we refer to it as *the* R–symmetry.

In table 2.2 we present the SUSY generators, (Poincaré) algebras, and automorphism groups, for the different D's (we write only one period in D mod 8). 'CC' stands for *'central charges'*, namely scalar charges generating symmetries commuting with all the other symmetries of the theory. Their transformation under the automorphism group can be read directly from the algebra[5]. By the *automorphism group*, Aut_R, we mean the compact subgroup preserving the natural metric. In the table $Sp(\mathcal{N})$ stands for the compact (unitary) form of the symplectic group[6] having a defining representation of dimension $\mathcal{N}$ (in particular $\mathcal{N}$ should be even). The reader may find more details in ref.[**60**], and also refs.[**61, 63, 62**].

The last line in the tables is very important: represents the centralizer of the given spin group, namely the automorphism of an *irreducible* representation. It is a division algebra by the Schur's lemma. The automorphism group for the minimal supersymmetry ($\mathcal{N} = 1$, except for the symplectic case in which it is $\mathcal{N} = 2$) equals the group of elements of unit norm in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ respectively: $\mathbb{Z}_2$, $U(1)$, or $Sp(2) \simeq SU(2)$. *This means that the irreducible representation commutes with the identity and none, one or three complex structures, respectively, which generate the corresponding division algebra* [**63**]. This correspons to a very deep fact, expecially emphasized by Kugo and Townsend, ref.[**64**]: increasing the space–time dimension D, the physical structures repeat themselves except that they get defined over bigger and bigger division algebras according to the scheme $\mathbb{R} \to \mathbb{C} \to \mathbb{H}$. This is manifest, for instance, in the structure of the Lorentz group for $D = 3$, 4 and 6, which are, respectively, (see appendix , §. 4)

$$SL(2,\mathbb{R}), \quad SL(2,\mathbb{C}), \quad SL(2,\mathbb{H}), \tag{1.4}$$

as well as in the corresponding little groups for massive particles

$$SU(2,\mathbb{R}), \quad SU(2,\mathbb{C}), \quad SU(2,\mathbb{H}). \tag{1.5}$$

[5] In Poincaré supersymmetry — in sharp contrast to AdS or conformal SUSY — the central charges Z should be invariant under all the actual symmetries of the theory. On the other hand, the SUSY algebra dictates that the Z's are in non–trivial representations of the automorphism group. Hence, in presence of non–trivial central charges, only the subgroup of the automorphism group which leaves Z invariant may be an R–symmetry.

[6] Notations $USp(\mathcal{N})$, $Sp(\mathcal{N}/2)$, or $SU(\mathcal{N}/2,\mathbb{H})$ (*i.e.* the unitary $\mathcal{N}/2 \times \mathcal{N}/2$ matrices with quaternionic entries) are also used.

TABLE 2.2. Supercharges and superalgebras in diverse dimensions

D	3	4
supercharges	Majorana	Weyl
reality	$(Q^a_\alpha)^\dagger = Q^a_\alpha \quad (*)$	$(Q^a_\alpha)^\dagger = \overline{Q}_{\dot\alpha\, a}$
superalgebra	$\{Q^a_\alpha, Q^b_\beta\} = 2\delta^{ab}(\Gamma^\mu\Gamma_0)_{\alpha\beta}P_\mu + \text{CC}$	$\{Q^a_\alpha, \overline{Q}_{\dot\beta\, b}\} = 2\delta^a_{\ b}\,(\sigma^\mu)_{\alpha\dot\beta}P_\mu$ $\{Q^a_\alpha, Q^b_\beta\} = \epsilon_{\alpha\beta}\, Z^{ab}$
automorphism	$SO(\mathcal{N})$	$U(\mathcal{N})$
centralizer	$\mathbb{R}$	$\mathbb{C}$

D	5	6
supercharges	(pseudo)symplectic–Majorana	symplectic–Majorana–Weyl
reality	$(Q^a_\alpha)^\dagger = \Omega_{ab}B_\alpha{}^\beta\, Q^b_\beta$	$(Q^a_\alpha)^\dagger \equiv \bar{Q}_{\dot\alpha\, a} = \Omega_{ab}B_{\dot\alpha}{}^\beta\, Q^b_\beta$
superalgebra	$\{Q^a_\alpha, Q^b_\beta\} = 2\Omega^{ab}(\Gamma^\mu C)_{\alpha\beta}P_\mu + \text{CC}$	$\{Q^a_\alpha, Q^b_\beta\} = 2\Omega^{ab}(\Sigma^\mu)_{\alpha\beta}P_\mu$ $\{Q^a_\alpha, \overline{Q}^b_{\bar\beta}\} = C_{\alpha\bar\beta}Z^{ab}$
automorphism	$Sp(\mathcal{N})$	$Sp(\mathcal{N}_R)\times Sp(\mathcal{N}_L)$
centralizer	$\mathbb{H}$	$\mathbb{H}$

D	7	8
supercharges	symplectic–Majorana	Weyl
reality	$(Q^a_\alpha)^\dagger = \Omega_{ab}B_\alpha{}^\beta\, Q^b_\beta$	$(Q^a_\alpha)^\dagger = \overline{Q}_{\dot\alpha\, a}$
superalgebra	$\{Q^a_\alpha, Q^b_\beta\} = 2\Omega^{ab}(\Gamma^\mu C)_{\alpha\beta}P_\mu + \text{CC}$	$\{Q^a_\alpha, \overline{Q}_{\dot\beta\, b}\} = 2\delta^a_{\ b}\,(\Sigma^\mu)_{\alpha\dot\beta}P_\mu$ $\{Q^a_\alpha, Q^b_\beta\} = C_{\alpha\beta}\, Z^{ab}$
automorphism	$Sp(\mathcal{N})$	$U(\mathcal{N})$
centralizer	$\mathbb{H}$	$\mathbb{C}\ \ [\mathbb{H}]$

D	9	10
supercharges	(pseudo)Majorana	Majorana–Weyl
reality	$(Q^a = C(\bar{Q}^a)^t$	$Q^a_\alpha)^\dagger = Q^a_\alpha$
superalgebra	$\{Q^a_\alpha, Q^b_\beta\} = 2\delta^{ab}(\Gamma^\mu C)_{\alpha\beta}P_\mu + \text{CC}$	$\{Q^a_\alpha, Q^b_\beta\} = 2\delta^{ab}(\Sigma^\mu)_{\alpha\beta}P_\mu$ $\{Q^a_\alpha, \overline{Q}^b_\beta\} = \text{CC}$
automorphism	$SO(\mathcal{N})$	$SO(\mathcal{N}_R)\times SO(\mathcal{N}_L)$
centralizer	$\mathbb{R}$	$\mathbb{R}$

Notes: (1) CC stands for 'Central Charges'. (2) $(*)$ holds in a Majorana rep. In a general rep. we have $(Q^a)^\dagger = BQ^a$, with $B \equiv C\Gamma^0$, where C is the charge–conjugation matrix. The same expression holds for the B matrices appearing in the reality conditions in $D = 5, 6, 7$; however in the symplectic case we have $B^*B = -1$, while in the Majorana one $B^*B = +1$.

This is also true for the supersymmetric interactions: as we shall see, the minimal SUSY (scalar) models in $D = 3, 4$, and 6 are based, respectively, on real, complex, and quaternionic differential geometry. Their superalgebras correspondingly have 2, 4, 8 supercharges (twice the dimension of the associated division algebra). All this is very beautiful and satisfactory except that one feels that something is missing. One would like to continue the series to 10 dimensions, in correspondence with SUSY theories with 16 supercharges. On the other hand we know that there exists a fourth (and last) division algebra, the octonions $\mathbb{O}$. The obvious guess would be a relation

TABLE 2.3. Physically allowed $\mathcal{N}$

D	allowed extensions
3	$\mathcal{N} = 1, 2, 3, 4, 5, 6, 8, \mathbf{9}, \mathbf{10}, \mathbf{12}, \mathbf{16}$
4	$\mathcal{N} = 1, 2, 3, 4, \mathbf{5}, \mathbf{6}, \mathbf{8}$
5	$\mathcal{N} = 2, 4, \mathbf{6}, \mathbf{8}$
6	$(\mathcal{N}_R, \mathcal{N}_L) = (2, 0), (2, 2), (\mathbf{4}, \mathbf{0}), (\mathbf{4}, \mathbf{2}), (\mathbf{6}, \mathbf{0}), (\mathbf{4}, \mathbf{4}), (\mathbf{6}, \mathbf{2}), (\mathbf{8}, \mathbf{0})$
7	$\mathcal{N} = 2, \mathbf{4}$
8	$\mathcal{N} = 1, \mathbf{2}$
9	$\mathcal{N} = 1, \mathbf{2}$
10	$(\mathcal{N}_R, \mathcal{N}_L) = (1, 0), (\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{0})$
11	$\mathcal{N} = \mathbf{1}$

Numbers in bold face correspond to extended supersymmetries possible only in the local case (SUGRA).

like $Spin(1, 9) \sim SL(2, \mathbb{O})$, but this cannot be true at its face value since the octonions are not associative, and hence we cannot define a matrix algebra over them. Yet it is true that for $D = 10$ Majorana–Weyl spinors are *octonionic* in nature, being related to $\mathbb{O}$ much in the same way as the Majorana ones in $D = 3$ to $\mathbb{R}$, the Weyl ones in $D = 4$ to $\mathbb{C}$, and the symplectic–Weyl fermions in $D = 6$ to $\mathbb{H}$ (see appendix , §.).

For our purposes, the important lesson of this section is that there are 'special' dimensions, namely $D = 3, 4, 6, 10$, in which the very nature of the spinors induces on a supersymmetric theory (respectively) a real, complex, quaternionic, or (morally) 'octonionic' structure (the last one being, more properly, a real structure with a peculiar triality property[7]). The $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ sequence is important in many ways in theoretical physics, expecially for superstrings and branes (see *e.g.* [**65**]).

Then

GENERAL LESSON 1.1. *Susy in $D = 3, 4, 6, 10$ has a natural 'uniform' structure based, respectively, on $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and — in a suitable sense — $\mathbb{O}$.*

Not all $\mathcal{N}$'s can be realized in local FT with fields having spins ≤ 1 (≤ 2 in the SUGRA case). See table 2.3 for the list of allowed ones. Below we shall recover this result from the geometric viewpoint.

Enough with the algebra. It is time to study physics (and geometry). We start by considering a simple prototypical example, the $D = 2$ case, as a warm–up.

[7] See appendix §..

2. A little warm–up: $D = 2$

2.1. The algebra. In the Weyl–Majorana notation (1 component spinors), the $(\mathcal{N}_+,\mathcal{N}_-)$ SUSY algebra reads

$$\begin{aligned} &\{Q^a_+, Q^b_+\} = 2\,\delta^{ab}\,P_+ \\ &\{Q^{a'}_-, Q^{b'}_-\} = 2\,\delta^{a'b'}\,P_- \\ &\{Q^a_+, Q^{b'}_-\} = Z^{ab'} \\ &a, b = 1, 2, \cdots, \mathcal{N}_+, \qquad a', b' = 1, 2, \cdots, \mathcal{N}_-, \end{aligned} \tag{2.1}$$

where $Q^a_\pm$ are Hermitean and Z^{ab} are central charges. We refer to the left–right symmetric case, $(\mathcal{N},\mathcal{N})$, as $\mathcal{N}$–extended SUSY. From eqn.(2.1) we see that the supercharges $Q^a_\pm$ have dimension 1/2. So, acting on a field, they increase its (engineering) dimension by 1/2.

2.2. $\mathcal{N} = 1$**.** We start by considering the $\mathcal{N} = 1$ models. For semplicity, we shall limit ourselves to parity–invariant theories (so, in the general Lagrangian (2.1) of chapt. 1 we set to zero the two–form coupling which is a peculiar feature of $D = 2$ and does not generalize to $D > 2$).

Let ϕ^i be the scalars fields appearing in the Lagrangian $\mathcal{L}$ of a supersymmetric model. We *define* the spin–1/2 fields $\psi^i_\pm$ by

$$\psi^i_\pm = [Q_\pm, \phi^i].$$

Then the algebra implies

$$\{Q_\pm, \psi^i_\pm\} = i\partial_\pm \phi^i \tag{2.2}$$

$$\{Q_\mp, \psi^i_\pm\} = iF^i \tag{2.3}$$

where the second equation is just the definition of what we mean by the real scalar F^i. It has (engineering) dimension 1 and hence is not propagating (auxiliary field).

From chap. 1 we know that the ϕ^i's are local coordinates in some target manifold $\mathcal{M}$. From the definition, eqn.(2.2), we see that the $\psi^i_\pm$'s take value in the tangent bundle $T\mathcal{M}$. [$*$To be *pedantic:* $\psi^i_\pm$ takes values in the bundles $\mathcal{V}_\pm := \mathcal{L}^{\pm 1} \otimes \Phi^* T\mathcal{M} \to \Sigma$, where the line bundle $\mathcal{L}$ is obtained by analitic continuation of an Euclidean spin–bundle $\mathcal{L}_E$, *i.e.* a line–bundle with the property that $\mathcal{L}^2_E = K$, the canonical bundle. To fix $\mathcal{L}_E$ requires, in general, a choice of spin–structure in the (Euclidean) space–time. To save print, we usually leave *implicit* the spacetime part $\mathcal{S}_R$ of the fields' bundles (cfr. GENERAL LESSON 2.1 of chapt. 1)].

In particular the $\psi^i_\pm$ transform as vectors under general field redefinitions, $\phi^i \mapsto \varphi^i(\phi)$,

$$\psi^i_\pm \mapsto (\partial\varphi^i/\partial\phi^j)\,\psi^j_\pm. \tag{2.4}$$

Then:

GENERAL LESSON 2.1. *In a supersymmetric theory, the spin–1/2 fermions (which are* SUSY *partners of the scalars) live in the (pull–back of the) tangent bundle $T\mathcal{M}$ (twisted by the appropriate space–time spin bundle).*

We added a proviso in the statement to the effect that not all the Fermi fields present in the Lagrangian need to appear in the transformations $\delta\phi^i$.

This result is essentially independent of the dimension D of space–time, although in higher dimension one needs to be slightly more precise, see secs.5,8.

The above observation suffices, in view of our discussion in chap. 1, to uniquely fix the kinetic terms of the ψ^i's, as well as the couplings of the form $\psi^i_\pm \psi^j_\pm \,\partial_\mp \phi^k$, in terms of the Riemannian geometry of $\mathcal{M}$. If the theory contains only scalars and fermions, this determines the Lagrangian $\mathcal{L}$ up to the scalar potential $V(\phi)$, the Yukawa couplings $Y_{ij}(\phi)\,\psi^i_\pm \psi^j_\mp$ and the 4–Fermi interactions (compare with eqn.(2.1) of chapt. 1).

We start by considering the SUSY models which are, at the bosonic level, pure σ–models; their action is the one written in eqn.(1.6) of chapt. 1 plus fermionic terms. This means that we set, for the moment[8], $V(\phi) = 0$. In a rigid SUSY theory, $V(\phi) = 0 \Rightarrow Y_{ij}(\phi) = 0$ as well. *Why?* If $V(\phi) = 0$ any constant scalar configuration[9], $\phi^i = \text{const.}$, is a classical vacuum around which we can expand the theory (classically!). The scalars' mass^2–matrix in such a vacuum, $(m^2)_i^{\ j}$, is proportional to $\partial_i \partial_k V\, g^{kj} \equiv 0$, identically for all constant ϕ^i's. The algebra (2.1) implies that the boson and fermion masses are equal around all vacua. The fermions' mass^2 is[10] $Y_{ik}(\phi)\, Y^{kj}(\phi)$, so

$$V(\phi) \equiv 0 \quad \Rightarrow \quad Y_{ik}(\phi)\, Y^{kj}(\phi) \equiv 0 \quad \Rightarrow \quad Y_{ij}(\phi) \equiv 0.$$

This also implies that the bosonic part of the auxiliary scalar F^i vanishes[11]. Taking into account its dimension and scaling with the volume of $\mathcal{M}$, we see that it should have the form $F^i = A^i_{jk}(\phi)\, \psi^j_+ \psi^k_-$. The coefficient function is easily obtained by requiring that the SUSY variation of diff–invariant expressions like $f_i(\phi)\, \psi^i$ be diff–invariant. In fact

$$\begin{aligned} \{Q_\mp, f_i \psi^i_\pm\} &= (\partial_j f_i) \psi^j_\mp \psi^i_\pm + i f_i F^i = \\ &= (\nabla_j f_i) \psi^j_\mp \psi^i_\pm + (\Gamma^k_{ji}\, \psi^j_\mp \psi^i_\pm + i F^k) f_k \\ &= (\nabla_j f_i) \psi^j_\mp \psi^i_\pm \qquad \Rightarrow \quad F^k = -i\, \Gamma^k_{ji}\, \psi^j_\mp \psi^i_\pm . \end{aligned} \tag{2.5}$$

In the $V = 0$ case, it remains to determine only the 4–Fermi coupling, $S_{ijkl}(\phi)\ \psi^i_+ \psi^j_+ \psi^k_- \psi^l_-$. One can get it by a two–line computation, but we prefer to argue geometrically, in the spirit of this course. $S_{ijkl}(\phi)$ should be a *covariant* 4–tensor on $\mathcal{M}$ made out of the metric g_{ij} and its derivatives. It should vanish for a flat metric. It should be antisymmetric in the first two indices, as well as in the last two (because the ψ^i anticommute[12]). By parity

[8] As already mentioned, the b_{ij} parity–violating terms are set to zero. The logic of the arguments remains valid if $b_{ij} \neq 0$, but then we have two covariant tensors, R_{ijkl} and H_{ijk}, and the geometry is somewhat richer. H_{ijk} behaves very much as a torsion. The reader may work out the details of the general case as an exercise, or just read the relevant references, see [**21**].

[9] That is any map $\Sigma_0 \to \mathcal{M}$, where the 'space–time' Σ_0 is a point.

[10] Indices are raised/lowered with the help of the metrics g_{ij} and g^{ij} as customarily in Riemannian geometry.

[11] $0 = \delta\mathcal{L}|_{\phi^i=\text{const}} = F^i\, \frac{\delta \mathcal{L}_F}{\delta \psi^i}\, \epsilon + 3 - \text{fermions}$.

[12] Here $\psi^i_\pm$ are Grassmannian Lagrangian fields (integration variables inside the path integral) not field operators!

symmetry, it should be invariant under the interchange of the two pairs of indices. Under the scaling $g_{ij} \to \hbar^{-1} g_{ij}$, it should scale like the metric itself — since $1/\hbar$ multiplies the full classical action, 4–fermions included. This last condition — in view of the vanishing of S_{ijkl} in flat space — already says that S_{ijkl} should be linear in the Riemann tensor without any covariant derivative. (Compare §. 1.1.1 of chapt.1). Of course we know that there is only one such 4–tensor with the given symmetries, namely the Riemann tensor itself. Then

$$S_{ijkl}(\phi) = c\, R_{ijkl}(\phi) \tag{2.6}$$

for some *universal* constant c. As a further check, let us show that the 4–Fermi coupling — whatever it is — should satisfy the second Bianchi identity. Consider the 5–Fermi part of the variation of the Lagrangian $\delta\mathcal{L}|_{5F}$. It receives contributions only from the variation of the 4–Fermi term. Then

$$0 = \delta_+\Big(S_{ijkl}\psi_+^i\psi_+^j\psi_-^k\psi_-^l\Big)\Big|_{5F} = (\nabla_m S_{ijkl})\psi_+^m\psi_+^i\psi_+^j\psi_-^k\psi_-^l \;\Rightarrow\; \nabla_{[m}S_{ij]kl} = 0.$$

The complete Lagrangian and SUSY transformations now read

$$\mathcal{L} = -\frac{1}{2}g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + \frac{i}{2}g_{ij}\bar{\psi}^i\gamma^\mu\nabla_\mu\psi^j + c\,R_{ijkl}\,\bar{\psi}^i\psi^k\,\bar{\psi}^j\psi^l \tag{2.7}$$

$$\delta\phi^i = \bar{\epsilon}\psi^i \tag{2.8}$$

$$\delta\psi^i = -i\gamma^\mu\partial_\mu\psi^i\epsilon - \Gamma^i_{jk}\,\bar{\epsilon}\psi^j\,\psi^k. \tag{2.9}$$

Our next task is to compute c. As always in these lectures, we have two choices, either we apply the above SUSY transformations to the Lagrangian, eqn.(2.7), and determine the c which makes it invariant, or we construct a *deep theory* predicting its value.

We shall follow the second path. Our next subject is the relation between SUSY (and supersymmetric Lagrangians) and the *topology* of the target manifold $\mathcal{M}$.

3. Susy and the topology of $\mathcal{M}$

3.1. The ψ's as differential forms. From the previous discussion we see that, geometrically speaking, the $\psi_\pm^i$'s behave very much as the differentials of the coordinates, $d\phi^i$. In fact they have also the same anticommuting algebra

$$d\phi^i \wedge d\phi^j = -d\phi^j \wedge d\phi^j \quad \text{(exterior form algebra)} \tag{3.1}$$

$$\psi_\pm^i\,\psi_\pm^j = -\psi_\pm^j\,\psi_\pm^i \quad \text{(Grassmann algebra)} \tag{3.2}$$

So, an operator containg only fermions of a given chirality, say +, can be interpreted as a differential form on $\mathcal{M}$ (twisted, as above, by the appropriate power of the spin bundle) by the rule:

$$A_{i_1 i_2\cdots i_p}(\phi)\,\psi_+^{i_1}\psi_+^{i_2}\cdots\psi_+^{i_p} \longleftrightarrow A_{i_1 i_2\cdots i_p}(\phi)\,d\phi^{i_1}\wedge d\phi^{i_2}\wedge\cdots\wedge d\phi^{i_p}. \tag{3.3}$$

To make a long story short[13], we compactify the model on a circle S^1 keeping only the zero modes, that is, we reduce the theory to 1 dimension

[13] We shall tell the long story in due turn.

and do Supersymmetric Quantum Mechanics[14]. The Lagrangian becomes

$$L=\frac{1}{2}g_{ij}\frac{d\phi^i}{dt}\frac{d\phi^j}{dt}+ig_{ij}\bar{\psi}^i\gamma^0D_t\psi^j+c\,R_{ijkl}\bar{\psi}^i\psi^k\bar{\psi}^j\psi^l \tag{3.4}$$

where c is a constant to be determined. It is convenient to rewrite L in a basis with γ^0 diagonal

$$L=\frac{1}{2}g_{ij}\frac{d\phi^i}{dt}\frac{d\phi^j}{dt}+ig_{ij}\psi^{*i}\gamma^0D_t\psi^j+3c\,R_{ijkl}\psi^{*i}\psi^{*j}\psi^k\psi^l. \tag{3.5}$$

The fermions now correspond to operators in a Hilbert space with standard CCR

$$\{\psi^i,\psi^{*j}\}=g^{ij}(\phi),\qquad \{\psi^i,\psi^j\}=\{\psi^{*i},\psi^{*j}\}=0. \tag{3.6}$$

As customarily, we take as reference state the Clifford vacuum $|0\rangle$ defined by the condition $\psi^i|0\rangle$. Then a generic state in the Hilbert space has the form[15]

$$\Big(\Psi(\phi)+\Psi_i(\phi)\,\psi^{*i}+\cdots+\Psi_{i_1i_2\cdots i_n}(\phi)\,\psi^{*i_1}\cdots\psi^{*i_n}\Big)|0\rangle, \tag{3.7}$$

where the coefficient–wave functions $\Psi_{i_1\cdots i_p}(\phi)\,\psi^{*i_1}\cdots\psi^{*i_p}$ are square–sommable (with respect to the appropriate measure) and antisymmetric in the indices. Hence a state is a (square–sommable) differential form on $\mathcal{M}$, and the Hilbert space is $\mathcal{H}\simeq\Lambda^\bullet(\mathcal{M})$, where the grading $\bullet$ by form–degree corresponds physically to the grading by the Fermi number F of the state. If $\omega\in\Lambda^\bullet(\mathcal{M})$ is a differential form, we note by $|\omega\rangle\in\mathcal{H}$ the corresponding state. Eqns.(3.6) imply

$$\langle\omega_1|\omega_2\rangle=\int_{\mathcal{M}}\omega_2\wedge*\overline{\omega}_1 \tag{3.8}$$

where $*$ is the Hodge dual (with respect g_{ij}) and the overbar stands for complex conjugation. We have two supercharges, Q and $Q^\dagger$, which, in the Schroedinger picture act as the following differential operators

$$Q|\omega\rangle=|d\omega\rangle \tag{3.9}$$

$$Q^\dagger|\omega\rangle=|\delta\omega\rangle, \tag{3.10}$$

where d is the de Rham differential and $\delta\equiv -*d*$ is its Hermitean adjoint (with respect to the inner product (3.8)). The Hamiltonian is (by definition) $2H=\{Q,Q^\dagger\}\to d\delta+\delta d=\Delta$, *i.e.* the usual Hodge Laplacian. Therefore, the states of zero–energy (the vacua) are precisely the harmonic forms $\eta\in\mathbb{H}^\bullet(\mathcal{M})$, that is forms satisfying

$$\Delta\eta=0.$$

[14] We follow ref.[**22**].

[15] We set $n=\dim\mathcal{M}$.

3.2. Susy and topological index theorems. If $\mathcal{M}$ is compact, the harmonic forms are in one–to–one correspondence with the de Rham cohomology classes[16], $H_d^\bullet(\mathcal{M}) \simeq \mathbb{H}^\bullet(\mathcal{M})$. Hence the Witten index

$$\mathrm{Tr}\Big[(-1)^F e^{-\beta H}\Big] = \mathrm{Tr}_{\text{harmonic}}(-1)^F = \sum_{p=0}^{n}(-1)^p B_p = \chi(\mathcal{M}), \tag{3.11}$$

where $B_p = \dim \mathbb{H}^p(\mathcal{M})$ are the Betti numbers of $\mathcal{M}$, and $\chi(\mathcal{M})$ is its Euler character. On the other hand, computing the LHS *via* the path integral, we get

$$\begin{aligned}\chi(\mathcal{M}) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathcal{M}}\prod_i d\psi^{*i}\, d\psi^i \exp\Big(-3c\, R_{ijkl}\psi^{*i}\psi^{*j}\psi^k\psi^l\Big) = \\ &= \frac{(-6c)^{n/2}}{(2\pi)^{n/2}\,(n/2)!}\int_{\mathcal{M}}\epsilon_{a_1a_2\cdots a_n}R^{a_1a_2}\wedge R^{a_3a_4}\wedge\cdots\wedge R^{a_{n-1}a_n} = \\ &= (12c)^{n/2}\,\chi(\mathrm{M}),\end{aligned} \tag{3.12}$$

since only one term in the expansion of the exponential has the right form–degree (or the exact number of fermionic zero–modes) to give a non–zero contribution. In the last line we used the Gauss–Bonnet formula for the Euler characteristic, compare eqns.(10)(18) of [**28**] (we write the Riemann tensor as a two–form $\frac{1}{2}R_{ij}{}^{ab}d\phi^i\wedge d\phi^j$ with values in End$(T\mathcal{M})$, alias with two flat indices in the the vielbein formulation of GR).

Thus we predict

$$c = \frac{1}{12}. \tag{3.13}$$

Of course, this is a *very silly* way to compute the coefficients in the Lagrangian. But that was not the main reason we did the above analysis. Certainly we wanted to stress that c is not *some* coefficient one gets from *some* computation, but it has a deep meaning. However the real motivation was to introduce — in the context of a very simple situation — the idea that the structure of SUSY is deeply related to algebraic and differential topology. Since our emphasis is on the geometry of the target space $\mathcal{M}$, we could not allow ourselves to ignore the role of the *topology* of $\mathcal{M}$, that is of its ‘geometry in the large’. Moreover, much of the computability of physically interesting quantities in supersymmetric theories stems from this SUSY–topology connection. During the last quarter of a century, this connection has grown to an impressively powerful tool, both in physics and mathematics, namely *Topological Field Theory.* Already in our simple SQM framework ($D = 1$), we can get all the ‘classical’ index theorems (Hirzebruch, Riemann–Roch–Hirzebruch [**36**], Atiyah–Singer [**37, 38**], *ect.*), just

[16] The harmonic form is just the form in the given cohomology class with the smaller norm $\langle\eta|\eta\rangle^{1/2}$. The proof of the statement in the text is a standard one–line argument: $\langle\eta|\Delta|\eta\rangle = \langle\eta|(d\delta+\delta d)|\eta\rangle = \|d|\eta\rangle\|^2 + \|\delta|\eta\rangle\|^2$, so $\Delta|\eta\rangle \Leftrightarrow d|\eta\rangle = \delta|\eta\rangle = 0$. Let $|\eta + d\xi\rangle$ another closed form in the same cohomology class:

$$\langle\eta + d\xi|\eta + d\xi\rangle = \langle\eta|\eta\rangle + \langle d\xi|d\xi\rangle + \langle\xi|\delta\eta\rangle + \langle\delta\eta|\xi\rangle = \langle\eta|\eta\rangle + \langle d\xi|d\xi\rangle.$$

by inserting the suitable operator $\mathcal{O}$ in the path–integral, that is by replacing

$$\mathrm{Tr}[(-1)^F e^{-\beta H}] \to \mathrm{Tr}[(-1)^F e^{-\beta H}\,\mathcal{O}]$$

(see ref.[**22**]) and repeating word–for–word the argument we used above for the Gauss–Bonnet theorem. Higher dimensional supersymmetric field theories lead to more sophisticated geometric and topological invariants (just to mention a few: *elliptic genera* [**39**], *Gromov–Witten invariants* [**40**], *Donaldson–Witten invariants* [**41**], ...). See ref.[**42**] for a review.

Last but not least, with the topological viewpoint we had set the stage for our next subject.

3.3. ∗ Adding a scalar potential. You may object to my claim that, in supersymmetry, everything is geometric (in the sense of standard geometric structures on $\mathcal{M}$), by purporting that the geometric viewpoint will never predict the form of the scalar potential $V(\phi)$ and Yukawa terms which preserve SUSY.

I assume that you already know the structure of these interaction in a supersymmetric theory. In the course on supersymmetry you have certainly learned about the superspace (or tensor calculus) approach, the superpotential W, and all that. I will try to reproduce those results (and also something more general) from the present viewpoint.

In the setting of §. 3.1, the addition of Yukawa couplings and scalar potential amounts to a deformation of the supercharges

$$Q \to Q + \Delta Q, \qquad Q^\dagger \to \Delta Q^\dagger,$$

which preserves their Fermi–grading and the SUSY algebra, by operators ΔQ, $\Delta Q^\dagger$ not containing derivatives of the fields (otherwise their anticommutator will produce new derivative couplings in the Hamiltonian).

Without loss of generality, we can study such no–derivative deformations in the dimensionally reduced theory. In the Schroedinger language of §. 3.1, we have to deform d to a new differential operator $\tilde{d}$ which maps even/odd forms into, respectively, odd/even ones, which squares to zero, $\tilde{d}^2 = 0$, and which differs from d by an operator which do not contain derivatives. There are (basically) two possibilities. We can add to d the operator $\xi\wedge$ which acts on a form ω by multiplying it by the odd form ξ: $\omega \mapsto \xi \wedge \omega$, or we can add the adjoint operator $*\eta*$ which *contracts* the form ω with the odd–degree form η. However (cfr. the begining of §. 2.2) we *defined* the fermions ψ^i to be the SUSY transformations of the scalars which, in the present set up, means that ξ and η must be 1–forms.

3.3.1. *The superpotential and Morse theory.* Consider the first possibility, $\tilde{d} = d + \xi$. We require

$$0 = (d + \xi)^2 = d\xi, \tag{3.14}$$

and hence (locally on $\mathcal{M}$) we must have $\xi = dW$ for a certain (real) function W, which you recognize to be the *superpotential.* In this way we recover what you already know from superspace approach (or other methods). Again, the Hamiltonian is given by the deformed Laplacian

$$H = \tilde{\Delta} \equiv \tilde{d}\tilde{\delta} + \tilde{\delta}\tilde{d} \qquad \text{where} \quad \tilde{\delta} = - * \tilde{d} *, \tag{3.15}$$

and the supersymmetric vacua are its (normalizable) eigenforms associated to the zero eigenvalue.

EXERCISE 3.1. From the above equation, work out the details of the scalar potential and Yukawa couplings for a given W.

Assume that $\mathcal{M}$ is compact[17]. Since $\tilde{d}^2 = 0$ we can define the $\tilde{d}$–cohomology groups $\tilde{H}^\bullet(\mathcal{M})$ as the $\tilde{d}$–closed $\bullet$–forms, $\tilde{d}\omega = 0$ modulo the $\tilde{d}$–exact ones, $\omega \sim \omega + \tilde{d}\varrho$. Just as in the de Rham case, the zero–energy states are in one–to–one correspondence with the $\tilde{H}^\bullet(\mathcal{M})$ classes. Since we can switch on the superpotential continuously, $d \to d + t\,dW$, with $t \in [0,1]$, we expect that the dimensions of these cohomology group do not jump, and if they jump this happens because a pair of states — one bosonic one fermionic — gets (equal) non–zero energy[18]. Therefore

$$\chi(\mathcal{M}) = \sum_p (-1)^p \dim \tilde{H}^p(\mathcal{M}),$$

is independent of the deformation, and hence equal to the Euler characteristic (= Witten index [**43, 44**] for a σ–model with compact target). *Can you see the magic?*

As $t \to \infty$ the scalar potential becomes larger and larger, and the zero–energy wave–forms get exponentially small away from the points on $\mathcal{M}$ where $V(\phi)$ vanishes. Since, as you verified in exer. 3.1,

$$V(\phi) = \frac{1}{2} t^2\, g^{ij}\, \partial_i W\, \partial_j W,$$

the points where $V(\phi) = 0$ are precisely the critical points of the superpotential, namely the points where $\partial W/\partial \phi^i = 0$. Thus, we are claiming that one can compute the cohomology of a manifold $\mathcal{M}$ by taking a smooth function $W\colon \mathcal{M} \to \mathbb{R}$, and doing *local* computations in the vicinity of its critical points. *Sound magics!* But it is exactly the content of Morse theory [**45, 8**]. You may find more details about the applications of SUSY to Morse theory in the original paper by Witten [**34**].

I hope to have convinced you that also the superpotential W has some interesting geometry in it. (Even more interesting when $\mathcal{N} > 1$).

3.3.2. *The second possibility: equivariant* SUSY*.* The second possibility is more subtle[19]: $\tilde{d} = d + i_v$, where $v = v^i \partial_i$ is some vector field on $\mathcal{M}$, and the inner product operator i_v is a derivation acting on forms as $*e_{\hat{v}}*$, where $e_{\hat{v}}\colon \xi \mapsto \hat{v} \wedge \xi$ stands for the exterior product by the form $\hat{v} = g_{ij} v^i d\phi^j$. [With respect the inner product in eqn.(3.8), i_v is the adjoint of $e_{\hat{v}}$].

17 This assumption is not necessary. The argument goes trough also in the non–compact case if we consider the L^2–cohomology, namely if we work with square–summable forms/wave–functions. However, in topology, one rarely is interested in the L^2–cohomology.

18 The states of non-zero energy are organized into Fermi–Bose pairs of equal energy. In fact, let $\mathcal{H}^\perp \subset \mathcal{H}$ be the subspace of non–zero energy states. On this subspace the operator $H^{-1} \equiv \tilde{\Delta}^{-1}$ exists and is continuous. Acting on this subspace, the operator $A := (Q + Q^\dagger)H^{-1} \equiv (\tilde{d} + \tilde{\delta})\tilde{\Delta}^{-1}$ maps bosons to fermions (and *viceversa)*; it commutes with H, and $A^2 = 1$.

19 Part of the subtlety stems from the fact that these models cannot be obtained from usual $\mathcal{N} = 1$ superspace.

Now,

$$(d+i_v)^2 = d\,i_v + i_v\,d = \pounds_v \qquad \text{(the Lie derivative on forms)} \tag{3.16}$$

is certainly not zero. However, assume that v is not a generic vector field, but it is a Killing vector K of $\mathcal{M}$. K generates a $U(1)$ isometry group which — in absence of scalar potential — is a symmetry of the supersymmetric theory (provided one transforms the fermions in the right way, namely as elements of $T\mathcal{M}$). This $U(1)$ symmetry is generated by a charge Z which, in the Schroedinger picture, is represented by the differential operator $i\mathcal{L}_v$ (acting on the wave–forms ω). Then (3.16) is equivalent to

$$\{Q, Q\} = -2iZ,$$

which merely states that Z is a *central charge* of the SUSY algebra[20]. Thus we have found a first example of a SUSY system with non–trivial central charges. Such a model was formulated in $D = 2$ SUSY in ref.[**29**], and analyzed in the TFT context in refs.[**30**][**33**]. In the SQM set–up it was consider by Witten in the second part of ref.[**34**] and by Alvarez–Gaumé at the end of ref.[**22**]. The nicer treatment is presented in the wonderful paper ref.[**32**]; their interest was to formulate the equivariant version of cohomology under the action of some group on the given manifold $\mathcal{M}$ and to prove a generalized version of the Lefschetz fixed point formula [**52, 53**].

The Lefschetz formula is obtained in the following way. From the commutator $\{\widetilde{d}, \widetilde{d}^\dagger\} = 2H$ one computes the potential

$$V(\phi) = \frac{1}{2} g_{ij} K^i K^j. \tag{3.17}$$

Rescaling $K^i \to tK^i$,and taking $t \to \infty$, the vacuum wave-forms concentrate exponentially in the vicinity of the zeros of the Killing vector K^i, that is on the fixed point of the $U(1)$ isometry. Since the vacuum wave–forms capture the cohomology of $\mathcal{M}$, in this way we end up with computing the cohomology of $\mathcal{M}$ by local analysis around the fixed–points. But this is precisely what a fixed–point formula aims to. Again, SUSY allows to recover a deep mathematical theorem in very easy (and very physical) terms.

REMARK. Of course the above two mechanisms can be combined together. The most general deformation of the supercharges which produces a potential is $d \to d + i_K + e_{dW}$. This squares to the central charge $\pounds_K$ provided, $i_k dW \equiv \pounds_K W = 0$, *i.e.* provided W is a $U(1)$ invariant function.

3.3.3. *The general Lagrangian in $D = 2$.* We are ready to write down the general $D = 2$ Lagrangian with potentials. The only coupling still to fix is the Yukawa matrix Y_{ij}. It must be of the generic form $\nabla_i \partial_j W + \nabla_i K_j$. From this expression we see that the superpotential W contributes to the

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20 As you may know, the BPS bound states that $E \geq |Z|$ and hence the vacua, having zero energy, should be $U(1)$ invariant. But the vacua — at the zeroth order in the deformation i_K — are just the harmonic forms on $\mathcal{M}$ (assuming it is compact). Then we have implicitly proven the following important theorem of differential geometry:

THEOREM 3.1 ([**71, 72, 73**]). *$\mathcal{M}$ compact. The harmonic forms on $\mathcal{M}$ are invariant under the isometry group* Iso($\mathcal{M}$).

symmetric part of the Yukawa matrix Y_{ij}, while the Killing vector to the antisymmetric. Thus, reverting to the 2–component notation we get

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + \frac{i}{2}g_{ij}\bar{\psi}^i\gamma^\mu\nabla_\mu\psi^j + \frac{1}{12}R_{ijkl}\,\bar{\psi}^i\psi^k\,\bar{\psi}^j\psi^l - \\ &-\frac{1}{2}g^{ij}\Big(\partial_i W\,\partial_j W + K_i K_j\Big) - \nabla_i\partial_j\,\bar{\psi}^i\psi^j - \nabla_i K_j\,\bar{\psi}^i\gamma_5\psi^j,\end{aligned} \tag{3.18}$$

and

$$\delta\phi^i = \bar{\epsilon}\psi^i \tag{3.19}$$

$$\delta\psi^i = -i\gamma^\mu\partial_\mu\psi^i\epsilon - \Gamma^i_{jk}\,\bar{\epsilon}\psi^j\,\psi^k - g^{ij}\partial_j W\,\epsilon - K^i\,\gamma_5\epsilon. \tag{3.20}$$

4. Extended supersymmetry in $D = 3$

Our next task is to generalize the findings of sect. 2.2 to the case of $\mathcal{N} > 1$. It would be natural to continue to work in $D = 2$ space–time dimensions, since this is both the dimension in which the arguments below were originally formulated (see refs.[**24, 25, 26**]) and the dimension in which the supersymmetric σ–models are power–counting renormalizable (in fact *finite*, for $\mathcal{N}$ large enough) and hence sensible at the quantum level. We shall work, instead, in $D = 3$ dimensions. The reason is twofold. Aestetically, $D = 3$ is the dimension associated to $\mathbb{R}$. Pedagogically, we wish to treat rigid and local SUSY together and on the same footing, as we think that the two theories are more conveniently developed together, emphasizing their common properties and contrasting them with respect to their structural differences. At least this is the natural path if one looks at these theories from a geometrical perspective. Now, for supergravity $D = 2$ is a tricky exceptional dimension, since the gravitino ψ_μ reduces, in a conformal gauge, to a pure gamma–trace $\gamma_\mu\psi$, as it is well–known from the quantization of the fermionic string *á la* Polyakov [**54**]. In $D = 3$, on the contrary, SUGRA looks like much as in all higher dimensions (although it is still 'trivial' since neither the graviton nor the gravitini propagate). In the *rigid* case, $D = 2$ left–right symmetric models with $(1, 1)$ SUSY have the same geometrical structure as their $D = 3$, $\mathcal{N} = 1$ counterparts; therefore — if you feel better — you can rephrase the discussion below in $D = 2$, by a straightforward dimensional reduction. Nothing will change.

We assume that the Lagrangian is (re)written in the canonical form, in the sense of sect. 6 of chapter 1. All bosonic propagating degrees of freedom are represented by scalars ϕ^i, and the spin–1/2 fields χ^i are in one–to–one correspondence with the scalars[21]. In addition to ϕ^i, χ^j the models possibly have gauge vectors A^x_μ, spin–3/2 gravitini ψ^A_μ, and a metric $g_{\mu\nu}$, all of which do not propagate local degrees of freedom (in particular, the derivatives of the A^x_μ's enter the Lagrangian only trough Chern–Simons terms, see §. 6.2 of chapt. 1 and the beginning of sec. 5. The supersymmetric models with Chern–Simons terms are described (in some detail) in sec.. and ...

[21] We speak of the propagating spin–1/2. In certain formalisms there may be additional *auxiliary* spin–1/2's. We assume they are already integrated out.

The $D = 3$ algebra is[22]

$$\begin{aligned} &\{Q^A_\alpha, Q^B_\beta\} = 2\,\delta^{AB}(\gamma^\mu\gamma^0)_{\alpha\beta}\,P_\mu + Z^{AB}\,\epsilon_{\alpha\beta} \\ &A, B = 1, \cdots, \mathcal{N}, \qquad \alpha, \beta = 1, 2. \end{aligned} \tag{4.1}$$

This algebra has an automorphism group $\mathrm{Aut}_R \equiv SO(\mathcal{N})_R$ under which the supercharges Q^A_α transform in the vector representation and the Z^{AB} in the adjoint. $SO(\mathcal{N})_R$ needs not to be a symmetry of the physical theory. If a subgroup $\mathcal{R} \subset SO(\mathcal{N})_R$ is actually a symmetry, we shall refer to it as *the* R–symmetry.

4.1. Rigid Susy. In §. 2.2 we saw, in $\mathcal{N} = 1$ SUSY, that the fermions are tangent vectors on $\mathcal{M}$,

$$\delta\phi^i = \bar{\epsilon}\chi^i \quad \Rightarrow \tag{4.2}$$

$$\Rightarrow \quad J_{\mu\alpha} = g_{ij}(\gamma^\nu\gamma_\mu\chi^i)_\alpha\,\partial_\nu\phi^j + \text{non derivative terms}, \tag{4.3}$$

where $J_{\mu\alpha}$ is the super–current ($Q_\alpha \equiv \int J_{0\alpha}\, d^2x$).

Rigid $\mathcal{N} > 1$ SUSY is a special case of $\mathcal{N} = 1$. Indeed, you can always fix your attention on one supercharge, and *forget about* the other $\mathcal{N} - 1$ ones. This is *not true* in the local case, since $\mathcal{N}$–extended supergravity is defined to have $\mathcal{N}$ species of gravitini, ψ^A_μ, $A = 1, 2, \ldots, \mathcal{N}$.

4.1.1. *The parallel complex structures.* Let $\mathcal{L}$ be the Lagrangian of a (rigid) $\mathcal{N}$–extended model with scalar fields ϕ^i. Applying the above remark, we fix the attention on Q^1_α, and use eqn.(4.2) to define a preferred basis for the spin-1/2 fields, χ^i.

Next, we consider the other $\mathcal{N} - 1$ supercharges Q^a_α. For each of them we have an equation like eqn.(4.2), $\delta\phi^i = \bar{\epsilon}_a\,\chi^{a\,i}$. Since the χ^i's form a basis of the spin–1/2 fields[23], we must have $\chi^{a\,i} = f^{a\,i}{}_j\chi^j$ for certain coefficients $f^{a\,i}{}_j(\phi)$, in general ϕ–dependent. Geometrically the $f^{a\,i}{}_j$'s ($a = 2, \cdots, \mathcal{N}$) are $(1, 1)$ tensor fields on $\mathcal{M}$, that is sections of $T\mathcal{M} \otimes T^*\mathcal{M}$.

The SUSY algebra puts severe restrictions on the $f^{a\,i}{}_j$'s. From

$$\{Q^A_\alpha, J^B_{0\beta}\} = \delta^{AB}\delta_{\alpha\beta}\,g_{ij}\,\partial^\mu\phi^i\partial_\mu\phi^j + \text{terms with at most 1 derivative}, \tag{4.4}$$

we infer

$$(f^a)^t g + g f^a = 0 \qquad\qquad a = 2, 3, \cdots, \mathcal{N} \tag{4.5}$$

$$(f^a)^t g f^b + (f^b)^t g f^a = 2\,\delta^{ab}\,g \qquad\qquad a, b = 2, 3, \cdots, \mathcal{N} \tag{4.6}$$

that is the tensors f^a obey the Clifford algebra

$$f^a f^b + f^b f^a = -2\,\delta^{ab}, \tag{4.7}$$

and $f^a_{ij} \equiv g_{ik} f^{a\,k}{}_j$ are antisymmetric. On the other hand, we could have chosen another supercharge, say Q^2_α, as the reference one to define the Fermi fields's basis χ^i. Of course, the Lagrangian cannot depend on our arbitrary choice. Hence we must have,

$$\forall a: \quad g_{ij}\,\bar{\psi}^i\gamma^\mu D_\mu\psi^j = g_{ij}\,\bar{\psi}^{a\,i}\gamma^\mu D_\mu\psi^{a\,j} \qquad \text{not summed over } a! \tag{4.8}$$

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[22] We adopt the Majorana rep. $\gamma^0 = i\sigma_2$, $\gamma^1 = \sigma_1$, $\gamma^2 = \sigma_3$, and $C = \gamma^0$. One has $\bar{\psi} = \psi^t\gamma^0$.

[23] **!!** Here is precisely the point where we need that the Lagrangian is in the 'canonical' form, that is that all propagating vectors have been dualized.

which implies

$$D_\mu f^{a\,i}{}_j = 0, \tag{4.9}$$

that is *the tensors* $f^{a\,i}{}_j$ *are parallel* (with respect to the Christoffel connection on $T\mathcal{M} \otimes T^*\mathcal{M}$).

GENERAL LESSON 4.1. *In order to have $\mathcal{N}$ extended supersymmetry, on $\mathcal{M}$ there must exist $\mathcal{N}-1$* PARALLEL *2–forms f^A_{ij} such that the associated $f^a \in \mathrm{End}(T\mathcal{M})$ satisfy the Clifford algebra*

$$f^a f^b + f^b f^a = -2\,\delta^{ab}.$$

*In particular $\mathcal{N}$–*SUSY *requires*

$$\begin{gathered}\dim_{\mathbb{R}} \mathcal{M} = \mathbf{N}(\mathcal{N})\, m \qquad \textit{where} \;\; m \in \mathbb{N} \;\; \textit{and} \\ \mathbf{N}(k) := \begin{cases} \mathbf{N}(k+8) = 16\,\mathbf{N}(k) \\ \mathbf{N}(1) = 1,\ \mathbf{N}(2) = 2,\ \mathbf{N}(3) = \mathbf{N}(4) = 4, \\ \mathbf{N}(5) = \mathbf{N}(6) = \mathbf{N}(7) = \mathbf{N}(8) = 8. \end{cases}\end{gathered} \tag{4.10}$$

The last statement is just the usual formula for the dimension of a Clifford module (= the rank of the Dirac 'matrices').

The tensors $f^{a\,i}{}_j \in \mathrm{End}(T\mathcal{M})$ are called *complex structures*, since[24] they have square -1 (cfr. definition 1.1(3)).

The existence of a set of parallel complex structures implies quite strong constraints on the geometry of $\mathcal{M}$. In fact the identity

$$0 = [D_\mu, D_\nu] f^{a\,i}{}_j = R_{\mu\nu}{}^i{}_k f^{a\,k}{}_j - R_{\mu\nu}{}^l{}_j f^{a\,i}{}_l, \tag{4.11}$$

shows that the each f^a puts an algebraic constraint on the curvature — hence on the geometry — of the manifold $\mathcal{M}$. Therefore *only certain target geometries are compatible with $\mathcal{N}$–extended supersymmetries.* As we increase $\mathcal{N}$ we get more and more 'special' geometries. Understanding the geometries which appear at a given $\mathcal{N}$, and their physical implications, is the fundamental aim of the present lectures.

4.1.2. *The automorphism group $SO(\mathcal{N})_R$.* In rigid supersymmetry $SO(\mathcal{N})_R$ acts as a global group which cannot be gauged[25].

The $\mathcal{N}(\mathcal{N}-1)/2$ matrices Σ^{AB} $(A, B = 1, 2, \cdots, \mathcal{N})$

$$\begin{aligned} &\Sigma^{1a} = -\Sigma^{a1} := f^a, \\ &\Sigma^{ab} := \frac{1}{2}\Big(f^a f^b - f^b f^a\Big) \qquad a, b, = 2, 3, \cdots, \mathcal{N}, \end{aligned} \tag{4.12}$$

generate the Lie algebra[26] $\mathfrak{spin}(\mathcal{N})$. The fermions χ^i are in a spinorial representation of this $\mathfrak{so}(\mathcal{N})$ algebra (reducible, in general) of *definite chirality.*

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[24] According to the mathematical jargon, we should call the f^A *almost* complex structures. However, we shall show in Part 2 that they are *integrable* and hence true complex structures. In this chapter we shall be sloppy with language, and refer to them simply as *complex structures.*

[25] Otherwise the commutator $[\Lambda(x)_{ab} R^{ab}, Q^c] = \Lambda_{ca} Q^a$ of a local R–transformation and a global supersymmetry would produce a local SUSY transformation.

[26] Think of the f^a's as Dirac matrices in a $\mathcal{N}$–dimensional space of signature $(+, -, -, \cdots, -)$. But read the following section for more precise statements.

The $Spin(\mathcal{N})$ transformation acting on the fermions

$$\chi \mapsto \exp\left(\Lambda_{AB}\Sigma^{AB}\right)\chi \tag{4.13}$$

may or may not be a symmetry of the Lagrangian L. Even when we have an R–symmetry, the actual invariance may correspond to a combination of a transformation of the above kind and an isometry $\in \mathrm{Iso}(\mathcal{M})$, acting on the field as

$$\delta\phi^i = \Lambda_m K^{m\,i} \tag{4.14}$$

$$\delta\chi^i = \pounds_{\Lambda_m K^m}\,\psi^i + \Lambda_m \Theta^m_{AB}(\Sigma^{AB})^i{}_j\chi^j \tag{4.15}$$

where K^m are Killing vectors on $\mathcal{M}$ and Θ^m_{AB} are suitable coefficients (a kind of 'embedding tensor': we shall do the general theory of such objects in chapter 7). This tensor is subject to the condition of closure of the algebra.

EXERCISE 4.1. Write down the condition on Θ from the closure of the R–symmetry algebra. HINT: The Lie derivative with respect a Killing vector K of a parallel two form is again a parallel two form.

4.1.3. *Algebraic properties of the complex structures.* We need to understand better the algebraic properties of the complex structures $f^{a\,i}{}_j$, since they are crucial for constructing SUSY/SUGRA models. Unfortunately, the treatment in the SUGRA literature is <u>not</u> particularly correct.

We shall denote by $\mathbb{C}l(m)$ the universal[27] Clifford algebra generated over $\mathbb{R}$ by elements Γ^a $(a = 1, 2, \dots, m)$ satisfying

$$\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = -2\delta^{ab}. \tag{4.16}$$

In the APPENDIX you find the proof of the following

THEOREM 4.1. *We have the following isomorphisms of $\mathbb{R}$–algebras*

$$\mathbb{C}l(8k) \simeq \mathbb{R}(2^{4k}), \qquad \mathbb{C}l(8k+1) \simeq \mathbb{C}(2^{4k}), \tag{4.17}$$

$$\mathbb{C}l(8k+2) \simeq \mathbb{H}(2^{4k}), \qquad \mathbb{C}l(8k+3) \simeq \mathbb{H}(2^{4k}) \oplus \mathbb{H}(2^{4k}), \tag{4.18}$$

$$\mathbb{C}l(8k+4) \simeq \mathbb{H}(2^{4k+1}), \qquad \mathbb{C}l(8k+5) \simeq \mathbb{C}(2^{4k+2}), \tag{4.19}$$

$$\mathbb{C}l(8k+6) \simeq \mathbb{R}(2^{4k+3}), \quad \mathbb{C}l(8k+7) \simeq \mathbb{R}(2^{4k+3}) \oplus \mathbb{R}(2^{4k+3}), \tag{4.20}$$

($\mathbb{A}(n)$ denotes the algebra of $n \times n$ matrices with entries in the algebra $\mathbb{A}$).

I stress that for $m = 8k+3$ and $m = 8k+7$ (corresponding to $\mathcal{N} = 4n$, $n \in \mathbb{N}$) the Clifford algebra $\mathbb{C}l(m)$ is *not* isomorphic to a matrix algebra[28] but to the *direct sum* of two matrix algebras.

In the APPENDIX one shows also

THEOREM 4.2. *Let $\mathbb{C}l^0(\mathcal{N})$ be the subset of $\mathbb{C}l(\mathcal{N})$ of the elements* even *under the involution $\Gamma^a \mapsto -\Gamma^a$. Then*

$$\mathbb{C}l^0(\mathcal{N}) \simeq \mathbb{C}l(\mathcal{N}-1). \tag{4.21}$$

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[27] For details, including the proof of the universality property, see APPENDIX B and references therein.

[28] That is, we have to give up the physicists' folklore that the 'Dirac matrices' are matrices: in some circumstance it is more correct to look at them as *pairs* of matrices (although one can always write a pair of matrices as a bigger matrix in block–diagonal form; but this is merely a *homo*morphism not an *iso*morphism).

In view of this, our GENERAL LESSON 4.1 can be restated as

GENERAL LESSON 4.2. *Let $\mathcal{M}$ be the scalars' manifold of a $D = 3$ $\mathcal{N}$–extended supersymmetric theory (either rigid or local). Then $T\mathcal{M}$ is an (ungraded) module of $\mathbb{C}l^0(\mathcal{N})$.*

The modules of $\mathbb{C}l^0(\mathcal{N})$ are described in Atiyah, Bott and Shapiro, see TABLE 2 in ref. [**74**]. The dimension of an irreducible module is given by $\mathbf{N}(\mathcal{N})$, where $\mathbf{N}(k)$ is the function defined in eqn.(4.10). However, thanks to the subtlety we mentioned before, a general $\mathbb{C}l^0(\mathcal{N})$ module has the following structure

$$\underbrace{M \oplus M \oplus \cdots \oplus M}_{p \text{ times}} \qquad \text{for } \mathcal{N} \not\equiv 0 \mod 4 \tag{4.22}$$

$$\underbrace{M \oplus M \oplus \cdots \oplus M}_{p \text{ times}} \oplus \underbrace{\widetilde{M} \oplus \widetilde{M} \oplus \cdots \oplus \widetilde{M}}_{q \text{ times}} \qquad \text{for } \mathcal{N} \equiv 0 \mod 4, \tag{4.23}$$

where $M, \widetilde{M}$ are (non–isomorphic) irreducible modules. Precisely for $\mathcal{N} = 4n$ there are *two* inequivalent such irreducible module, whereas in the other dimensions there is just one. $\widetilde{M}$ is the *twisted* module. If we view M and $\widetilde{M}$ as *graded*[29] $\mathbb{C}l(\mathcal{N}-1)$–modules, $M = M^0 \oplus M^1$, the twisted module $\widetilde{M}$ is obtained simply by interchanging the two summands, $\widetilde{M} = M^1 \oplus M^0$ ([**74**], **pro. (5.5)**).

This means (for $\mathcal{N} \neq 4k$, say) that we can introduce an orthornormal frame $\{e^i_{\eta\, r}\, \partial_i\}$ in $T\mathcal{M}$, $[\eta = 1, 2, \dots, \mathbf{N}(\mathcal{N})$, and $r = 1, 2, \dots, \dim \mathcal{M}/\mathbf{N}(\mathcal{N})]$, such that

$$f^a\, e_{\eta\, m} = (\Gamma^a)_\eta{}^\xi\, e_{\xi\, m},$$

where the Γ^a's are the standard (numerical) Dirac matrices of $\mathbb{C}l(\mathcal{N}-1)$. As in §. 2.1.2 of chapt. 1, the frame $e_{\eta\, r}$ can be seen as a *bundle isomorphism*

$$T\mathcal{M} \simeq S \otimes \mathcal{U},$$

where S is a rank $\mathbf{N}(\mathcal{N})$ vector bundle with structure group $Spin(\mathcal{N})_R$, acting as in eqn.(4.13), and $\mathcal{U}$ is the vector bundle 'associated with the index r'. Therefore our GENERAL LESSON can be restated in still another form, more in the spirit of §. 2.1.2 of chapt. 1:

GENERAL LESSON 4.3. *Let $\mathcal{M}$ be the scalars' manifold of an $\mathcal{N}$–extended* SUSY/SUGRA *model in $D = 3$.*

If $\mathcal{N} \not\equiv 0 \mod 4$, on $\mathcal{M}$ there are two vectors bundles S and $\mathcal{U}$, of respective ranks $\mathbf{N}(\mathcal{N})$ and $(\dim \mathcal{M})/\mathbf{N}(\mathcal{N})$, so that

$$T\mathcal{M} \simeq S \otimes \mathcal{U}. \tag{4.24}$$

If $\mathcal{N} = 0 \mod 4$, we have two rank $\mathbf{N}(\mathcal{N})$ vector bundles, S and $\tilde{S}$, and two vector bundles $\mathcal{U}, \widetilde{\mathcal{U}}$, with $\text{rank}\, \mathcal{U} + \text{rank}\, \widetilde{\mathcal{U}} = (\dim \mathcal{M})/\mathbf{N}(\mathcal{N})$, such that

$$T\mathcal{M} \simeq S \otimes \mathcal{U} \,\oplus\, \widetilde{S} \otimes \widetilde{\mathcal{U}}, \tag{4.25}$$

($\mathcal{U}$ or $\widetilde{\mathcal{U}}$ may be trivial).

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[29] As before, the grading, $M = M^0 \oplus M^1$ corresponds to the even/odd elements under the involution $\Gamma^a \mapsto -\Gamma^a$, that is M^0 (resp. M^1) is the subspace of elements which may be written as a linear combination of products of an *even* number of Γ^a's (resp. *odd*).

The bundles S, $\widetilde{S}$ have structure group $Spin(\mathcal{N})_R$. In rigid SUSY *the $Spin(\mathcal{N})_R$ connection — and hence the bundles S, $\widetilde{S}$ — are* flat[30]*. (See sect. 5 for the corresponding statement in* local SUSY*).*

NOTE ADDED. As discussed in the class, the bundles S, $\widetilde{S}$ need to be flat for $\mathcal{N} > 2$, but the case $\mathcal{N} = 2$ is special. In fact, what we actually have proven is that the Σ^{AB} are parallel, that is that the $Spin(\mathcal{N})_R$ connection is trivial *in the adjoint representation.* For $\mathcal{N} = 2$, $Spin(2)_R$ is Abelian, and the adjoint representation is the trivial one. Hence, in this case, we cannot conclude that S itself is flat. (The cases in which *it is* flat will correspond to some special geometries, but, in general, S is not flat).

The splitting of $T\mathcal{M}$ into two distinct summands for $\mathcal{N} = 0 \mod 4$ is a consequence of the previous subtlety in the structure of Clifford modules.

The above subtlety is well understood in $\mathcal{N} = 4$ SUSY/SUGRA. It corresponds to the statement that the scalars' manifold — in general — splits in a product space $\mathcal{M} \times \widetilde{\mathcal{M}}$, with $T\mathcal{M} \simeq S \otimes \mathcal{U}$ and $T\widetilde{\mathcal{M}} \simeq \widetilde{S} \otimes \widetilde{\mathcal{U}}$. The scalars living in the two spaces have different properties under SUSY (as the discussion above implies) and — according to the standard jargon — the scalars in $\mathcal{M}$ are said to belong to *hypermultiplets* and those of $\widetilde{\mathcal{M}}$ to *twisted hypermultiplets.*

[For the *cognoscenti*: assume our $D = 3$ SUGRA is obtained by compactifying Type IIA (or Type IIB) superstring on a Calabi–Yau space X times a circle S^1. Then one factor space parameterize the complex structures of X and the other its complexified Kähler moduli: the interchange of the two $\mathbb{H}$'s in $\mathbb{C}l(3) \simeq \mathbb{H} \oplus \mathbb{H}$ thus corresponds to the replacement of X with its *mirror* Calabi–Yau $\widetilde{X}$. I hope this example will explain why I bothered so much to enter in the above subtleties].

What is not usually realized, is that, algebraically speaking, the same 'twisted multiplet' phenomenon applies to $\mathcal{N} = 8, 12, 16, \dots$ also[31]. In fact

EXERCISE 4.2. Write down a rigid $\mathcal{N} = 8$–SUSY invariant model in $D = 3$ with *both* $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ non–trivial.

4.2. An example: $D = 3$ σ–models. In the case of σ–models with only 'metric' interactions (that is in absence of superpotential and gauge interactions) the statement 4.1 has in fact an inverse:

PROPOSITION 4.1 (SUSY enhancement). *If the metric g_{ij} on the target manifold $\mathcal{M}$ admits $\mathcal{N} - 1$ parallel complex structures which generate a Clifford algebra, eqn.*(4.7)*, the $\mathcal{N} = 1$ supersymmetry of the corresponding σ–model*

$$\mathcal{L} = -\frac{1}{2} g_{ij} \partial^\mu \phi \partial_\mu \phi^j + \frac{i}{2} g_{ij} \bar{\psi}^i \gamma^\mu D_\mu \psi^j - \frac{1}{24} R_{ijkl}\, \bar{\psi}^i \gamma_\mu \psi^j\, \bar{\psi}^k \gamma^\mu \psi^l \qquad (4.26)$$

is enhanced to $\mathcal{N}$–extended supersymmetry. Moreover $\mathcal{N} = 3 \ \Rightarrow\ \mathcal{N} = 4$. If $\mathcal{M}$ is irreducible, the R–symmetry is $SO(2)$ for $\mathcal{N} = 2$ and $SO(3)$ for $\mathcal{N} = 3, 4$.

[30] See, however, the NOTE ADDED below.

[31] This is the discrepancy I noted before, when I stated that the usual treatment is not *that* correct.

In presence of a superpotential $W(\phi)$, the condition for the enhancement of supersymmetry is that there exist $\mathcal{N} - 1$ functions $W^a(\phi)$ such that

$$\partial_i W = f^{a\,j}{}_i \partial_j W^a \qquad \textit{Not summed over } a! \tag{4.27}$$

(the generalized Cauchy–Riemann conditions).

The proof in elementary. The replacement $\psi^i \leftrightarrow f^{a\,i}{}_j\,\psi^j$ leave invariant the metric and hence all couplings (in particular the 4–Fermi one). Hence if $J_\mu = \gamma^\nu \gamma_\mu \psi^i\, g_{ij}\, \partial_\nu \phi^j + \cdots$ is a conserved supercharge, so is $J^a_\mu = \gamma^\nu \gamma_\mu (f^a \psi)^i\, g_{ij}\, \partial_\nu \phi^j + \cdots$. Now assume we have two anticommuting complex structures f^1, f^2. It follows from the theory of the Pauli matrices that $f^3 = f^1 f^2$ is a third complex structure which anticommutes with the f^1 and f^2. $SO(\mathcal{N})_R$ acting on the fermions as in eqn.(4.13) is a symmetry. For even $\mathcal{N}$, the ψ's transform as *Weyl* spinors of $Spin(\mathcal{N})$. In particular, for $\mathcal{N} = 4$ — assuming $\mathcal{M}$ irreducible — they transform according to the irreducible representation $(\mathbf{2}, \mathbf{1})$ of $Spin(4) \simeq SU(2) \times SU(2)$; thus the second $SU(2)$ acts trivially on all fields. The effective R–symmetry[32] is thus $SU(2)$.

The last statement is the condition under which the replacement $\psi^i \leftrightarrow f^{a\,i}{}_j\,\psi^j$ does not spoil the SUSY invariance of the Yukawa and potential terms. In the case of $\mathcal{N} = 2$, just one complex structure, the condition reduces to the usual Cauchy–Riemann equation whose general solution is $W = \mathrm{Re}\, h(\phi)$, $W^1 = \mathrm{Im}\, h(\phi)$ with $h(\phi)$ holomorphic[33]. In the cases $\mathcal{N} = 3, 4$ the condition is very restrictive.

REMARK. The above proposition is <u>not</u> true in presence of Chern–Simons couplings. Generically an $\mathcal{N} = 3$ CS theory is <u>not</u> enhanced to $\mathcal{N} = 4$. (In fact it happens only under quite special conditions). Moreover a $\mathcal{N} = 4$ CS model may have the full $Spin(4)$ R–symmetry acting non–trivially on the fields. However the condition under which SUSY enhancement *does happen* are well known, and I will return to it (in chapter 8) after having developed the necessary geometric tools.

5. Local extended supersymmetry ($D = 3$)

In the supergravity case, SUSY is a local symmetry, that is the spinorial Grassmann parameter $\epsilon^A_\alpha(x)$, $(A = 1, \cdots, \mathcal{N})$, depends on the space–time coordinate x. To each SUSY parameter it is associated a vector–spinor gauge

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[32] This is true for a *generic* $\mathcal{N} = 4$ metric. If the given metric has isometries with certain special properties, the R–symmetry may be enhanced to a larger group. See sect.. for how this happens in the special case of a σ–model which is actually conformal invariant. Compare with the discussion at the end of §. 4.1.2.

[33] Here we are cheating a bit: we have not proven yet (but it is true) that all parallel complex structures are integrable. Granted this, the argument is perfectly rigorous.

field ψ_μ^A which under a SUSY transformation behaves us[34]

$$\delta\psi_\mu^A = \widehat{\mathcal{D}}_\mu \epsilon^A \equiv D_\mu\epsilon^A + \mathcal{Q}^A_{\mu\ B}\,\epsilon^B + \mathcal{M}^A{}_B\,\gamma_\mu\epsilon^A, \tag{5.1}$$

where D_μ is the standard covariant derivative acting on spinors, and $\mathcal{Q}^A_{\mu\ B}$ and $\mathcal{M}^A{}_B$ are some model–dependent covariant expressions depending on the various fields. Local supersymmetry implies general covariance, since the anticommutator of two x–dependent supersymmetry is an x–dependent translation, hence a general reparametrization of the space–time coordinates. This, in particular, means that the metric $h_{\mu\nu}(x)$ is one of the fields in the theory.

$D = 3$ is special since the metric $h_{\mu\nu}$, the gravitinos ψ_μ^I and the vector fields (with CS kinetic terms) A_μ^x do not propagate *independent* local degrees of freedom. In fact the Hilbert–Einstein action $\int\sqrt{-h}R$, the Rarita–Schwinger one $\int\bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho$, and the (pure) Chern–Simons term lead to equations of motion of the form

$$\left.\begin{matrix} R_{\mu\nu\rho\lambda} \\ D_{[\mu}\psi_{\nu]} \\ F_{\mu\nu}\end{matrix}\right\} = \text{sources},$$

so the field is pure–gauge away from the sources.

To be concrete, we preliminary describe the *pure* $\mathcal{N}$–extended SUGRA's in some detail.

5.1. Pure $D = 3$ $\mathcal{N}$-supergravity. Pure supergravity in $D = 3$ exists for all $\mathcal{N}$'s.

The field content is the metric vielbein $e_\mu{}^a$ ($h_{\mu\nu} = e_\mu{}^a e_\nu{}^b\eta_{ab}$, with $\eta_{ab} = \mathrm{diag}(-1,+1,+1)$), the spin–connection $\omega_\mu^a \equiv \epsilon^{abc}\omega_{\mu\,bc}$, and $\mathcal{N}$ gravitini ψ_μ^A ($A = 1, 2\ldots,\mathcal{N}$)). The Lagrangian is the obvious one, Einstein–Hilbert plus Rarita–Schwinger,

$$\mathcal{L}_{\mathrm{SG}} = -\frac{1}{2}\epsilon^{\mu\nu\rho}\Big\{e_\mu{}^a R_{\nu\rho\,a}(\omega) + i\bar\psi_\mu^A D_\nu\psi_\rho^A\Big\} \tag{5.2}$$

where

$$R_{\mu\nu}{}^a = \partial_\mu\omega_\nu{}^a - \partial_\nu\omega_\mu{}^a + \epsilon^{abc}\omega_{\mu\,b}\,\omega_{\nu\,c} \tag{5.3}$$

$$D_\mu\psi = \left(\partial_\mu + \frac{1}{2}\omega_\mu{}^a\gamma_a\right)\psi \tag{5.4}$$

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \epsilon^{abc}\omega_{\mu\,b}\,e_{\nu\,c}\,. \tag{5.5}$$

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[34] This is the most general expression you may write compatible with the symmetries. Indeed, since ϵ^A is a spinor, $\hat{\mathcal{D}}_\mu$ is an element of the Clifford algebra $\mathbb{C}l(1,2)$. The matrices $\mathbf{1}$ and γ_μ make a basis of that algebra, so we must have

$$(\widehat{\mathcal{D}}_\mu)^A{}_B = \delta^A{}_B\,D_\mu + \mathcal{Q}^A_{\mu\ B} + \mathcal{A}_\mu{}^{\nu\,A}{}_B\,\gamma_\nu + \mathcal{M}^A{}_B\,\gamma_\mu$$

the term $\mathcal{A}_\mu{}^{\nu\,A}{}_B\gamma_\nu$ has the same Lorentz structure as the spin–connection present in the covariant derivative D_μ (which, in $D = 3$, can be rewritten as a term with only 1 gamma–matrix). In order not to mix space–time symmetries with internal one, it should be proportional to $\delta^A{}_B$. Then can be absorbed by a redefinition of the spin–connection $\omega_\mu{}^\nu$, which , for us, is an auxiliary field to be eliminated trough its equations of motion *(1.5 formalism,* see below). Then, in presence of matter, the ω equations of motion get new terms.

Here the spin connection is meant to be eliminated trough its (algebraic) equations of motion[35], which just state that the torsion is a bilinear in the gravitini

$$D_{[\mu}e^a_{\nu]} = i\frac{1}{4}\bar{\psi}^A_\mu\gamma^a\psi^A_\nu. \tag{5.6}$$

The Lagrangian is invariant under

$$\delta e_\mu{}^a = \frac{i}{2}\bar{\epsilon}^A\gamma^a\psi^A_\mu \tag{5.7}$$

$$\delta\psi^A_\mu = D_\mu\epsilon^A. \tag{5.8}$$

Indeed[36]

$$\begin{aligned}\delta\mathcal{L} &= -\frac{i}{4}\epsilon^{\mu\nu\rho}\Big\{\bar{\epsilon}^A\gamma^a\psi^A_\mu\, R_{\nu\rho\, a} + 4\bar{\psi}^A_\mu D_\nu D_\rho\,\epsilon^A\Big\} \\ &= -\frac{i}{4}\epsilon^{\mu\nu\rho}\Big\{\bar{\epsilon}^A\gamma^a\psi^A_\mu\, R_{\nu\rho\, a} + R_{\nu\rho\, a}\,\bar{\psi}^A_\mu\gamma^a\epsilon^A\Big\} \equiv 0 \quad \text{(by the Grassmann property)}\end{aligned}$$

where we used the definition of the curvature

$$[D_\mu, D_\nu]\,\epsilon = \frac{1}{2}R_{\mu\nu\, a}\,\gamma^a\epsilon.$$

The above theory is a Topological Field Theory [**67**]. Indeed the $e_\mu{}^a$ and ψ^A_μ equations of motion set the corresponding gauge–invariant curvatures to zero. In fact, $D = 3$ $\mathcal{N}$–SUGRA may be seen as a standard Chern–Simons theory for the appropriate superalgebra [**67**].

REMARK. In $D = 3$ pure supergravity, we may have any $\mathcal{N}$. By analogy with $D = 4$, one may have expected (naively) that $\mathcal{N} \leq 16$. But the usual SUSY algebra representation argument does not apply in $D = 3$ since there are no particle (local propagating states) to which to apply it. However it is true that, adding matter, we would get $\mathcal{N} \leq 16$ for a (locally) non–trivial theory. But this will be more a theorem in pure geometry than in supersymmetry!!

5.2. Coupling to a non–linear σ–model. We couple to the above pure SUGRA a σ–model with scalars ϕ^i parametrizing some manifold $\mathcal{M}$ and their spin–1/2 superpartners χ^i. The ϕ^i and the χ^i are the true propagating degrees of freedom[37].

Up to higher order couplings, we expect a Lagrangian having the general structure

$$\mathcal{L} \;\sim\; \mathcal{L}_{SG} + \mathcal{L}_{\sigma-\text{model}} + \mathcal{L}_{\text{inter.}}.$$

The interaction Lagrangian, $\mathcal{L}_{\text{inter.}}$, at the linearized level in the gravitino fields should have the Noether form: $\sqrt{-h}\,(\text{gauge field})_\mu\,(\text{current})^\mu$, that is

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[35] This is called first order formalism.

[36] One has

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta e_\mu{}^a}\delta e_\mu{}^a + \frac{\delta\mathcal{L}}{\delta\omega_\mu{}^a}\delta\omega_\mu{}^a + \frac{\delta\mathcal{L}}{\delta\psi_\mu{}^A}\delta\psi_\mu{}^A,$$

but, since we assuming that $\omega_{\mu\, a}$ satisfies its eqn. of motion, $\delta\mathcal{L}/\delta\omega_{\mu\, a} \equiv 0$, and we do not need to vary the spin–connection in $\mathcal{L}$. (This trick is known as *the 1.5 formalism* in the SUGRA jargon).

[37] Non–propagating, auxiliary, scalar and spin–1/2 fields, if any, are assumed to have been integrated away.

$\sqrt{-h}\,\bar{\psi}^A_\mu\, J^{\mu A}$ (J^A_μ is the σ–model supercurrent). In fact this extends to the full non–linear coupling

$$\frac{\delta\mathcal{L}_{\text{inter.}}}{\delta\bar{\psi}^A_\mu} = \text{properly supercovariantized supercurrent,} \tag{5.9}$$

by the SUSY analogue of the Gauss' law.

Thus, to have a consistent theory, the matter σ–model must have (at least) $\mathcal{N}$ supersymmetries (*i.e.* $\mathcal{N}$ conserved supercurrents at the zero order in the supergravity fields). From the SUSY variation $2\,\delta\phi^i = \bar{\epsilon}^A f^{a\,i}{}_j \chi^j$, which we take as the definition[38] of both the ψ^i's and the $f^{A\,i}{}_j$'s, we learn that on $\mathcal{M}$ there should be $(\mathcal{N}-1)$ complex structures[39], $f^{a\,i}{}_j$, generating a Clifford algebra as in eqns.(4.5)–(4.7). To write more symmetric formulae we write[40]

$$f^{A\,i}{}_j = \begin{cases} \delta^i{}_j & A=1 \\ f^{a\,i}{}_j & A=2,3,\dots,\mathcal{N}. \end{cases} \tag{5.10}$$

The situation in much the same as in the rigid case, §. 4.1, *with two major differences:* the one already mentioned in the discussion following eqn.(4.3), and the fact that, in the local case, the automorphism group, $SO(\mathcal{N})_R$ may be a local symmetry, and in fact it *should be* a local symmetry. This is already manifest from eqn.(5.1): $\mathcal{Q}^A_{\mu\,B}$ may be seen as a *composite* connection for $SO(\mathcal{N})_R$. $\mathcal{Q}^A_{\mu\,B}$ cannot vanish, as one realizes by going to the linearized theory and using rigid SUSY representations (and as we shall prove momentarily).

The general form for $\mathcal{Q}^A_{\mu\,B}$ is

$$\mathcal{Q}_\mu{}^{AB} = \partial_\mu\phi^i\, Q_i{}^{AB}(\phi) + \text{fermions}, \tag{5.11}$$

where Q^{AB}_i is naturally interpreted as an $SO(\mathcal{N})_R$ connection over $\mathcal{M}$, according to the GENERAL LESSON 2.1 of chapt. 1. This connection defines a principal bundle $\mathcal{P}\to\mathcal{M}$. We denote by $V(\mathcal{P})$, $\mathrm{Adj}(\mathcal{P})$, and $S_\pm(\mathcal{P})$ the vector bundles over $\mathcal{M}$ associated to $\mathcal{P}$ trough, respectively, the vector, the adjoint and the ($\pm$ chirality) spinor representations of $SO(\mathcal{N})$.

The SUSY parameters $\epsilon^A(x)$ rotate under (local) $SO(\mathcal{N})_R$ transformations as vectors. Since $2\,\delta\phi^i = \bar{\epsilon}_A(f^A\chi)^i$, so do the f^A. Therefore the $(f^A\chi)^i$'s are sections of the vector bundle

$$T\mathcal{M}\otimes V(\mathcal{P})\to\mathcal{M}. \tag{5.12}$$

I stress that the objects $(f^A\chi)^i$ are the covariant fermionic fields; to define the χ^i's we have to choose what we mean by the 'first' supercharge (the one

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[38] I have changed the normalization of the spin–1/2 fields, $\chi^i \mapsto \frac{1}{2}\chi^i$ in order to make easy the comparison with the existing literature, which in $3D$ SUGRA has such a factor of 1/2.

[39] *Pedantic remark.* In the present case they should be called *almost*–complex structures according to the mathematical jargon. We permit ourselves some abuse of language. In Part 2, in the context of geometry, we shall use the terminology the way mathematicians do.

[40] $SO(\mathcal{N})$ indices are raised/lowered with the invariant metric δ_{AB}.

associated to the identity) and this is a (non–covariant) $SO(\mathcal{N})_R$ gauge–choice[41].

Since $V(\mathcal{P})$ is not flat, the f^a's cannot be covariantly constant as tensors. To understand the property which, in the local case, replaces covariant constance we have to study the χ's kinetic terms. But before we compute the curvature of $V(\mathcal{P})$.

REMARK. The following computation is not at all necessary in order to construct the theory. The result of the following computation will be deduced (and in many different ways) from purely geometric considerations in Part 2 below. However, it may be didactically convenient to do a concrete computation (confirming the general geometric expectations).

5.2.1. *The $SO(\mathcal{N})_R$ curvature.* The connection ${Q_i}^{AB}$ appears both in the SUSY transformation of the gravitini, eqn.(5.1), and in their kinetic terms

$$\mathcal{L}_{RS} = -\frac{i}{2}\epsilon^{\mu\nu\rho}\,\bar{\psi}^A_\mu\,\mathcal{D}_\nu\,\psi^A_\rho \tag{5.13}$$

$$\mathcal{D}_\mu\psi^A_\nu := D_\mu\psi^A_\nu + \partial_\mu\phi^i\,{Q_i}^{AB}\,\psi^B_\nu. \tag{5.14}$$

where D_μ, as before, is the spin–connection covariant derivative. Now (suppressing $\mathfrak{so}(\mathcal{N})_R$ matrix indices to simplify the notation)

$$\begin{aligned}\mathcal{D}_\mu\mathcal{D}_\nu =& D_\mu D_\nu + \big(D_\mu\partial_\nu\phi^i\big)Q_i + \partial_\nu\,\phi^i\partial_\mu\phi^j\,(\nabla_j Q_i) + \\ &+ \big(\partial_\nu\phi^i Q_i D_\mu + \partial_\mu\phi^i Q_i D_\nu\big) + \partial_\mu\phi^i\partial_\nu\phi^j\,Q_iQ_j,\end{aligned} \tag{5.15}$$

so

$$[\mathcal{D}_\mu,\mathcal{D}_\nu] = \mathcal{R}_{\mu\nu} + \partial_\mu\phi^i\partial_\nu\phi^j\Big(\partial_iQ_j - \partial_jQ_i + Q_iQ_j - Q_jQ_i\Big) \tag{5.16}$$

where $\mathcal{R}_{\mu\nu}$ is the Riemann tensor (seen as a two–form taking values in $\mathfrak{so}(2,1)$); the term in the parenthesis multiplying $\partial_\mu\phi^i\partial_\nu\phi^j$ is precisely the *curvature* on the space $\mathcal{M}$ of the $\mathfrak{so}(\mathcal{N})$ connection ${Q_i}^{AB}$. We write ${P_{ij}}^{AB}$ for this curvature 2–form. Then, writing $\mathcal{D} = dx^\mu\,\mathcal{D}_\mu$ for the covariant exterior derivative, we have

$$2\,\mathcal{D}^2 = \mathcal{R} + \Phi^* P. \tag{5.17}$$

Let $\mathcal{L}_{EH}$ and $\mathcal{L}_{RS}$ be the Einstein–Hilbert and gravitino kinetic terms, respectively. The computation in §. 5.1 implies

$$\begin{aligned}\delta(\mathcal{L}_{EH} + \mathcal{L}_{\mathcal{RS}}) &= -\frac{i}{2}\epsilon^{\mu\nu\rho}\Big([D_\nu, D_\rho] - \mathcal{R}_{\nu\rho}\Big)\,\bar{\psi}^A_\mu\epsilon^B \\ &= -\frac{i}{2}\epsilon^{\mu\nu\rho}\partial_\nu\phi^i\partial_\rho\phi^j\,{P_{ij}}^{AB}\,\bar{\psi}^A_\mu\epsilon^B.\end{aligned} \tag{5.18}$$

This term can be cancelled only by the variation of the Nother coupling which, in view of eqn.(4.3), must have the form

$$\frac{i}{2}e\,g_{ij}\bar{\psi}^A_\mu\gamma^\nu\partial_\nu\phi^i\,\gamma^\mu {f^{Aj}}_k\chi^k + \text{higher order in the fermions.} \tag{5.19}$$

[41] Since $SO(\mathcal{N})_R$ is now gauged, such a choice would break gauge–invariance and then make the geometric structure less manifest. However the $\mathcal{N}(\mathcal{N}-1)\dim\mathcal{M}/2$ fields $(f^A\chi)^i$ are not independent since we have only $\dim\mathcal{M}$ physical fermions.

This form of the supercurrent fixes also the form of the SUSY variation of χ^i to be

$$\delta\chi^i = \frac{1}{2} g^{ij} g_{kl}\, \gamma^\mu \partial_\mu \phi^k\, f^{A\,l}{}_j\, \epsilon^A + \text{covariantizing fermions.} \tag{5.20}$$

The term (5.18) of $\delta\mathcal{L}$ must be cancelled by the contribution obtained by varying the fermion χ^i in the Noether coupling (5.19); using eqn.(5.20), we get

$$\frac{i}{4} e\big(\bar\psi^A_\mu \gamma^\nu \gamma^\mu \gamma^\rho \epsilon^B\big)\, g_{ij} g^{kh} g_{lm}\, \partial_\nu \phi^i \partial_\rho \phi^l\, f^{A\,j}{}_k\, f^{B\,m}{}_h, \tag{5.21}$$

inserting the identity

$$\gamma^\nu \gamma^\mu \gamma^\rho = \epsilon^{\nu\mu\rho} + \text{traces}, \tag{5.22}$$

we see that the variation (5.21) cancels the one in eqn.(5.18) provided the curvature of the bundle $V(\mathcal{P}) \to \mathcal{M}$ is equal to

$$P_{ij}{}^{AB} = \frac{1}{2} \Sigma^{AB}{}_{ij} \tag{5.23}$$

where

$$\Sigma^{AB} \equiv -\frac{1}{2}\Big(g f^A g^{-1} (f^B)^t g - g f^B g^{-1} (f^A)^t g \Big) \tag{5.24}$$

is the tensor defined in eqn.(4.12), which is a section of

$$\wedge^2 T^*\mathcal{M} \otimes \mathrm{Adj}(\mathcal{P}).$$

Thus

$$P_{ij}{}^{AB} = -\frac{1}{4}\Big(g f^A g^{-1} (f^B)^t g - g f^B g^{-1} (f^A)^t g \Big) \tag{5.25}$$

a formula which is reminescent of, say, the tt^* curvature [**69**], and of many other celebrated topics in both mathematics and physics (Toda field theory, variations of Hodge structures, $2D$ CFT, Hitchin equations for YM and TFT,...). They all have the generic *minus a commutator* form:

$$(\text{curvature}) = -[C, C^\dagger],$$

where the C's are *semi*parallel tensors. This is far from being a coincidence, it is rather deep. However, the SUGRA case has some additional peculiar structure[42] which makes the geometry even more elegant.

The Bianchi identity gives

$$\mathcal{D}_{[i} \Sigma^{AB}{}_{jk]} = 0 \tag{5.26}$$

where the covariant derivative $\mathcal{D}_i$ is (target) Christoffel and $\mathfrak{so}(\mathcal{N})_R$ covariant.

[42] For a recent application to a physical problem see ref. [**70**].

5.2.2. *χ^is' kinetic terms.* In the discussion around eqn.(5.12), we learned that an $SO(\mathcal{N})_R$ covariant kinetic term for the χ's must have the form[43]

$$-\frac{i}{2\mathcal{N}}\, e\, g_{ij}\, (f^A\bar{\chi})^i\gamma^\mu\, \mathcal{D}_\mu(f^A\chi)^j, \tag{5.27}$$

where $\mathcal{D}_\mu$ is spin–connection, target Christoffel, and target $\mathfrak{so}(\mathcal{N})_R$ covariant

$$\mathcal{D}_\mu(f^A\chi)^i = \big(\partial_\mu + \frac{1}{2}\omega_\mu^m\gamma_m\big)(f^A\chi)^i + \partial_\mu\phi^j\Big\{\Gamma^i_{jk}(f^A\chi)^k + Q_i^{AB}(f^B\chi)^i\Big\}. \tag{5.28}$$

Consistency of the kinetic terms requires

$$\mathcal{D}_\mu(f^A\chi)^i \equiv f^A\,\mathcal{D}_\mu(f^1\chi)^i, \tag{5.29}$$

that is[44]

$$\nabla_i f^A + Q_i^{AB} f^B \equiv Q_i^{1b} f^A f^b. \tag{5.30}$$

The $A = a$ component of this equation can be recast in the form

$$\nabla_i\Sigma^{1a} + Q_i^{ab}\,\Sigma^{1b} + Q_i^{1b}\,\Sigma^{ba} = 0. \tag{5.31}$$

But this is just a component of a generally covariant equation. So we must have in full generality

$$\mathcal{D}_i\Sigma^{AB} \equiv \nabla_i\Sigma^{AB} + Q_i^{AC}\,\Sigma^{CB} + Q_i^{BC}\,\Sigma^{AC} = 0, \tag{5.32}$$

that is the 2–forms Σ_{ij}^{AB}'s — and hence the curvature P_{ij}^{AB} — *are covariantly constant with respect to the Christoffel + $SO(\mathcal{N})_R$ connections.* In particular, the Bianchi identity (5.26) is verified. This is the condition which replaces, in the local case, the rigid SUSY requirement that the f^A's (and hence *a fortiori* the Σ^{AB}'s) are *parallel* (in the Riemannian sense).

As a consequence of $\mathcal{D}_i\Sigma^{AB} = 0$, all tensors constructed out of the Σ^{AB}'s which are $SO(\mathcal{N})_R$ singlets are *parallel* for the Christoffel connection. This remark applies, in particular, to the central tensor[45] $\mathbf{K}$ of $\mathcal{N} = 4$ SUGRA.

Finally, note that the χ^i's kinetic terms may be written, canonically but less covariantly, in the form

$$-\frac{i}{2}\, e\, g_{ij}\, \bar{\chi}^i\gamma^\mu\, \widetilde{\mathcal{D}}_\mu\chi^j, \tag{5.33}$$

where (cfr. eqn.(5.29))

$$\widetilde{\mathcal{D}}_\mu\chi^i = D_\mu\chi^i + \partial_\mu\phi^j\big(\Gamma^i_{jk}\chi^k + Q_i^{1a}{f^a}^j{}_k\big)\chi^k. \tag{5.34}$$

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[43] $e \stackrel{\text{def}}{=} \det e_\mu^a \equiv \sqrt{-h}$.

[44] ∇_i stands for the covariant derivative with respect to the Christoffel symbols of the target Riemannian space $\mathcal{M}$.

[45] $\mathbf{K}$ is defined as follows. Recall that, for $\mathcal{N} = 4k$, one has

$$T\mathcal{M} \simeq \mathcal{S}\otimes\mathcal{U} \,\oplus\, \widetilde{\mathcal{S}}\otimes\widetilde{\mathcal{U}}.$$

Then $\mathbf{K}$ is the element of $\mathrm{End}(T\mathcal{M})$ which is $+1$ on the first direct summand and -1 on the second. For $\mathcal{N} = 4$, $\mathbf{K}$ is proportional to

$$\mathrm{Pf}(\Sigma^{AB}) \equiv \epsilon_{ABCD}\Sigma^{AB}\Sigma^{CD} \in \mathrm{End}(T\mathcal{M}).$$

5.2.3. *The 4–Fermi coupling.* We repeat in the present $D = 3$ context the exercise we did in §. 2.2 for the $D = 2$ σ–model. The most general form of the coupling of 4 spin–1/2 fields is

$$A_{ijkl}[g]\ \bar{\chi}^i\gamma^\mu\chi^k\ \bar{\chi}^j\gamma_\mu\chi^l \tag{5.35}$$

for some covariant tensor $A_{ijkl}[g]$ on $\mathcal{M}$, depending only on the metric g and its derivatives, and having the following properties:

$$A_{ijkl} = -A_{jikl} = -A_{ijlk} = A_{klij} \tag{5.36}$$

$$\nabla_i A_{jklm} + \nabla_j A_{kilm} + \nabla_k A_{ijlm} = 0. \tag{5.37}$$

We replace the metric by $g_{ij} \to \lambda\, g_{ij}$ and take the limit $\lambda \to 0$, that is *infinite* σ–model coupling. In this limit the gravitational interactions are negligible, and we should recover our old result. Recalling the scaling properties of the various polynomials in the curvatures and their covariant derivatives (see §. **1.**1.1.1) we see that the general form of the 4 χ^i's coupling should be

$$A_{ijkl}[g] = -\frac{1}{4!} e\, R_{ijkl} + a\, e\, \big(g_{ik}g_{jl} - g_{il}g_{jk}\big), \tag{5.38}$$

for some *universal* constant a. Although this equation is quite important for us, the actual value of a is not that interesting from the geometric viewpoint (it can be fixed to any desidered value by a rescaling of the metric). Therefore we shall not loose our precious time in computing it. For the curious, $a = 1/16$ in some standard convention [**49**]. In order not to leave the impression that a cannot be determined geometrically, I leave the determination as an exercise (with a hint).

EXERCISE 5.1 (for the unbeliever). Predict a from geometry. HINT: take the metric to be the standard metric on $P\mathbb{C}^N$ with the God given normalization[46] (volume) $= (4\pi)^N/N!$ [*e.g.* $\mathrm{Vol}(P\mathbb{C}^1) \equiv \mathrm{Vol}(S^2) = 4\pi$], and replace it in eqn.(5.38). What happens for $a = 1/6$? *Does a bell ring?*

5.2.4.∗ *The other couplings.* It remains to determine a few couplings. They all belong to $\mathcal{L}_{\text{inter.}}$, contain at least one gravitino ψ^A_μ, and can be obtained from integrating eqn.(5.9). To do this, define the supercovariant derivative of ϕ^i to be[47]

$$\widehat{\partial}_\mu\phi^i := \partial_\mu\phi^i - \frac{1}{2}\bar{\psi}^A_\mu (f^A\chi)^i. \tag{5.39}$$

The *'properly supercovariantized supercurrent'* in the RHS of eqn.(5.9) is, quite obviously, the rigid supercurrent with the replacement $\partial_\mu\phi^i \to \widehat{\partial}_\mu\phi^i$. Then the integral is

$$\mathcal{L}_{\text{inter}} = \frac{1}{4}\, e\, g_{ij}\, (f^A\bar{\chi})^i\gamma^\mu\gamma^\nu\big(\partial_\nu\phi^j + \widehat{\partial}_\nu\phi^j\big)\psi^A_\mu. \tag{5.40}$$

This completes the Lagrangian.

[46] See sec.6 for why God gave us precisely this normalization.

[47] That is the derivative which commutes with local SUSY transformations.

5.3. Summary of results. It is time to summarize the results for *local* $\mathcal{N}$–SUSY in $D = 3$.

We wrote the most general Lagrangian in absence of vector fields and scalar potential. However, from the absolute generality of the arguments, it is obvious that *our findings are valid even in presence of vectors (after they have being dHS dualized to the CS form)*, except that — if gauge couplings are presents — the scalars' manifold $\mathcal{M}$ should realize the gauge group G as a group of isometries. The gauge coupling is then obtained by adding the gauge connection in the covariant derivatives, trough the corresponding Killing vectors K^i, as we discussed in §. 1.2.1 of chapt. 1. For $\mathcal{N} > 1$ the SUSY completion of the gauge interactions also require (specific) Yukawa couplings and a scalars' potential $V(\phi)$ (and, of course, the appropriate Chern–Simons terms). For the rest, *the kinetic terms and the* 4 χ^i *interactions are not affected by the gaugings* (either believe me, or prove it as an easy exercise). And these are the couplings we are momentarily interested in. This should not be interpreted as saying that in the gauge (or Yukawa) couplings there is no interesting differential geometry: on the contrary, the interplay between the Killing vectors K^i, generating the gauge transformations, and the 'parallel' tensors f^A and Σ^{AB} is one of the deepest tricks in the trade; this piece of differential geometry is crucial to understand which gaugings, Yukawas and potentials are allowed in a $\mathcal{N}$–extended SUSY/SUGRA. But before adressing that (more advanced) aspect of the geometry, we have to understand more f^A and Σ^{AB} as *geometric structures* on $\mathcal{M}$. We certainly shall return to the gauged version of the theory when armed with weapons powerful enough (in chapters 7 and 8).

Anyhow, here is what we learned in this section:

GENERAL LESSON 5.1. *Let $\mathcal{M}$ be the scalars' manifold of a 3D theory with $\mathcal{N}$–extended* local *supersymmetry (in its 'canonical' form, that is all vectors dHS dualized to CS couplings). There exists a $Spin(\mathcal{N})_R$ principal bundle $\mathcal{P} \to \mathcal{M}$, with a* covariantly constant *curvature, P_{ij}^{AB}*

$$\mathcal{D}_i \, P_{jk}^{AB} = 0. \tag{5.41}$$

The endomorphisms

$$\mathbf{1} := \partial_i \otimes d\phi^i, \;\; \textit{and} \;\; L^{AB} := P^{AB\,i}{}_j \, \partial_i \otimes d\phi^j \in \mathrm{End}(T\mathcal{M}) \tag{5.42}$$

generate a subalgebra of $\mathrm{End}(T\mathcal{M})$ *isomorphic to the* even *Clifford subalgebra $\mathbb{C}l^0(\mathcal{N})$; under this isomorphism the curvature endomorphisms L^{AB} are mapped to the standard generators $\frac{1}{2}\Gamma^{AB}$ of $\mathfrak{spin}(\mathcal{N})$. Hence*

$$T\mathcal{M} \simeq \begin{cases} S(\mathcal{P}) \otimes \mathcal{U} & \mathcal{N} \not\equiv 0 \mod 4 \\ S(\mathcal{P}) \otimes \mathcal{U} \oplus \widetilde{S}(\mathcal{P}) \otimes \widetilde{\mathcal{U}} & \mathcal{N} \equiv 0 \mod 4, \end{cases} \tag{5.43}$$

and this decomposition is preserved by the parallel transport *on* $\mathcal{M}$. *For* $\mathcal{N}=0 \mod 4$*, the two* $Spin(\mathcal{N})$ *bundles have opposite Euler classes*[48]

$$e\big(S(\mathcal{P})\big)=-e\big(\widetilde{S}(\mathcal{P})\big). \tag{5.44}$$

Finally, the 4 χ^i*'s coupling has the form*

$$A_{ijkl}[g]=-\frac{1}{4!}e\,R_{ijkl}+a\,e\,\big(g_{ik}g_{jl}-g_{il}g_{jk}\big). \tag{5.45}$$

If you don't understand something in the above, *don't worry!* This is just motivation (that is *propaganda*). We shall return to this structure when studying the geometry of the problem, and then everything will be clearer (I hope).

Anyhow, SUGRA is quite of a geometrical structure!!

6.* *Hey, this is Algebraic Geometry!*

I cannot resist to state here a first consequence of our GENERAL LESSON 5.1. We shall return to it in Part 2 (and, more in detail, in Part 3), so don't worry if you do not grasp the technicalities. It is the *message* which matters.

In case someone has got the impression that GENERAL LESSON 5.1 is not an astonishing and powerful result, I just state without a proof (it will be given in Part 3; the curious *conoscenti* may find a sketch of it in the footnote[49]) what it says in the *simplest possible case,* namely $\mathcal{N}=2$ and $\mathcal{M}$ compact. The result is true, word–for–word, also in $D=4$ $\mathcal{N}=1$ SUGRA. In fact, already in the simplest situation, the corollary to GENERAL LESSON 5.1 contains some of the major achievements of mankind in thirty centuries of geometry:

COROLLARY 6.1. *Let* $\mathcal{M}$ *be the scalars' manifold of a* 3*D* $\mathcal{N}=2$ SUGRA *(in canonical form*[50]*). Assume* $\mathcal{M}$ *compact. Then* $\mathcal{M}$ *is a* projective algebraic variety *(over* $\mathbb{C}$*). Conversely, all projective algebraic varieties are target spaces of some* $\mathcal{N}=2$ SUGRA.

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[48] *Proof.* The Euler class of the first bundle is proportional to the class of the $\mathcal{N}$–form $\epsilon_{A_1A_2\cdots A_\mathcal{N}}\rho^{A_1A_2}\wedge\cdots\rho^{A_{\mathcal{N}-1}A_\mathcal{N}}$, where $\rho^{AB}=P^{AB}|_{S(\mathcal{P})}$. This form is proportional to $\epsilon_{A_1A_2\cdots A_\mathcal{N}}\sigma^{A_1A_2}\wedge\cdots\sigma^{A_{\mathcal{N}-1}A_\mathcal{N}}$ where $\sigma^{AB}=\Sigma^{AB}|_S$. σ^{AB}, viewed as an element of $\mathrm{End}\big(S\otimes\mathcal{U}\big)$, has the form $\Gamma_S^{AB}\otimes\mathbf{1}$ with Γ_S^{AB} the generators of $\mathbb{C}l^0(\mathcal{N})$ acting on the irreducible module S. The Euler class for $\widetilde{S}(\mathcal{P})$ is obtained from that of $S(\mathcal{P})$ by interchanging the two irreducible representations. Explicitly, this amounts to the 'parity' transformation ([**74**], pro.(5.5))

$$\Gamma^{1a}\leftrightarrow-\Gamma^{1a}\qquad\Gamma^{ab}\leftrightarrow\Gamma^{ab}.$$

From the previous formulae, it is obvious that e is 'parity–odd' (see the explicit ϵ tensor). Hence

$$e(S)=-e(\widetilde{S}).$$

[49] *Proof.* For $\mathcal{N}=2$, $SO(\mathcal{N})$ is Abelian. The curvature of the $SO(2)$ bundle $V(\mathcal{P})$, *i.e.* the 2–form $P\equiv P^{12}$, is closed by the Abelian Bianchi identity. From the GENERAL LESSON, the 2–form $\omega_{ij}\equiv g_{ik}f^k{}_j$, being proportional to P, is also closed. Since $f^2=-1$, f is an (integrable) complex structure, and ω is a Kähler form. It is proportional to the curvature of the $SO(2)$ bundle $V(\mathcal{P})$, and hence $\mathcal{M}$ is Hodge. Then the corollary follows from the Kodaira embedding theorem [**76, 77**].

[50] This assumption may be weakened.

Explicitly, there is an embedding

$$\kappa \colon \mathcal{M} \hookrightarrow P\mathbb{C}^M, \tag{6.1}$$

[that is a smooth (in fact holomorphic) map which is an isomorphims onto its image], such that the metric on $\mathcal{M}$ is the pull–back of *twice*[51] the canonical one on $P\mathbb{C}^N$.

For those who think this to be just a math fancy without physical interest, I present three different physical interpretations of the corollary.

Remark (physical interpretation 1). The corollary states that the Newton constant is quantized (at least for $\mathcal{M}$ compact). See ref. [**78**] (they discuss it in the context of $D = 4$, $\mathcal{N} = 1$ sugra).

Remark (physical interpretation 2). The quantization above is, in fact, Dirac's monopole charge quantization. The curvature $P \equiv P^{12}$ may be seen as an Abelian field–strength in target space $\mathcal{M}$, and its flux on a closed surface $\mathcal{C} \subset \mathcal{M}$ measures the number of monopoles trapped inside $\mathcal{C}$. Then corollary describe the magnetic monopole configuration in target space.

Remark (physical interpretation 3). By a theorem of Chow, all (analytic) submanifolds of $P\mathbb{C}^M$ are algebraic, that is the zero loci of a finite numner of homogeneous polynomials $P_r(Z^I)$, $r = 1, 2, \dots, R$, $I = 1, 2, \dots, M$. Then all $\mathcal{N} = 2$ sugra's with $\mathcal{M}$ compact can be written as the model with target space $P\mathbb{C}^M$, provided we add a superpotential[52] of the form

$$W = \sum_r \Lambda_r \, P_r(Z^I),$$

where the Λ's are Lagrange multiplier (chiral) superfields. What is the physical point? Think of a physical model with superpotential

$$W = \sum_r \big\{ M_r \, \Lambda_r^2 + \Lambda_r \, P_r(Z^I) \big\},$$

where the M_r are *large* masses. The vacuum moduli has a branch

$$\Lambda_r = P_r(Z^I) = 0$$

on which the Λ_r are *very* massive. The (low energy) physics on this branch is equivalent to that of the original model with target space $\mathcal{M}$. That is: the geometrical structure is precisely the one needed in order to reproduce itself at different energy scales.

Mathematically, the fact that $\mathcal{M}$ is projective is quite deep.

Remark. The situation described in this section looks similar to the one in §. 4.1 of chapt. 1 (the 'modular' map). In fact we have

$$\Sigma \xrightarrow{\ \Phi\ } \mathcal{M} \xrightarrow{\ \kappa\ } \frac{SU(M+1)}{U(1) \times SU(M)} \simeq P\mathbb{C}^M, \tag{6.2}$$

[51] The canonical metric is the $U(1)$ curvature contracted with the complex structure f. In the present case, P_{ij} is *one–half* f, so the metric is twice the curvature.

[52] Cheating here! I have not defined it! (But the reader understands, I suppose...).

hence in both cases we have that the target space $\mathcal{M}$ is embedded in some symmetric space G/H, and the metrics, connections, bundles, etc. are obtained from those of G/H *via* pull–back. The geometry of G/H is completely specified in terms of the Lie groups G and H, so one has only to understand which maps κ's are allowed. This 'functorial' approach into the Lie group category is one of the most promising avenues toward a full (geometric) understanding of SUGRA.

REMARK. Unfortunately, despite the above beautiful characterizations, the interesting models for the phenomenological applications (of $D = 4$ $\mathcal{N} = 1$ SUSY) have necessarily *non–compact* $\mathcal{M}$. In fact, on a compact $\mathcal{M}$ there is no *non–trivial* superpotential W. (For the *cognoscenti:* let $\mathcal{L}$ denote the very ample bundle whose curvature is the properly normalized Kähler form. SUSY requires W to be a holomorphic section of $\mathcal{L}^{-3}$, and hence $W = 0$ by the Kodaira vanishing theorem).

7.∗ Conformal supersymmetry ($D = 3$)

This is an aside, also missplaced. It would be a natural part of chapt.... since it has little to do with SUSY and it hold even for $\mathcal{N} = 0$. But, perhaps, it is worth to discuss here, before we leave our $D = 3$ *diet* supersymmetry. We discuss it in the context of $\mathcal{N} = 1$; we shall generalize it to extended SUSY after having constructed the relevant geometric machinery.

STILL TO BE WRITTEN

8. Supersymmetry in $D = 4, 6$ dimensions

Sugar, finally! Of course, we are mainly interested in $D \geq 4$ physics. To justify our previous discussions, we have to show that the structures we found in $D = 3$ apply — with few modifications — in all dimensions.

From the properties of the supercharges in the diverse dimensions it is manifest that the most interesting dimensions D are those at which we have a *jump of structure,* namely $D = 3, 4, 6, 10$. Of course, we are interested in other dimensions as well, but we focus first on these ones.

From the discussion in §. 1.2, we expect that SUSY in *four* and *six* dimensions behaves much like in $D = 3$, except that now the underlying geometry is complex, or respectively quaternionic, in nature. This is particularly evident in the σ–models (*i.e.* models with only scalar and spin–1/2 fields). We begin our discussion with these models. Then, in sect. 9 we add the vector fields and analyze the geometry of the SUSY gauge theories.

* * *

We know from SUSY representation theory that the σ–models are possible in $D = 4$ only for $\mathcal{N} = 1, 2$, in $D = 6$ only for $(\mathcal{N}_L, \mathcal{N}_R) = (2, 0)$, and in $D = 10$ never. However, here we pretend *ignorance* and study general $\mathcal{N}$'s. It is for the geometry to say, *a posteriori,* which $\mathcal{N}$ may be realized and which not.

8.1. Six dimensional σ–models. The basic fermions in $D = 6$ are symplectic–Majorana–Weyl. To make the SUSY automorphism group $\mathrm{Aut}_R \equiv Sp(\mathcal{N})_R$ manifest, we choose a symplectic basis for the spin–1/2 fields. In such a basis the ψ^m_α's ($m = 1, 2, \dots, 2M$) satisfy the reality condition[53]

$$(\psi^m_\alpha)^\dagger \stackrel{\text{def}}{=} \bar{\psi}_{\dot{\alpha}\, m} = \Omega_{mn}\, B_{\dot{\alpha}}{}^{\beta}\, \psi^n_\beta. \tag{8.1}$$

where Ω_{mn} is a *constant* $2M \times 2M$ symplectic matrix. The Fermi kinetics terms then have the canonical form

$$-\frac{i}{2}\Omega_{mn}\, \psi^m_\alpha\, \partial^{\alpha\beta}\, \psi^n_\beta. \tag{8.2}$$

In the same way, we take a symplectic basis for the supercharges and SUSY parameters

$$(\epsilon^A_{\dot{\alpha}})^\dagger \stackrel{\text{def}}{=} \bar{\epsilon}_{\alpha\, A} = \epsilon_{AB}\, B_\alpha{}^{\dot{\beta}}\, \epsilon^B_{\dot{\beta}}. \tag{8.3}$$

Let ϕ^i be a set of (real) scalar fields which parameterize (locally) the scalar's manifold $\mathcal{M}$. We must have[54]

$$\delta\phi^i = \gamma^i_{A\,m}(\phi)\, \bar{\epsilon}^A \psi^m \tag{8.4}$$

for certain field–dependent coefficients $\gamma^i_{A\,m}$. The reality condition implies

$$(\gamma^i_{A\,m})^* = \epsilon^{AB}\, \Omega^{mn} \gamma^i_{B\,n}. \tag{8.5}$$

Then the supercurrent has the form[55]

$$\Gamma^\mu \Gamma_\nu \psi^m\, \gamma^i_{A\,m}\, g_{ij}\, \partial_\nu \phi^j + \dots \tag{8.6}$$

whose SUSY variation implies the Clifford–like property (by the same argument we used in the $D = 3$ case)

$$\gamma^i_{A\,m}\, g_{ij}\, \gamma^j_{B\,n} + \gamma^i_{B\,m}\, g_{ij}\, \gamma^j_{A\,n} = 2\, \epsilon^{AB}\, \Omega_{mn}. \tag{8.7}$$

The variation $\delta\phi^i$ transforms under a diffeomorphism as a section of $T\mathcal{M}$. Then, geometrically, $\gamma^i_{A\,m}$ is a bundle map

$$\mathcal{S} \otimes \mathcal{U} \to T\mathcal{M}, \tag{8.8}$$

where $\mathcal{S}$ is a rank $\mathcal{N}$ vector bundle related to the index A, and $\mathcal{U}$ is a rank $2M$ bundle associated with the index m.

The reality condition (8.5) requires the structure groups of these two bundles to be contained in $Sp(\mathcal{N})$ and $Sp(2M)$, respectively. The SUSY generators carry the index A; thus the structure group $Sp(\mathcal{N})$ of $\mathcal{S}$ is, in fact the automorphism group of the $D = 6$, *left* SUSY algebra.

Eqn.(8.7) says that the map (8.8) has rank $2\mathcal{N}M$. Of course this is a contradiction, unless $\mathcal{N} = 2$, since each pair of symplectic–Majorana–Weyl spinors is equivalent to *four* real degree of freedom, so the equality of

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53 We adhere to the standard SUGRA convention that Hermitian conjugation is represented by raising/lowering the $Sp(2M)$ indices.

54 One raises and lowers the indices with the rules $\chi^A = \epsilon^{AB}\chi_B$, $\chi_A = \epsilon_{BA}\chi^B$, $\epsilon_{AB}\epsilon^{BC} = -\delta_A{}^C$ and the same rules for the indices m, n using the matrix Ω_{mn}.

55 To avoid confusion with the $\gamma^i_{A\,m}$'s, we write Γ_μ for $D = 6$ Dirac matrices.

Bose/Fermi degrees of freedom requires $\dim\mathcal{M} = 4\,M$ and hence $\mathcal{N} = 2$. Then (8.8) becomes an isomorphism

$$T\mathcal{M} \simeq \mathcal{S}\otimes\mathcal{U}. \tag{8.9}$$

Therefore $\gamma^i_{A\,m}$ is like a vielbein in General Relativity, converting the 'curved' index i into the flat *bi*–index $A\,m$, except that here the bundle structure group is also reduced from the $SO(4M)$ to the subgroup $Sp(2)\times Sp(2M)$, as it is manifest from the fact that the metric in 'flat' indices is (cfr. eqn.(8.7))

$$g_{ij}\,\gamma^i_{A\,m}\gamma^j_{B\,n} = \epsilon_{AB}\,\Omega_{mn}, \qquad \epsilon^{AB}\,\Omega^{mn}\gamma^i_{A\,m}\gamma^j_{B\,n} = g^{ij}. \tag{8.10}$$

8.1.1. *Parallel structures.* In rigid SUSY the R–group $Sp(\mathcal{N})$ cannot be gauged. Hence, by the GENERAL LESSON 2.3 of chapt. 1, $\mathcal{S}$ should correspond to a *parallel structure.* The $Sp(2)_R$ connection vanishes, and all flat–index tensors of the form (say)

$$T_{A_1A_2\cdots A_k}\,\Omega_{m_1n_2}\cdots\Omega_{m_rn_r}, \qquad (T_{A_1A_2\cdots A_k}\ \text{a numerical tensor})$$

are *covariantly constant.*

This, in particular, applies to the tensors $L^a_{AB}\,\Omega_{mn}$, where $L^a_{AB} = L^a_{BA}$ ($a = 1,2,3$) are the generators of $\mathfrak{sp}(2)\simeq\mathfrak{su}(2)$ in the defining representation. Converting to curved indices with the help of the γ's, we construct 3 *covariantly constant* (hence closed) 2*–forms*

$$(\omega^a)_{ij} = \gamma_i^{\ A\,m}\,\gamma_j^{\ A\,m}\,L^a_{AB}\,\Omega_{mn}. \tag{8.11}$$

The situation is very much like the one in $D = 3$, with the parallel 2–forms Σ^{AB} associated with the generators of $\mathfrak{so}(\mathcal{N})_R$ replaced by the parallel 2–forms ω^a associated with the generators of $\mathfrak{sp}(2)_R$. Indeed, apart for the fact that only $\mathcal{N} = 2$ is consistent, the structure we find is *exactly the same* as in $D = 3$, with the $SO(\mathcal{N})_R$ structures (bundles, covariantly constant forms, *ect.*) replaced by $Sp(\mathcal{N})_R$ structures. This exactly corresponds to the philosophical principle that we pass from one space–time dimension to the other by a change of the 'ground field' $\mathbb{R}\leftrightarrow\mathbb{H}$.

8.2. Four dimensions. Now the general pattern should be clear. In $4D$ we have complex structures replacing the real ones: Weyl fermions and $U(\mathcal{N})_R$ automorphism groups instead of Majoranas and $SO(\mathcal{N})_R$'s. Indeed, $D = 4$ corresponds to $\mathbb{C}$ in the $\mathbb{R}\leftrightarrow\mathbb{C}\leftrightarrow\mathbb{H}\leftrightarrow\mathbb{O}$ sequence.

The isomorphisms in GENERAL LESSON 5.1 is replaced by

$$T\mathcal{M}\otimes\mathbb{C}\simeq\mathcal{S}\otimes\mathcal{U}\ \oplus\ \overline{\mathcal{S}}\otimes\overline{\mathcal{U}} \tag{8.12}$$

where $\mathcal{S}$ is a *flat*[56] $U(\mathcal{N})_R$ bundle and $\overline{\mathcal{S}}$ is its complex conjugate. Corresponding, we expect a set of $\mathcal{N}^2$ *parallel* two–forms, $\omega^a_{ij}\,d\phi^i\wedge d\phi^j$, with the property that the associated endomorphisms $\omega^{a\,i}{}_j$ generate the Lie algebra $\mathfrak{u}(\mathcal{N})_R\subset\mathrm{End}(\mathcal{S})$. In fact the story is slightly subtler, since the actual tangent bundle is not $T_{\mathbb{C}}\mathcal{M} := T\mathcal{M}\otimes\mathbb{C}$ but a real subspace. By construction,

[56] More precisely, the $SU(\mathcal{N})_R$ part of the connection is flat. The $U(1)_R$ is not restricted, by the same reasons we discussed in $D = 3$ for the $Spin(2)_R\simeq U(1)_R$. The remark in this FOOTNOTE is *absolutely crucial.*

$T_{\mathbb{C}}\mathcal{M}$ is a complex vector bundle with a natural real structure; recall the definition 1.1: a real structure is an *anti*linear involution

$$\varrho\colon T_{\mathbb{C}}\mathcal{M} \to T_{\mathbb{C}}\mathcal{M}, \qquad \varrho^2 = 1. \tag{8.13}$$

The real tangent space $T\mathcal{M} \equiv [[T_{\mathbb{C}}\mathcal{M}]]$ is the ϱ invariant subspace of $T_{\mathbb{C}}\mathcal{M}$. The actual parallel two–forms correspond to the generators of $\mathfrak{u}(\mathcal{N})_R$ which are *ϱ–even*

$$\varrho^*\omega^a = \omega^a, \tag{8.14}$$

whereas the ϱ–odd ones are 'projected out'.

EXAMPLE. Consider an $\mathcal{N} = 2$ σ–model in $D = 4$ obtained from a $D = 6$ one by dimensional reduction. Since (in a pure σ–model) $\mathcal{M}$ is invariant under dimensional reduction, from the discussion in §. 8.1 we learn that the target space has a parallel $Sp(2)$ structure. We have, in general, only three parallel forms, transforming in the adjoint of $Sp(2)$. Given $Sp(2) \simeq SU(2)$ we recover the correct $D = 4$ structure, but there is no parallel form associated to the $\mathfrak{u}(1)$ generator: it has been projected out by the reality condition, which in this case is the quaternionic one, eqn.(8.1).

REMARK. As the previous example illustrates, the parallel structures of a σ–model should be invariant under dimensional reduction. In (Q)FT dimensional reduction is a 'structure transporting map', like a functor in category theory. In the case at hand, by pushing down the $D = 4$ models to three dimensions, we learn that the $U(\mathcal{N})$ structures should be equivalent to the $SO(2\mathcal{N})$ ones. The embedding $U(\mathcal{N}) \hookrightarrow SO(2\mathcal{N})$, by itself, is not enough since, in $D = 3$, the flat two–forms should make a *complete* representation of the *full* $SO(2\mathcal{N})$. This constraint has only *two solutions:* $\mathcal{N} = 1$, thanks to the isomorphism $U(1) \simeq SO(2)$, and $\mathcal{N} = 2$. In the second case the three parallel forms coming from $D = 4$ make the $(\mathbf{3}, \mathbf{1})$ representation of $SO(4) \simeq SU(2) \times SU(2)$.

This result, of course, corresponds to the well–know fact that, in $D = 4$, SUSY σ–models exist only for $\mathcal{N} = 1, 2$.

9. Susy gauge theories in $D = 4$, 5

Theories with vectors introduce additional geometric structures, as we saw already at the purely bosonic level in §. 4.1 of chap. 1. Here we limit ourselves to the most interesting case (geometrically, physically and phenomenologically), namely $D = 4$ (with an addendum about $D = 5$). We shall return to vectors in general dimensions, in Part 3, when armed will all the necessary tools. Recall that the goal of the present chapter is merely to introduce and motivate the necessity of the tools themselves.

It is convenient to analyze the basic geometric structures with the gauge coupling constants set to zero (that is all vectors are taken to be Abelian). Having done that, it will be relatively easy to switch on the YM coupling g. In fact the gauge interactions are themselves described by geometrical objects of the $g = 0$ theory: Killing vectors, momentum maps, ect., as we saw in §. 1.2.1 of chapt. 1 and will seen in detail in chapters 7 and 8.

The field content of the model is: scalars ϕ^i $(i = 1, 2, \dots, n)$, spin–1/2 (Weyl) fermions ψ^m_α $(m = 1, 2, \dots, F)$ and (Abelian) vectors A^u_μ $(u = 1, 2, \dots, V)$. Of course, we must have $F = V + 2\,n$.

9.1. Gauge–matter splitting. As the reader already knows from the superspace approach, or from the theory of SUSY representations on fields (supermultiplets), in general there may be two kinds of spin–1/2 fields: those in the same supermultiplet with the vectors *(gauge* supermultiplets), and those belonging to supermultiplets containing only spin–0 and spin–1/2 fields *(matter* supermultiplets). In the geometric language, the vectors' SUSY transformations, $\delta A^u_\mu = \bar{\epsilon}_A \gamma_\mu \lambda^{A\,u}$, split the Fermi bundle $\mathcal{V} \to \mathcal{M}$ into the direct sum

$$\mathcal{V} \;\simeq\; \Lambda \oplus \Lambda^\perp \longrightarrow \mathcal{M},$$

where Λ is the bundle of the fermions $\lambda^{A\,u}$ appearing in the δA^u_μ *(gaugini)*, whereas $\Lambda^\perp$ corresponds to the fermions which are orthogonal to the $\lambda^{A\,u}$'s (with respect the metric defined by the kinetic terms).

The splitting $\Lambda \oplus \Lambda^\perp$ is preserved by the parallel transport on $\mathcal{M}$ (since SUSY is rigid, the decomposition into SUSY representations is also rigid). As in $D = 3$, we fix our attention on an $\mathcal{N} = 1$ subalgebra. The scalars' transformation, under this subalgebra, $\delta\phi^i = \bar{\epsilon}_1 \psi^i$ defines a bundle *monomorphims*

$$T\mathcal{M} \xrightarrow{\;j\;} \mathcal{V} \;\simeq\; \Lambda \oplus \Lambda^\perp \tag{9.1}$$

$$\Rightarrow \quad T\mathcal{M} \simeq \big(\Lambda/\mathrm{coker}\, j\big) \oplus \Lambda^\perp, \tag{9.2}$$

where we used the fact that $\Lambda^\perp \subset j(T\mathcal{M})$ in order to guarantee the positivity of the kinetic terms of the $\psi \in \Lambda^\perp$ (in down–to–earth language: these fermions are not the superpartners of a vector, hence they should be the superpartners of a scalar). Geometrically, the crucial point is that the splitting in eqn.(9.2) is preserved by the parallel transport on $\mathcal{M}$, for reasons we have already discussed many times.

This is quite a strong condition, geometrically speaking. We shall show[57] that it implies that the scalars' manifold is a Riemannian product

$$\mathcal{M} = \mathcal{M}_{\text{gauge}} \times \mathcal{M}_{\text{matter}} \tag{9.3}$$

with a direct sum metric

$$ds^2 = g^{(1)}_{mn}(x)\, dx^m dx^n + g^{(2)}_{ab}(y)\, dy^a dy^b \tag{9.4}$$

where $g^{(1)}$ (resp. $g^{(2)}$) is a metric on $\mathcal{M}_{\text{gauge}}$ (resp. $\mathcal{M}_{\text{matter}}$).

The geometry of $\mathcal{M}_{\text{matter}}$ is, of course, that of an $\mathcal{N}$–extended σ–model, already discussed in §. 8.

It remains to analyze $\mathcal{M}_{\text{gauge}}$. As the reader already knows from representation theory, SUSY gauge theories exist in $D = 4$ for three values of $\mathcal{N}$, namely 1, 2, and 4. The $\mathcal{N} = 1$ gauge supermultiplet has no scalars, so $\mathcal{M}_{\text{gauge}} =$ (a point), and there is no geometry to study[58]. We start with the $\mathcal{N} = 4$ case.

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[57] In chapter 3. See de Rham's theorem.

[58] This is not correct: although the metric geometry of $\mathcal{M}$ is just the one of an $\mathcal{N} = 1$ σ–model, the vectors' kinetic terms define a 'magnetic susceptibility' map

$$\mu\colon \mathcal{M} \to Sp(2V, \mathbb{R})/U(V) \;\simeq\; \mathfrak{H}_V \qquad \text{(Siegel's upper half–space)},$$

9.2. $\mathcal{N} = 4$ gauge theories. We consider only Lagrangians *quadratic* in the field–strenghts $F^u_{\mu\nu}$.

The $\mathcal{N} = 4$ geometry is so constrained that we have not to work much to determine it. Schematically, the SUSY tansformations should have the generic form

$$\delta A^u_{\alpha\dot\alpha} = \bar\epsilon_{\dot\alpha\, A}\, \lambda^{A\, u}_\alpha + \text{H.c.}$$
$$\delta\lambda^{A\, u}_\alpha = E_i^{\;AB\, u}(\phi)\, \partial_\alpha{}^{\dot\alpha}\phi^i\; \bar\epsilon_{\dot\alpha\, B} + D^{uv\, A}{}_B\, F^v_{\alpha\beta}\, \epsilon^{\beta\gamma}\, \epsilon^B_\gamma,$$

where $E_i^{\;AB\, u}(\phi)$ is the analogue of $\gamma_i^{A\, m}$ in §. 8.1. From the SUSY algebra and the reality of the A^v_μ's one gets[59].

$$E_i^{\;AB\, u} = -E_i^{\;BA\, u} \tag{9.5}$$

$$E_{i\, AB}{}^u \overset{\text{def}}{=} (E_i^{\;AB\, u})^* = -\frac{1}{2}\epsilon_{ABCD}\, E_i^{\;CD\, u}. \tag{9.6}$$

Just as in §. 8.1, we interprete this result as a bundle isomorphism

$$T\mathcal{M}_{\text{gauge}} \;\simeq\; \mathcal{S}_{\mathbf{6}} \otimes \mathcal{U} \tag{9.7}$$

where $\mathcal{S}_{\mathbf{6}}$ is the vector bundle associated to the representation **6** of $SU(4)_R \sim SO(6)$ and $\mathcal{U}$ is a real bundle. Again, $\mathcal{S}_{\mathbf{6}}$ is flat (otherwise we would be forced to gauge $SU(4)_R$, but that would contradict rigid SUSY) and hence for all constant symmetric 6×6 matrices $S_{AB\; CD}$ the symmetric tensor[60]

$$\hat S_{ij} := E_i^{\;AB\, u}\, E_j^{\;CD\, v}\, \mathcal{N}_{uv}\, S_{AB\; CD},$$

should be *parallel.* This is very strong constraint on the geometry of $\mathcal{M}$, and — as we shall see in chapt. 3 — it suffices to fix its metric and hence all the couplings. Hence, in this case, there is no need of a more in–depth analysis.

9.3. $\mathcal{N} = 2$ gauge theories in $D = 4$. Geometrically speaking, the rigid $\mathcal{N} = 2$ gauge theories in $D = 4$ are among the most interesting field theories. In these lectures we shall return to them may times (expecially in Part 3), looking at them from different points of view, and giving alternative interpretations of their remarkable geometrical structure. In the present section, we are just trying to motivate *which* are the relevant geometrical issues worth studying in order to construct and understand these SUSY gauge theories. The full-fledged theory is best discussed in the context of SUGRA — since there we have a larger supply of allowed $\mathcal{N}$'s to play with. Therefore, here I will construct the $\mathcal{N} = 2$ geometrical structures on $\mathcal{M}_{\text{gauge}}$ using an

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and one has to describe the geometric conditions on the map μ in order the model to be supersymmetric. We know that $T_{\mathbb{C}}\mathcal{M} \simeq T \oplus \bar T$; the same argument (applied to the action of $U(V)$ on $T\mathfrak{H}_V$) gives $T\mathfrak{H}_V = \mathfrak{h} \oplus \bar{\mathfrak{h}}$. The condition will turn out to be $\mu_* T \subset \mathfrak{h}$ (translation for the *cognoscenti:* μ is holomorphic).

[59] This is well–known from the linear theory. In all SUSY models based on a curved target space, the structures on the tangent space are just those of the linear theory. It follows from the 'target space equivalence principle', corl. 1.1 of chapt. 1.

[60] As in § 4.1 of chap. 1 $\mathcal{N}_{uv}$ is the matrix appearing in the vectors' kinetic terms.

ad hoc, poor man's style, approach. Unfortunately, the discussion is less elementary than I hoped, so the reader may prefer to skip it.

There are two fundamental structures. The first one is easily obtained by the 'forgetful trick': Forget one of the two supercharges, and consider the model just as an $\mathcal{N}=1$ supersymmetric theory. The result of §. 8.2. applies, hence

$$T_{\mathbb{C}}\mathcal{M} \simeq F \oplus \overline{F} \qquad (\text{where } F \simeq \mathcal{S}\otimes\mathcal{U}), \tag{9.8}$$

that is on $\mathcal{M}$ there is one *parallel* complex structure. Correspondingly, the cotangent bundle also splits $T^*_{\mathbb{C}}\mathcal{M} \simeq F^\vee \oplus \overline{F}^\vee$. Given a (complex) 1–form ξ, that is a section of of $T^*_{\mathbb{C}}\mathcal{M}$, we can decompose it as

$$\xi = \xi\big|_{F^\vee} + \xi\big|_{\overline{F}^\vee}. \tag{9.9}$$

$\xi\big|_{F^\vee}$ (resp. $\xi\big|_{\overline{F}^\vee}$) is called the $(1,0)$ [resp. $(0,1)$] part of the form ξ. We apologize for anticipating here some results from Parts 2 and 3: there we show that *parallel* (almost)complex structures are *integrable,* that is, there exist on $\mathcal{M}$ holomorphic functions z^i ($i=1,2,\dots,\dim\mathcal{M}/2$) such that their differentials dz^i (resp. $d\bar{z}^{\bar{i}}$) span $F^\vee$ (resp. $\overline{F}^\vee$), *i.e.* they form a basis for the $(1,0)$ (resp. $(0,1)$) differential forms. More generally, a differential form Ξ, of degree $n=p+q$, is called a form of *type* (p,q) if, in holomorphic coordinates, takes the form

$$\Xi_{i_i i_2\cdots i_p\,\bar{\jmath}_1\bar{\jmath}_2\cdots\bar{\jmath}_q}\, dz^{i_1}\wedge dz^{i_2}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{\bar{\jmath}_1}\wedge\cdots\wedge d\bar{z}^{\bar{\jmath}_q}.$$

The second structure is subtler. It is a *real* structure, and is best described from the real point of view. Thus we revert to Majorana (= Hermitean) supercharges[61].

The formula[62]

$$-i\epsilon_{AB}\; C^{\alpha\beta}\,\{Q^A_\alpha,[Q^B_\beta,A^x_\mu]\} = Q^x_i(\phi)\,\partial_\nu\phi^i \tag{9.10}$$

defines a set of V real 1–forms, $\theta^x \equiv Q^x_i(\phi)\,d\phi^i$ on $\mathcal{M}$, (here $x=1,2,\dots,V$).

In sect. 4.1 of chapt. 1, we learned that the theory has a dual formulation in terms of the dual vectors B_y (defined by $G_y = dB_y$). The dual of eqn.(9.10) reads

$$-i\epsilon_{AB}\; C^{\alpha\beta}\,\{Q^A_\alpha,[Q^B_\beta,B_{y\,\mu}]\} = P_{y\,i}(\phi)\,\partial_\nu\phi^i \tag{9.11}$$

with $\widetilde{\theta}_y \equiv P_{y\,i}\,d\phi^i$ is *another* set of V real 1–forms on $\mathcal{M}$. By linearity of eqns.(9.10)(9.11), the vector of 1–forms

$$\Theta^X \equiv \begin{pmatrix}\theta^x\\ \widetilde{\theta}_y\end{pmatrix} \tag{9.12}$$

transforms canonically under $Sp(2V,\mathbb{R})$

$$\begin{pmatrix}\theta^x\\ \widetilde{\theta}_y\end{pmatrix} \longrightarrow \mathbf{S}\begin{pmatrix}\theta^x\\ \widetilde{\theta}_y\end{pmatrix}, \qquad \text{where } \mathbf{S}\in Sp(2V,\mathbb{R}). \tag{9.13}$$

[61] C stands for the Majorana representation charge conjugation matrix. It satisfies $C^t=-C$.

[62] As the reader already knows, in the linerarized theory the RHS is proportional the gradient of a scalar belonging to the vector supermultiplet; in the curved theory we should allow for a vielbein–like tensor to convert 'curved' indices into 'flat' ones, exactly as we did in sect. 8 for the σ models.

Consider the associated (real) symplectic two–form

$$\omega := \theta^x \wedge \widetilde{\theta}_x. \tag{9.14}$$

As you learned in esercise 4.2 of chapt. 1, this symplectic structure cannot be degenerate, otherwise the energy–momentum tensor $T_{\mu\nu}$ would also be degenerate. Then $\omega^V \neq 0$. Since $\dim_{\mathbb{R}} T^*\mathcal{M} = 2V$, the the 1–forms Θ^X ($X = 1, 2, \dots, 2V$) must span the cotangent bundle $T^*\mathcal{M}$. Thus Θ^X defines a bundle isomorphism

$$T^*\mathcal{M} \;\simeq\; \mathcal{S}, \tag{9.15}$$

where $\mathcal{S}$ is a vector bundle with structure group $Sp(2V, \mathbb{R})$. It is a very special kind of bundle: it is a *flat* one, that is the curvature of its connection vanishes identically. This follows from the fact that only *rigid* $Sp(2V, \mathbb{R})$ rotations are allowed, *i.e.* the matrix $\mathbf{S}$ in eqn.(9.13) should be locally independent of the point[63] $\phi \in \mathcal{M}$; therefore the $Sp(2V, \mathbb{R})$ holonomy, $P\exp\big(-\int_C A_{Sp(2V,\mathbb{R})}\big)$, cannot change under a continuous deformation of the path C. Then, if C is contractible, the holonomy equals 1, which means that $A_{Sp(2V,\mathbb{R})}$ is locally *pure gauge* (= no curvature).

We write $\boldsymbol{\nabla}_i$ for the $Sp(2V, \mathbb{R})$ covariant derivative, and $\boldsymbol{\nabla} \equiv d\phi^i \boldsymbol{\nabla}_i$ for the associate exterior derivative acting on differential forms. The flatness condition is

$$\boldsymbol{\nabla}^2 = 0. \tag{9.16}$$

The coframe Θ^X is the symplectic analog of the usual orthogonal coframe (vielbein forms) of General Relativity (see APPENDIX A): here the usual orthogonal (symmetric) pairing δ_{ab} is replaced by the symplectic (skew–symmetric) one Ω_{XY}, and the $\mathfrak{so}(n)$ spin–connection 1–form by an $\mathfrak{sp}(2V, \mathbb{R})$ connection form $\boldsymbol{\varpi}^X{}_Y \in \Omega^1\big(\mathfrak{sp}(2V, \mathbb{R})\big)$ [this holds in a general symplectic coframe, in our PARTICULAR coframe Θ^X's the connection $\boldsymbol{\varpi}^X{}_Y$ vanishes by construction]. Then we have the symplectic versions of the Cartan's curvature and structural equations:

$$d\Theta^X + \boldsymbol{\varpi}^X{}_Y \wedge \Theta^Y = \boldsymbol{T}^X \qquad \text{(symplectic torsion)} \tag{9.17}$$

$$d\boldsymbol{\varpi}^X{}_Y + \boldsymbol{\varpi}^X{}_Z \wedge \boldsymbol{\varpi}^Z{}_Y = \boldsymbol{R}^X{}_Y \quad \text{(symplectic curvature).} \tag{9.18}$$

We already know that the symplectic curvature vanishes. What about the symplectic torsion?

LEMMA 9.1. *The symplectic torsion vanishes identically* $\boldsymbol{T}^X \equiv 0$.

There are many ways of seeing this. By far the simplest method is to use $\mathcal{N} = 2$ superfields. However let us sketch a poor man's argument showing that the vanishing of the symplectic torsion on $\mathcal{M}$ is just the push–forward to the target space $\mathcal{M}$ of gauge–invariance in space–time Σ. Again, $\Phi\colon \Sigma \to \mathcal{M}$ is the scalars' field configuration, seen as a map.

Indeed, eqn.(9.10) can be rewritten as ($A^x \equiv A^x_\mu\, dx^\mu$)

$$\Phi^*\theta^x = -i\epsilon_{AB}\, C^{\alpha\beta}\, \{Q^A_\alpha, [Q^B_\beta, A^x]\,\}. \tag{9.19}$$

[63] However we can (rather we should!!) have non–trivial monodromies at the global level.

The symmetry of the index contractions allows to rewrite the RHS, with the help of the super–Jacobi identity, in the form

$$\Phi^*\theta^x = -\frac{i}{2}\big[\epsilon_{AB}\, C^{\alpha\beta}\,\{Q^A_\alpha, Q^B_\beta\}, A^x\big].$$

The RHS is formally zero, since the anticommutator inside the bracket is a central charge, and a massless gauge vector cannot be charged under a central charge. The apparent paradox is easily solved: the SUSY algebra is realized only up to gauge transformations, that is only on *gauge invariant* operators. But then

$$\Phi^*\, d\theta^x \equiv d\Phi^*\theta^x = -\frac{i}{2}\Big[\epsilon_{AB}\, C^{\alpha\beta}\,\{Q^A_\alpha, Q^B_\beta\}, F^x\Big] \equiv 0,$$

since $F^x = dA^x$ *is* gauge–invariant. Then $d\theta^x = 0$, and, by $Sp(2V,\mathbb{R})$ covariance,

$$\mathbf{T}^X \equiv d\Theta^X \equiv d\begin{pmatrix}\theta^x\\ \widetilde{\theta}_y\end{pmatrix} = 0. \tag{9.20}$$

The two geometric structures should be compatible with each other. First of all, the parallel symplectic 2–form of the second structure, ω (see eqn.(9.14)), and the parallel symplectic 2–form $g_{ik}\, f^k{}_j\, d\phi^i \wedge d\phi^j$ should be the same (if properly normalized). This is not a big deal: geometrically, we cannot have two non–proportional *commuting* symplectic forms on an (irreducible) manifold. Physically, the contraints on the fermions' kinetic terms we get by relating them to the scalars' ones, should match with those we obtained by relating them to the vectors' kinetic terms. In particular, it is easy to see directly that the canonical symplectic two form ω should be of type $(1,1)$ with respect to the flat complex structure: in fact ω should remain invariant[64] under the action of the structure subgroup $U(1)_R$ of the bundle F in eqn.(9.8). Alternatively, one may extract this result from the consistency of the canonical form of the energy–momentum tensor with SUSY.

The second consistency condition is deeper. One may state it as the requirement

$$\boldsymbol{\nabla}\left(\Theta^X\big|_{(1,0)}\right) = 0, \tag{9.21}$$

know as the *special Kähler condition.*

COROLLARY 9.1 (to lemma 9.1). *The special Kähler condition holds.*

PROOF. From eqn.(9.19), we have

$$\Phi^*\left(\theta^x\big|_{(1,0)}\right) = -i\epsilon_{AB}\epsilon^{\alpha\beta}\,\{Q^A_\alpha, [Q^B_\beta, A^x]\,\},$$

where now Q^A_α denotes *Weyl* (2–components) supercharges. Again, the anticommutator inside the bracket is a central charge, and hence

$$\boldsymbol{\nabla}\left(\theta^x\big|_{(1,0)}\right) = d\left(\theta^x\big|_{(1,0)}\right) = 0.$$

□

REMARK. The $(1,0)$ form is $\boldsymbol{\nabla}$–closed but <u>not</u> $\boldsymbol{\nabla}_i$–parallel.

[64] I make the story short, we have to return to these geometrical requirements anyhow.

The geometry resulting from the above two structures, and their compatibility condition, is called *special Kähler geometry.* By the logic of the subject, we have to postpone its discussion after that of the of the nonspecial Kähler geometries. However, for future reference here we sumarize our finding and elaborate on some of their first consequences.

9.4.∗ A first glance to (rigid) special geometry.

SUMMARY 9.1. *On the manifold* $\mathcal{M}_{\text{gauge}}$ *of a* $D = 4$, $\mathcal{N} = 2$ *gauge theory we have the following:*

(1) *First structure:*
  (a) *a Riemannian metric* g, *with its Christoffel connection* ∇_i*;*
  (b) *a* parallel *complex structure* ${f^i}_j$ *satisfying*

$$g_{ik}{f^k}_j = -g_{jk}{f^k}_i, \quad {f^i}_k{f^k}_j = -{\delta^i}_j, \quad \nabla_k{f^i}_j = 0. \tag{9.22}$$

(2) *Second structure:*
  (a) *a* flat, torsionless $Sp(2V, \mathbb{R})$ *connection* $\boldsymbol{\nabla}_i$ *on* $T^*\mathcal{M}_{\text{gauge}}$ *(equivalently on* $T\mathcal{M}_{\text{gauge}}$*).*

(3) *Compatibility conditions:*
  (a) *the canonical parallel symplectic two form* ω *of* (2) *is equal to* $g_{ik}{f^k}_j\, d\phi^i \wedge d\phi^j$*;*
  (b) *(the special Kähler condition) Let* $\Pi_{(1,0)}$ *be the projector onto the* $(1, 0)$ *forms. Then*

$$\boldsymbol{\nabla}\Pi_{(1,0)} = 0. \tag{9.23}$$

This definition of what we mean by *special geometry* is expressed in the language of D. Freed ref.[**79**] (in a footnote he says to have learned it from ref.[**80**]).

As I have already said, we shall study these special geometries many times in the following, from different viewpoints and with different applications in mind. However, here we want to make two preliminary points. First point: the above geometry locally coincides with the one you would obtain using $\mathcal{N} = 2$ superfields in terms of a holomorphic pre–potential $\mathcal{F}$ (as you know from M. Bertolini's course). The second point is more suggestive: consider the second geometrical structure, the one inherited from the vectors. When invariantly considered this structure turns out to have the same, tt^*–like, form that we got in $D = 3$ *local* supersymmetry, and that we shall find also in $D = 4$ (or any other dimensions) SUGRA. In a general SUGRA theory, we shall find *two* such geometrical structures — one induced by the automorphism group of the SUSY algebra and the other induced by the (generalized) duality group. The two together will fix all the couplings. The fact that they are two manifestation of the same *abstract* geometry (which can be described in a countless number of ways and languages) is rather magic.

9.4.1. *The prepotential.* The fact that $\boldsymbol{\nabla}_i$ is both flat and torsionless implies that there exist (local) Darboux coordinates, namely local functions (q^x, p_y) such that $\theta^x = dq^x$, $\widetilde{\theta}_y = dp_y$ and

$$\omega = dq^x \wedge dp_x. \tag{9.24}$$

On the other hand, let z^i be holomorphic coordinates for $\mathcal{M}$. we must have $\Pi_{(1,0)}dq^x = A^x_i\, dz^i$, $\Pi_{(1,0)}dp_y = B_{i\,y}dz^i$, for certain functions A^x_i and $B_{i\,y}$. The special Kähler condition requires $\boldsymbol{\nabla}\Pi_{(1,0)}dq^x = \boldsymbol{\nabla}\Pi_{(1,0)}dp_y = 0$; decomposing into type we get

$$\overline{\partial}A^x_i = \overline{\partial}B_{i\,y} = 0, \tag{9.25}$$

$$\partial(A^x_i\, dz^i) = \partial(B_{i\,y}\, dz^i) = 0, \tag{9.26}$$

where $\overline{\partial}$, ∂ are the Dolbeault exterior differrentials[65]. Then $A^x_i\, dz^i$ and $B_{i\,y}dz^i$ are ∂–closed *holomorphic* forms. Then, by the Poincaré lemma [see footnote 65], there exist holomorphic functions $Z^x(z)$, $W_y(z)$ on $\mathcal{M}$ such that

$$\Pi_{(1,0)}dq^x \;=\; \frac{1}{2}\,dZ^x \quad\Rightarrow\quad dq^x = \mathrm{Re}(dZ^x) \tag{9.27}$$

$$\Pi_{(1,0)}dp_y = -\frac{1}{2}\,dW_y \quad\Rightarrow\quad dp_y = -\mathrm{Re}(dW_y). \tag{9.28}$$

The symplectic form $\omega = -\frac{1}{4}(dZ^x + d\overline{Z}^x) \wedge (dW_x + d\overline{W}_x)$ has pure type $(1,1)$ hence its $(2,0)$ part vanishes

$$dW_x \wedge dZ^x = 0 \quad\Rightarrow\quad d(W_x\, dZ^x) = 0 \quad\Rightarrow\quad W_x\, dZ^x = d\mathcal{F}, \tag{9.29}$$

for some holomorphic $\mathcal{F}$ function (again the Poincaré lemma). $\mathcal{F}$ is known as *the superpotential.* Choosing the Z^x's as the (local) holomorphic coordinates, we write

$$W_x = \frac{\partial\mathcal{F}}{\partial Z^x}, \tag{9.30}$$

and ω is

$$\omega = \frac{i}{2}\mathrm{Im}\left(\frac{\partial\mathcal{F}}{\partial Z^x\partial Z^y}\right) dZ^x \wedge d\overline{Z}^y, \tag{9.31}$$

which is the standard formula.

9.4.2. *The curvature.* The *flat* connection $\boldsymbol{\nabla}_i$ cannot be metric (in general) since it takes value in the Lie algebra $\mathfrak{sp}(2V,\mathbb{R}) \not\subset \mathfrak{so}(2V)$. However, the difference of two connections is a (Lie–algebra–valued) one–form, so we may write

$$\boldsymbol{\nabla}_i = \nabla_i + \widehat{\mathbf{A}}_i \qquad \widehat{\mathbf{A}} \in \Omega^1\big(\mathfrak{sp}(2V,\mathbb{R})\big). \tag{9.32}$$

Where ∇_i is the Christoffel connection. ω is parallel for both connections, ∇_i and $\boldsymbol{\nabla}_i$, and $\nabla\Pi_{(1,0)} = \boldsymbol{\nabla}\Pi_{(1,0)} = 0$.

The connection $\boldsymbol{\nabla} = d + \mathbf{A}$ has, in 'curved' indices and complex notation, the following components:

$$(\mathbf{A}_i)^j{}_k, \quad (\mathbf{A}_{\bar{\imath}})^j{}_k, \quad (\mathbf{A}_i)^{\bar{\jmath}}{}_{\bar{k}}, \quad (\mathbf{A}_{\bar{\imath}})^{\bar{\jmath}}{}_{\bar{k}}, \tag{9.33}$$

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[65] We shall discuss these operators in detail in chapt. **??**. However, here is their definition and basic properties.

DEFINITION: $\overline{\partial} \equiv d\bar{z}^{\bar{\imath}}\frac{\partial}{\partial\bar{z}^{\bar{\imath}}} = d|_{(0,1)}$ and $\partial \equiv d|_{(1,0)}$. Obviously $d = \partial + \overline{\partial}$. Acting on a (p,q) form $\alpha \in \Omega^{(p,q)}(\mathcal{M})$, ∂ increases p by one, $\partial\colon \Omega^{(p,q)}(\mathcal{M}) \to \Omega^{(p+1,q)}(\mathcal{M})$ while $\overline{\partial}\colon \Omega^{(p,q)}(\mathcal{M}) \to \Omega^{(p,q+1)}(\mathcal{M})$. Decomposing $0 = d^2 = (\partial + \overline{\partial})^2$ into type we get

$$\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

From the fact that $\partial^2 = 0$ (resp. $\overline{\partial}^2 = 0$) we deduce the ∂ (resp. $\overline{\partial}$)–*Poincaré lemma:* if a form $\alpha \in \Omega^{(p,q)}(\mathcal{M})$, with $p \geq 1$, is ∂–closed, $\partial\alpha = 0$, *locally* there exists a form $\beta \in \Omega^{(p-1,q)}(\mathcal{M})$ such that $\alpha = \partial\beta$. The same holds with $\partial \leftrightarrow \overline{\partial}$. Of course, *globally* such a form β may not exist.

$$(\mathbf{A}_i)^j{}_{\bar k}, \quad (\mathbf{A}_{\bar\imath})^j{}_{\bar k}, \quad (\mathbf{A}_i)^{\bar\jmath}{}_k, \quad (\mathbf{A}_{\bar\imath})^{\bar\jmath}{}_k, \tag{9.34}$$

those in the first line commute with the complex structure, whereas those in the second one *anti*commute. Since in complex coordinates the metric is simply $g_{i\bar\jmath} = i\omega_{i\bar\jmath}$ and $g_{\bar\imath j} = -i\omega_{\bar\imath j} = i\omega_{j\bar\imath}$, and $\omega_{i\bar\jmath}$ is $\boldsymbol{\nabla}$–parallel, it is obvious[66] that the first $d + \mathbf{A}|_{\text{first line}}$ is a metric connection, which — being torsionless — should coincide with the Christoffel one. Thus the tensor $\widehat{\mathbf{A}}$ should be the sum of the anticommuting components in the second line. [In fact many components vanishes. The non zero ones[67] are $(\mathbf{A}_i)^j{}_k$ and $(\mathbf{A}_{\bar\imath})^{\bar\jmath}{}_{\bar k}$ for the Christoffel part, and $(\mathbf{A}_i)^{\bar\jmath}{}_k$, $(\mathbf{A}_{\bar\imath})^j{}_{\bar k}$ for the F part]. We shall write $\mathbf{B} \equiv \widehat{\mathbf{A}}|_{(1,0)}$ so $\widehat{\mathbf{A}} = \mathbf{B} + \overline{\mathbf{B}}$.

Now let us compute the Riemannian curvature $R = \nabla^2$. It takes values in the same Lie algebra as the metric connection, and hence it commutes with f. Thus — by the usual symmetries of the Riemann tensor — it should be a $(1, 1)$ form. One has

$$0 = \boldsymbol{\nabla}^2\big|_{(1,1)} = (\nabla+\mathbf{B}+\overline{\mathbf{B}})^2\big|_{(1,1)} = R+\mathbf{B}\wedge\overline{\mathbf{B}}+\overline{\mathbf{B}}\wedge\mathbf{B}+\nabla(\mathbf{B}+\overline{\mathbf{B}})\big|_{(1,1)} \tag{9.35}$$

the first three terms in the RHS commute with f while the last one *anti*commutes. So it must vanish separately. We get

$$R_{i\bar\jmath} = -\big[\mathbf{B}_i, \overline{\mathbf{B}}_{\bar\jmath}\big]. \tag{9.36}$$

Again we find the curvature of the scalars' manifold to be *minus* the commutator of geometric tensors which have peculiar 'flatness' properties. In the present case

$$\nabla_i \overline{\mathbf{B}}_{\bar\jmath} = \overline{\nabla}_{\bar\imath}\mathbf{B}_j = 0. \tag{9.37}$$

---

[66] Indeed, in complex coordinates the complex structure

$$f^\alpha{}_\beta = i\delta^\alpha{}_\beta, \quad f^{\bar\alpha}{}_{\bar\beta} = -i\delta^{\bar\alpha}{}_{\bar\beta}, \quad f^{\bar\alpha}{}_\beta = f^\alpha{}_{\bar\beta} = 0,$$

is constant. So, using the real notation,

$$\begin{aligned}\boldsymbol{\nabla}_h g_{ij} &= \boldsymbol{\nabla}_h\big(\omega_{ik}\, f^k{}_j\big) = \omega_{ik}\boldsymbol{\nabla}_h f^k{}_j = \\ &= -\omega_{ik}\Big(\boldsymbol{A}_h{}^k{}_l\, f^l{}_j - f^k{}_l\boldsymbol{A}_h{}^l{}_j\Big) = -\omega_{ik}\Big(\widetilde{\boldsymbol{A}}_h{}^k{}_l\, f^l{}_j - f^k{}_l\,\widetilde{\boldsymbol{A}}_h{}^l{}_j\Big) = \\ &= 2\,\omega_{ik}\, f^k{}_l\widehat{\boldsymbol{A}}_h{}^l{}_j = 2\, g_{il}\widehat{\boldsymbol{A}}_h{}^l{}_j = \\ &= g_{il}\widehat{\boldsymbol{A}}_h{}^l{}_j + g_{jl}\widehat{\boldsymbol{A}}_h{}^l{}_i\end{aligned}$$

which is equivalent to $\nabla_h g_{ij} = 0$. Alternatively, in the complex notations,

$$\begin{aligned}0 = \boldsymbol{\nabla}_\alpha\omega_{\beta\bar\gamma} &= \partial_\alpha\omega_{\beta\bar\gamma} - (A_\alpha)_\beta{}^\gamma\omega_{\gamma\bar\gamma} - (A_\alpha)_\beta{}^{\bar\delta}\omega_{\bar\delta\bar\gamma} - (A_\alpha)_{\bar\beta}{}^{\bar\delta}\omega_{\beta\bar\delta} - (A_\alpha)_{\bar\gamma}{}^{\delta}\omega_{\beta\delta} = \\ &= \partial_\alpha\omega_{\beta\bar\gamma} - (A_\alpha)_\beta{}^\gamma\omega_{\gamma\bar\gamma} - (A_\alpha)_{\bar\beta}{}^{\bar\delta}\omega_{\beta\bar\delta},\end{aligned}$$

which is $\nabla_\alpha g_{\beta\bar\gamma} = 0$.

[67] This is equivalent to the statement that $\boldsymbol{\nabla}|_{(0,1)} = \overline{\partial}$ acting on $(1, 0)$ vectors (that is vector belonging to the bundle F. Indeed, consider a function $f(q^x, p_y)$ of the *flat* Darboux coordinates. We have

$$\frac{\partial}{\partial Z^w} f(q^x, p_y) = \frac{1}{2}\left(\frac{\partial}{\partial q^w} - \frac{\partial^2\mathcal{F}}{\partial Z^w\partial Z^u}\frac{\partial}{\partial p_u}\right) f\left(\frac{Z^x + \overline{Z}^x}{2}, \frac{W_y + \overline{W}_y}{2}\right)$$

therefore

$$2\,\boldsymbol{\nabla}\frac{\partial}{\partial Z^w}\bigg|_{(1,0)} = -\frac{\partial^3\mathcal{F}}{\partial Z^x\partial Z^w\partial Z^u}\, dZ^x \otimes \frac{\partial}{\partial p_u}\bigg|_{(1,0)} = 0,$$

so $\boldsymbol{\nabla} f^w\, \partial/\partial Z^w|_{(1,0)} = (\overline{\partial} f^w)\,\partial/\partial Z^w$.

The lesson we learn is that these tt^*–like structures arise precisely because, beside the metric connection ∇, there is an 'improved' one $\boldsymbol{\nabla} = \nabla + \widehat{\mathbf{A}}$, which is not metric but it is *flat.* This should hold also for the connections associated to the SUSY automorphism group which have the same tt^* structure, as we saw in $D = 3$.

9.5.* Susy gauge theories in $D = 5$. The scalars' manifold $\mathcal{M}$ of an $\mathcal{N} = 2$ SUSY gauge theory has two geometric structures: the complex one and the *special* one. We can ask whether there are *pure* special geometries, without 'additional' ingredients. The answer is yes: five dimensional $\mathcal{N} = 2$ gauge theories have a purely real special geometry. This is related to the fact that the corresponding SUSY algebra automorphism group — $Sp(1)$ — is pseudoreal (= quaternionic).

We have two motivations to briefly discuss these models: special geometry itself, in its purest version, and the fact that $D = 5$ is the lowest dimension in the series $D = 6k + 5$, a very interesting sequence of dimensions. The second entry is 11... We like to say something about the SUSY physics in such dimensions.

STILL TO BE WRITTEN

10. $D = 4$ supergravity

We close this introductory chapter by adding some caffeine to our soft drink. We discuss some first geometrical aspect of $D = 4$ extended supergravity. The structure is similar to the one we found in $D = 3$, but now the graviton, the gravitini and the gauge vectors do propagate physical states. At first sight, the theory looks quite a mess; however, once one has understood its internal logic, it appears to be rather elegant and (almost) simple.

This section is meant as *an appetizer.* We shall return to $D = 4$ SUGRA after the development of the necessary tools in Part 2. Our immediate goal is to extract from the physics of local SUSY enough geometric data to uniquely determine the relevant couplings in the Lagrangian $\mathcal{L}$ of any $\mathcal{N}$–extended supergravity (with arbitrary matter).

The field content is: the GR vielbein e^a_μ — related to the spacetime metric by the usual relation $h_{\mu\nu} = e^a_\mu\, e^b_\nu\, \eta_{\mu\nu}$ — the spin–3/2 gravitini, ψ^A_μ, $(A = 1, 2, \dots, \mathcal{N})$, the vector fields, A^v_μ, $(v = 1, 2, \dots, V)$, the spin–1/2 fermions χ^m, $(m = 1, 2, \dots, M)$, and the scalars ϕ^i $(i = 1, 2, \dots, n)$. As usual in $4D$ SUGRA, we shall adopt the following convention [**81**]: fermions with upper indices have chirality $+1$, while those with lower indices have chirality -1, namely

$$\gamma_5\, \psi^A_\mu = \psi^A_\mu, \quad \gamma_5\, \psi_{A\,\mu} = -\psi_{A\,\mu}, \tag{10.1}$$

$$\gamma_5\, \chi = \chi^m, \qquad \gamma_5\, \chi_m = -\chi_m, \tag{10.2}$$

$$\bar{\psi}_{A\,\mu} = (\psi^A_\mu)^\dagger \gamma^0 \qquad \bar{\chi}_m = (\chi^m)^\dagger \gamma^0. \tag{10.3}$$

Without loss of generality, we may assume that the fermions correspond to a 'flat' basis, *i.e.* their kinetic terms have the form

$$-i\, e\, \overline{\chi}_m \not{D} \chi^m + \cdots$$

The vectors have Yang–Mills like kinetic terms

$$-\frac{1}{4} i\, e\, \mathcal{N}_{xy}\, h^{\mu\rho} h^{\nu\sigma}\, F^{x\,+}_{\mu\nu}\, F^{y\,+}_{\rho\sigma} + \text{H.c.}$$

and correspondingly the transformation of the gravitini has the general form:

$$\begin{aligned} \delta\psi^A_\mu =& D_\mu \epsilon^A + \mathcal{Q}^{A}_{\mu\ B}\, \epsilon^B + \mathcal{T}^{AB}_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu \epsilon_B + \\ &+ \mathcal{M}^{AB} \gamma_\mu \epsilon_B + \mathcal{K}^{A}_{\nu\ B} \gamma^\nu \gamma_\mu \epsilon^B + \mathcal{J}_\mu{}^{\nu\, AB} \gamma_\nu \epsilon_B \end{aligned} \tag{10.4}$$

where, in analogy with eqn.(10.1), we adopted the convention [**81**] that $\mathcal{F}^{AB}_{\rho\sigma}$ with *upper* indices refers to the *anti*selfdual component, while we write the selfdual part with *lower* ones, $\mathcal{F}_{\rho\sigma\, AB} \stackrel{\text{def}}{=} (\mathcal{F}^{AB}_{\rho\sigma})^*$.

In this section we consider three basic geometric structures:

(R–bundles): the $U(\mathcal{N})_R$ bundles, connections and curvatures and their first implications for the geometry of $T\mathcal{M}$;

(duality): the $Sp(2V, \mathbb{R})$ bundles, connections and curvatures, and their compatibility with the $U(\mathcal{N})_R$ ones;

(Z–map): the 'central charge' geometry, and its relations with the $U(\mathcal{N})_R$ and $Sp(2V, \mathbb{R})$ bundles.

In future chapters we shall address two other structures: the one connected with the *gaugings* of an isometry subgroup $\subset \text{Iso}(\mathcal{M})$, and the one related to the scalars' potential. We prefer to discuss them having at our disposal the geometric technologies to be developed in Part 2 below. These structures, taken together, are more than sufficient to determine all couplings and SUSY transformations. In fact, the study of the three topics of the present section is already very pleonastic: the geometry of $\mathcal{M}$ is already *uniquely determined* by the first issue (the $U(\mathcal{N})_R$ bundles); in fact, as we shall see in Part 2, the most elementary considerations about the **($\boldsymbol{R}$–bundles)** suffice to fix all derivative couplings.

10.1. The $U(\mathcal{N})_R$ bundles. The automorphism group of the SUSY algebra, in $D = 4$, is $U(\mathcal{N})_R$. However, the $U(1)$ part acts trivially on the scalars of the massless supermultiplets which are PCT self–conjugate: we have already seen this in §. 8.2 for the $\mathcal{N} = 2$ matter multiplet (hypermultiplet), and in §. 9 for the $\mathcal{N} = 4$ *gauge* supermultiplet. The same argument works in $\mathcal{N} = 8$ SUGRA for the *supergravity* multiplet.

The analysis we did in §. 4.1 and §. 5 for the three dimensional case, and in §. 8.1, §. 8.2 and §. 9 for $D = 4, 6$, can be repeated here, with essentially the same conclusions. According to the philosophy of §. 1.2, we have just to replace the geometric structure defined over the ground field $\mathbb{R}$ with the corresponding ones based on $\mathbb{C}$. In particular, on the scalars' manifold, $\mathcal{M}$, there are bundles with structure groups $U(1)_R \times SU(\mathcal{N})_R$.

10.1.1. *Tangent bundle isomorphisms.* As in sect. 5, the gravitini SUSY transformations, eqn.(10.4), define an $U(\mathcal{N})_R$ connection on $\mathcal{M}$, $Q^{A}_{i\ B}(\phi)$, by the rule

$$\mathcal{Q}^{A}_{\mu\ B} = Q^{A}_{i\ B}\, \partial_\mu \phi^i + \cdots .$$

This defines, generically, an $U(1)_R$–principal bundle $\mathcal{P}_1 \to \mathcal{M}$ and an $SU(\mathcal{N})_R$–principal bundle $\mathcal{P}_\mathcal{N} \to \mathcal{M}$. From these two bundles we construct vector bundles associated to any given representation of $U(1)_R \times SU(\mathcal{N})_R$. As in sect. 5, we are particularly interested in the *bundle isomorphisms* between the tangent bundle $T\mathcal{M}$ to bundles with canonical $U(1)_R \times SU(\mathcal{N})_R$ structure. There are two ways of doing this: one either goes trough the detailed analysis, as we did in $D = 3$, or uses a short–cut: by the 'target space equivalence principle' (corollary 1.1 of chapt. 1) the linear structure on $T_\phi\mathcal{M}$ is exactly the one dictated by the SUSY representations at the *linear* level. (So one may be naive, but only at the infinitesimal level).

The first element is the splitting of $T\mathcal{M}$ into the direct sum of two *real* bundles; a phenomenon we already observed in $D = 3$ for $\mathcal{N} = 0 \mod 4$, and in $D = 4$ for $\mathcal{N} = 2$ (rigid). By any one of the two methods mentioned earlier[68], it is easy to see that the splitting appears only in two SUGRA's: $\mathcal{N} = 2$ (just as in the rigid case) and $\mathcal{N} = 4$. Geometrically, this splitting arises in the following way: consider the curvature $(P^{(1)}_{kl})^i{}_j$ of the $U(1)_R$ connection. It is a 2–form with values in $\mathfrak{u}(1)_R \subset \mathrm{End}(T\mathcal{M})$. Let

$$\Pi^i{}_j = \left(P^{(1)\,kl}\right)^i{}_m \left(P^{(1)}_{kl}\right)^m{}_j. \tag{10.5}$$

The above splitting is simply[69]

$$T\mathcal{M} \simeq \mathrm{Ker}\,\Pi \oplus \mathrm{Im}\,\Pi, \tag{10.6}$$

since $(P^{(1)}_{kl})_{ij}$ is *anti*symmetric, Π is symmetric, and the splitting in eqn.(10.6) is actually an *orthogonal* decomposition of $T\mathcal{M}$. Then, for $\mathcal{N} = 2, 4$, $T\mathcal{M}$ has the structure

$$T\mathcal{M} \simeq [[\mathcal{L} \otimes \mathcal{U}]] \oplus [[\Theta \otimes \mathcal{B}]]. \tag{10.7}$$

here $\mathcal{L}$ is the complex line bundle [70] associated to the charge $+1$ representation of $U(1)_R$, and $\mathcal{U} \simeq \mathcal{L}^{-1} \otimes \mathrm{Im}\,\Pi$, is a complex bundle with a structure group commuting with $U(1)_R \subset SO(\mathrm{Im}\,\Pi)$. Θ is an $SU(\mathcal{N})_R$ bundle associated to the *real* representation **6** for $\mathcal{N} = 4$, and to the *quaternionic* one **2** for $\mathcal{N} = 2$ (compare with the rigid case, §. 8.2). Physical consistency requires the structure group of the bundle $\mathcal{B}$ to be a subgroup of $SO(\mathrm{Ker}\,\Pi)$ *commuting* with the $SU(\mathcal{N})_R$ gauge transformations. Thus the structure group of $\mathcal{B}$ must be a subgroup of $\mathcal{C}[SU(\mathcal{N})_R]$, the *centralizer* of $SU(\mathcal{N})_R$ in $SO(\mathrm{Ker}\,\Pi)$. More precisely, for $\mathcal{N} = 2$ the structure group of $\mathcal{B}$ is (contained in) $Sp(\dim \mathrm{Ker}\,\Pi/2)$ and the tangent space $T\mathcal{M}$ belongs to the defining representation (we saw this in §. 8.1); whereas in the $\mathcal{N} = 4$ case the structure group is (a subgroup of) $SO(V)$, again in the defining representation. Anticipating a result we shall present in chapt.3 (and that we already used in eqn.(9.3)) we deduce that in these cases the scalars' manifold is a product

$$\mathcal{M} = \mathcal{M}_{\mathrm{Im}\,\Pi} \times \mathcal{M}_{\mathrm{Ker}\,\Pi} \tag{10.8}$$

$$\text{where} \quad T\mathcal{M}_{\mathrm{Im}\,\Pi} \simeq [[\mathcal{L} \otimes \mathcal{U}]], \quad T\mathcal{M}_{\mathrm{Ker}\,\Pi} \simeq [[\Theta \otimes \mathcal{B}]]. \tag{10.9}$$

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[68] The easiest one is to look to the linear representation (lectures by M. Bertolini) and apply target space equivalence principle.

[69] Recall that if $\sigma\colon \mathcal{A} \to \mathcal{B}$ is a bundle map, $\mathrm{Ker}\,\sigma \subset \mathcal{A}$ and $\mathrm{Im}\,\sigma \subset \mathcal{B}$ are *sub–bundles*. See sec. **0.**5 of ref. [**77**].

[70] *Definition:* Complex line bundle = vector bundle with fiber isomorphic to $\mathbb{C}$.

In down–to–earth terms, we may think of $\mathcal{M}_{\mathrm{Ker}\,\Pi}$ to be parameterized by the 'matter' scalars' (hypermultiplets for $\mathcal{N} = 2$, and vector multiplets for $\mathcal{N} = 4$).

For $\mathcal{N} = 1, 3, 5, 6$, the structure of the tangent bundle is simpler

$$T\mathcal{M} \otimes \mathbb{C} = \mathcal{L} \otimes \mathcal{U} \ \oplus \ \mathcal{L}^{-1} \otimes \mathcal{U}^{\vee}, \tag{10.10}$$

with $\mathcal{L}$ the $U(1)_R$ line bundle of the appropriate charge, and $\mathcal{U}$ a complex bundle with structure group contained in the semisimple part of the centralizer of $U(1)_R$ in $SO(\dim \mathcal{M})$. This structure group is $SU(\dim \mathcal{M}/2)$ for $\mathcal{N} = 1$, $SU(\mathcal{N})_R$ for $\mathcal{N} = 5, 6$, and $SU(3) \times SU(\dim \mathcal{M}/6)$ for $\mathcal{N} = 3$. In the last three cases $\mathcal{U}$ is associated with the representations $\overline{\mathbf{5}}$, $\mathbf{15}$, and $\overline{\mathbf{3}}$, respectively.

The $\mathcal{N} = 7$ case is special. On the face value, it looks similar to the previous ones. Naively one would expect to get $T\mathcal{M} \simeq [[\mathcal{L} \otimes \mathcal{U}]]$, with $\mathcal{U}$ a $SU(7)_R$ bundle associated to the representation $\mathbf{35}$. However, we shall see in Part 2 that *no Riemannian manifold $\mathcal{M}$ can have a tangent bundle isomorphic to a vector bundle with such a (proper) structure group and representation.* Therefore geometry leaves us with only one possible conclusion: there is no $\mathcal{N} = 7$ SUGRA.

In $\mathcal{N} = 8$ SUSY, we must have the isomorphism

$$T\mathcal{M} \simeq \Theta, \tag{10.11}$$

where Θ is the vector bundle with structure group $SU(8)_R$ associated to the *real* irreducible representation of dimension $\mathbf{70}$.

We shall see in Part 2 that it is a basic fact of Riemannian geometry that there are precisely *three* simply–connected manifolds with this property: they correspond, respectively, to positive, negative and zero scalar curvature. The same statement holds for the $T\mathcal{M}$–isomorphisms we got for $\mathcal{N} = 5, 6$; while for $\mathcal{N} = 3, 4$ there are precisely three manifolds in *each (allowed) dimension* compatible with the given tangent bundle isomorphisms *(alias* reduction of structure group). Again they correspond to the three possible signs of the curvature. Since, as we shall see, the $SU(\mathcal{N})_R$ curvature should be *negative,* we have a *single* Riemannian space with the given properties for $\mathcal{N} \geq 3$. Then the full non–linear couplings of (ungauged) $D = 4$ SUGRA are uniquely determined[71]. Thus, the other geometrical structures we shall present in this section — although quite interesting on their own ground — are, in a sense, just mathematical consequences of the bundle structure we already discussed above. We stress that this extremely power result is nothing else that the combined statement of the linear representation theory and the 'target space equivalence principle' for general Lagrangian couplings.

In tables 10.3, 2.5 we present the complete list of tangent bundle isomorphism.

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[71] This may seem miraculous at first. But recall (§. 2 of chap. 1) that for us all the couplings in $\mathcal{L}$ are bundle morphisms. Thus, in down–to–earth language what we are doing is to take one particular coupling from $\mathcal{L}$, interpret it as a bundle morphism, and then relate all the other coupling to it by supersymmetry; since all these couplings are also bundle morphisms, the SUSY relation between the various terms in $\mathcal{L}$ is more easily worked out in the differential–geometric language.

10.1.2. *Construction of the tables.* One starts from the (massless) representations of the $\mathcal{N}$–extended SUSY algebras in D space–time dimensions (here $D = 4$ but the argument is pretty universal) as listed, *e.g.*, in the table 6 of reference [**60**]. On looks for scalars (that is states of zero helicity) in the various multiplets and sees the representations of $SU(\mathcal{N})$ they belong[72]:

	multiplet	$SU(\mathcal{N})_R$ representations
$\mathcal{N} = 1$	matter (chiral)	$\mathbf{1}_{+1} \oplus \mathbf{1}_{-1}$
$\mathcal{N} = 2$	matter (hyper)	$\mathbf{2}_0$
	Yang–Mills	$\mathbf{1}_{+2} \oplus \mathbf{1}_{-2}$
$\mathcal{N} = 3$	Yang–Mills	$\overline{\mathbf{3}}_{+2} \oplus \mathbf{3}_{-2}$
$\mathcal{N} = 4$	Yang–Mills	$\mathbf{6}$
	gravitational	$\mathbf{1}_{+3} \oplus \mathbf{1}_{-3}$
$\mathcal{N} = 5$	gravitational	$\overline{\mathbf{1}}_{+3} \oplus \mathbf{1}_{-3}$
$\mathcal{N} = 6$	gravitational	$\overline{\mathbf{15}}_{+3} \oplus \mathbf{15}_{-3}$
$\mathcal{N} = 7$	gravitational	$\overline{\mathbf{35}}_{+3} \oplus \mathbf{35}_{-3}$
$\mathcal{N} = 8$	gravitational	$\mathbf{70}$

Finally, one invoke the *target space equivalence principle* to interprete the above table as saying that

$$T\mathcal{M} \simeq \left[\left[\bigoplus_\alpha \mathcal{L}_\alpha \otimes \mathcal{S}_\alpha \otimes \mathcal{U}_\alpha \right]\right], \tag{10.12}$$

where $\mathcal{L}_\alpha$, $\mathcal{S}_\alpha$ and $\mathcal{U}_\alpha$ are vector bundles over $\mathcal{M}$ with structure groups, respectively, $U(1)_R$, $SU(\mathcal{N})_R$ and $\mathcal{C}[U(\mathcal{N})_R]$, associated to the various representations α of $U(\mathcal{N})_R$ appearing in the above table for the given $\mathcal{N}$.

10.1.3. *Computing the $U(\mathcal{N})_R$ curvatures.* In this subsection we compute the curvature of the R–symmetry connection along the lines we followed in §. 5 for the $D = 3$ case. As we mentioned at the end of the previous subsection, strictly we need to know only the *sign* of the curvature. In fact, we already know even that: the sign is easily determined by reducing to $D = 3$. We shall be sketchy. The SUSY transformations of scalars have the form

$$\delta\phi^i = K^{i\,A}{}_m(\phi)\,\bar{\epsilon}_A \chi^m + \text{H.c.}, \tag{10.13}$$

for some tensor $K^{i\,A}_m$; we saw above that this tensor is a kind of vielbein, in particular it is covariantly constant (with respect to the three connections acting on its various indices). We set

$$K_i^m{}_A \overset{\text{def}}{=} g_{ik}\, K^{k\,m}{}_A \overset{\text{def}}{=} g_{ik}\,(K^{k\,A}{}_m)^*. \tag{10.14}$$

[72] The normalization of the $U(1)_R$ charge is arbitrary.

Eqn.(10.13) requires the (chiral) supercurrent to have the form

$$J^A_\mu = \gamma^\rho \gamma_\mu \chi^m \, g_{ij}(\phi) \, K^{i\,A}{}_m(\phi) \, \partial_\rho \phi^j + \cdots \tag{10.15}$$

thus the fermion SUSY transformation should contain the term

$$\delta \chi_m = \gamma^\mu \epsilon_A \, K^{i\,A}_m \, g_{ij} \, \partial_\mu \phi^j + \cdots \tag{10.16}$$

This form of the SUSY transformations — exactly as in $D = 3$ and in rigid SUSY — implies a Clifford–like property

$$\big[K^A (K^B)^\dagger + (K^B)^* (K^A)^t\big]_{ij} = 2\,\delta^A{}_B \, g_{ij}. \tag{10.17}$$

The gravitino equation of motion[73][74]

$$i\gamma^{\mu\nu\rho} \, \mathcal{D}_\nu \psi^A_\rho = J^{A\,\mu}, \tag{10.18}$$

must transform in the right way under SUSY. The LHS of eqn.(10.18) gives

$$i\gamma^{\mu\nu\rho} \, \mathcal{D}_{[\nu} \mathcal{D}_{\rho]} \epsilon^A + \text{fermions}, \tag{10.19}$$

which contains a term linear in the $U(\mathcal{N})_R$ curvature $P_{ij}{}^A{}_B$

$$\cdots + i\,\gamma^{\mu\nu\rho} \, \partial_{[\nu} \phi^i \, \partial_{\rho]} \phi^j \, P_{ij}{}^A{}_B \, \epsilon^B + \cdots \tag{10.20}$$

The term proportional to $\gamma^{\mu\nu\rho} \partial_{[\nu} \phi^i \partial_{\rho]} \phi^j$'s in the RHS of eqn.(10.18) reads

$$\gamma^{\mu\nu\rho} \partial_{[\nu} \phi^i \partial_{\rho]} \phi^j \left(-\frac{1}{2} \Big[g_{ik} K^{k\,A}{}_m \, K^m{}_{j\,B} - g_{jk} K^{k\,A}{}_m \, K^m{}_{i\,B} \Big] \right). \tag{10.21}$$

Thus

$$P_{ij}{}^A{}_B = -\frac{1}{2} \Big[K^A \, (K^B)^\dagger - (K^B)^* (K^A)^t \Big]_{ij}. \tag{10.22}$$

From this equation we infer

$$P_{ij}{}^A{}_B = -P_{ji}{}^A{}_B \tag{10.23}$$

$$(P_{ij}{}^A{}_B)^* = -P_{ij}{}^B{}_A, \tag{10.24}$$

that is $P \in \Omega^2 \left(\mathfrak{u}(\mathcal{N})_R \right)$ (a Lie algebra valued 2–form). In particular, the curvature is not zero. In fact it is negative.

Since $\mathcal{D}_i K^{j\,A}{}_m = 0$, one has

$$\mathcal{D}_i P^A{}_B = 0 \tag{10.25}$$

that is the $U(\mathcal{N})_R$ curvature is *covariantly* constant with respect the Levi–Civita plus $\mathfrak{u}(\mathcal{N})_R$ connections.

General lesson 10.1. *In $D = 4$* SUGRA *the tangent bundle $T\mathcal{M}$ has a reduced structure group $U(1)_R \times SU(\mathcal{N})_R \times \mathcal{C}[U(\mathcal{N})_R]$ (here $\mathcal{C}[U(\mathcal{N})_R]$ is the centralizer of $U(\mathcal{N})_R$ in $SO(\dim \mathcal{M})$), according to the isomorphisms listed in the tables at the end of the chapter. The $U(\mathcal{N})_R$ curvature is given by eqn.(10.22). The curvature tensor $P_{ij}{}^A{}_B$ is* COVARIANTLY *constant (with respect the total connection).*

[73] The derivative $\mathcal{D}_\mu$ is covariant with respect all the relevant local symmetries.

[74] This argument is equivalent to the variation of $\mathcal{L}$.

10.2. Duality bundles and connections. If we have V vectors, we should have a map

$$\mu\colon \mathcal{M} \to \frac{Sp(2V,\mathbb{R})}{U(V)} \simeq \mathfrak{H}_V, \tag{10.26}$$

describing the scalar–vector couplings. As in the case of rigid $\mathcal{N}=2$ case, over $\mathfrak{H}_V$ there are natural bundles: the flat $Sp(2V,\mathbb{R})$ bundle, $\mathcal{F}$, the tautological bundle $\mathcal{T}$, and the the quotient bundle $\mathcal{F}/\mathcal{T}$. All of these may be pulled back to $\mathcal{M}$. We are interested in the relation between these bundles and $T\mathcal{M}$, as well the $U(\mathcal{N})_R$ bundles discussed above. In this section $\mathcal{V}$ stands for the $U(\mathcal{N})_R$ bundle associated to the fundamental representation (the one corresponding to the left–handed gravitini).

Our starting point is the bundle map

$$\xi\colon\ \wedge^2\mathcal{V}^\vee\otimes\mu^*\mathcal{F}\to T^*\mathcal{M}, \tag{10.27}$$

generalizing the one we found in rigid special geometry, eqn.(9.10). Concretely,

$$\left(L_{AB},\begin{pmatrix}A^x_\mu\\ B_{y\,\mu}\end{pmatrix}\right)\mapsto L_{AB}\,C^{\alpha\beta}\,\delta^A_\alpha\delta^B_\beta\begin{pmatrix}A^x_\mu\\ B_{y\,\mu}\end{pmatrix}\equiv L_{AB}\begin{pmatrix}q_i^{AB\,x}\\ p^{AB}_{y\,i}\end{pmatrix}\partial_\mu\phi^i+\cdots \tag{10.28}$$

where $\delta^A_\alpha(\cdot)$ replaces in the local case the operator $-i[Q^A_\alpha,\cdot\}$ we used in the rigid case[75]. Again, defining the vector of forms

$$\Xi^{AB,X}=\begin{pmatrix}q_i^{AB\,x}\,d\phi^i\\ p^{AB}_{y\,i}\,d\phi^i\end{pmatrix}, \tag{10.29}$$

we have the 'pushed–forward gauge–invariance' condition,

$$\boldsymbol{\nabla}\Xi^X=0, \tag{10.30}$$

where now the exterior differential $\boldsymbol{\nabla}$ is covariantized with respect both the flat $Sp(2V,\mathbb{R})$ connection and the $U(\mathcal{N})_R$ one. In particular, $\boldsymbol{\nabla}$ is *flat:*

$$\boldsymbol{\nabla}^2=0. \tag{10.31}$$

From the 'equivalence principle' we learn that the bundle map ξ is:

- trivial for $\mathcal{N}=1$;
- an epimorphism on the factor manifold $\mathcal{M}_{\mathrm{Im}\,\Pi}$ for $\mathcal{N}=2$;
- an epimorphism for $\mathcal{N}\geq 3$.

Hence, for $\mathcal{N}\neq 1$,

$$T\mathcal{M}_{(\mathrm{Im}\,\Pi)}\simeq(\wedge^2\mathcal{V}^\vee\otimes\mu^*\mathcal{F})/\mathrm{Ker}\,\xi. \tag{10.32}$$

Then $T\mathcal{M}_{(\mathrm{Im}\,\Pi)}$ inherits a connection $\widehat{\boldsymbol{\nabla}}\equiv d+\widehat{\mathbf{A}}$ from $\boldsymbol{\nabla}$ by the canonical projection

$$\widehat{\boldsymbol{\nabla}}\eta=\boldsymbol{\nabla}\eta\quad\mathrm{mod}\ \mathrm{Ker}\,\xi, \tag{10.33}$$

and again we decompose the connection $\widehat{\mathbf{A}}$ as

$$\widehat{\mathbf{A}}=\underbrace{\frac{1}{2}(\mathbf{A}-\mathbf{A}^\dagger)}_{\mathfrak{u}(\mathcal{N})_R\oplus\,\mathfrak{u}(V)\ \text{part}}+\underbrace{\frac{1}{2}(\mathbf{A}+\mathbf{A}^\dagger)}_{\text{non-compact part}}\ . \tag{10.34}$$

[75] Compare with the discussion around eqn.(3.11) of ref.[**13**].

As in special geometry, the relation between the metric connection D and $\widehat{\boldsymbol{\nabla}}$ is given by

$$\widehat{\boldsymbol{\nabla}} = D + \frac{1}{2}(\mathbf{A} + \mathbf{A}^\dagger). \tag{10.35}$$

Therefore

General lesson 10.2. *For $\mathcal{N} \neq 1$, one reconstructs the full geometry of $\mathcal{M}$ ($\mathcal{M}_{\mathrm{Im}\,\Pi}$ for $\mathcal{N} = 2$) from the map ξ.*

We stop here. We shall no pursue further this subject here. We shall return to it in chapt.4 with the help of more powerful techniques.

Remark. As we already know from the rigid case, the structure of the connection in eqn.(10.35) implies that

$$(\text{the metric curvature}) = -[C, C^\dagger] \tag{10.36}$$

for some C. Thus we recover, from an *a priori* totally unrelated argument, the basic tt^*–like structure of the Riemann curvature of $\mathcal{M}$.

It remains to discuss the case $\mathcal{N} = 1$. If $V \neq 0$, we have a map μ as in eqn.(10.26). We need to determine its properties, that is the compatibily condition with the geometric structures on $\mathcal{M}$. In fact on $\mathcal{M}$ we have only the (almost)complex structure f, which — as in sect. 6 — is related to the curvature of a complex line bundle $\mathcal{L}$. Also the coset $Sp(2V, \mathbb{R})/U(V)$, being the Siegel's hupper half–space, is a complex manifold with a complex structure I. We claim the compatibility condition is

$$\mu_* f = I\, \mu_*, \qquad \text{or dually,} \qquad \mu^* I^t = f^t \mu^*, \tag{10.37}$$

i.e. complex structure goes to complex structure. In fact, both complex structures are manifestations of the $U(1)_R$ automorphism group, and both ways of implementing a $U(1)_R$ transformation should give the same answer.

10.3. Z–map. In General Relativity one can define the Poincaré algebra generators P_μ, $M_{\mu\nu}$ only for space–times which are asymptotic to flat Minkowski space at infinity. Then one defines the ADM momentum 4–vector P_μ in terms of an integral at spatial infinity. Under certain circumstances which we do not review here, the same holds [**82, 83**] for the other generators of the *super*Poincaré algebra: supercharges and central charges. For central charges this boils down to the following simple rule: take the gravitino transformations, eqn.(10.4), and consider the complex two form $\mathcal{T}^{AB}_{\rho\sigma}$. Then

$$Z^{AB} = \int_{\text{spatial } \infty} \mathcal{T}^{AB}_{\mu\nu}\, dx^\mu \wedge dx^\nu. \tag{10.38}$$

The SUSY algebra implies that Z^{AB} — if well–defined — is the central charge operator. We must have

$$\mathcal{T}^{AB}_{\mu\nu} = H^{AB}_x(\phi)\, F^{x\,-}_{\mu\nu} + (\text{fermions}) \tag{10.39}$$

for a certain tensor $H^{AB}_x(\phi)$. The central charge Z^{AB} is a linear combination of the Abelian charges $\int_\infty F^x$ with coefficients $H^{AB}_x(\phi_\infty)$, where ϕ^i_∞ is the (constant) value of the scalars at spatial infinity.

We may describe the situation in the following geometric terms: over $\mathcal{M}$ we have a flat $Sp(2V, \mathbb{R})$ vector bundle $\mathcal{F}$ on which the field–strength $\begin{pmatrix} F^{x\,-} \\ G_y^- \end{pmatrix}$ takes values[76]. Inside $\mathcal{F}$ there is a subbundle, $\mathcal{Z}$, of rank $\mathcal{N}(\mathcal{N}-1)/2$, whose fiber $\mathcal{F}_\phi$ over the point $\phi \in \mathcal{M}$ is the subspace spanned by the $\mathcal{T}^{AB}_{\mu\nu}$. Concretely, the coefficients H^{AB}_x define a map $H \colon \mathcal{F}^\vee \to \mathcal{Z}^\vee$ dual to the embedding map $H^\vee \colon \mathcal{Z} \to \mathcal{F}$.

The geometrical problem is then to describe how the linear space $\mathcal{Z}_\phi$ moves inside the fiber $\mathcal{F}_\phi$ as we move our base point $\phi \in \mathcal{M}$. Physically, we are expanding the theory around a classical 'vacuum', $\phi^i = \phi^i_\infty$, and asking which linear combinations of the vector fields A^x_μ are, in this particular 'vacuum', the 'graviphotons[77]' and which are the 'matter' vectors. The question is meaningful only for $2 \leq \mathcal{N} \leq 4$, since for $\mathcal{N} = 1$ there are no graviphotons, and for $\mathcal{N} \geq 5$ there is no 'matter'.

Again, for $2 \leq \mathcal{N} \leq 4$, the knowledge of the monomorphism $H^\vee \colon \mathcal{Z} \to \mathcal{F}$ is sufficient to fully determine the Lagrangian for (ungauged) SUGRA. As in §. 9 and §. 10.2, the flat $Sp(2V, \mathbb{R})$ connection $\boldsymbol{\nabla}$ defines a connection $\widehat{\boldsymbol{\nabla}}$ on $\mathcal{Z}$ and a connection $\widehat{\widehat{\boldsymbol{\nabla}}}$ on $\mathcal{F}/\mathcal{Z}$

$$\widehat{\boldsymbol{\nabla}}\eta = \boldsymbol{\nabla}\eta\big|_{\Omega^1(\mathcal{Z})} \tag{10.40}$$

$$\widehat{\widehat{\boldsymbol{\nabla}}}\varpi = \boldsymbol{\nabla}\varpi \mod \Omega^1(\mathcal{Z}). \tag{10.41}$$

As before, we can decompose the connections $\widehat{\boldsymbol{\nabla}}$ and $\widehat{\widehat{\boldsymbol{\nabla}}}$ into two parts, according to the their symmetry under †, and so define a metric connection and a Clifford–like tensor.

The map $H^\vee$ will turn out to have a very deep interpretation in terms of algebraic geometry. In the case $\mathcal{N} = 2$ it will be related to the holomorphic geometry of the Calabi–Yau and to mirror symmetry.

* * *

Enough for the motivations!

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[76] The usual *pedantic* remark applies here too: *strictly speaking* it is a section of $\wedge^2 T^*\Sigma \otimes \Phi^*\mathcal{F}$; for our present purposes we may ignore the space–time part.

[77] A *graviphoton* is a vector field belonging to the supermultiplet containing the graviton. It is not an invariant notion, as the discussion in the text indicates; the concept of *graviphoton* makes sense in the linearized theory, but depends on the configuration around which we linearize.

TABLE 2.4. Scalars' manifold tangent bundle isomorpshims for $\mathcal{N} \leq 4$ supergravity in $D = 4$

- $\underline{\mathcal{N} = 1}$

$$T\mathcal{M} \simeq [[\mathcal{L} \otimes \mathcal{B}]],$$

where $\mathcal{L}$ is the line bundle with canonical $U(1)_R$ connection, and $\mathcal{B}$ is a complex vector bundle *with structure group the normalizer of the complex structure f in $SO(n)$, isomorphic to $U(n/2)$:*

$$T_p\mathcal{M} \otimes \mathbb{C} \simeq \left(\frac{\mathbf{n}}{\mathbf{2}}\right)_{+\mathbf{1}} \oplus \left(\frac{\overline{\mathbf{n}}}{\mathbf{2}}\right)_{-\mathbf{1}} \qquad \text{as rep. of } SU(n/2) \times U(1).$$

- $\underline{\mathcal{N} = 2}$

$$T\mathcal{M} \simeq [[\mathcal{L} \otimes \mathcal{B}]] \oplus [[\mathcal{H} \otimes \mathcal{U}]],$$

where $\mathcal{L}$ is the line bundle with canonical $U(1)_R$ connection, $\mathcal{H}$ is a rank 2 bundle with canonical $SU(2)_R$ connection, $\mathcal{B}$ is a complex vector bundle *with structure group the semisimple part of the normalizer of the complex structure f in $SO(2V)$, isomorphic to $SU(V)$, and $\mathcal{U}$ is a bundle with structure group the normalized of $SU(2)_R$ in $SO(n-2V)$, which is isomorphic to $Sp((n-2V)/2)$:*

$$T_p\mathcal{M} \otimes \mathbb{C} \simeq (\mathbf{V}, \mathbf{1}, \mathbf{1})_{+\mathbf{2}} \oplus \left(\overline{\mathbf{V}}, \mathbf{1}, \mathbf{1}\right)_{-\mathbf{2}} \oplus \left(\mathbf{1}, \mathbf{2}, \frac{\mathbf{n} - \mathbf{2V}}{\mathbf{2}}\right)_{\mathbf{0}}$$

as a rep. of $U(1)_R \times SU(v) \times SU(2)_R \times Sp((n-2v)/4)$.

- $\underline{\mathcal{N} = 3}$

$$T\mathcal{M} \simeq [[\mathcal{L} \otimes \mathcal{B}]],$$

where $\mathcal{L}$ is a vector bundle with canonical $U(3)_R$ connection, and $\mathcal{B}$ is a complex vector bundle *with structure group the semisimple part of the normalizer of the $U(3)_R$ in $SO(n)$, isomorphic to $SU(n/6)$:*

$$T_p\mathcal{M} \otimes \mathbb{C} \simeq \left(\mathbf{3}, \frac{\overline{\mathbf{n}}}{\mathbf{6}}\right)_{-\mathbf{2}} \oplus \left(\overline{\mathbf{3}}, \frac{\mathbf{n}}{\mathbf{6}}\right)_{+\mathbf{3}},$$

as rep. of $SU(3)_R \times SU(n/6) \times U(1)_R$.

- $\underline{\mathcal{N} = 4}$

$$T\mathcal{M} \simeq [[\mathcal{L}]] \oplus \big(\mathcal{R} \otimes \mathcal{V}\big),$$

where $\mathcal{L}$ is the line bundle with canonical $U(1)_R$ connection, $\mathcal{R}$ is a rank 6 bundle with canonical $SU(4)_R \simeq SO(6)$ connection, and $\mathcal{V}$ is a real vector bundle *with structure group the normalizer of $SO(6)$ in $SO(6v)$, isomorphic to $SO(v)$:*

$$T_p\mathcal{M} \otimes \mathbb{C} \simeq (\mathbf{1}, \mathbf{1})_{+\mathbf{3}} \oplus (\mathbf{1}, \mathbf{1})_{-\mathbf{3}} \oplus (\mathbf{6}, \mathbf{v})_{\mathbf{0}}$$

as a rep. of $SO(6)_R \times SO(v) \times U(1)_R$.

$* \quad * \quad *$

NOTE: statements in *italics* will be proven in chapt.. In bundles of the form $\mathcal{A}_R \otimes \mathcal{B}$, the structure groups of $\mathcal{B}$ is SO, SU or Sp if the R–symmetry bundle $\mathcal{A}_R$ is, respectively, real, complex or quaternionic.

TABLE 2.5. Scalars' manifold tangent bundle isomorpshims for $\mathcal{N} \geq 5$ supergravity in $D = 4$

- $\underline{\mathcal{N} = 5}$

$$T\mathcal{M} \simeq [[\mathcal{L} \otimes \mathcal{B}]],$$

where $\mathcal{L}$ is the line bundle with canonical $U(1)_R$ connection, and $\mathcal{B}$ is a complex vector bundle with $SU(5)_R$:

$$T_p\mathcal{M} \otimes \mathbb{C} \simeq \left(\overline{\mathbf{5}}\right)_{+\mathbf{3}} \oplus \ (\mathbf{5})_{-\mathbf{3}} \quad \text{as rep. of } SU(5)_R \times U(1)_R.$$

- $\underline{\mathcal{N} = 6}$

$$T\mathcal{M} \simeq [[\mathcal{L} \otimes \mathcal{B}]],$$

where $\mathcal{L}$ is the line bundle with canonical $U(1)_R$ connection, and $\mathcal{B}$ has canonical $SU(6)_R$ connection,:

$$T_p\mathcal{M} \otimes \mathbb{C} \simeq (\mathbf{15})_{+\mathbf{3}} \oplus \ (\mathbf{15})_{-\mathbf{3}}$$

as a rep. of $SU(6)_R \times U(1)_R$.

- $\underline{\mathcal{N} = 7}$

$$T\mathcal{M} \overset{?!}{\simeq} [[\mathcal{L} \otimes \mathcal{B}]] \quad \textbf{impossible}!!,$$

there is NO Riemannian manifold whose tangent bundle is isomorphic to such a bundle with structure group $U(1)_R \times SU(7)_R$ and representations $(\mathbf{35})_{+3}$ plus conjugate.

- $\underline{\mathcal{N} = 8}$

$$T\mathcal{M} \simeq \mathcal{W},$$

where $\mathcal{W}$ is the $SU(8)_R$ bundle associated to the $SU(8)_R$ connection *via* the **70** representation.

* * *

NOTE: statements in *italics* will be proven in chapt.. In bundles of the form $\mathcal{A}_R \otimes \mathcal{B}$, the structure groups of $\mathcal{B}$ is SO, SU or Sp if the R–symmetry bundle $\mathcal{A}_R$ is, respectively, real, complex or quaternionic.

Part 2

Geometry
(Answers from the BOOK)

CHAPTER 3

Parallel structures and holonomy

In chapt. 2 we found that $\mathcal{N}$–extended SUSY/SUGRA requires the presence of certain *parallel* tensors on the Riemannian manifold $\mathcal{M}$:

in rigid susy: *(i)* the $(\mathcal{N}-1)$ complex structures $f^a \in \mathrm{End}(T\mathcal{M})$; *(ii)* the associated 2–forms $\omega^a := g_{ik}\,(f^a)^k{}_j\, d\phi^i \wedge d\phi^j$; more generally, *(iii)* all tensors in the Clifford algebra $\mathbb{C}l_0(\mathcal{N}) \subset \mathrm{End}(T\mathcal{M})$;

in local sugra: (i) $\mathcal{P}(P_{i_1 j_1}, P_{i_2 j_2}, \dots, P_{i_k j_k})$, where $P \in \Omega^2(\mathfrak{aut}_R)$ is the Aut_R–projection of the Riemann curvature, and

$$\mathcal{P}\big(\cdot,\cdot,\dots,\cdot\big)\colon\ \odot^k\,\mathfrak{aut}_R \to \mathbb{C}$$

is any Aut_R–invariant k–linear map. In particular, *(ii)* we have the parallel $2k$–forms

$$\mathcal{P}\big(P \wedge P \wedge \cdots \wedge P\big),$$

which (up to normalization) represent the Chern–Weil characteristic classes [**84, 188**] of the gravitino vector bundle $\Psi \to \mathcal{M}$.

There may be more. Depending on the particular isomorphim

$$T\mathcal{M} \simeq \text{(bundles with reduced structure group)}$$

which is appropriate for the given $\mathcal{N}$ and D (see chapt. 2), we may construct many other parallel tensors on $\mathcal{M}$. In fact, a fundamental principle of differential geometry — see sec. 1 below — states that the existence of parallel tensors is equivalent to the reduction of the holonomy group for $T\mathcal{M}$.

Therefore, we begin this chapter by studying the implications for the metric geometry of $\mathcal{M}$ of the existence of (many) parallel structures.

Certain nice geometries will play a crucial rôle in the analysis: their tangent bundles are isomorphic to vector bundles whose curvature is *precisely* MINUS ONE QUARTER *the commutator of some natural one–forms in* $\Omega^1(\mathrm{End}(T\mathcal{M}))$. This is precisely the structure (tt^*–like, as we called it) we found in chapt. 2 for the curvature of the local SUSY automorphism symmetry, Aut_R, for *all* $\mathcal{N}$ and *all* D !!

* * *

General references for this chapter are: [**85, 86, 87, 88, 89**].

1. The holonomy group

In this section $\mathcal{M}$ is a connected Riemannian manifold equipped with the metric g and the Levi–Civita connection[1] ∇, the unique connection which is both metric and torsionless.

1.1. Definitions. Let $\phi \in \mathcal{M}$ be a point, and C a piecewise smooth loop starting and ending at ϕ. We denote by $W(C)$ the *parallel transport* along C (*i.e.* the Wilson line of the connection ∇). Since the connection is metric, $W(C)$ is an element of the orthogonal group[2] $O(T_\phi)$. If we have two such loops, C_1 and C_2, we can define their product

$$C_1 \cdot C_2(t) = \begin{cases} C_2(2t) & 0 \le t \le 1/2 \\ C_1(2t-1) & 1/2 \le t \le 1, \end{cases} \tag{1.1}$$

and we have $W(C_1 \cdot C_2) = W(C_1) \cdot W(C_2)$ as elements of $O(T_\phi)$. Then

Definition 1.1. The *holonomy group* of $\mathcal{M}$ at ϕ, $\mathrm{Hol}(\phi)$ is the subgroup of $O(T_\phi)$ generated by all the $W(C)$'s, where C runs through all piecewise smooth loops on $\mathcal{M}$ based at ϕ. The *restricted holonomy group,* $\mathrm{Hol}^0(\phi)$ is the subgroup generated by the $W(C_0)$'s, where C_0 are the contractible loops.

Let us change the base point from ϕ to $\widetilde{\phi}$. Fix a path $C_{\widetilde{\phi}\phi}$ from ϕ to $\widetilde{\phi}$. Then

$$\mathrm{Hol}(\widetilde{\phi}) = W\big(C_{\widetilde{\phi}\phi}\big)\, \mathrm{Hol}(\phi)\, W\big(C_{\widetilde{\phi}\phi}\big)^{-1} \tag{1.2}$$

$$\mathrm{Hol}^0(\widetilde{\phi}) = W\big(C_{\widetilde{\phi}\phi}\big)\, \mathrm{Hol}^0(\phi)\, W\big(C_{\widetilde{\phi}\phi}\big)^{-1}, \tag{1.3}$$

so the holonomy groups are independent of the point ϕ up to isomorphism. Then we shall speak of *the* holonomy groups of $\mathcal{M}$ and denote them as $\mathrm{Hol}(g)$, $\mathrm{Hol}^0(g)$, respectively.

Here we are interested only in the (much simpler) group $\mathrm{Hol}^0(g)$. Of course, $\mathrm{Hol}(g) \equiv \mathrm{Hol}^0(g)$ for a simply–connected manifold. Hence, replacing $\mathcal{M}$ with its universal Riemannian covering if necessary, we assume $\mathcal{M}$ to be simply–connected. In this case

$$\mathrm{Hol}(g) \subset SO(\dim \mathcal{M}). \tag{1.4}$$

Suppose on $\mathcal{M}$ there is a tensor field $T \in \otimes^k T\mathcal{M} \otimes \otimes^l T^*\mathcal{M}$ which is invariant under parallel transport, *i.e.* for every $\phi_1, \phi_2 \in \mathcal{M}$ and *every* path $C_{\phi_2\phi_1}$ from ϕ_1 to ϕ_2 one has

$$W^*(C_{\phi_2\phi_1})\, T(\phi_1) = T(\phi_2). \tag{1.5}$$

where $W^*(C) = W(C)^{\otimes k} \otimes (W(C^{-1})^t)^{\otimes l}$ denotes the parallel transport (Wilson integral) in the representation appropriate for the tensor T. Then, by the definition 1.1, the tensor $T(\phi)$ at ϕ is invariant under the group $\mathrm{Hol}(g) \subset O(T_\phi)$ (again acting in the appropriate tensor representation).

[1] As usual in the differential–geometric language, we identify a connection A and its associated covariant derivative $d + A$ since one determines uniquely the other.

[2] To save print, we write simply T_ϕ for the fiber $T_\phi\mathcal{M}$ of the tangent bundle at the point ϕ. $O(T_\phi)$ is the group of endomorphism of T_ϕ which preserves the fiber metric.

Conversely, given a tensor on $T_{\phi_0}\mathcal{M}$ which is invariant under $\mathrm{Hol}(\phi_0)$, we can construct a tensor field on $\mathcal{M}$ which is invariant by parallel transport by defining $T(\phi)$, for all $\phi \in \mathcal{M}$, by the formula (1.5). Obviously $T(\phi)$ is independent of the choice of path $C_{\phi\phi_0}$. On the other hand, we known from the general theory of Riemannian geometry that a tensor T is invariant under parallel transport if and only if its covariant derivative vanishes, $\nabla_i T = 0$. Therefore

FUNDAMENTAL PRINCIPLE 1.1. *Let $\mathcal{M}, g$ be a Riemannian manifold. The following three properties are equivalent:*

1. *there exists on $\mathcal{M}$ a tensor field T of type (k, l) which has zero covariant derivative $\nabla_i T = 0$;*
2. *there exists on $\mathcal{M}$ a tensor field of type (k, l) which is invariant under parallel transport;*
3. *there esists a point $\phi \in \mathcal{M}$ and a tensor $T_0 \in (T_\phi)^{\otimes k} \otimes \big(T_\phi^\vee\big)^{\otimes l}$ which is invariant under the appropriate holonomy representation* $\mathrm{Hol}(\phi)$.

Thus, the geometric problem of finding all the parallel structures (tensors with vanishing covariant derivative) on a given manifold $\mathcal{M}$ — which is *our* basic concern — is equivalent to the algebraic problem of finding the invariants of the holonomy group $\mathrm{Hol}(g)$. Therefore, our program reduces to the following:

1. find all Lie groups G and representations R_G, such that G is the holonomy group of some Riemannian manifold $\mathcal{M}$, with G acting on $T\mathcal{M}$ according to the R_G representation;
2. classify all invariants ($\equiv$ parallel structures) for given (G, R_G), and see for which pairs (G, R_G) these parallel structures coincide with the ones required by SUSY/SUGRA;
3. determine the metric geometries corresponding to each (G, R_G) and understand their properties.

Thus we are left with two basic questions:

- **(A):** *Which Lie groups G may be holonomy groups of a Riemannian manifold? Which are the allowed representations of G on $T_\phi\mathcal{M}$?*
- **(B):** *Given the holonomy group* $\mathrm{Hol}(g)$*, what can we say about the metric geometry of $\mathcal{M}$?*

We start with **(A)**. As a first step, we consider the holonomy of the 'trivial' situations, namely the Riemannian products $\mathcal{M}_1 \times \mathcal{M}_2$, *i.e.* the target spaces of *decoupled* σ–models.

1.2. Riemannian products. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Riemannian manifolds with metrics g_1 and g_2, respectively. By the *Riemannian product* of the two manifolds we mean the product manifold $\mathcal{M}_1 \times \mathcal{M}_2$ equipped with the metric $g_1 \oplus g_2$, that is, in local coordinates,

$$g_1(x)_{ij}\, dx^i\, dx^j + g_2(y)_{ab}\, dy^a\, dy^b.$$

The subbundles of $T(\mathcal{M}_1 \times \mathcal{M}_2)$ corresponding to vectors tangent, respectively, to $\mathcal{M}_1$ and $\mathcal{M}_2$ are *involutive*[3]. The corresponding integral submanifolds are, obviously, of the form $\{\phi_1\} \times \mathcal{M}_2$ and $\mathcal{M}_1 \times \{\phi_2\}$ and are also *totally geodesic submanifolds*[4].

Definition 1.2. A Riemann manifold $\mathcal{M}$ is (resp. *locally) reducible* if it is (resp. locally) isometric to a Riemann product.

The two tangent subbundles $T\mathcal{M}_{1,2} \subset T(\mathcal{M}_1 \times \mathcal{M}_2)$ are manifestly invariant under the holonomy group $\mathrm{Hol}(g_1 \oplus g_2)$. Since we assume $\mathcal{M}_{1,2}$ to be simply–connected (and hence orientable) we have two forms ϵ_1 and ϵ_2, corresponding, respectively, to the volume forms of the two factor manifolds which are *parallel*. By the general principle 1.1, $\mathrm{Hol}(g_1 \oplus g_2)$ is a subgroup of[5] $SO(n_1 + n_2)$ leaving invariant the two complementary and orthogonal subspaces $T\mathcal{M}_{1,2}$. Thus $\mathrm{Hol}(g_1 \oplus g_2) \subseteq SO(n_1) \times SO(n_2) \subset SO(n_1 + n_2)$. In fact

$$\mathrm{Hol}(g_1 \oplus g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2) \tag{1.6}$$

since $W(C) = W(\pi_1 C)_1 \times W(\pi_2 C)_2$, where $\pi_{1,2} \colon \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_{1,2}$ are the natural projections. This result has a local converse:

Theorem 1.1. *Let $\mathcal{M}, g$ be a Riemannian manifold and $\phi \in \mathcal{M}$ a point. Let $T_0 \subset T_\phi \mathcal{M}$ the subspace on which $\mathrm{Hol}(\phi)$ acts trivially, and let $T_0^\perp$ its orthogonal complement. Since $\mathrm{Hol}(\phi) \subset O(T_\phi)$, $T_0^\perp$ can be decomposed into the orthogonal sum of irreducible representations of the holonomy group*

$$T_0^\perp = T_1 \oplus T_2 \oplus \cdots \oplus T_k. \tag{1.7}$$

Then $\mathcal{M}, g$ is locally a Riemannian product $g_0 \oplus g_1 \oplus \cdots \oplus g_k$ where g_0 is flat. Moreover $\mathrm{Hol}(\phi)$ is a direct sum of representation

$$\mathrm{Hol}(\phi) = H_1 \times H_2 \times \cdots \times H_k \tag{1.8}$$

where $H_l \subseteq O(T_l)$ acting irreducibly on T_l and trivially on T_j, $j \neq l$.

Proof. Consider the corresponding orthogonal splitting of the tangent bundle, $T\mathcal{M} \simeq \oplus_{l=0}^k \mathcal{T}_l$, where $\mathcal{T}_l$ is obtained by parallel transporting T_l on $\mathcal{M}$. Each subbundle is involutive: in fact if $X \in \mathcal{T}_l$, also $\nabla_Y X \in \mathcal{T}_l$, since $\mathcal{T}_l$ is preserved by parallel transport; hence $X, Y \in \mathcal{T}_l \Rightarrow [X,Y] \equiv \nabla_X Y - \nabla_Y X \in \mathcal{T}_l$. By Frobenius' theorem[6], the $\mathcal{T}_l$'s are integrable; then $\mathcal{M}$ is *locally* diffeomorphic to $\mathcal{F}_0 \times \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$, where $T\mathcal{F}_l \simeq \mathcal{T}_l$. Since $T\mathcal{F}_l \perp T\mathcal{F}_j$ for $l \neq j$, the metric g has the 'block–diagonal' form $g_0 \oplus g_1 \oplus \cdots \oplus g_k$. Let $x^{(l,\alpha)}$

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[3] *Definitions:* **(1)** a subbundle $\mathcal{D} \subset T\mathcal{M}$ is said to be *involutive* if, given any two vector fields in $\mathcal{D}$, $X = X^i \partial_i$ and $Y = Y^i \partial_i$, their Lie bracket $[X,Y] = \pounds_X Y \in \mathcal{D}$; **(2)** a subbundle $\mathcal{D} \subset T\mathcal{M}$ is said to be *integrable* if for each point $\phi \in \mathcal{M}$ there is a submanifold $\iota \colon \mathcal{S} \to \mathcal{M}$, with $\phi \in \mathcal{S}$, such that $\mathcal{D}\big|_{\mathcal{S}} = T\mathcal{S}$.

The *Frobenius theorem* [**90**] states that $\mathcal{D}$ *is integrable if and only if it is involutive.* For the dual statement in terms of differential forms, see *e.g.* ref. [**91**].

[4] *Definition:* A Riemann submanifold $\iota \colon \mathcal{S} \to \mathcal{M}$ (equipped with the metric $\iota^* g$ induced by the immersion ι) is a *totally geodesic submanifold* if all the geodesics of $\mathcal{S}$ are also geodesics of the ambient space $\mathcal{M}$. Equivalently, the *second fundamental form* of $\mathcal{S}$ vanishes [**94**].

[5] We have set $n_{1,2} = \dim \mathcal{M}_{1,2}$ to save print.

[6] See footnote 3 on page 96.

be local coordinates on $\mathcal{F}_l$. Invariance under parallel transport of the $T\mathcal{F}_l$'s requires[7]

$$\Gamma^{(l,\alpha)}_{(j,\beta)\,(k,\gamma)} = 0 \quad \text{unless} \;\; l = j = k.$$

Now, for $j \neq l$

$$\partial_{(j,\alpha)} g_{(l,\beta)\,(l,\gamma)} = g_{(l,\gamma)\,(l,\delta)}\,\Gamma^{(l,\delta)}_{(j,\alpha)\,(l,\beta)} + \big(\beta \leftrightarrow \gamma\big) = 0,$$

and hence the metric–block g_l is a function of the $x^{(l,\alpha)}$'s only. □

This local result has a global version.

THEOREM 1.2 (de Rham). *If a Riemannian manifold $\mathcal{M}, g$ is complete, simply–connected, and if its holonomy acts reducibly on $T\mathcal{M}$, then $\mathcal{M}, g$ is a Riemannian product.*

1.3. Holonomy and curvature. The variation of a vector $v \in T_\phi\mathcal{M}$ under parallel transport along the infinitesimal parallelogram spanned by ϵx^i and ϵy^i is given by (see R. Percacci lecture notes §. **6.5**):

$$\Delta v^i = -\epsilon^2\, x^k\, y^l\, R_{kl}{}^i{}_j\, v^j$$

where $R_{kl}{}^i{}_j$ is the Riemann tensor. Hence the Lie algebra $\mathfrak{hol}(\phi)$ of $\mathrm{Hol}(\phi) \subset O(T_\phi)$ certainly contains all the curvature endomorphisms $R(x,y) \in \mathrm{End}(T_\phi)$ for all $x, y \in T_\phi$. Let ϕ, ϕ' two points in $\mathcal{M}$, and $C_{\phi'\phi}$ a path connecting them. We construct the following loop $\widetilde{C}$ starting and ending at ϕ. First we go to ϕ' along $C_{\phi'\phi}$, then we do an infinitesimal loop around ϕ' of sizes $\epsilon\, x^i$, $\epsilon\, y^j$, and then we return back trough $C^{-1}_{\phi'\phi}$. The variation of the vector $v^i \in T_\phi$ under the parallel transport along the circuit $\widetilde{C}$ is given by

$$\Delta v^i = -\epsilon^2 \left(W(C_{\phi'\phi})^{-1}\, R(x,y)'\, W(C_{\phi'\phi})\right)^i{}_j\, v^j$$

where $R(x,y)' = x^k y^l R_{kl}(\phi') \in \mathrm{End}(T_{\phi'})$ is the Riemann tensor at the point ϕ'. Hence

$$\begin{gathered} W(C_{\phi'\phi})^{-1}\, R(x,y)'\, W(C_{\phi'\phi}) \in \mathfrak{hol}(\phi), \\ \forall\ \phi' \in \mathcal{M}, \ \ \forall\ C_{\phi'\phi}, \ \ \text{and} \ \ \forall\ x, y \in T_{\phi'}, \end{gathered} \tag{1.9}$$

which, in particular, implies

THEOREM 1.3 (Nijenhuis [**92**]). *$R_{ijk}{}^l \in \mathfrak{hol}$ and $\nabla_m R_{ijk}{}^l \in \mathfrak{hol}$.*

Hence the Lie algebra $\mathfrak{hol}(g)$ contains all the curvature endomorphisms at every point of $\mathcal{M}$, and for all pairs of tangent vectors, everything pulled back to a fixed point ϕ along all possible parallel transportations. It is quite a big deal, but it is all:

THEOREM 1.4 (Ambrose–Singer [**93**]). *The Lie algebra $\mathfrak{hol}(\phi)$ is exactly the subalgebra of $\mathfrak{so}(T_\phi\mathcal{M})$ generated by all the elements in eqn.*(1.9).

This theorem is true for any connection, not only for a Riemannian one. For, say, Yang–Mills theory, this fact is well–known, and it is physically encoded in the loop–equations. The physical idea may be formalized into a rigorous proof [**92**]. Here is a proof for the more mathematically–minded:

[7] We use the fact that the Christoffel connection is torsion–less (that is *symmetric*).

PROOF ∗. Fix an orthonormal frame e_0^a at some point $\phi_0 \in \mathcal{M}$. Consider the space P of all frames e^a over $\mathcal{M}$ obtained by parallel–transporting the frame e_0^a along all curves starting from ϕ. P is a principal bundle over $\mathcal{M}$ with structure group $\mathrm{Hol}(\phi_0)$. Consider the map

$$\alpha\colon P \otimes \wedge^2 T\mathcal{M} \to \mathfrak{so}(n),$$
$$(\phi, e^a, x, y) \mapsto x^i\, y^j\, R_{ijk}{}^l\, e^{a\,k}\, e^b{}_l \;\in\; \mathfrak{so}(n)$$

where $R_{ijk}{}^l$ is the Riemann tensor at the (generic) point $\phi \in \mathcal{M}$. The theorem is equivalent to the statement

$$\mathfrak{hol}(\phi_0) = \mathrm{Im}\,\alpha \equiv \mathfrak{a}.$$

Consider the subbundle $\mathcal{E}$ of TP of the vector fields of the form

$$v^i\left(\frac{\partial}{\partial \phi^i} - \Gamma^l_{ik}\, e^{a\,k}\, \frac{\partial}{\partial e^{a\,l}}\right) + \omega_a{}^b\, e^{a\,k}\frac{\partial}{\partial e^{b\,k}}$$

where $v^i(\phi)$ is a vector field on $\mathcal{M}$ and $\omega_a{}^b$ is a field taking value in the Lie algebra $\mathfrak{a}$. By construction $\mathcal{E}$ is involutive. Hence, by Frobenius' theorem, there exists an integral manifold. Let L be the maximal integral manifold through the base point (ϕ_0, e_0^a). By the construction of P, one has $L = P$ and hence $\mathfrak{a} = \mathfrak{hol}$. □

2. Symmetric Riemannian spaces

At the face value, question **(A)** of §.1.1 has a very easy answer:

> *All compact groups G are the holonomy group of some Riemannian manifold.*

Indeed, take as Riemannian space G itself with the (unique up to overall normalization) metric which is invariant under both right and left translations[8]. Then[9] $\mathrm{Hol}(G) = G$.

However, this is quite a special example. All connections, curvatures, *ect.* are constructed using the commutators[10] in the Lie algebra $\mathfrak{g}$ of G. In particular,

$$\nabla_i\, R_{jkl}{}^m = 0, \tag{2.1}$$

and the Riemann tensor *itself* is PARALLEL.

To get an interesting classification theorem for the Riemannian holonomy groups (and the corresponding representations), we have to leave apart such 'cheap' algebraic examples, namely manifold in which the Riemann tensor is, essentially, a constant numerical tensor.

Therefore we have preliminary to study (and classify) the manifolds in which the full curvature tensor is parallel in order to separate these peculiar instances from the rest.

2.1. Definitions. For completeness, we begin by reviewing some geometrical facts.

[8] This metric always exists for G compact: see *e.g.* [**94**] **prop. 26.2** as well as chapter 5 below.

[9] This will be shown in sect. 2.3.

[10] *Hoy, hoy... a bell rings...*

2.1.1. *Geodesic symmetries.* Let $\mathcal{M}$ be a Riemannian manifold[11] and $\phi_0 \in \mathcal{M}$ a point. The *exponential map*

$$\exp_{\phi_0} \colon T_{\phi_0}\mathcal{M} \to \mathcal{M} \tag{2.2}$$

associates to each vector $X \in T_{\phi_0}\mathcal{M}$ the point $\gamma_X(1) \in \mathcal{M}$, where

$$\gamma_X(\tau) \colon [0,1] \to \mathcal{M} \tag{2.3}$$

is the unique geodesic starting at ϕ_0 and having initial velocity

$$\left.\frac{d\gamma_X}{d\tau}\right|_{\tau=0} = X \in T_{\phi_0}\mathcal{M}. \tag{2.4}$$

We write $\exp(X)$ for the image of the exponential map. $\exp(\cdot)$ is a *local* diffeomorphism from a neighborhood of the origin in $T_{\phi_0}\mathcal{M}$ to a neighborhood of ϕ_0 in $\mathcal{M}$. *A priori,* $\exp(\cdot)$ is defined only locally near ϕ_0. However, if $\mathcal{M}$ is *complete*[12], the exponential map is defined *globally* [but it is not a diffeomorphism any more, since for $\|X\| \geq \rho$ (the injectivity radius [**95**]) $\exp(X)$ is not longer injective].

DEFINITION 2.1. By the *geodesic symmetry* s_{ϕ_0} at the point ϕ_0 we mean the map (defined only locally, unless $\mathcal{M}$ is *complete*)

$$s_{\phi_0} \colon \exp(X) \mapsto \exp(-X), \tag{2.5}$$

i.e. the exponential image of a 'parity' transformation in the tangent space at ϕ_0.

REMARK. If $\mathcal{M}$ is complete and also *simply–connected,* s_ϕ is globally an isometry, see [**94**], **Theorem 5.1**.

DEFINITION 2.2. A Riemannian manifold $\mathcal{M}$ is *locally symmetric* if for each point ϕ the geodesic symmetry s_ϕ (which is defined only locally) is an isometry. $\mathcal{M}$ is *symmetric* if for each point ϕ the geodesic symmetry is a globally defined isometry. Equivalently, $\mathcal{M}$ is symmetric if for all $\phi \in \mathcal{M}$ there is an isometry s_ϕ of $\mathcal{M}$ such that:

$$s_\phi(\phi) = \phi, \qquad s_{\phi *}\big|_\phi = -\mathrm{Id}_{T_\phi\mathcal{M}}. \tag{2.6}$$

Hence, a *complete, connected, simply connected* locally symmetric manifold is (globally) symmetric [**94**] **Theorem 5.1**.

[11] More generally, an affine connection space.

[12] The equivalence between the different notions of *completeness* is the following:

THEOREM 2.1 (Hopf–Rinow). *For a Riemannian $\mathcal{M}$ the following are equivalent:*

1. *$\mathcal{M}$ is geodesically complete (i.e. each maximal geodesic $\gamma_{\phi\,X}(\tau)$ is defined for τ on the entire real axis $\mathbb{R}$);*
2. *$\mathcal{M}$ is complete as a metric space (i.e. the Cauchy sequences converges);*
3. *the bounded subset of $\mathcal{M}$ are relatively compact.*

If one (hence all) of these conditions holds, given two points $\phi, \phi' \in \mathcal{M}$ there exists at least one geodesic connecting them.

2.1.2. *Parallel curvatures.* The relevance of the symmetric spaces for us stems from the following:

PROPOSITION 2.1. *A Riemannian manifold $\mathcal{M}$ is* locally symmetric *if and only if its Riemann tensor is* PARALLEL:

$$\nabla_i R_{jkl}{}^m = 0. \tag{2.7}$$

PROOF. Let $\mathcal{M}$ be symmetric. The Riemann tensor is invariant under any isometry[13], so $R = s_\phi^* R$. By eqn.(2.6),

$$s_\phi^* \nabla R\big|_\phi = -\nabla R\big|_\phi = 0. \tag{2.8}$$

Conversely, let R be covariantly constant. Fix a point ϕ. The *normal coordinates* x^i centered at ϕ (see refs.[**97, 98, 99, 100, 26**]) are defined by taking an orthonormal basis e_i on $T_\phi\mathcal{M}$ and parameterizing (locally) $\mathcal{M}$ in the form $\exp_\phi[x^i\, e_i]$. In these coordinates the metric[14] has a Taylor expansion with coefficients given by covariant expressions in the Riemann tensor and its covariant derivatives computed at the base point ϕ

$$\begin{aligned} g_{ij}(x) = \delta_{ij} &- \frac{1}{3} R_{ikjl}\, x^k\, x^l + \frac{1}{6}\nabla_s R_{iklj}\, x^k\, x^l\, x^s + \\ &+ \left(\frac{1}{20}\nabla_s\nabla_t R_{iklj} + \frac{2}{45} R_{ikl}{}^m R_{istm}\right) x^k\, x^l\, x^s\, x^t + \cdots \end{aligned} \tag{2.9}$$

if R_{ijkl} is parallel, the coefficients in the RHS are products of Riemann tensors, $R_{i_1j_1k_1l_1}\, R_{i_1j_1k_1l_1} \cdots R_{i_1j_1k_1l_1}$, with the indices contracted either with δ^{kl} or with x^l. By index counting, all terms have an *even* number of x^l's. Hence from eqn.(2.9)

$$g_{ij}(-x) = g_{ij}(x), \tag{2.10}$$

that is the geodesic symmetry at ϕ is a (local) isometry. Since ϕ was arbitrary, $\mathcal{M}$ is (locally) symmetric. □

2.2. Cartan theorem. So, (for a complete, simply–connected manifold) the conditions of *being symmetric* and of *having a parallel Riemann tensor* are equivalent. But there is a lot more. We have the wonderful

THEOREM 2.2 (Cartan). *$\mathcal{M}$ a* complete *Riemannian manifold. $\mathcal{M}$ is (globally) symmetric if and only if there is a homogeneous space G/H with G a connected Lie group, H a compact subgroup of G, and there is an involutive automorphism σ of the group G for which, if S denotes the fixed set of σ and S_e its connected component of the identity, one has $S_e \subset H \subset S$. The symmetric metric of G/H is invariant under G.*

REMARK. Not all *homogeneous* spaces G/H are symmetric. Symmetry requires the existence of an involutive automorphism σ of G having the properties stated in the theorem. σ is part of the definition of the symmetric structure on the manifold G/H, and an isomorphism of symmetric manifolds

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[13] We omit the indices in the Riemann tensor and covariant derivative.

[14] 2$^{\text{nd}}$ CARTAN THEOREM (**prop. E.III.7** of ref.[**100**]): *The coefficients of the Taylor expansion of* $\exp_\phi^* g(x)$ *near* $x = 0 \in T_\phi\mathcal{M}$ *are* UNIVERSAL *polynomials in the covariant derivatives of the curvature tensor at the point* ϕ.

is defined to be an isometry preserving σ. In particular, certain manifolds can be written as *homogeneous* spaces in more than in one way. For example:

$$S^{2n+1} = \frac{SO(2n+2)}{SO(2n+1)} = \frac{U(n+1)}{U(1)\cdot U(n)}, \quad S^6 = \frac{SO(7)}{SO(6)} = \frac{G_2}{SU(3)}, \ \textit{ect.} \tag{2.11}$$

On the contrary, the *symmetric* space G/H representation (if it exists) is always *unique*. This will be evident from the proof of the theorem.

Before proving the theorem, we establish two simple lemmas.

Consider the isometry group $\mathrm{Iso}(\mathcal{M})$. It is a Lie group[15].

LEMMA 2.1. *$\mathcal{M}$ complete and symmetric. The isometry group* $\mathrm{Iso}(\mathcal{M})$ *acts* TRANSITIVELY *on $\mathcal{M}$.*

That is given two points $\phi, \phi' \in \mathcal{M}$, there exists $g \in \mathrm{Iso}(\mathcal{M})$ such that $\phi' = g \cdot \phi$.

PROOF. By Hopf–Rinow (cfr. footnote 12 on page 99) there is a geodesic passing trough the two points. Let $\widetilde{\phi}$ the middle point on the geodesic arc between ϕ and ϕ'. $s_{\widetilde{\phi}}$ is an isometry mapping ϕ into ϕ' (and *viceversa).* □

LEMMA 2.2. *Each smooth manifold $\mathcal{X}$ on which a Lie group G acts smoothly and transitively is diffeomorphic to the quotient manifold G/H, where H is the stabilizer of an arbitrary point $p_0 \in \mathcal{X}$. The diffeomorphism*

$$\varphi\colon G/H \to \mathcal{X}$$

is given by

$$\varphi(g\,H) = g \cdot p_0, \qquad g \in G. \tag{2.12}$$

Obvious. [If you do not find it obvious, see ref. [**94**] **lemma 9.3**].

PROOF. (of Cartan theorem) **(1)** Let G be a Lie group, H a Lie subgroup, and σ an involutive automorphism of G such that

$$\mathrm{Fix}(\sigma)_e \subset H \subset \mathrm{Fix}(\sigma).$$

We have to show that G/H equipped with σ is a *symmetric space.* Since the quotient is smooth, it is enough to show that for all points $p \in G/H$ there is a geodesic isometry s_p. A point in the coset has the form gH for $g \in G$. Let $p_i = g_i H$, $i = 1, 2$. We set

$$s_{p_1}\, p_2 = g_1\, \sigma(g_1^{-1}\, g_2) H. \tag{2.13}$$

It is easy to check that $s_{p_1}^2 = 1$. It is an isometry since both multiplication by an element $g \in G$ (on the left) and σ are isometries[16].

(2) Conversely, assume $\mathcal{M}$ is complete and symmetric. Let $G = \mathrm{Iso}(\mathcal{M})$. Fix a point $\phi_0 \in \mathcal{M}$, and let $H \subset G$ be the stabilizer of ϕ_0. We define a map $\sigma\colon G \to G$ by

$$\sigma(g) = s_{\phi_0}\, g\, s_{\phi_0}. \tag{2.14}$$

[15] This is the content of the *Myers–Steenrod* theorem [**96**].

[16] In facts one constructs the homogeneous coset metric using the Lie algebras of G and H, and hence all algebraic automorphism should be isometries. See chapter 5 where the invariant metrics are discussed in detail.

σ is an involutive automorphism of G. For any automorphism $\rho\colon \mathcal{M} \to \mathcal{M}$, and any point $\phi \in \mathcal{M}$, we have

$$\rho \circ s_\phi \circ \rho^{-1} = s_{\rho(\phi)}.$$

Hence for $\forall\, h \in H$, we have $h\, s_{\phi_0}\, h^{-1} = s_{h\cdot\phi_0} = s_{\phi_0}$, so

$$\sigma(h) = s_{\phi_0}\, h\, s_{\phi_0} = s_{\phi_0}^2\, h = h \quad \Rightarrow \quad H \subset \mathrm{Fix}(\sigma).$$

Let $A \in T_e(\mathrm{Fix}(\sigma))$. $\sigma_* A = A$, and therefore $\sigma(\exp \tau A) = \exp(\tau A)$, for all $\tau \in \mathbb{R}$. Now

$$s_{\phi_0}((\exp \tau A)\phi_0) = (s_{\phi_0}\, \exp \tau A)\phi_0 = (\exp \tau A\, s_{\phi_0})\phi_0 = (\exp \tau A)\phi_0,$$

that is the point $(\exp \tau A)\phi_0$ is a fixed point of s_{ϕ_0}. But ϕ_0 is an isolated fixed point, and hence $(\exp \tau A)\phi_0 = \phi_0$. So $\exp \tau A \in H$, which means $\mathrm{Fix}(\sigma)_e \subset H$. Thus

$$\mathrm{Fix}(\sigma)_e \subset H \subset \mathrm{Fix}(\sigma).$$

The map in eqn.(2.12)

$$\varphi\colon G/H \to \mathcal{M} \tag{2.15}$$

is a diffeomorphism. Now, from eqns.(2.13)(2.14), for all $g_i \in G$,

$$\begin{aligned}\varphi(s_{g_1}\, g_2 H) &= \varphi\big(g_1 \sigma(g_1^{-1}\, g_2)H\big) = g_1 \sigma(g_1^{-1}\, g_2) \cdot \phi_0 = \\ &= g_1\, s_{\phi_0}\, g_1^{-1}\, g_2\, s_{\phi_0} \phi_0 = s_{g_1\, \phi_0}\, g_2\, s_{\phi_0} \phi_0 = \\ &= s_{g_1\, \phi_0}(g_2\, \phi_0)\end{aligned}$$

so the diffeomorphism φ is an automorphism of symmetric spaces. The argument also shows that the representation G/H is unique. □

2.2.1. *Lie algebraic constructions.* We can prove Cartan's theorem from a different viewpoint, perhaps more convenient. We ask ourselves: *given a manifold $\mathcal{M}$ with a* PARALLEL CURVATURE, *can we find directly two groups G and H such that $\mathcal{M} \simeq G/H$?*

Yes! Fix a point $\phi_0 \in \mathcal{M}$. Let $\mathfrak{m} = T_{\phi_0}\mathcal{M}$. For $x, y \in \mathfrak{m}$, $R(x, y) \in \mathfrak{so}(\mathfrak{m}) \subset \mathrm{End}(\mathfrak{m})$. Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{so}(\mathfrak{m})$ generated by the $R(x, y)$ for all $x, y \in \mathfrak{m}$. I claim that the direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ has a natural structure of Lie algebra precisely iff the curvature is parallel. In fact, define the bracket

$$[x, y] = R(x, y) \in \mathfrak{h}, \qquad x, y \in \mathfrak{m} \tag{2.16}$$

$$[r, x] = r \cdot x \in \mathfrak{m} \qquad x \in \mathfrak{m},\ r \in \mathfrak{h} \subset \mathrm{End}(\mathfrak{m}) \tag{2.17}$$

$$[r, s] = r \cdot s - s \cdot r \in \mathfrak{h} \subset \mathrm{End}(\mathfrak{m}) \qquad r, s \in \mathfrak{h}. \tag{2.18}$$

We have to check the Jacobi identities:

$$\begin{aligned}&\big([[x, y], z] + [[y, z], x] + [[z, x], y]\big)^i = \\ &\qquad = -\big(R_{klm}{}^i + R_{lmk}{}^i + R_{mkl}{}^i\big) x^k\, y^l\, z^m \equiv 0\end{aligned}$$

$$[[x, r], s] + [[r, s], x] + [[s, x], r] \equiv [r, s] \cdot x + (s \cdot r - r \cdot s) \cdot x \equiv 0$$

$$\begin{aligned}&[[x, y], r] + [[y,r], x] + [[r, x], y] = \\ &\qquad = R(x, y)\, r - r\, R(x, y) - R(r \cdot y, x) + R(r \cdot x, y),\end{aligned}$$

only the last one is non–trivial. r is a linear combination of endomorphism of $\mathfrak{m}$ of the form $R(w,z)$, so the last Jacobi identity is equivalent to

$$\begin{aligned}\big(R_{ij}{}^{m}{}_{p}\,R_{kl}{}^{p}{}_{n} - R_{kl}{}^{m}{}_{p}\,R_{ij}{}^{p}{}_{n} - R_{kl}{}^{p}{}_{j}\,R_{pi}{}^{m}{}_{n} + R_{kl}{}^{p}{}_{i}\,R_{pj}{}^{m}{}_{n}\big) \equiv \\ \equiv x^i y^j w^k z^l\,[\nabla_k, \nabla_l]\,R_{ij}{}^{m}{}_{n} = 0,\end{aligned}$$

since the Riemann tensor is parallel. $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively. The involution σ_* acts on the Lie algebra $\mathfrak{g}$ as the automorphism

$$\sigma_*\big|_{\mathfrak{h}} = \mathrm{Id}, \qquad \sigma_*\big|_{\mathfrak{m}} = -\mathrm{Id}. \tag{2.19}$$

The exponential map of $(\mathfrak{g} \mod \mathfrak{h}) \simeq T_e(G/H)$ and that of $T_{\phi_0}\mathcal{M}$ coincide, then the metric written in normal coordinates, see eqn.(2.9), is the same for G/H and $\mathcal{M}$. Hence the two manifolds are (locally) isometric.

* * *

We shall discuss symmetric spaces in greater detail in chapt. 5: models, realizations, metrics, connections, *ect.* Here we are only interested at them for their peculiar rôle in the general theory of holonomy groups.

This is our next subject.

2.3. The holonomy group of a symmetric space. As before, for simplicity we replace all manifolds with their universal covers, so, we assume $\mathcal{M}$ to be *simply connected.* By the de Rham theorem (theorem 1.2) we can limit ourselves to *irreducible* manifolds without loss of information.

Now the question is: What is the holonomy group of a symmetric manifold G/H?

PROPOSITION 2.2. *The holonomy group* $\mathrm{Hol}(G/H)$ *of an irreducible, simply–connected symmetric space G/H is equal to H, and its action on $T(G/H)$ is induced by the adjoint representation of G.*

PROOF. By the Ambrose–Singer theorem 1.4, $\mathfrak{hol}(G/H)$ is generated by all the parallel transports of the endomorphism $R(x,y)$. Since the Riemann tensor of a symmetric space is invariant under parallel transport, $\mathfrak{hol}(G/H)$ is generated by the curvature at a fixed point, say at $e\,H$, that is it is generated by the endomorphism of $x^k\,y^l\,R_{kl}{}^{i}{}_{j}$ of $T_e(G/H) \simeq \mathfrak{m}$, compare with §. 2.2.1. By the analysis we did there, $\mathfrak{h} \subset \mathrm{End}(\mathfrak{m})$ is precisely the span of the endomorphisms $R(x,y)$. Hence $\mathfrak{h} = \mathfrak{hol}(G/H)$. □

2.4. Rank and transitive actions on spheres. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as in §. 2.2.1. Consider the maximal Abelian subalgebras of $\mathfrak{m}$. By a theorem of Cartan, two such algebras are conjugated under the adjoint action of H. Their common dimension is called the *rank* of the symmetric space.

For our present purposes, the following result is of interest

PROPOSITION 2.3. *A symmetric space G/H has rank* 1 *if and only if the action of H on the unit tangent sphere in $T_e(G/H) \simeq \mathfrak{m}$ is* TRANSITIVE.

PROOF. Let $x \in \mathfrak{m}$, and $H \cdot x$ its H–orbit in $\mathfrak{m}$. The tangent space to the orbit in x is, by definition, given by the elements $[\mathfrak{h}, x]$. Suppose the action of H is <u>not</u> transitive. Then there is a vector, y, not proportional to x, which is orthogonal to the orbit of x at x, namely

$$0 = \langle [\mathfrak{h}, x], y\rangle = \langle \mathfrak{h}, [x, y]\rangle \quad \text{(by invariance of the Killing metric on } \mathfrak{g}).$$

$[x, y]$ is an element of $\mathfrak{h}$ which is orthogonal to all elements of $\mathfrak{h}$ and hence zero. Then we have at least two non–zero commuting elements of $\mathfrak{m}$, that is rank $(G/H) \geq 2$. □

The compact symmetric space of rank 1 are:

$$S^n = \frac{SO(n+1)}{SO(n)} \qquad P\mathbb{C}^n = \frac{SU(n+1)}{U(1)\times SU(n)} \tag{2.20}$$

$$P\mathbb{H}^n = \frac{Sp(2n+2)}{Sp(2)\times Sp(2n)} \qquad \frac{F_4}{SO(9)} \tag{2.21}$$

i.e. the spheres and the projective spaces over $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$.

3. Berger's theorem

We are reduced to the problem of classifying the holonomy groups of (simply–connected) Riemannian manifolds which are *irreducible* and <u>not</u> *symmetric*. This is precisely the content of Berger's theorem. We shall give two different statements of this theorem[17] and a third equivalent statement in the form of a corollary:

THEOREM 3.1 (Berger's theorem [**101**]). *Let $\mathcal{M}$, g be a Riemannian manifold of dimension n. Assume* $\mathrm{Hol}^0(g)$ *is irreducible and* $\nabla_i R_{ijkl} \not\equiv 0$. *Then* $\mathrm{Hol}^0(g)$ *is one of the following:*

$SO(n)$	
$U(n/2)$	n even
$SU(n/2)$	n even
$Sp(2)\times Sp(n/2)$	$n = 4m$
$Sp(n/2)$	$n = 4m$
G_2	$n = 7$
$Spin(7)$	$n = 8$.

THEOREM 3.2 (Simons' version [**102, 103**]). *Let $\mathcal{M}$, g be a Riemannian manifold and assume* $\mathrm{Hol}^0(g)$ *to be irreducible. Then either* $\mathrm{Hol}^0(g)$ *is transitive on the unit sphere in $T_\phi\mathcal{M}$, or $\mathcal{M}$ is a locally symmetric space of rank $r \geq 2$.*

COROLLARY 3.1. *Let $\mathcal{M}$ be Riemannian irreducible. Let $T_{i_1 i_2\cdots i_l}$ be a* symmetric parallel *tensor which is not proportional to $g_{(i_1 i_2} g_{i_3 i_4}\cdots g_{i_{l-1} i_l)}$. Then $\mathcal{M}$ is a symmetric space of rank ≥ 2.*

COROLLARY 3.2. *Let $\mathcal{M}$ be a (simply–connected) irreducible Riemannian manifold, Assume one of the following tangent bundle ismorphims (preserved by parallel transport):*

[17] Here we limit ourselves to the case of metric with Euclidean signature (*i.e.* positive–definite). The theorem is formulated for metrics of general signatures, as well as for non–metric, but still torsionless, connections. See sect.6 below.

TABLE 3.1. Transitive groups acting on spheres and homogeneous representations G/K of the S^m's

Sphere	G	K
S^{n-1}	$SO(n)$	$SO(n-1)$
S^{2n-1}	$U(n)$	$U(n-1)$
S^{2n-1}	$SU(n)$	$SU(n-1)$
S^{4n-1}	$Sp(2) \times Sp(2n)$	$Sp(2) \times Sp(2n-2)$
S^{4n-1}	$U(1) \times Sp(2n)$	$U(1) \times Sp(2n-2)$
S^{4n-1}	$Sp(2n)$	$Sp(2n-2)$
S^6	G_2	$SU(3)$
S^7	$Spin(7)$	G_2
S^{15}	$Spin(9)$	$Spin(7)$

(1) $(\mathbb{C} \otimes T\mathcal{M})\Big|_{(1,0)} \simeq \mathcal{V}_1 \otimes \mathcal{V}_2$ *with* $2 < \operatorname{rank} \mathcal{V}_1 \leq \operatorname{rank} \mathcal{V}_2$*;*

(2) $(\mathbb{C} \otimes T\mathcal{M})\Big|_{(1,0)} \simeq \wedge^k \mathcal{V}$ *with* $2 \leq k \leq \operatorname{rank} \mathcal{V}/2$*;*

then $\mathcal{M}$ *is symmetric.*

REMARK. Theorem 3.2 is more conceptual. Theorem 3.1 follows from theorem 3.2 and the well–known list of Lie groups which act transitively on spheres S^{n-1}, [**106, 107, 108**]. They corresponds to the groups G such that there is a representation $S^{n-1} = G/K$ for some isotropy subgroup $K \subset G$ *(warning:* G/K, in general, is homogeneous but not symmetric). See table 3.1. Comparing that table with thm. 3.1, we see that two groups which *do* act transitively on some sphere do not appear in the list of holonomy groups for non–symmetric Riemannian manifolds, namely $U(1) \times Sp(2m)$ and $Spin(9)$. In facts, a manifold with $\mathrm{Hol}^0 = Spin(9)$ is isometric to $P\mathbb{O}^2 \simeq F_4/Spin(9)$ or its non–compact dual (see chapt. 5), while for $U(1) \times Sp(2m)$ the Bianchi identity shows that the $U(1)$ part of the curvature is identically zero. We do not show this here, since it will follow from a computation we shall do below for the Quaternionic–Kähler manifolds. Strictly speaking, it is a computation we already did: it is just the computation of the $SU(2)_R$ curvature in $D = 4$ $\mathcal{N} = 2$ SUGRA (trivially: if the curvatures are commutators, the Abelian part should be zero).

REMARK. The first corollary is a useful diagnostics to determine if a given manifold $\mathcal{M}$ is in fact symmetric. Often in physics we have some natural symmetric tensors on $\mathcal{M}$ (the Yukawa couplings, say) and if we know that one of them is parallel we have identified the metric *exactly.*

The second corollary is a weak version of the theorem (the poor man's Berger theorem) which is easy to prove (see APPENDIX D). For the applications to SUSY/SUGRA this simple result is essentially sufficient.

There are three proofs of Berger's theorem [**101, 102, 103**]. No one is particularly short, nor easy, or illuminating. Sketchs of the Simon's proof can be found in [**86, 87**]. Besides, we have the proof of the more general classification theorem of general torsionless holonomy groups [**104**]. It is deep an elegant, but it requires a course of its own. I will resist the temptation of giving a proof. However in SUGRA one does not need the full power of the Berger's theorem; a weaker statement is sufficient. This weaker result corresponds to the very first lemmas in the proof of the theorem. The 'poor man's version' of the theorem is presented in APPENDIX D.

Notice that — with the only exception of $n = 7$ and $\mathrm{Hol}^0 = G_2$ — in *odd* dimension we have only one possible holonomy group, namely $SO(n)$.

All holonomy groups allowed by Berger's theorem are actually realized in some non–symmetric manifold $\mathcal{M}$. The manifolds with a given holonomy group have got names:

DEFINITION 3.1. A manifold $\mathcal{M}$ with holonomy group

$$\mathrm{Hol}^0 \subseteq \left\{\begin{matrix} SO(n) \\ U(m) \\ SU(m) \\ Sp(2)\times Sp(2m) \\ Sp(2m) \end{matrix}\right\} \text{ is called } \left\{\begin{matrix} \text{Riemannian} \\ \text{Kähler} \\ \text{Calabi–Yau} \\ \text{Quaternionic–Kähler} \\ \text{hyper–Kähler} \end{matrix}\right\}.$$

It is called *proper* Kähler — resp. Calabi–Yau, Quaternionic–Kähler, hyper–Kähler — if the inclusion $\subseteq$ is actually a strict equality $=$. The manifolds with $\mathrm{Hol}^0 = G_2$ and $Spin(7)$ are said to have *exceptional* holonomy.

REMARK. From the group inclusions we get the inclusions of sets:

$$\text{(hyperKähler man.)} \subset \text{(Calabi–Yau man.)} \subset \text{(Kähler manifolds)}$$
$$\text{(hyperKähler man.)} \subset \text{(Quaternionic–Kähler manifolds)}.$$

REMARK. From the definition it follows that the symmetric spaces G/H with $H = U(1) \times K$ are, in particular Kählerian (they are known as the Hermitean–symmetric spaces). Analogously, the symmetric spaces with $H = SU(2) \times K$ and holonomy representation $(\mathbf{2}, V)$ are Quaternionic–Kähler (they are known as Wolf spaces [**110**]). The only (non–hyperKähler) spaces which are both Kählerian and Quaternionic–Kählerian are the symmetric spaces

$$\frac{SU(2,k)}{U(1)\times SU(2)\times SU(k)} \quad \text{and} \quad \frac{SU(2+k)}{U(1)\times SU(2)\times SU(k)}.$$

The study of the geometry of manifolds with holonomy in the Berger list is one of the focus of the present lectures. It is crucial for SUSY/SUGRA and the superstring (as well as for general theoretical physics) in many different (and often unexpected) ways.

For future reference, it is useful to list the normalizers $\mathrm{Norm}(\mathrm{Hol}^0)$ and centralizers $\mathcal{C}(\mathrm{Hol}^0)$ of the diverse holonomy groups in $O(n)$.

PROPOSITION 3.1 ([**86, 101, 109**]). *For the symmetric spaces the normalizer of the holonomy representation is always a finite extension of* Hol^0

(and hence the centralizer is the Abelian part of Hol^0 *up to a finite group). For the Berger's groups we have:*

$$\begin{aligned}
&\mathrm{Norm}(SO(n)) = O(n), && \mathcal{C}(SO(n)) = \mathbb{Z}_2\\
&\mathrm{Norm}(U(m)) = U(m), && \mathcal{C}(U(m)) = U(1)\\
&\mathrm{Norm}(SU(m)) = U(m), && \mathcal{C}(SU(m)) = U(1)\\
&\mathrm{Norm}(Sp(2)\times Sp(2m)) = Sp(2)\times Sp(2m), && \mathcal{C}(Sp(2)\times Sp(2m)) = \mathbb{Z}_2\\
&\mathrm{Norm}(Sp(2m)) = Sp(2)\times Sp(2m), && \mathcal{C}(Sp(2m)) = Sp(2)\\
&\mathrm{Norm}(Spin(7)) = Spin(7), && \mathcal{C}(Spin(7)) = \mathbb{Z}_2\\
&\mathrm{Norm}(G_2) = G_2, && \mathcal{C}(G_2) = \mathbb{Z}_2.
\end{aligned}$$

4. Parallel forms on $\mathcal{M}$

We started our analysis from the existence of certain parallel tensors (in particular, forms) on $\mathcal{M}$; then we used the FUNDAMENTAL PRINCIPLE to convert this into a requirement about the holonomy group of $\mathcal{M}$ (which corresponds to the one obtained from the 'target space equivalence' isomorphisms). It is time to study the issue sistematically. So we ask: *what are the parallel tensors (apart for the metric and the Levi–Civita tensor* ϵ*) for each holonomy group allowed by the Berger theorem?*

We already know from corollary 3.1 that the parallel tensors cannot be symmetric.

PROPOSITION 4.1. *Let* $\mathcal{M}$ *be irreducible and non–symmetric. Then the parallel forms of degree* $1 \leq n \leq \dim\mathcal{M}-1$*, are:*

1. $\mathrm{Hol}^0 = SO(n)$*: none;*
2. $\mathrm{Hol}^0 = U(m)$*: the 2–form* $\omega := g_{ik}f^k{}_j d\phi^i\wedge d\phi^j$ *(called the Kähler form) and its (exterior) powers* ω^k*;*
3. $\mathrm{Hol}^0 = SU(m)$*: the Kähler form, its powers, a* $(m,0)$ *complex volume form* ε *and its* $(0,m)$ *conjugate* $\bar{\varepsilon}$*;*
4. $\mathrm{Hol}^0 = Sp(2m)$*: the three Kähler 2–forms* $\omega^a := g_{ik}f^{a\,k}{}_j d\phi^i\wedge d\phi^j$ *and all the polynomial algebra generated by them*[18]*.*
5. $\mathrm{Hol}^0 = Sp(2)\times Sp(2m)$*: the 4–form* $\Theta := \omega^a\wedge\omega^a$ *and its powers.*
6. $\mathrm{Hol}^0 = G_2$*: a 3 form* ϕ *and its dual 4–form* $*\phi$*.*
7. $\mathrm{Hol}^0 = Spin(7)$*: a self–dual 4 form* ϕ*.*

PROOF. Elementary group theory. Exercise. □

REMARK. The structure is quite constrained. For instance, assume that in an irreducible non–symmetric manifold $\mathcal{M}$, we have a parallel 2–form. We wish to show that this form comes from a parallel *complex structure* $f\in\mathrm{End}(T\mathcal{M})$, $f^2=-1$, trough the formula $\omega_{ij} = g_{ik}f^k_j$. Indeed consider $L_{ij} := \omega_{ik}\omega_{jl}\,g^{kl}$; it is symmetric and non–vanishing (its trace is $\|\omega\|^2$). Hence, by corl. 3.1, L_{ij} should be proportional to g_{ij}. Normalize ω so that the constant of proportionality is 1. Then $(\omega)_i{}^k(\omega^t)_k{}^j = \delta_i^j$, and since $\omega^t = -\omega$, $\omega^2 = -1$, *i.e.* it is a complex structure. By the same argument, if on $\mathcal{M}$

[18] Of course, there are relations in this algebra. We shall discuss this topic in chapt.

there are n linearly–independent parallel 2–forms, the corresponding skew–symmetric endomorphism f^a should generate a $\mathbb{C}l(n)$ algebra.

5. Parallel spinors and holonomy

The above results about parallel tensors may be generalized to parallel spinors. In this section, $\mathcal{M}$ is a Riemannian n–fold with a spin structure, and $\mathcal{S}_\pm$ are the corresponding spin bundles (of given chirality for n even). As we already mentioned, they are the vectors bundles associated to the principal $Spin(n)$ bundle trough the fundamental spinorial representations (the spin structure being precisely the uplift of the usual Riemannian $SO(n)$–principal bundle given by the connection ∇ to a principal bundle with fiber its double cover $Spin(n)$; this uplift is always possible provided a certain $\mathbb{Z}_2$ cohomology class, the *second Stiefel–Whitney class,* vanishes [**84**]). Th Levi–Civita connection on $T\mathcal{M}$ induces a connection on $\mathcal{S}_\pm$, called the *spin connection*

$$D^{\mathcal{S}} = \partial + \frac{1}{4}\omega_{ab}\,\Gamma^{ab}, \tag{5.1}$$

where ω_{ab} is the $SO(n)$ connection and $\Gamma^{ab} = \Gamma^{[a}\Gamma^{b]}$ are the Dirac matrices generating $Spin(n)$. For simply–connected manifolds one has $\mathrm{Hol}(D^{\mathcal{S}}) = \mathrm{Hol}(\nabla)$, otherwise it may be a double cover.

A smooth section ψ of $\mathcal{S}_\pm$ is called a *parallel spinor* if

$$D^{\mathcal{S}}\psi = 0. \tag{5.2}$$

Again we have an integrability condition,

$$0 = (D^{\mathcal{S}})^2\psi = R\,\psi,$$

which, as in the FUNDAMENTAL PRINCIPLE 1.1, says that ψ *is parallel if and only if it is invariant under* $\mathrm{Hol}(g) \subset Spin(n)$.

To find the number of (linearly independent) parallel spinors $N_\pm(H)$ for a given holonomy group $\mathrm{Hol}(g) = H$ (and, for n even, a given chirality $\pm$), the only thing we have to do is to decompose the spinor representations of $Spin(n)$ into irreducible representations of the subgroup H and count how many times we get the trivial representation, that is[19]

$$N_\pm(H) = \int_H \mathrm{Tr}_{\mathcal{S}_\pm}(h)\,dh, \tag{5.3}$$

where dh is the Haar measure[20] on $Spin(n)$ normalized so that the total volume is 1.

We have the following

THEOREM 5.1 (M. Y. Wang [**123**]). *Let $\mathcal{M}$ be a (connected simply–connected) irreducible spin Riemannian n–fold with $n \geq 3$. Let $N_\pm$ be the dimensions of the space of parallel spinors in $\mathcal{S}_\pm$. If $N_+ + N_- \geq 1$ one of the following holds*[21]*:*

---

[19] The equality of the two sides follows from the orthogonality of the characters of the irreducible representations of compact Lie groups.

[20] See chapter 5 for technical details.

[21] For an appropriate choice of the orientation of the manifold $\mathcal{M}$; for the opposite orientation $N_+ \leftrightarrow N_-$.

(i): $n = 4m$, Hol $= SU(2m)$, *and* $N_+ = 2$ *and* $N_- = 0$;
(ii): $n = 4m$, Hol $= Sp(2m)$, *and* $N_+ = m + 1$ *and* $N_- = 0$;
(iii): $n = 4m + 2$, Hol $= SU(2m + 1)$, *and* $N_+ = 1$ *and* $N_- = 1$;
(iv): $n = 7$, Hol $= G_2$, *and* $N = 1$;
(v): $n = 8$, Hol $= Spin(7)$, *and* $N_+ = 0$ *and* $N_- = 1$.

Before going to the proof, let us establish a pair of crucial lemmas.

LEMMA 5.1. *Assume that on the Riemannian manifold* $\mathcal{M}$ *there is a (non–vanishing) parallel spinor,* $D_i^{\mathcal{S}}\psi = 0$. *Then* $\mathcal{M}$ *is* Ricci–flat, $R_{ij} = 0$.

PROOF. Take the integrability condition

$$0 = 4\left[D_i^{\mathcal{S}}, D_j^{\mathcal{S}}\right]\psi = R_{ijkl}\,\Gamma^{kl}\,\psi$$

and contract it with Γ^j. Using the identity

$$\Gamma^j\Gamma^{kl} = \Gamma^{jkl} - \delta^{jk}\,\Gamma^l + \delta^{jl}\,\Gamma^k,$$

you get

$$R_{ijkl}\,\Gamma^{jkl}\psi = -2\,R_{ij}\Gamma^j\psi. \tag{5.4}$$

The LHS vanishes by the first Bianchi identity. So

$$0 = g^{ij}\,(R_{ik}\Gamma^k)\,(R_{jl}\Gamma^l)\psi = g^{ij}g^{kl}\,R_{ik}\,R_{jl}\,\psi \equiv \|R_{ij}\|^2\,\psi, \tag{5.5}$$

and hence $R_{ij} = 0$. □

LEMMA 5.2. ***(a)*** *Let* $\mathcal{M}$ *be an irreducible symmetric* n*–fold with* $n \geq 2$. *Then* $N_+ + N_- = 0$.

(b) *More generally, any* symmetric *space admiting a* non–zero *parallel spinor is* FLAT.

PROOF. Assume $\mathcal{M}$ admits a non–zero parallel spinor ψ. By lemma 5.1, $\mathcal{M}$ should be Ricci–flat. Comparing with the explicit Cartan's classification (see chapter 5), we see that the only complete, simply–connected, irreducible, symmetric, Ricci–flat manifold is $\mathbb{R}$ which has dimension 1 while we assume $n \geq 2$. Relaxing the condition of irreducibility, we see that $\mathcal{M}$ should be locally isometric to $\mathbb{R}^n$. □

PROOF. (of theorem 5.1) **(1)** By lemma 5.2,

$$N_+ + N_- \geq 1 \quad \Rightarrow \quad \mathcal{M} \text{ is not } \underline{\text{symmetric}}.$$

Then the holonomy Hol(g) of $\mathcal{M}$ should be in the Berger's list.

We check each group H in the list (recall that the parallel spinors ψ correspond to the trivial H–representations contained in $\mathcal{S}_\pm$):

(2) If Hol$(g) = SO(n)$ we have no parallel spinors since $\mathcal{S}_\pm$ are non–trivial *irreducible* representations of $\mathfrak{so}(n)$.

(3) Consider Hol$(g) = U(1) \times SU(m)$. In this case we can split the $Spin(n)$ gamma matrices into $(1,0) \oplus (1,0)$ type, namely Γ^i, $\Gamma^{\bar{\imath}} \equiv (\Gamma^i)^\dagger$, so that the Clifford algebra reads

$$\Gamma^i\Gamma^{\bar{\jmath}} + \Gamma^{\bar{\jmath}}\Gamma^i = 2\delta^{i\bar{\jmath}} \tag{5.6}$$

$$\Gamma^i\Gamma^j + \Gamma^j\Gamma^i = 0, \quad i,j = 1,2,\ldots,m. \tag{5.7}$$

Then the representation spaces $\mathcal{S}_\pm$ are constructed by taking a Clifford vacuum ψ_0 which satisfies $\Gamma^{\bar{\imath}}\psi_0 = 0$ for all $\bar{\imath}$. The spinor spaces $\mathcal{S}_\pm$ are spanned by the vectors

$$\Gamma^{i_1 i_2 \cdots i_l}\,\psi_0, \tag{5.8}$$

with l *even* and *odd*, respectively, for $\mathcal{S}_+$ and $\mathcal{S}_-$. The $U(1)$ charge of the vector in eqn.(5.8) is $(l - m/2)$ (by 'PCT symmetry'). There are only two vectors which are invariant under $SU(m)$, namely

$$\psi_0 \qquad \text{and} \qquad \Gamma^{1\,2\,3\,\cdots\,m}\,\psi_0\,. \tag{5.9}$$

Their $U(1)$ charges are $\pm m/2$, so no component of $\mathcal{S}$ is invariant under the full $U(m)$.

(4) However the previous calculation shows that we have two parallel spinors if $\mathrm{Hol}(g) = SU(m)$, that is ψ_0 and $\Gamma^{1\,2\,3\,\cdots\,m}\,\psi_0$. They have the same chirality if m is even, and opposite chirality if m is odd.

(5) $\mathrm{Hol}(g) = Sp(2m) \subset SU(2m) \subset U(2m)$: from these group embeddings, we see that eqn.(5.8) still holds. But the Ω_{ij}–traces are now $Sp(2m)$–invariants (here Ω_{ij} is the symplectic matrix). Writing $\omega := \frac{1}{2}\Omega_{ij}\,\Gamma^{ij}$, we have the following list of $Sp(2m)$ singlets:

$$\psi_0,\ \omega\,\psi_0,\ \omega^2\psi_0,\ \cdots\ \omega^m\psi_0. \tag{5.10}$$

In total we have $m+1$ singlets all of the same chirality.

(6) $\mathrm{Hol}(g) = Sp(2) \times Sp(2m)$: we give two proofs of $N_+ + N- = 0$.

(6a) We computed the $Sp(2)$ curvature of a generic Quaternionic–Kähler manifold under the name, say, of the $Spin(3)_R$–curvature for an $\mathcal{N} = 3$ SUGRA in $D = 3$. That explicit formula implies (see chapt. 2) that an irreducible *strictly* Quaternionic–Kähler manifold is *never* Ricci–flat. So $\mathrm{Hol} = Sp(2) \times Sp(2m)$ is ruled out by lemma 5.1.

(6b) The $\big(Sp(2) \times Sp(2m)\big)$–invariant spinors should be, in particular, $Sp(2m)$–invariant. So they should be in the list (5.10). Let us classify the spinors in (5.10) according to the representations of the group

$$Sp(2) \times Sp(2m) \subset Spin(4m).$$

$\omega \equiv \frac{1}{2}\Omega_{ij}\Gamma^{ij}$ is manifestly a generator of $Spin(4m)$ commuting with $Sp(2m)$ hence, by definition, a generator of $Sp(2) \simeq SU(2)$. Since it has chirality $+2$, it should correspond to the raising generator L_+ of $SU(2)$. Then the $(m+1)$ $Sp(2m)$–invariant spinors belong to the *irreducible* spin $m/2$ representation of

$$Sp(2) \simeq SU(2) \equiv \{\text{centralizer of } Sp(2m) \text{ in } Spin(4m)\}.$$

Since the representation is *irreducible*, it does not contain any singlet.

(7) $\mathrm{Hol}(g) = G_2 \subset Spin(7)$. The (unique) spinorial representation, the $\mathbf{8}$ of $Spin(7)$, decomposes as $\mathbf{7} \oplus \mathbf{1}$ under G_2, so $N = 1$.

(8) $\mathrm{Hol}(g) = Spin(7) \subset Spin(8)$. By $Spin(8)$ triality (see appendix C for details), $\mathcal{S}_+ = \mathcal{S}|_{Spin(7)}$, whereas $\mathcal{S}_- = \mathbf{7} \oplus \mathbf{1}$. So $N_+ = 0$ and $N_- = 1$. □

REMARK. The proof is more interesting that the result itself. In particular, we have proven the following interesting

THEOREM 5.2. *Let $\mathcal{M}$ be a Calabi–Yau manifold (that is $\mathrm{Hol}(g) = SU(m)$). Denote by $\Omega^{(*,0)} = \oplus_{p=0}^m \Omega^{(p,0)}$ the ring of smooth $(p,0)$ forms, $0 \leq p \leq m$, and $\mathcal{S}$ the space of smooth spinor fields (sections of the spin* *bundle). Let $\psi_0 \in \mathcal{S}$ be a parallel spinor as in the previous theorem (unique up to multiplication by $\lambda \in \mathbb{C}$). Then*

$$\Omega^{(*,0)} \simeq \mathcal{S} \qquad \textit{isomorphic as } \Omega^{(*,0)} \textit{ graded modules} \tag{5.11}$$

$$\phi_{i_1 i_2 \cdots i_p}\, dz^{i_1} \wedge dz^{i_2} \wedge \cdots dz^{i_p} \overset{\sim}{\longmapsto} \phi_{i_1 i_2 \cdots i_p}\, \Gamma^{i_1 i_2 \cdots i_p}\, \psi_0. \tag{5.12}$$

Under the above isomorphism, the Dirac operator $\not{D}$ is mapped into the Kähler–Dirac operator

$$\not{D} \leftrightarrow \partial + \overline{\mathfrak{d}} \tag{5.13}$$

where $\overline{\mathfrak{d}}$ is the adjoint of the Dolbeault operator ∂. In particular,

$$\text{(parallel } (p,0) \text{ forms)} \ \leftrightarrow\ \text{(parallel spinors)}.$$

Proof. The Dirac operator is $\Gamma^i D_i + (\Gamma^i D_i)^\dagger$. One has

$$\begin{aligned}
\Gamma^i D_i\big(\phi_{i_1 i_2 \cdot i_p}\, \Gamma^{i_1 i_2 \cdots i_p}\, \psi_0\big) &= \\
&= (D_i \phi_{i_1 i_2 \cdot i_p}) \Gamma^{i\, i_1 i_2 \cdots i_p}\, \psi_0 + \phi_{i_1 i_2 \cdot i_p}\, \Gamma^{i\, i_1 i_2 \cdots i_p}\, D_i \psi_0 = \\
&= (\partial \phi)_{i_0 i_1 i_2 \cdots i_p} \Gamma^{i_0 i_1 i_2 \cdots i_p}\, \psi_0,
\end{aligned}$$

so $\partial \leftrightarrow \Gamma^i D_i$. On the other hand, the forms' antilinear map $\phi \mapsto *\phi^*$ corresponds on spinors to $\psi \mapsto \psi^\dagger$, so the two notions of adjoint coincide and

$$\overline{\mathfrak{d}} = (\partial)^\dagger \leftrightarrow (\Gamma^i D_i)^\dagger.$$

□

For the phenomenological implications of this result see Witten, ref.[**23**].

Exercise 5.1. Let $\Omega^{(*,*)}$ be the space of *all* (smooth) differential forms on a manifold $\mathcal{M}$ (not necessarily Calabi–Yau). One has

$$\Omega^{(*,*)} \simeq \mathcal{S} \otimes \overline{\mathcal{S}}.$$

Use this isomorphism to deduce from the classification of the parallel spinors the previous theorem on the classification of parallel forms.

6.∗ *G*–structures on manifolds and Spencer cohomology

Berger's holonomy theorem is, in fact, only a special case of a more general theorem (largely due to Berger himself) which classify all *torsionless* affine holonomy groups. If we restrict to *metric* connections we get the result of sect. 3, but we can consider more general connections on $T\mathcal{M}$. We introduce this more general viewpoint not only to present a different language, possibly more deep, to adress the questions we already discussed: The point is that in SUGRA/SUSY we *do have* natural geometric structures which are torsionless but non–metric, *e.g.* the *special geometry* of sect.9 or, more generally, the vector–coupling geometry we introduced in Part 1. It would be nice if a unified language will allow us to discuss the geometrical aspects we reviewed before and the other, non–metric, ones.

The theory of *G*–structures on a manifold is the better framework to discuss geometric structures in differential geometry. It is quite elegant.

6.1. Definitions. We start with the basics. $\mathcal{M}$ is a smooth manifold with a linear connection on $T\mathcal{M}$ which is not required to descend from a metric (in general, $\mathcal{M}$ is not equipped with a metric).

DEFINITION 6.1. Let $\mathcal{M}$ be a manifold of dimension n, and $\mathcal{E}$ the frame bundle of $T\mathcal{M}$, which is an $GL(n,\mathbb{R})$ principal bundle. Let G be a Lie subgroup of $GL(n,\mathbb{R})$. A G–structure on $\mathcal{M}$ is a principal subbundle $\mathcal{P}\subset\mathcal{E}$ with fiber G.

REMARK. A large family of structures on $\mathcal{M}$ can be described in terms of G–structures. For instance: *(i)* an *orientation* of $\mathcal{M}$ is an $GL^+(n,\mathbb{R})$–structure; *(ii)* a *metric* is an $SO(n)$–structure; *(iii)* an *almost complex structure* is an $GL(n/2,\mathbb{C})$–structure; *(iv)* an *almost Hermitean structure* is an $U(n/2)$–structure; *(v)* an *almost symplectic structure* is an $Sp(n,\mathbb{R})$–structure; *etc. etc.*

Given a connection (even non–metric) ∇ on $T\mathcal{M}$ we can define the notions of parallel transport of a vector, and hence its holonomy group $\mathrm{Hol}(\nabla)$, torsion, and curvature[22]. Then

DEFINITION 6.2. A connection ∇ on $T\mathcal{M}$ is said to be *compatible with the G–structure $\mathcal{P}$* if $\mathrm{Hol}(\nabla)\subseteq G$.

PROPOSITION 6.1. *Given a connection ∇ on $T\mathcal{M}$ the set of G–structures $\mathcal{P}$ compatible with ∇ is given by the coset space*

$$\{a\in GL(n,\mathbb{R})\colon a\,\mathrm{Hol}(\nabla)\,a^{-1}\subseteq G\}/G. \tag{6.1}$$

Now the main question is: *How many* TORSION–FREE *connections ∇ are there compatible with a given G–structure $\mathcal{P}$?*

If ∇ and ∇' are two connections of $\mathcal{P}$, their difference $\alpha=\nabla'-\nabla\in C^\infty(\mathrm{adj}(\mathcal{P})\otimes T^*\mathcal{M})$ is a tensor ${\alpha_{ij}}^k$ and

$$T(\nabla')_{ij}{}^k=T(\nabla)_{ij}{}^k-{\alpha_{ij}}^k+{\alpha_{ji}}^k$$

hence — if a torsionless connection on $\mathcal{P}$ exists, all the other are in 1–1 correspondence with the $\alpha\in C^\infty(\mathrm{adj}(\mathcal{P})\otimes T^*\mathcal{M})\cap C^\infty(\Omega^2(T\mathcal{M}))$. We can rephrase the situation as a cohomology problem (Spencer cohomology).

6.2. Spencer cohomology. Let V a vector space and $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{gl}(V):=V\otimes V^*$. Define recursively the following $\mathfrak{g}$–modules

$$\begin{aligned}\mathfrak{g}^{(-1)}&=V\\ \mathfrak{g}^{(0)}&=\mathfrak{g}\\ \mathfrak{g}^{(k)}&=[\mathfrak{g}^{(k-1)}\otimes V^*]\cap[V\otimes\odot^{k+1}V^*],\quad k=1,2,3,\dots\end{aligned}$$

and define the map

$$\partial\colon\mathfrak{g}^{(k)}\otimes\wedge^{l-1}V^*\to\mathfrak{g}^{(k-1)}\otimes\wedge^lV^*$$

[22] We recall that the torsion $T(\nabla)$ and curvature $R(\nabla)$ are defined as follows ($v,w,z\in C^\infty(T\mathcal{M})$):

$$\begin{aligned}T(\nabla)\cdot(v\wedge w)&=\nabla_vw-\nabla_wv-[v,w],\\ R(\nabla)(z\otimes v\wedge w)&=\nabla_v\nabla_wz-\nabla_w\nabla_vz-\nabla_{[v,w]}z.\end{aligned}$$

as antisymmetrization on the last l indices[23] Since $\partial^2 = 0$, we can define the *Spencer cohomology groups* $H^{k,l}(\mathfrak{g})$ as

$$H^{k,l}(\mathfrak{g}) = \ker \partial\big|_{\mathfrak{g}^{(k-1)}\otimes\wedge^l V^*} \Big/ \operatorname{Im} \partial\big|_{\mathfrak{g}^{(k)}\otimes\wedge^{l-1} V^*}.$$

Given a connection ∇ on $\mathcal{P}$ its torsion $T(\nabla) \in T\mathcal{M} \otimes \wedge^2 T\mathcal{M}$. By the previous computation, the difference between the torsion of two such connection is $T(\nabla') - T(\nabla) \in \partial\big(\mathfrak{g} \otimes T^*\mathcal{M}\big)$. Hence the class $[T(\nabla)] \in H^{0,2}(\mathfrak{g})$ is independent of the particular connection, and depends only on the principal bundle $\mathcal{P}$. *The class* $[T(\nabla)] \in H^{0,2}(\mathfrak{g})$ *is called the* INTRINSIC TORSION *of the principal* G*–bundle* $\mathcal{P}$. Thus

PROPOSITION 6.2. *The* G*–structure* $\mathcal{P}$ *admits a torsionless connection* ∇ *if and only if its Spencer class in* $H^{0,2}(\mathfrak{g})$ *vanishes. In this case, the space of torsionless compatible connections is isomorphic to* $(\mathfrak{g} \otimes V^*)/\partial\mathfrak{g}^{(1)}$.

REMARK. The natural G–structures are the torsionless ones: *(i)* a torsionless $GL(n/2, \mathbb{C})$–structure is a *complex* structure; *(ii)* a torsionless $U(n/2)$–structure is a *Kähler* structure, *(iii)* a torsionless $Sp(n, \mathbb{R})$ is a *symplectic* structure, *ect. ect.*

Having the definitions of curvature tensor R and connection ∇, we can define what we mean by a *symmetric* G–connection by the condition that the curvature is covariantly constant. Hence we ask again for the classification of irreducible holonomy representations of torsionfree non–symmetric G–compatible connections. If $G \subseteq O(n)$ the connection is metric, and we get back the previous Berger–Simons classification. But the possibility of non–metric connections open us a whole new world. The general classification was initiaded by Berger himself, which wrote down a list of groups and representations, stating that it contained all possible holonomies *but* a finite number of *exotic* ones. Finally, after contributions by many people, the list was completed by S. Merkulov and L. Schwachhöfer in 1998 [**104**]; the list of *exotic holonomies* turned out to contain an infinite number of groups/representations. Technically, one has to compute all the relevant Spencer cohomology groups; Merkulov and Schwachhöfer do this by using a deep generalization of Simon's 'transitivity on the unit sphere' idea, reintepreting the various group representations in the light of the Bott–Borel–Weil theorem [**105**], that is, they look to the question as a quantum mechanical problem (with some SUSY needless to say).

[23] By abuse of notation, we identify the vector bundles and the corresponding sheaf of C^∞ sections.

CHAPTER 4

susy/sugra Lagrangians and U–duality

The above geometrical results give us powerful constraints on the couplings of a sypersymmetric theory. They fully determine the complete non–linear Lagrangian for any *(ungauged)* SUSY/SUGRA.

In this chapter we present the general picture, valid for all D's, and discuss in detail the cases $D = 3$ and 4.

1. Determination of the scalars' manifold $\mathcal{M}$

In chapt. 2 we found that, roughly speaking, the scalars' manifold tangent space $T\mathcal{M}$ of a SUSY/SUGRA theory is isomorphic (locally) to a tensor product $\mathcal{S} \otimes \mathcal{U}$, where the vector bundle $\mathcal{S}$ has structure group Aut_R, the SUSY automorphism group. This product structure is invariant under parallel transport on $\mathcal{M}$. Moreover, we found that the Aut_R curvature vanishes in rigid SUSY, whereas is given by a tt^*–like formula in local SUGRA. The specifics of Aut_R depend on the particular D and $\mathcal{N}$ at hand, but otherwise the situation is pretty universal.

In the language of chapt. 3 (see §. 3 below for a different one) these results are stated as a condition on the holonomy group $\mathrm{Hol}^0(\mathcal{M})$, namely

$$\mathrm{Hol}^0(\mathcal{M}) \subseteq \begin{cases} \mathrm{Aut}_R \times \mathcal{C}(\mathrm{Aut}_R) & \text{SUGRA} \\ \mathcal{C}(\mathrm{Aut}_R) & \text{rigid SUSY}, \end{cases} \tag{1.1}$$

where $\mathcal{C}(\mathrm{Aut}_R)$ is the *centralizer* of Aut_R in $SO(\dim \mathcal{M})$.

The specific representations of Aut_R on $T\mathcal{M}$ are listed for the various D's and $\mathcal{N}$'s in chapt. 2; for the present purposes we need only to know that they never contain the trivial representation.

1.1. Rigid supersymmetry.

In particular, for *rigid* SUSY, eqn.(1.1) implies $\mathrm{Aut}_R \subseteq \mathcal{C}(\mathrm{Hol}^0)$. Comparing with proposition 3.1 of chapt. 3, we get

COROLLARY 1.1. *In rigid* SUSY*:*

- **(1):** *$\mathcal{M}$ irreducible and not symmetric is possible only for*[1]*: $\mathcal{N} \leq 4$ in $D = 3$, $\mathcal{N} \leq 2$ in $D = 4$, and $(\mathcal{N}_L, \mathcal{N}_R) = (2, 0)$ in $D = 6$. (Or, more intrinsically, if the total number of supercharges $\mathfrak{N} \leq 8$);*
- **(2):** *The scalars' manifold is:*
  - *<u>Riemannian</u> for: $D = 3$, $\mathcal{N} = 1$;*
  - *<u>Kähler</u> for*
    - *$D = 3$, $\mathcal{N} = 2$;*
    - *$D = 4$, $\mathcal{N} = 1$*
    - *$\mathcal{M}_{\text{gauge}}$ in $D = 4$, $\mathcal{N} = 2$;*

[1] We write only the D's in the $\mathbb{R} \leftrightarrow \mathbb{C} \leftrightarrow \mathbb{H} \leftrightarrow \mathbb{O}$ sequence to save print. The reader may add the other dimensions if he/she wishes.

- *hyper–Kähler for:*
  - $D = 3$, $\mathcal{N} = 3, 4$;
  - $\mathcal{M}_{\text{matter}}$ *in* $D = 4$, $\mathcal{N} = 2$;
  - $D = 6$, $(\mathcal{N}_L, \mathcal{N}_R) = (2, 0)$.

(3): *In all other cases* $\mathcal{M}$ *is* FLAT.

PROOF. Let $\mathfrak{C}$ be the Lie algebra of $\mathcal{C}(\text{Hol}^0)$. If $\mathcal{M}$ is irreducible and not symmetric, its holonomy group should be in the Berger list. The corresponding $\mathcal{C}(\text{Hol}^0)$ are listed in proposition **3.**3.1. One has $\mathfrak{aut}_R{}' \subseteq \mathfrak{C}$, where $\mathfrak{aut}_R{}' \subseteq \mathfrak{aut}_R$ is the Lie subalgebra acting *effectively* on $T\mathcal{M}$. Comparing proposition **3.**3.1 with the list of the groups Aut_R for the diverse D's and $\mathcal{N}$'s given in §. **2.**1.2 the first part of the corollary follows.

Indeed, $\mathcal{C}$ is trivial, except for *i)* $\text{Hol}^0 = U(m), SU(m)$, in which case it is $\mathfrak{u}(1)$, and *ii)* $\text{Hol}^0 = Sp(2m)$ in which case it is $\mathfrak{sp}(2) \simeq \mathfrak{su}(2)$. $\mathfrak{aut}_R$ is, respectively, $\mathfrak{so}(\mathcal{N})$, $\mathfrak{u}(\mathcal{N})$, and $\mathfrak{sp}(\mathcal{N})$ in $D = 3, 4, 6$. $\mathfrak{aut}_R$ is *simple,* and hence $\mathfrak{aut}_R{}' \equiv \mathfrak{aut}_R$ in all cases, but:

(1) $D = 3$, $\mathcal{N} = 4$, where $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Thus $\mathfrak{aut}_R{}' \simeq \mathfrak{su}(2)$ or $\mathfrak{aut}_R{}' \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In the second case, from the Atiyah–Bott–Shapiro classification of Clifford modules (cfr. chapter 2) we know that $\mathcal{M}$ is *reducible*;

(2) $D = 4$ where $\mathfrak{aut}_R \simeq \mathfrak{su}(\mathcal{N}) \oplus \mathfrak{u}(1)$. By the de Rham theorem, the subspace of $T\mathcal{M}$ on which $\mathfrak{su}(\mathcal{N})$ acts trivially corresponds to a factor space $\mathcal{M}_0$ of $\mathcal{M} \simeq \mathcal{M}_0 \times \widetilde{\mathcal{M}}$ and this is possible, in $D = 4$ only for $\mathcal{N} = 2$.

Then the condition

$$\mathfrak{aut}_R{}' \subset \mathfrak{C} \tag{1.2}$$

has only the solutions listed above. $\mathcal{M}$ is Kähkler or hyperKähler according if $\mathfrak{C}$ is $\mathfrak{u}(1)$ or $\mathfrak{sp}(2)$.

It remains to prove the last statement. For pairs $(D, \mathcal{N})$ not in the list, $\mathcal{M}$ cannot be irreducible and non–symmetric. Assume $\mathcal{M}$ to be *irreducible symmetric.* By proposition **3.**3.1, $\mathfrak{C}$ is

$$\mathfrak{C} = \begin{cases} \mathfrak{u}(1) & \text{if } \ H = U(1) \times K, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

The irreducible symmetric spaces with $\mathfrak{C} = \mathfrak{u}(1)$ are precisely those which are also Kähler manifolds. Since $\mathfrak{aut}_R \subseteq \mathfrak{C} \subset \mathfrak{u}(1)$, in *rigid* SUSY, an irreducible symmetric space is possible only for those D's and $\mathcal{N}$'s with an *Abelian* Aut_R. Comparing with the tables in §. **2.**1.2, this gives $D = 3$, $\mathcal{N} = 1, 2$ and $D = 4$, $\mathcal{N} = 1$. In facts, for $D = 3$, $\mathcal{N} = 1$ all Riemannian manifolds will do, and the irreducible symmetric spaces certainly are Riemannian; analogously, in the other two cases any Kählerian manifold is allowed, wheter symmetric or not. These spaces are just special instances of those listed in the first part of the corollary.

For $(D, \mathcal{N})$ not in the list, we remain with only one possibility: $\mathcal{M}$ is a *reducible* manifold. By de Rham's theorem, $\mathcal{M}$ should have the form (going to the universal cover)

$$\mathcal{M} = (\text{flat}) \times \mathcal{M}_1 \times \cdots \times \mathcal{M}_k$$

with the $\mathcal{M}_l$'s non–flat irreducible. Since the holonomy acts separately on each factor space, the previous arguments apply to each irreducible $\mathcal{M}_l$, which then should be trivial. It remains the flat factor. □

REMARK. In particular, $\mathcal{M}$ should be *flat* for:

- $(\mathcal{N}_L, \mathcal{N}_R) = (2,2)$ in $D = 6$;
- $\mathcal{N} = 4$ in $D = 4$;
- $5 \leq \mathcal{N} \leq 8$ in $D = 3$.

REMARK. Corollary 1.1 contains only the conditions on $\mathcal{M}$ following from the Aut_R–structure of $T\mathcal{M}$ (equivalently, from the existence of $(\mathcal{N}-1)$ *parallel* complex structures, cfr. the FUNDAMENTAL PRINCIPLE **3.**1.1). In principle there may be other restrictions on $\mathcal{M}$. In facts, as we shall see, the above corollary fully characterizes the allowed $\mathcal{M}$'s with only one exception, $\mathcal{N} = 2$ $D = 4$, where we have the additional restrictions coming from 'special geometry', see sect. 9 of chapt. 2. However, even this case is completely determined by the above result, if only we require consistency with reduction to $D = 3$ (recall: dimensional reduction is a 'structure transporting' morphism). The allowed Kähler manifolds for $\mathcal{M}_{\text{gauge}}$ in $D = 4$ $\mathcal{N} = 2$ are precisely those which become hyper–Kähler in $D = 3$ after duality (*i.e.* in the *diet* form).

1.2. $\mathcal{M}$ is supergravity.

The corresponding result for SUGRA is[2]:

PROPOSITION 1.1. *Let $\mathcal{M}$ be the universal covering space of the scalars' manifold of a $\mathcal{N}, D$ SUGRA. Then:*

(a): *$\mathcal{M}$ is* IRREDUCIBLE *except for:*
  - **(i):** *$\mathcal{N} = 1, 2, 4$ in $D = 3$;*
  - **(ii):** *$\mathcal{N} = 1, 2, 4$ in $D = 4$;*
  - **(iii):** *$(2,0), (2,2)$ in $D = 6$;*

(b): *$\mathcal{M}$ is* SYMMETRIC *except for:*
  - **(i):** *$D = 3$, $\mathcal{N} = 1$: $\mathcal{M}$ is Riemannian;*
  - **(ii):** *$D = 3$, $\mathcal{N} = 2$: $\mathcal{M}$ is Kähler (in fact Hodge);*
  - **(iii):** *$D = 3$, $\mathcal{N} = 3, 4$: $\mathcal{M}$ is Quaternionic–Kähler;*
  - **(iv):** *$D = 4$, $\mathcal{N} = 1$: $\mathcal{M}$ is Kähler (in fact Hodge);*
  - **(v):** *$D = 4$, $\mathcal{N} = 2$: $\mathcal{M}_{\text{matter}}$ is Quaternionic–Kähler, while $\mathcal{M}_{\text{gauge}}$ is Kähler[3] (in fact Hodge);*
  - **(vi):** *$D = 6$, $(2,0)$: $\mathcal{M}_{\text{`matter'}}$ is Quaternionic–Kähler[4];*

(c): *in all other cases we have an* IRREDUCIBLE SYMMETRIC *Riemannian manifold of the form*

$$\frac{G}{\mathrm{Aut}_R \times K} \quad \text{with} \quad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{aut}_R \oplus \mathfrak{k} \quad \text{and} \quad \mathfrak{m} \simeq T_e\mathcal{M},$$

and the adjoint action of $\mathfrak{aut}_R$ on $\mathfrak{m}$

$$[R^a, X^j] = f^{a\,j}{}_k X^k \qquad R^a \in \mathfrak{aut}_R, \quad X^i \in \mathfrak{m}$$

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[2] Again, we quote the results only for $D = 3, 4, 6, 10$. The reader may complete the list by the same procedure.

[3] As far as holonomy is concerned. We have also the *special geometry* constraints, see previous footnote and sect. 3 below.

[4] In absence of chiral two forms.

is given by the representations in the two tables at the end of chapt. 2.

(d): *item* **(c)** *completely fixes $\mathcal{M}$ (for a given dimension) to be the symmetric space G/H listed for the diverse D and $\mathcal{N}$ in table 4.1.*

REMARK. In the table we have also inserted a reference to the historical papers where the geometry of the coset G/H was deduced for the given SUGRA. You can appreciate the enormous amount of work which was historically needed (in particular in $D = 4$) to establish the results we got by free! (I have not quoted the papers reporting the numberless aborted effort to get these results by direct methods).

PROOF. **(a)**. We have only to compare the isomorphisms of chapt. 2 with de Rham's theorem 1.2. For $D = 3$ the holonomy group is $Spin(\mathcal{N})_R \times K$, with $Spin(\mathcal{N})_R$ acting on $T\mathcal{M}$ according to a spinorial representation. If $Spin(\mathcal{N})$ is simple[5], de Rham theorem implies $\mathcal{M}$ to be irreducible. The only exceptions are $Spin(1) = \{e\}$, $Spin(2) = U(1)$ (Abelian), and $Spin(4) \simeq SU(2)_1 \times SU(2)_2$. In the last case we may have $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ with $\mathrm{Hol}(\mathcal{M}_i) = SU(2)_i \times K_i$. Physically, one manifold is parameterized by hypermultiplets and the other by twisted hypermultiplets. In $D = 4$ the same applies: $\mathrm{Aut}_R = U(1) \times SU(\mathcal{N})$, and the only possible factorization is in a manifold of $U(1)$ holonomy and one of $SU(\mathcal{N})$; the holonomy representation on $T\mathcal{M}$ may decompose only if the representation of $SU(\mathcal{N})_R$ on $T\mathcal{M}$ is *real* or *pseudoreal* (quaternionic). The first case happens for $\mathcal{N} = 4$ and the second for $\mathcal{N} = 2$. For $\mathcal{N} = 1$ $SU(1)_R$ is trivial and hence we have no restriction on the Kähler manifold $\mathcal{M}$.

(b) Just compare the list of holonomies (or, equivalently, bundle isomorphisms) in chapt. 2 with Bergers' theorem and definition **3**.3.1.

(b.1) As a side remark, notice that $\mathfrak{hol}(\mathcal{M}) = \mathfrak{aut}_R$ implies $\mathcal{M}$ to be (locally) *symmetric*. Indeed, from our computations in chapt. 2, we know that the $\mathfrak{aut}_R$ part of the curvature, $R\big|_{\mathfrak{aut}_R}$ is covariantly constant. Then $R \equiv R\big|_{\mathfrak{aut}_R}$ implies $DR = 0$, *i.e.* $\mathcal{M}$ symmetric. Thus, for instance, in $D = 3$, $\mathcal{N} = 7$, the possibility $\mathrm{Hol}^0(\mathcal{M}) = Spin(7)$ and $\mathcal{M}$ a $Spin(7)$–manifold is ruled out[6].

(c) The first statement follows from the explicit description of the holonomy for a symmetric manifold G/H and $H \equiv \mathrm{Hol}(\mathcal{M}) = \mathrm{Aut}_R \times K$. The second one is the relation between the holonomy representation of G/H and the bundle isomorphisms of chap. 2 which we deduced from 'target space equivalence principle'.

(d) The irreducible symmetric manifolds were classified by E. Cartan. Once we know that the relevant manifolds are irreducible symmetric, we

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[5] From the de Rham theorem one infers, in particular, that if $\mathrm{Hol}(\mathcal{M})$ acts reducibly, $\mathrm{Hol}(\mathcal{M}) = \mathrm{Hol}(\mathcal{M}_1) \times \mathrm{Hol}(\mathcal{M}_2)$. Then, *if* $\mathrm{Hol}(\mathcal{M}) = G \times K$, *with G simple, and no subspace of $T\mathcal{M}$ is invariant under G, $\mathcal{M}$ should be* IRREDUCIBLE. PROOF: assume (absurt) $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$. Since G is simple, $G \subset \mathrm{Hol}(\mathcal{M}_1)$ or $G \subset \mathrm{Hol}(\mathcal{M}_2)$. But then G acts trivially either on $\mathcal{M}_2$ or $\mathcal{M}_1$, contrary to the assumption. To get the result in the text, apply this result to $G = \mathrm{Aut}_R$.

[6] There are other reasons why a $Spin(7)$–manifold is excluded: for instance, a $Spin(7)$–manifold is necessarily *Ricci–flat,* whereas $\mathcal{N} \geq 5$ SUGRA requires $\mathcal{M}$ to be Einstein with a specific, *negative*, cosmological constant.

Table 4.1. Symmetric manifolds for extended SUGRA

D	$\mathcal{N}$	G/H	Notes
3	5	$Sp(4,2k)/\big(Sp(4)\times Sp(k)\big)$	[**49**]
3	6	$SU(4,k)/\big(SU(4)\times U(k)\big)$	[**49**]
3	7	none	$D=3$, $\mathcal{N}=7$ cannot exist!
3	8	$SO(8,k)/\big(SO(8)\times SO(k)\big)$	[**49**]
3	9	$F_{4(-20)}/SO(9)$	[**49**]
3	10	$E_{6(-14)}/\big(SO(10)\times SO(2)\big)$	[**49**]
3	11	none	$D=3$, $\mathcal{N}=11$ cannot exist!
3	12	$E_{7(-14)}/\big(SO(12)\times SO(3)\big)$	[**49**]
3	$13,14,15$	none	$D=3$, $\mathcal{N}=13,14,15$ cannot exist!
3	16	$E_{8(+8)}/SO(16)$	[**49**]
3	≥ 17	none	$D=3$, $\mathcal{N}\geq 17$ cannot exist!
4	3	$SU(3,k)/\big(SU(3)\times U(k)\big)$	[**111**]
4	4	$\frac{SU(1,1)}{U(1)}\times\frac{SO(6,k)}{SO(6)\times SO(k)}$	[**112**]
4	5	$SU(5,1)/U(5)$	[**11**]
4	6	$SO^*(12)/U(6)\equiv SO(6,\mathbb{H})/U(6)$	[**11**]
4	7	none	$D=4$, $\mathcal{N}=7$ cannot exist!
4	8	$E_{7(7)}/SU(8)$	[**11**]
4	≥ 9	none	$D=4$, $\mathcal{N}\geq 9$ cannot exist!
6	$(2,2)$	$\mathbb{R}\times\frac{SO(4,k)}{SO(4)\times SO(k)}$	
6	$(4,0)$	$SO(5,k)/\big(SO(5)\times SO(k)\big)$	[**114, 115**]
6	$(4,2)$	$SO(5,1)/SO(5)$	
6	$(6,0)$	$SU^*(6)/Sp(6)\equiv SL(3,\mathbb{H})/Sp(6)$	???
6	$(4,4)$	$SO(5,5)/\big(SO(5)\times SO(5)\big)$	[**114**]
6	$(6,2)$	$F_{4(4)}/\big(Sp(6)\times SU(2)\big)$	???
6	$(8,0)$	$E_{6(6)}/Sp(8)$	???
10	$(1,0)$	$\mathbb{R}$	[**116**]
10	$(1,1)$	$\mathbb{R}$	[**117**]
10	$(2,0)$	$SU(1,1)\times U(1)$	[**118**]

Notes: "???" stands for situation in which there is no massless supermultiplet containing the graviton, so they are theories (not constructed, to my knowledge) more general than SUGRA (if they exist at all !!). In the $D=6$ case we have assumed that no chiral two form is present.

take the tables, see *e.g.* [**86, 119, 120**], and look for cosets G/H with the right subgroup H and holonomy representation. We do this first for $D = 3$ and then for all $D \geq 4$.

(d.1) $D = 3$. For $\mathcal{N} \geq 5$, $\mathcal{M}$ is irreducible symmetric. It is a manifold of dimension $k\,\mathbf{N}(\mathcal{N})$, $k \in \mathbb{N}$ (cfr. GENERAL LESSON **2.**4.1) which has an holonomy of the form $Spin(\mathcal{N}) \times K$, where K acts orthogonally, unitarily or symplectically, according whether the Clifford algebra $\mathbb{C}l(\mathcal{N}-1)$ is real, complex, or quaternionic (the usual $\mathbb{R} \leftrightarrow \mathbb{C} \leftrightarrow \mathbb{H}$ story). Using the tables in sect. **2.**1.2 (or the theory presented in APPENDIX C), we see that $K \subseteq SO(k)$ for $\mathcal{N} = 7, 8, 9 \mod 8$, $K \subseteq U(k)$ for $\mathcal{N} = 2, 6 \mod 8$, and $K \subset Sp(k)$ for $\mathcal{N} = 3, 4, 5 \mod 8$.

The crucial point is that $T\mathcal{M}$ belongs to a *spinorial representation* of $Spin(\mathcal{N})$. In the Cartan classification, there are very few symmetric spaces having the holonomy acting in spinorial representation. Indeed, for a symmetric space G/H, we have (see §. **3.**2.2.1) $T_eG/H \simeq \mathfrak{m} := \mathfrak{g} \ominus \mathfrak{h}$, and the holonomy representation $\mathfrak{m}_{\mathfrak{hol}}$ is given by the decomposition of the adjoint representation of $\mathfrak{g}$ into representations of the subalgebra $\mathfrak{h}$

$$\mathfrak{g} = \mathrm{adj}(\mathfrak{h}) \oplus \mathfrak{m}_{\mathfrak{hol}}.$$

Now, the decomposition of the Lie algebra of a *classical group* never produces spinorial representations. Thus holonomy groups acting *via* spinorial representations can arise only in two ways. First, for small $\mathcal{N}$'s through the Lie algebra isomorphisms:

$$\mathfrak{spin}(2) \simeq \mathfrak{u}(1), \qquad \mathfrak{spin}(3) \simeq \mathfrak{su}(2) \simeq \mathfrak{sp}(2),$$
$$\mathfrak{spin}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2), \qquad \mathfrak{spin}(5) \simeq \mathfrak{sp}(4), \qquad \mathfrak{spin}(6) \simeq \mathfrak{su}(4),$$

as well as the triality automorphism of $Spin(8)$ (see APPENDIX C), which allow us to *re*interpret classical representations as spinorial ones. They gives the entries in the table for $\mathcal{N} = 5, 6, 8$.

Second, G may be an *exceptional* Lie group. In this case we have the esoteric *projective planes* based on the octonions[7] [**122**]:

$$\begin{array}{cc} \mathbb{O}P^2 & (\mathbb{C} \otimes \mathbb{O})P^2 \\ (\mathbb{H} \otimes \mathbb{O})P^2 & (\mathbb{O} \otimes \mathbb{O})P^2 \end{array}$$

which correspond to four pairs of symmetric spaces with genuine spinorial representation holonomies, namely (compare the symmetric space tables, refs.[**86, 119, 120**])

$$\frac{F_4}{Spin(9)}, \quad \frac{E_6}{Spin(10) \times SO(2)}, \quad \frac{E_7}{Spin(12) \times SU(2)}, \quad \frac{E_8}{Spin(16)}, \tag{1.4}$$

and their non–compact (= negatively curved) duals.

In chapt. 2 we computed the $Spin(\mathcal{N})$ curvature: it was *negative.* Hence we should keep the non–compact versions. The four non–compact octonionic spaces should be the $\mathcal{M}$'s for $\mathcal{N} = 9, 10, 12, 16$. No other SUGRA exists! In particular, any $D = 3$ SUGRA, having local propagating degrees of freedom, must have $\mathcal{N} \leq 16$! This constraint stems, ultimately, from the Hurwitz

[7] For reason explained in APPENDIX B only projective *planes* are well defined for octonions. The higher dimensional projective spaces over $\mathbb{O}$ do not exist.

theorem which states that the octonions $\mathbb{O}$ are the last normed division algebra [**122**].

(d.2) $D \geq 4$. In order to work all D's at once, we take the tables of *linear* SUSY representations in ref. [**60**], look for the quantum numbers of the scalar fields under Aut_R, invoke our blessed 'target space equivalence principle' to identify them as holonomy representations, and check which — if any — G/H space in Cartan's list has *that* holonomy representation. In doing this we use the Lie algebra isomorphisms listed in page 120. The method works smoothly even when $\mathcal{M}$ is a *reducible* symmetric manifold. For each $\mathcal{N}$ and D one finds either *two* G/H spaces with the right representation or none. The second case corresponds (magically?) to situations where the usual physicists' folklore predicts that no SUGRA exists. In the other cases, the two solutions form a dual pair of Cartan spaces[8]: a compact one (with positive curvature), called a *space of Type I* in Helgason's classification [**121**], and its non–compact negatively–curved companion, *space of Type III* in Helgason's classification. Since we know that the curvature of the Aut_R–connection is negative, we conclude for the Type III coset. In this way everything gets determined, up to a few scalars which split from $T\mathcal{M}$ because they are singlets of Aut_R. They are either a single real scalar — which corresponds to the trivial symmetric space $\mathbb{R}$ — or a complex one charged under $U(1)_R$. If the $U(1)$ curvature is constant[9] and negative, the complex scalars should parameterize $SU(1,1)/U(1)$, *i.e.* the upper half–plane. In this way one writes down the table 4.1 in less than two minutes. It is remarkable than one *never* finds more than one solution!

[As a matter of notation: The small numbers in parenthesis appended to certain groups in the table, say, (-20) in $F_{4(-20)}$ refer to the specific real form. Specifically, the number in parenthesis is the signature of the invariant Killing form for the given real fom $\equiv$ (the number of Lie algebra generators with positive metric) $-$ (the number of Lie algebra generators with negative metric)]. □

From the above results we get the

GENERAL LESSON 1.1. *In the following* SUGRA*:*

- $D = 3$ $\mathcal{N} \geq 5$*;*
- $D = 4$ $\mathcal{N} \geq 3$*;*
- $D = 6$ $\mathcal{N}_R + \mathcal{N}_L \geq 4$*;*
- $D = 10$ $(2,0)$ *(Type IIB)*

the scalars' sector is invariant under a big non–compact symmetry group G*, namely the isometry group* $G \equiv \text{Iso}(G/H)$*, with* G *as in table 4.1.*

REMARK. The above symmetry G was known as the *hidden symmetry*[10] of the given SUGRA model (*e.g.* for $\mathcal{N} = 8$ SUGRA in $D = 4$ it corresponds

[8] Symmetric spaces and all that will be discussed in detail in chapter 5. There you may find the proper definitions of all concepts used here, including the classification of symmetric spaces into types.

[9] Here we are flying over a few subtleties. In chapt. 2 we computed the $U(1) \simeq SO(2)$ curvature to have these two properties, but one can think of twisting the $U(1)$ by some other Abelian connection.

[10] 'Hidden' because people did a great effort to discover it.

TABLE 4.2. Symmetric manifolds for $\mathcal{N} \geq 4$, $D = 5$ SUGRA

$\mathcal{N}$	G/H
4	$\mathbb{R} \times \frac{SO(5,k)}{SO(5)\times SO(k)}$
6	$\frac{SU^*(6)}{Sp(6)}$
8	$\frac{E_{6(6)}}{Sp(8)}$

to the exceptional group $E_{7(7)}$). After the 'string duality' revolution of the mid '90, this symmetry changed name in *U–duality* [**124**].

EXERCISE 1.1. Check the manifold list for $D = 5$ $\mathcal{N} \geq 4$ SUGRA, given in table 4.2 below. Write down the tables for $D = 7, 8$.

2. Four χ's couplings

We could have obtained the above results by a somewhat more direct method. We shall sketch it for illustrative purposes.

2.1. $D = 3$**.** From the Ambrose–Singer theorem, we know that Hol^0 is generated by the curvature tensors. These tensors can be read directly in the 4–χ couplings which read (schematically)

$$a\, R_{ijkl}\, \bar{\chi}^i \gamma^\mu \chi^j\, \bar{\chi}^k \gamma_\mu \chi^l \qquad \text{(rigid)} \tag{2.1}$$

$$\Big(a\, R_{ijkl} - b\, g_{ik} g_{jl}\Big) \bar{\chi}^i \gamma^\mu \chi^j\, \bar{\chi}^k \gamma_\mu \chi^l \qquad \text{(local).} \tag{2.2}$$

2.1.1. *Rigid supersymmetry.* In rigid SUSY, if the scalars' potential[11] $V(\phi) \equiv 0$, the group Aut_R is actually a symmetry, by an argument we developed in chapt. 2. Aut_R acts on the fields as an isometry of $\mathcal{M}$ plus an Aut_R rotation of the fermions, say

$$\chi \mapsto \exp\left[\frac{1}{2} \alpha_{AB}\, \Sigma^{AB}\right] \chi,$$

in $D = 3$, or the analogue formula in $D > 3$. Now, if Aut_R is a symmetry of $\mathcal{L}$, it should be — in particular — a symmetry of the 4–Fermi coupling, *i.e.* the Riemann tensor must satisfy

$$R_{ijkl} = S_i^{\ m}\, S_i^{\ n}\, S_i^{\ p}\, S_i^{\ q}\, R_{mnpq} \qquad S_i^{\ j} \in \mathrm{Aut}_R. \tag{2.3}$$

But this means that the Lie algebra $\mathfrak{hol}$ spanned by $R(x, y) \in \mathrm{End}(T\mathcal{M})$ is invariant under the adjoint action of $\mathfrak{aut}_R$. Since the $S_i^{\ m}$ in the above equation are *constant matrices* all covariant derivatives $D^{(r)} R_{ijkl}$ are invariant, and hence by an adaption of the Ambrose–Singer argument, the Aut_R transformations *commute* with $\mathfrak{hol}$ and

$$\mathfrak{aut}_R \subseteq \mathfrak{C}. \tag{2.4}$$

[11] $V \equiv 0$ in the ungauged theory for $\mathcal{N} \geq 3$.

Thus we got back the previous result.

2.1.2. *Supergravity.* More interesting is the local case. Still eqn. (2.3) holds, but now the $S_i{}^j$ are ϕ–dependent matrices. Besides, eqn.(2.3) is true in *full generality,* because Aut_R, being a *gauge* symmetry, should be always *exact.*

By definition, a manifold $\mathcal{M}$ is (locally) symmetric if and only if eqn.(2.3) holds for all $S \in \mathrm{Hol}^0$. Thus, say, in $D=4$, $\mathcal{N} \geq 5$ SUGRA, where[12]

$$\mathrm{Hol}^0 \equiv \mathrm{Aut}_R,$$

we immediately conclude that $\mathcal{M}$ is a symmetric space, *without any need of the Berger theorem:* It is a direct consequence of eqn.(2.3), *i.e.* of the gauge invariance of $\mathcal{L}$. In general, however, we have

$$\mathfrak{hol} = \mathfrak{aut}_R \oplus \mathfrak{k} \tag{2.5}$$

and we can only conclude that the projection of the curvature endomorphism on $\mathfrak{aut}_R$ is covariantly constant

$$D_i R_{ijkl}|_{\mathfrak{aut}_R} = 0, \tag{2.6}$$

(as we saw in chapt. 2 by a direct computation[13] of the curvatures), and also *invariant* under the adjoint action of Aut_R. By Ambrose–Singer, the Riemann tensor is an element[14] of

$$\odot^2 \mathfrak{hol} \equiv \odot^2(\mathfrak{aut}_R \oplus \mathfrak{k}), \tag{2.7}$$

with $[\mathfrak{k}, \mathfrak{aut}_R] = 0$.

Let $\mathfrak{s} \subset \mathfrak{aut}_R$ be a *simple* subalgebra. From eqn.(2.7), the Riemann tensor decomposes, in general, in the following representations of $\mathfrak{s}$

$$\odot^2\mathfrak{s} \;\oplus\; \mathfrak{s} \otimes V_1 \;\oplus\; V_2 \tag{2.8}$$

(here V_1, V_2 stand for trivial $\mathfrak{s}$–modules). By Schur's lemma, (2.8) can be invariant under $\mathfrak{s}$ only if *(i)* $V_1 = 0$ and (ii) the first term in the RHS is proportional to the quadratic Casimir (with a *constant* coefficient[15]). Hence, in local SUSY, the Riemann tensor is a sum of the quadratic Casimirs of the simple factors of Aut_R, plus an element of the space $\odot^2(\mathfrak{k} \oplus \mathfrak{aut}_R\big|_{\text{Abelian}})$. Unless $\mathfrak{aut}_R$ is Abelian (the Kähler case) in which instance this is an empty condition, the Bianchi identities force *all* the components of the curvature

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[12] See chapt. 2.

[13] I stress that even the few computation we did are, logically speaking, not necessary in order to construct SUGRA. However, I think they served their illustrative purpose.

[14] $\odot^k V$ stands for the k–fold *symmetric* tensor product of V, also written as $\mathrm{Sym}^k V$ or $\vee^k V$.

[15] This is a consequence of the Bianchi identity: $0 = DR\big|_{\mathfrak{s}} = d\log\alpha \wedge R\big|_{\mathfrak{s}}$, where $R\big|_{\mathfrak{s}} = \alpha\, C_2 \in \odot^2\mathfrak{s}$.

tensor having this special form to be covariantly constant. Therefore the space is symmetric.

"Wait a minute! Are you kidding? What if $\mathrm{Aut}_R = SU(2)$*? $SU(2)$ IS non–Abelian, but the corresponding space is not symmetric, in general, it is just Quaternionic–Kähler!"*

"Ohps... I guess you are right..." Then we have proven a fundamental theorem:

THEOREM 2.1 (Salamon [**87**]). ***(1)*** *Let $\mathcal{M}$ be Quaternionic–Kähler with $T\mathcal{M} \simeq \mathcal{S} \otimes \Lambda$ ($\mathcal{S}, \Lambda$ have structure groups $Sp(2)$ and $Sp(2m)$, respectively). Then the Riemann tensor belong to the space*

$$\mathbb{R}\, R_0 \;\oplus\; \odot^4 \Lambda \tag{2.9}$$

where R_0 is the curvature tensor of the canonical projective space $P\mathbb{H}^n$.

(2) *In dimension $4n \geq 8$, $\mathcal{M}$ is Einstein (i.e. $R_{ij} = \lambda g_{ij}$) and it is Ricci–flat if and only if it is hyperKähler.*

PROOF. (1) is the condition of invariance under the gauge symmetry $Sp(2)_R$. (2) is the fact that $\odot^4\Lambda$ component of the Riemann tensor does not contain $\wedge^2\Lambda$ (unless $m = 1$), and hence does not contribute to the Ricci tensor which is $\lambda \in \mathbb{R}$ times that of $P\mathbb{H}^n$. Finally if, $\mathcal{M}$ is Ricci flat, $\lambda = 0$, and $\mathfrak{hol} \subset \mathfrak{sp}(2m)$, which is the definition of hyperKähler. □

GENERAL LESSON 2.1. *Comparing eqn.*(2.9) *with the computation of the Spin($\mathcal{N}$) curvature in chapt. 2, we see that, given any negatively curved Quaternionic–Kähler metric g, there is a (unique) re–scaling, $g \to g' \equiv \lambda\, g$, such that g' is the target space metric of an $\mathcal{N} = 4$ SUGRA theory, for an appropriate scale λ. This should be contrasted with the $\mathcal{N} = 2$ case, where the analogous argument shows that not all Kähler manifolds are allowed as target spaces, but only the subclass of Hodges one (see sect.**2...**).*

REMARK (about the theorem). In an exercise of chapt. 2 you were asked to compute — by geometric arguments — the coefficients a and b in eqn.(2.2). You may now check that the linear combination is precisely right so that the tensor in parethesis which couples to χ^4 is $\odot^4\Lambda$, that is, the effect of the SUGRA correction to the rigid coupling is to project out the $Sp(2)_R$ contribution to the curvature. The formula is exactly as in the rigid case, except that R_{ijkl} is replaced by Ω_{mnpq}, the $Sp(2m)$ curvature (which is the only one present in the rigid case!). And the $Sp(2m)$ curvature is $Sp(2)_R$ invariant by definition.

REMARK (about the theorem). The exception $\mathrm{Aut}_R = Sp(2)_R$ stems from the fact that the Bianchi identities amount to an antisymmetrization with respect three indices, so if the representation of Aut_R has less than dimension 3, it becomes an empty statement[16]. Except for that subtlety, the argument is correct, and for $\mathcal{N} > 2$ (in $D = 4$) we can get only symmetric spaces! (We got this result by Berger's theorem, but, as we have seen, more elementary arguments are enough to get the full answer for all the cases appearing in SUGRA.

[16] See also APPENDIX D.

2.2. $D \geq 4$. We approach the general case by the following argument. Assume we have a SUSY/SUGRA model in $D \geq 4$ dimensions. We can dimensionally reduce the model to $D = 3$ by assuming that the fields depend only on 3 coordinates, and rewriting all couplings in a manifestly $3D$ covariant way. In general the $3D$ scalars' manifold, $\mathcal{M}_3$, has more dimensions that the original one, $\mathcal{M}_D$, since we get new scalars from the internal components of tensor fields; moreover, to put the theory in canonical form, we have to dualize the various form–fields into scalars, increasing the dimension of $\mathcal{M}_3$. Thus, in general, $\mathcal{M}_D$ is just a *submanifold* immersed in $\mathcal{M}_3$

$$\mathcal{M}_D \hookrightarrow \mathcal{M}_3.$$

We ask: *what is the geometrical relation between the manifolds $\mathcal{M}_D$ and $\mathcal{M}_3$ (or, more generally, $\mathcal{M}_d$ for $3 \leq d < D$)?*

To answer this question, we do a little *gedanken experiment.* Consider our SUGRA theory (ungauged and with vanishing scalar's potential[17]) in D dimensions. Look for a solution of the equations of motion in which all fields are set to zero, except the scalars and the metric. We take the scalars to depend only on time t, $\phi^i = \phi^i(t)$. Moreover, we consider the adiabatic limit, that is we write $\phi^i = \phi^i(\epsilon\, t)$, and take $\epsilon \to 0$. In this limit we can forget about the gravitational back–reaction (since $T^{\mu\nu} = O(\epsilon^2)$), and the equations of motions reduce to

$$-\frac{d}{dt}\Big(G_{ij}\,\dot{\phi}^j\Big) + \Gamma^l_{ij}\,G_{lk}\,\dot{\phi}^j\,\dot{\phi}^k + O(\epsilon^3) = 0,$$

so the solutions are simply the geodesic on $\mathcal{M}_D$. Reduce to d dimensions by requiring the fields not to depend on $D - d$ spatial coordinates. The above solutions should be also solutions of the d dimensional theory (in the same adiabatic limit). But, in d dimensions the abiabatic–limit solutions are precisely the geodesics of $\mathcal{M}_d$. Hence under the embedding $\mathcal{M}_D \hookrightarrow \mathcal{M}_d$ geodesics go to geodesics. This is precisely the definition of a *totally geodesic submanifold.* Hence

GENERAL LESSON 2.2. *Let $\mathcal{M}_D$ and $\mathcal{M}_d$ be, respectively, the scalars' manifolds of a (Q)FT in D spacetime dimensions and of its (trivial) reduction to d dimensions. Then $\mathcal{M}_D$ is a* TOTALLY GEODESIC SUBMANIFOLD *of $\mathcal{M}_d$.*

REMARK. No supersymmetry required!

REMARK. In the above *gedanken experiment* we assumed that $V(\phi) \equiv 0$. But, of course, the relation between $\mathcal{M}_D$ and $\mathcal{M}_d$ cannot depend on the scalar potential, thus our conclusion is fairly general.

Now, returning to our problem, consider the 4-χ coupling from the $D = 3$ viewpoint (we recall that the χ's are defined to be the fermions taking value in $T\mathcal{M}_D$ in D dimensions):

$$\bar{\chi}^i\gamma^\mu\chi^j\,\bar{\chi}^k\gamma_\mu\chi^l\,(a\,g_{ik}g_{jl} - b\,R^{(3)}_{ijkl})\Big|_{T\mathcal{M}_D \subset T\mathcal{M}_3}, \tag{2.10}$$

[17] Is this a consistent assumption? *Yes!* If, in some SUGRA model, a certain gauging, or potential, was needed for consistency in D dimension, we would have the corresponding condition on the gauging and potential down in $D = 3$ which is not the case.

where $R^{(3)}$ is the curvature computed in $\mathcal{M}_3$. The main property of a totally geodesic submanifold is precisely that the curvature tensor of $T\mathcal{M}_D \subset T\mathcal{M}_3$ computed using the $\mathcal{M}_D$ metric is exactly the same as the one computed using the metric of $\mathcal{M}_3$. Thus, we can erase the superscript (3), and learn that the formula eqn.(2.2) is true (for the fermions spanning $T\mathcal{M}$) for SUGRA in *any* dimension. Therefore the discussion we did in $D = 3$ relating the 4–Fermi coupling to holonomy and hence metric geometry holds in all dimensions as well.

What about all the other 4–Fermi couplings (those involving gauginos, the dilatini/axionini, the spin–3/2 gravitini themselves). Do they have a geometrical description?

Of course they do. Consider the dimensional reduction map

$$\varrho_{D,3} \colon \mathcal{M}_D \hookrightarrow \mathcal{M}_3. \tag{2.11}$$

The fermions' vector bundle $\mathcal{F} \to \mathcal{M}_D$ is just $\varrho^*_{D,3} T\mathcal{M}_3$. This bundle contains all dynamical fermions, even the physical polarizations of the gravitini ψ^A_μ, $\mu = 3, 4, \dots, D-1$. *The full 4–Fermi coupling is just the (projective) curvature of the $\varrho^*_{D,3} T\mathcal{M}_3$ bundle!*

Computing the 4–Fermi terms by actually constructing $\varrho_{D,3}$ may be cumbersome, but once we know that they are *curvatures of certain natural bundles,* we have only to identify these bundles over $\mathcal{M}_D$ by comparing their Aut_R quantum numbers, much as we did in the previous section for the bundle $T\mathcal{M}$. This is particularly easy for $\mathcal{N} \geq 3$ where everything is Lie–algebraic, and the bundles are uniquely fixed by their representation content. Thus

GENERAL LESSON 2.3. *The 4–Fermi couplings are also uniquely determined by the holonomy group* $\mathrm{Hol}(\mathcal{M})$.

This $\varrho_{D,3}$ game may be played with the other couplings as well (Yukawas, in particular). The map $\varrho_{D,d}$ is so important to deserve a proper name: it is called *group disintegration,* see ref. [**149**].

3. Vector couplings in $D = 4$

3.1. $\mathcal{N} = 8$ sugra. To fix the ideas, let us start by considering the $\mathcal{N} = 8$ case (*i.e.* the model first constructed in ref. [**11**]). From linear representation theory, we know that there are 28 vectors, and hence — by §.of chapt. 2 — the field strengths take value in a (flat torsionless) $Sp(56, \mathbb{R})$–bundle $\mathcal{F}_{\mathbf{56}} \to \mathcal{M} = E_{7(7)}/SU(8)$ associated to the defining representation $\mathbf{56}$ of $E_{7(7)}$. Moreover, our pet 'equivalence principle' says that

$$\mathcal{F}_{\mathbf{56}} \simeq \wedge^2 \mathcal{S}_{\mathbf{8}} \ \oplus\ \wedge^2 \mathcal{S}^\vee_{\mathbf{8}}, \tag{3.1}$$

where $\mathcal{S}_{\mathbf{8}}$ is the $SU(8)$ bundle associated to the fundamental representation (that is to the gravitino bundle). In concrete, the bundle isomorphism (3.1) means that there is a (non–singular) vielbein $(U^{[A\,B]}_x, V_{x\ [A\,B]})$ ($x = 1, 2, \cdots, 56$; $A, B = 1, 2, \cdots, 8$) converting the curved $SU(8)$ indices, A, B, into flat $Sp(56, \mathbb{R})$ indices, x, y, which is *covariantly constant under the combined $Sp(56, \mathbb{R})$ and $SU(8)$ connections.* We can choose our frame $(U^{[A\,B]}_x, V_{x\ [A\,B]})$ so that the $Sp(56, \mathbb{R})$ connection vanishes. Then we have 56 covariantly constant sections of $\wedge^2\mathcal{S}_{\mathbf{8}} \oplus \wedge^2\mathcal{S}_{\mathbf{8}}^\vee$ on our manifold $E_{7(7)}/SU(8)$. The $\mathcal{S}_8$ bundle was constructed in chapter 2 where we computed its tt^*–like curvature. By the analysis in chapt. 3[18], this bundle and its connection are $E_{7(7)}$ covariant, and hence the 56 covariantly constant sections above should form a definite representation of $E_{7(7)}$. This representation, ρ, should be *real symplectic,* by consistency with $Sp(56, \mathbb{R})$.

The representation ρ on the 56 covariantly constant sections may be seen as a map

$$\rho\colon E_{7(7)} \to Sp(56, \mathbb{R}). \tag{3.2}$$

The 56×56 matrix representing an element $E \in E_{7(7)}$ is just the vielbein[19] $\mathcal{E}^{-1}$ we discussed at length in chapt. 1 in the context of general dualities:

$$\rho(E) \equiv \mathcal{E}^{-1}(E). \tag{3.3}$$

In facts we are exactly in the situation of the *prototypical example* of page 30: We have only to check that

$$\forall\; U \in SU(8) \subset E_{7(7)}\colon \qquad \rho(U) \in U(28) \subset Sp(56, \mathbb{R}), \tag{3.4}$$

and then the commutative diagram of page 30 will do all the work for us. But this is pretty obvious: the image under ρ of a compact subgroup should be a compact subgroup and hence contained in a maximal compact subgroup of the target group which is a $U(56)$. From another point of view, the 'downstairs' $SU(8)$ in the symmetric manifold $E_{7(7)}/SU(8)$ is equal to its holonomy group Hol. Under the holonomy group $\mathcal{F}_{\mathbf{56}}$ splits into $\wedge^2\mathcal{S}_{\mathbf{8}} \oplus \wedge^2\mathcal{S}_{\mathbf{8}}^\vee$; the action of $\mathrm{Hol} \equiv SU(8)$ on its representation **28** should be unitary and hence an element of $U(28)$.

Identifying an element of the group $E_{7(7)}$ with the 56×56 matrices which represent it in the faithful **56** representation, we can represent the points $\phi \in E_{7(7)}/SU(8)$ as the corresponding vielbein, $\mathcal{E}^{-1}$, with the proviso that two vielbeins $\mathcal{E}$ and $\mathcal{E}'$ do correspond to the same point in $\mathcal{M}$ if there exist an $U \in SU(8)$ such that

$$\mathcal{E} \sim \mathcal{E}' \equiv \rho_{\mathbf{28} \oplus \overline{\mathbf{28}}}(U)\, \mathcal{E}. \tag{3.5}$$

In term of the vielbein $\mathcal{E}$, the map $\mu\colon \mathcal{M} \to Sp(56, \mathbb{R})$ is the identity up to the identification (3.5). The scalars–vectors couplings are directly determined (as in chapter 1) by the vielbein, that is by the diagram on page 30. So also the vector couplings are elegantly predicted by geometry (without doing any *substantial* computation).

Notice that the commuting diagram on page 30 also guarantees that the E_7 transformations, which act as isometries of $\mathcal{M}$ and hence symmetries of the scalars (and fermions) couplings are also symmetries of the vectors' kinetic terms; however these symmmetries act as *dualities,* and hence they tipically are not invariances of the Lagrangian $\mathcal{L}$. In sect. 5 below we shall show that it is an exact symmetry of the full equations of motion (in ungauged SUGRA).

[18] I mean the construction of the holonomy of $\mathcal{S}_{\mathbf{8}}$ in terms of the Lie algebras $\mathfrak{g} \equiv \mathfrak{e}_7$ and $\mathfrak{h} \equiv \mathfrak{su}(8)$.

[19] Again, we take the inverse to convert a right action of E_7 into the more canonically looking left action.

TABLE 4.3. Representation content of the field–strengths $\mathcal{F}$ under the U–duality group G

$\mathcal{N}$	G	Representation
3	$SU(3,k)$	$[[\,\mathbf{3+k} \oplus \overline{\mathbf{3+k}}\,]]$
4	$SU(1,1)\times SO(6,k)$	$(\mathbf{2},\mathbf{6+k})$
5	$SU(5,1)$	$[[\,\mathbf{10}\oplus\overline{\mathbf{10}}\,]]$ (self–dual part of $\wedge^3 V_{\mathbf{6}}$)
6	$SO^*(12)$	$\mathbf{32}$ (the chiral spinor)
8	$E_{7(7)}$	$\mathbf{56}$ (the fundamental rep.)

3.2. Generalization to all $\mathcal{N}$. The generalization of our findings to all $\mathcal{N}$'s is obvious.

3.2.1. *$\mathcal{N}\geq 3$ SUGRA's.* Consider first the case $\mathcal{N}\geq 3$, in which — as we saw in sect. 1.2 — the scalars' manifold has the form G/H. In all such models the isometry group G acts on the field–strengths trough a real symplectic representation F of dimension $2n$. The representations F embeds $G\hookrightarrow Sp(2n,\mathbb{R})$, and we identify G with its image. The subgroup $H\subset G$, being compact, is mapped to $U(n)$, and again the commuting diagram on page 30 do its job. So all scalars–vectors couplings get determined (and are guaranteed to be G invariant): the 'magnetic susceptibility' map, μ, of section ... of chapt.1 is nothing else that the natural projection of the representation map ρ_F:

$$\begin{array}{ccc} \rho_F\colon\ G & \xrightarrow{\ i\ } & Sp(2n,\mathbb{R})\ \subset\ \mathrm{End}(\mathbb{R}^{2n}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{M}=G/\big(G\cap U(n)\big) & \xrightarrow{\ \mu\ } & Sp(2n,\mathbb{R})/U(n) \end{array} \tag{3.6}$$

and thus the couplings are determined once the representation content of F is known. But this is simplicity itself: we write the representations under G and H for the various $\mathcal{N}$ in table 4.3.

All other remarks for the $\mathcal{N}=8$ apply word–for–word. Then

GENERAL LESSON 3.1. *For $D=4$ $\mathcal{N}\geq 3$ ungauged* SUGRA*, with $\mathcal{M}=G/H$ as listed in table 4.1, the isometry group G of $\mathcal{M}$ is also a symmetry of the vectors' kinetic terms (in the sense of duality).*

3.2.2. *$\mathcal{N}=1,2$ SUGRA's.* The case of $\mathcal{N}=1$ was already treated in sect. of chapt. 2; all *holomorphic* maps $\mu\colon\mathcal{M}\to Sp(2V,\mathbb{R})/U(V)$ are allowed.

The case $\mathcal{N}=2$ is more interesting. Here we have the local version of special geometry (we gave a preliminary discussion of the rigid case in sect. of chapt. 2), called *projective special Kähler geometry.* In the particular case (the most interesting one for the applications) in which we have 'enough' symmetries — that is the group $\mathrm{Iso}(\mathcal{M})$ is transitive — the argument we used above for the $\mathcal{N}\geq 3$ case goes trough word–for–word for $\mathcal{N}=2$ too, and all couplings are fixed (and easy computed explicitly) once we know in which representations of $\mathrm{Iso}(\mathcal{M})$ the field–strengths $\mathcal{F}$ are.

The general case is similar, and again uniquely determined by geometrical considerations: however (in the generic model) these geometric arguments do not reduce to purely Lie–algebraic manipulations. Hence, for the sake of order and clarity, we prefer to postpone the discussion of the general $\mathcal{N} = 2$ model after the foundation of the relevant geometric theory, namely *projective special Kähler geometry,* see Part 3 below.

4. The gauge point of view

We have seen that in $\mathcal{N} \geq 3$ the scalar fields live on a coset manifold G/H. The number of scalar degrees of freedom is $\dim G - \dim H$, of course. We identify a (constant) value of the scalars ϕ^i with the *class* of the vielbein $\mathcal{E}(\phi)$ in the coset G/H. This is the mathematicians' language. Physicists are more clever. They say: "let us introduce a full set of $\dim G$ scalar fields, parameterizing the *group* G (a much easier object than the coset G/H)". Now the entries of the matrix $\mathcal{E}$ *are* our scalar fields. But in this way we get $\dim H$ too many degrees of freedom. "Well, let us introduce a *gauge* symmetry with $\dim H$ generators 'to eat' the unwanted degrees of freedom". This is easily done. As gauge group one takes H itself, and the gauge transformations act on the scalars by[20]

$$\mathcal{E}^{-1} \to \mathcal{E}^{-1}\, U(x) \qquad U(x) \in H. \tag{4.1}$$

The theory with target space the group manifold G and the subgroup H gauged as above is physically equivalent to the original one on the coset G/H. The last theory is just the first one in the 'unitary gauge', and all H–gauge–invariant observables are manifestly the same in the two formulations.

The gauge formulation, however, has formidable advantages. It is more intuitive (for a physicist, at least), and has more symmetry: The formalism now has an automorphism group which is $G_{\text{GLOBAL}} \times H_{\text{LOCAL}}$ acting on $\mathcal{E}^{-1}$ as follows

$$\mathcal{E}^{-1} \to g\, \mathcal{E}^{-1}\, U(x), \qquad g \in G_{\text{GLOBAL}}, \;\; U(x) \in H_{\text{LOCAL}}. \tag{4.2}$$

Having a formalism with so a big automorphism group is a tremendous technical asset. We shall adhere to it with entusiasm. Besides, working directly on the group manifold G is a major algebraic simplification. Stay tuned for further developments!

Note that G_{GLOBAL} may or may not be a symmetry of the complete Lagrangian[21]. On the contrary, H_{LOCAL} is *always* exact, since it was 'artificially' constructed to be a symmetry.

5. The complete Lagrangian: U–duality

In sect. 3 above we have not just computed the full non–linear scalar–vector couplings. We have done a lot more: we have proven that the G symmetry of the scalars' sector is also a symmetry of the vectors' sector, and hence of the full bosonic Lagrangian $\mathcal{L}^{\mathcal{N}\geq 3}_{\text{bos}}$ (in ungauged SUGRA!!).

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[20] Again I take the inverse $\mathcal{E} \to \mathcal{E}^{-1}$ to convert a right action into a more pleasant left action (just a matter of taste).

[21] It is in the *ungauged* case.

On the other hand, the arguments of sect. 2 imply that the 4–Fermi couplings are also G–invariant, as well as the Fermi kinetics terms (in fact their Aut_R–bundles are homogeneous bundles over G/H, and hence G covariant).

It appears that the full Lagrangian $\mathcal{L}_{\mathcal{N}\geq 3}$ of (ungauged) $\mathcal{N} \geq 3$ supergravity is invariant under the *hidden symmetry* (*a.k.a.* U–duality) group G. To fully establish this result we have to prove invariance of the last two classes of couplings present in $\mathcal{L}_{\mathcal{N}\geq 3}$ which we have not discussed yet. They are:

(1) couplings proportional to $\bar{\psi}^A_\mu\, \gamma^\nu \gamma^\mu\, \chi_{BCD}$;
(2) Pauli couplings with two fermions (spin 1/2 or 3/2) and a vector field strength.

We start by giving an *a priori* argument to the effect that also these terms should be G–invariant. Indeed, these couplings are sections of some bundles which are obtained as $\rho^*_{D,3}$–pullbacks of certain sections of the appropriate bundles in the $D = 3$ theory. In chapter 2 we constructed the *full* $D = 3$ (ungauged) Lagrangian, and found it to be invariant under *all* the isometries of $\mathcal{M}$. The pulled–back sections then are automatically $\mathrm{Iso}(\mathcal{M})$–invariant by construction[22].

However, let us be explicit. For notational defineteness, we take $\mathcal{N} \geq 5$, so we have only the gravitational multiplet; the extension to $\mathcal{N} = 3, 4$ being straighforward.

The couplings in (1), linear in the gravitini, are just the 'Nother' term, which couples the 'superconnection' ψ^a_μ to the supercurrent which is proportional to

$$\gamma^\nu \gamma_\mu\, \chi_{BCD}\; P_i^{ABCD}\, \partial_\nu \phi^i, \tag{5.1}$$

where, as in all discussions of chapt. 2, $P_i^{ABCD}(\phi)$ is the bundle isomorphism (again our pet 'target space equivalence principle'[23])

$$T\mathcal{M} \simeq \wedge^4 \mathcal{S}_{\mathcal{N}}, \tag{5.2}$$

which is exactly the argument which led us to the G/H structure in the first place! P_i^{ABCD} plays exactly the same role as γ_i^{Am} in the rigid $\mathcal{N} = 2$, $D = 6$ σ model (cfr. §. 8 of chapt. 2), and it generates a Clifford–algebra $\subset \mathrm{End}(T\mathcal{M})$, for reasons already reviewed too many times, (for the $\mathcal{N} = 8$ case, cfr. eqn.(4.16) of the first ref. [**81**]). In particular (up to the normalization that I have not kept track of) the G invariant metric on G/H is precisely

$$g_{ij} = P_i^{ABCD}\, (P_j^{ABCD})^*, \tag{5.3}$$

and thus the coupling P_i^{ABCD} should be invariant (up to a H gauge transformation acting on the capital indices $A, B, \dots$) under *any* isometry.

Let us now show explicitly that the Pauli couplings (2) are G–invariant. We return to eqn.(4.29) of chapt. 1, but now we replace the generic locution *'other fields'* in the RHS with the appropriate term, bilinear in the fermions, which enters in $G_{x\,\mu\nu} \equiv i * (\partial\mathcal{L}/\partial F^x_{\mu\nu})$ as a consequence of the

[22] Here we are sloppy. We have not proven that any isometry of $\mathcal{M}_D$ is induced by an isometry of the larger space $\mathcal{M}_3$ *via* the totally geodesic embedding $\varrho_{D,3} \colon \mathcal{M}_D \hookrightarrow \mathcal{M}_3$. However, for $\mathcal{M}$ symmetric, it is true; see chapter 5.

[23] That is, the capitol indices $A, B, C, \dots = 1, 2, \dots, \mathcal{N}$ are $U(\mathcal{N})_R$–indices.

Pauli couplings, linear in $F^x_{\mu\nu}$. In the $Sp(2V,\mathbb{R})$–covariant formalism, these terms are rewritten in the form of a **2V**–vector $\mathcal{K}_{\mu\nu}$; the constraint on the field–strengths (eqn.(4.29) of chapt. 1) now reads[24]

$$\frac{1}{2}(1-i\,\Omega)\,\mathcal{E}\,\mathcal{F}^+ = \mathcal{K}^+, \tag{5.4}$$

while the vectors' equations of motion are simply $d\,\mathcal{F}=0$. Since $(1-i\,\Omega)/2$ is a projector, consistency requires $(1-i\,\Omega)\mathcal{K}^+ = 2\mathcal{K}^+$, so only half the components of $\mathcal{K}$ are independent

$$\mathcal{K}^+ = \begin{pmatrix} K \\ iK \end{pmatrix}. \tag{5.5}$$

The *gauged* H symmetry acts on the LHS of eqn.(5.4) as in eqn.(4.2). (Recall: $H \subset U(V)$, so $U(x)\in H$ commutes with Ω). Then gauge invariance requires the RHS of (5.4) to transform in the same way

$$\mathcal{K}^+ \to U(x)\,\mathcal{K}^+. \tag{5.6}$$

To be more explicit, we write the matrix $\mathcal{E}$ as in eqn.(4.30) of chapt. 1

$$\mathcal{E} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{5.7}$$

The entries of $\mathcal{E}$ are related by various algebraic identities, reflecting the fact that the matrix $\mathcal{E}$ is an element of the group G. A matrix of the form (5.7) belongs to $H \equiv G \cap U(V)$ iff it satisfies the G group identities and it is of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \quad \text{with} \quad (A+iB)\in U(V),$$

so

$$\begin{pmatrix} K^\pm \\ \pm iK^\pm \end{pmatrix} \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix}\begin{pmatrix} K^\pm \\ \pm iK^\pm \end{pmatrix} \equiv \begin{pmatrix} (A\pm iB)K^\pm \\ \pm i(A\pm iB)K^\pm \end{pmatrix},$$

and K^+, K^- transform in conjugate representations of the gauge symmetry $H\subset U(V)$.

Define the H–gauge–invariant **2V**–vector of two–forms

$$\widehat{\mathcal{K}}_{\mu\nu} = \mathcal{E}^{-1}\,\mathcal{K}_{\mu\nu} \tag{5.8}$$

and the *improved* field strengths $\widehat{\mathcal{F}}_{\mu\nu} \equiv \mathcal{F}_{\mu\nu} - \widehat{\mathcal{K}}_{\mu\nu}$. The constraint now reads

$$(1-i\,\Omega)\,\mathcal{E}\,\widehat{\mathcal{F}}^+ = 0, \tag{5.9}$$

which is manifestly G invariant under

$$\widehat{\mathcal{F}}^+ \to g\,\widehat{\mathcal{F}}^+, \qquad \mathcal{E}^{-1}\to g\,\mathcal{E}^{-1}, \quad g\in G. \tag{5.10}$$

We see that the global G symmetry is induced by local H invariance. Hence the vectors' equations of motion are G invariant.

GENERAL LESSON 5.1. *In $\mathcal{N}\geq 3$, $D=4$* UNGAUGED SUGRA *the complete equations of motion are invariant under the non–compact 'hidden symmetry' G. G acts on the field strenghts $\mathcal{F}$ through a real symplectic representation F of* GENERALIZED DUALITY*, and all couplings are uniquely determined once we know the G–representation F.*

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[24] I omit space–time and G, H, indices. I hope that would not cause confusion.

REMARK. This results explains why the non–compact G symmetries are now called U–DUALITIES: in fact they act on the fields as dualities and are not symmetries of the Lagrangian but only of the equations of motion (the Lagrangian changes according to the canonical Hamilton–Jacobi formula, eqn. of chapt. 1). These dualities are quite important in the context of superstring/M–theory. In that case the theory has many other sectors, so the G symmetry is broken down to a discrete (arithmetic?) subgroup $G_{\mathbb{Z}}$. The good news is that this — *a priori* quite bizzare — symmetry is a good *quantum* invariance in the stringy framework.

We have completed the computation of the ungauged supergravity Lagrangians in $D = 4$. It remains only to *(i) gauge* a subgroup of $\mathrm{Iso}(\mathcal{M})$ to get gauged SUGRA's; *(ii)* dwell the two low–$\mathcal{N}$ cases ($\mathcal{N} = 1$ and 2); and *(iii)* to work out SUGRA in other (interesting) dimensions.

I have the feeling that somebody in the audience thinks that our *explicit* Lagrangian $\mathcal{L}_{\mathcal{N}}$ for $\mathcal{N} \geq 3$ is not explicit enough for his/her taste.

Well, I will be even *more* explicit in the next chapter!

6. U–duality, central charges and Grassmannians

Perhaps this is the right place to discuss the *central charge geometry* we introduced in §. 10.3 of chapt. 2. By far the most interesting case — both in terms of physical implications and mathematical structures — is the $\mathcal{N} = 2$ case. Again, we defer this more sophisticated case to Part 3, and here we limit ourselves to the $\mathcal{N} \geq 3$ SUGRA's which we can approach by simple Lie–algebraic techniques.

In order to discuss central charges, we have to be slightly more explicit in our treatment of U–duality.

To write simpler and nicer formulae, it is convenient to rewrite the real matrix $\mathcal{E}$ in a complex basis where the action of the compact subgroup $H \subset U(V)$ is diagonal. This change of basis has a deep geometrical meaning. So we shall spend some minute discussing it. If you feel the topic too pedantic, just jump ahead.

6.1. The Cayley transformation. [25]

We saw in §. 6.1 of chapt. 1 that the vectors' coupling constant symmetric space, $Sp(2V, \mathbb{R})/U(V)$, is the same as the *Siegel's upper half–space*

$$\mathfrak{H}_V = \{Z \in \mathbb{M}_V(\mathbb{C}) \mid Z^t = Z, \quad \mathrm{Im}\, Z > 0\}. \tag{6.1}$$

For $V = 1$, this is just the upper half–plane $\mathfrak{H} = \{z \in \mathbb{C}, z = x + iy, y > 0\}$, which we know so well from basic string theory. As it is well–know, the upper half–plane $\mathfrak{H}$ is conformally equivalent to the open unit disk

$$\mathfrak{D} = \{w \in \mathbb{C} \,|\, ww^* < 1\}.$$

In this second representation, the compact $U(1)$ isometry of $\mathfrak{H}$ acts simply as a rotation $w \mapsto e^{i\alpha} w$. The analogue of the disk $\mathfrak{D}$ for $V > 1$ is the space

$$\mathfrak{D}_V := \{W \in \mathbb{M}_V(\mathbb{C}) \mid W^t = W, \quad 1 - WW^* > 0\}. \tag{6.2}$$

[25] For more details on this topic see, *e.g.* ref. [**120**] pages 221–230, and expecially **proposition 31.4** and **proposition 31.7**.

The convenience of $\mathfrak{D}_V$ with respect $\mathfrak{H}_V$ is twofold. For our present purposes, it block–diagonalizes the representation of the subgroup $U(V) \in Sp(2V, \mathbb{R})$, leading to more transparent formulae; from a more general standpoint, it replaces the *unbounded* domain $\mathfrak{H}_V \subset \mathbb{C}^{V(V+1)/2}$ with a *bounded* domain $\mathfrak{D}_V \subset \mathbb{C}^{V(V+1)/2}$, allowing us to use the powerful machinery of bounded domain technology to study it [**135**].

To map $\mathfrak{H}_V$ into $\mathfrak{D}_V$, we conjugate $\mathcal{E}$ in $Sp(2V, \mathbb{C})$. That is, we let $\mathcal{E} \mapsto C\mathcal{E}C^{-1}$, where C, the Cayley transform, is the $Sp(2V, \mathbb{C})$ matrix

$$C = \sqrt{\frac{i}{2}} \begin{pmatrix} \mathbf{1}_V & \mathbf{1}_V \\ i\,\mathbf{1}_V & -i\,\mathbf{1}_V \end{pmatrix}. \tag{6.3}$$

The new $\mathcal{E}$ reads

$$\mathcal{E} = \begin{pmatrix} U & V \\ V^* & U^* \end{pmatrix}, \tag{6.4}$$

where

$$2\,U = (\mathcal{A} + \mathcal{D}) - i(\mathcal{B} - \mathcal{C}) \tag{6.5}$$

$$2\,V = (\mathcal{A} - \mathcal{D}) + i(\mathcal{B} + \mathcal{C}). \tag{6.6}$$

Now the action of H is simply matrix multiplication

$$V \mapsto h\,V, \qquad U \mapsto h\,U. \tag{6.7}$$

In particular, the projectors $P_\pm = \frac{1}{2}(1 \pm i\Omega)$ are now diagonal

$$P_\pm = \frac{1}{2}(1 \mp \Sigma_3).$$

In the Cayley basis $Sp(2V, \mathbb{R})$ is the set of complex matrices such that

$$\mathcal{E}^* = \Sigma_1 \mathcal{E} \Sigma_1, \qquad \mathcal{E}^t \Omega \mathcal{E} = \Omega, \tag{6.8}$$

where

$$\Sigma_1 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \qquad \Sigma_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

The symplectic property now reads

$$U^t V^* - V^\dagger U = 0 \tag{6.9}$$

$$U^t U^* - V^\dagger V = \mathbf{1}. \tag{6.10}$$

EXERCISE 6.1. Show that C maps the region (6.1) of $\mathbb{C}^{V(V+1)/2}$ into the region (6.2).

6.2. The H–covariant field–strength $\mathfrak{F}$. Using the above machinery, we can define two projections of the $2V$–vector $\mathcal{F}^+$

$$P_- \mathcal{E}\,\mathcal{F}^+ = \begin{pmatrix} K^+ \\ 0 \end{pmatrix} \tag{6.11}$$

$$P_+ \mathcal{E}\,\mathcal{F}^+ = \begin{pmatrix} 0 \\ \mathfrak{F}^+ \end{pmatrix} \tag{6.12}$$

here $\mathfrak{F}^+$ is the H–covariant version of the vectors' field–strengths. $\mathfrak{F}^+$ transforms in the opposite H–representation with respect to K^+ (that is, in the same representation as its conjugate K^-). In the old SUGRA jargon, in passing from $\mathcal{F}^+$ to $\mathfrak{F}^+$, we have converted the 'curved' $G \subset Sp(2V, \mathbb{R})$ indices into 'flat'[26] $H \subset U(V)$ indices using the Vielbein $\mathcal{E}$.

Solving the above linear equations, one gets

$$\mathfrak{F}^+ = (U^* V^{-1} - V^* U^{-1}) V (U - V)^{-1} U \, F^+. \tag{6.13}$$

where F^+ is the (complex) self–dual part of the Abelian field–strength dA (all indices soppressed in the notation). From eqns.(6.9), we get

$$\left.\begin{array}{c} U^* V^{-1} - (U^t)^{-1} V^\dagger = (U^t)^{-1} V^{-1} \\ V^* U^{-1} - (U^t)^{-1} V^\dagger = 0 \end{array}\right\} \Rightarrow \; (U^* V^{-1} - V^* U^{-1}) = (U^T)^{-1} V^{-1} \tag{6.14}$$

so, finally, the relation between the H–invariant field–strenghts $\mathfrak{F}$ and the Maxwell ones F reads

$$\mathfrak{F}^+ = (U^T)^{-1} (U - V)^{-1} U \, F^+, \tag{6.15}$$

which manifestly has the right transformation property under H, eqn.(6.7),

$$\mathfrak{F}^+ \mapsto (h^t)^{-1} \, \mathfrak{F}^+ \equiv h^* \, \mathfrak{F}^+. \tag{6.16}$$

In SUGRA, the local symmetry group, H, has the form

$$H \equiv \mathrm{Aut}_R \times \widetilde{H} = \begin{cases} U(1)_R \times SU(\mathcal{N})_R \times \widetilde{H} & \mathcal{N} \neq 8 \\ SU(8)_R & \mathcal{N} = 8. \end{cases}$$

and K^+, $\mathfrak{F}^-$ belong to the representations[27]

$$\left(\frac{\mathcal{N}(\mathcal{N}-1)}{2}, \mathbf{1} \right)_{+2} \oplus (\mathbf{1}, \mathbf{k})_0, \tag{6.17}$$

where k is the number of matter vectors ($k = 0$ for $\mathcal{N} \geq 5$).

6.3. Pauli couplings and central charges. Local H symmetry requires the structure of the Pauli couplings to be proportional to

$$e \, (K^+_{\mu\nu})^t \, \mathfrak{F}^{+\,\mu\nu} + \text{H.c.} \tag{6.18}$$

which can be also be written in the more suggestive form

$$e \, \mathcal{K}^t_{\mu\nu} \Omega \mathcal{E} \mathcal{F}^{\mu\nu}.$$

The equation $G^+ = i\partial\mathcal{L}/\partial F^+$, fixes the overall coefficient to be $-1/2$ [CHECK]. Now the 'target space equivalence principle' implies that K^+ is a bilinear in the fermions without any scalar; K^+ is exactly given by the formula one would get from the linear theory [**141**]. We are interested in the term bilinear in the gravitini ψ^A_μ. Gauge invariance, Fermi statistics, and covariance determine this term up to a numerical coefficient

$$K^{+\,AB}_{\mu\nu} \; \propto \; \bar{\psi}^A_\rho \, \gamma^{[\rho} \gamma_{\mu\nu} \gamma^{\sigma]} \psi^B_\sigma + \cdots \tag{6.19}$$

---

[26] Note that in the SUGRA jargon the indices of the *flat* G–bundle are called 'curved' while those of the *curved* H–bundle are called 'flat'. This twist is due to analogy with the language one uses for General Relativity.

[27] In a convention where the $U(1)_R$ charge of the left–handed gravitino is 1.

The variation of the action with respect to ψ^A_μ gives the supercurrent, from which one reads the SUSY transformation of the fields. Covariance fixes the term in $\delta\psi^A_\mu$ containing the vectors' field–strength to the form

$$\delta\psi^A_\mu = \mathcal{D}_\mu \psi^A + c\,\mathfrak{F}^{-\,AB}_{\rho\sigma}\gamma^{\rho\sigma}\gamma_\mu\epsilon_B + \cdots\,, \tag{6.20}$$

for certain (highly convention dependent) costant c. This formula is uniquely singled out by $U(\mathcal{N})_R \subset H$ covariance. In particular, $\mathfrak{F}^-$ is the unique object with precisely the right $U(\mathcal{N})_R$ properties to lead to a locally covariant formula.

Comparing with our previous discussion in chapt. 2, we arrive at the conclusion that

$$Z^{AB} = c \int\limits_{\text{spatial }\infty} \mathfrak{F}^{-\,AB} \tag{6.21}$$

$$= c\left((U^{-1})^\dagger (U^* - V^*)^{-1} U^*\right)^{AB}{}_{x} \int\limits_{\text{spatial }\infty} F^{-\,x}. \tag{6.22}$$

that is the central charges are linear combination of the Maxwell electric and magnetic fluxes at infinity with coefficients which are given by the value at infinity (assumed to be constant, otherwise the charges make no sense) of the matrix in front of the integral in the second line.

In the linearized theory, the graviphotons are precisely the gauge vectors whose fluxes equal the central charges (that is the vectors associated to the internal symmetries part of the SUSY algebra). Here we see that the particular linear combinations of the A^m_μ's which play this rôle is background dependent. As we move in the target space $\mathcal{M}$, the $\mathcal{N}(\mathcal{N}-1)/2$–dimensional linear space spanned by the 'graviphotons' *moves* inside the (flat) V–dimensional space of all vectors. Thus the 'graviphotons' define a map from $\mathcal{M}$ to the Grassmannian

$$\mathrm{Gr}\left(\frac{\mathcal{N}(\mathcal{N}-1)}{2}, V, \mathbb{F}\right),$$

of $\mathcal{N}(\mathcal{N}-1)/2$–planes in $\mathbb{F}^V$. Here $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, according to the values of D, $\mathcal{N}$.

Again, the Grassmannian is a *symmetric space*. Thus the 'central charge geometry' defines yet another map of the form

$$\zeta\colon \mathcal{M} \to \mathrm{Gr}\left(\frac{\mathcal{N}(\mathcal{N}-1)}{2}, V, \mathbb{F}\right) \tag{6.23}$$

from $\mathcal{M}$ into *a symmetric space.* It is the third such map, after the one defined by the $T\mathcal{M}$ 'equivalence principle' isomorphism and the one defined by vectors' duality. Each of these maps, in principle, suffices to fix all the couplings in the Lagrangian. Luckily they lead to the same results — even if they are *different* structures.

6.4. Example: $\mathcal{N} = 4$. In this case the Vielbein matrix $\mathcal{E}$ is the tensor product of an $SU(1,1)$ matrix and an $SO(6,k)$ matrix.

$$\mathcal{E} = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_2^* & \phi_1^* \end{pmatrix} \otimes \left(L^{AB}{}_x, L^m{}_x\right) \tag{6.24}$$

where $A, B = 1, \dots, 6$, $m = 1, \dots, k$, $x = 1, 2, \dots k+6$, and $|\phi_1|^2 - |\phi_2|^2 = 1$. The $SO(6,k)$ matrix L has the properties

$$L^{AB}{}_x = (L_{AB\,x})^* = \frac{1}{2}\epsilon^{ABCD} L_{CD\,x} \qquad \text{(reality)} \tag{6.25}$$

$$-L^{AB}{}_x L_{AB\,y} + L^m{}_x L^m{}_y = \eta_{xy} \qquad \text{(`Vielbein' property)} \tag{6.26}$$

$$\text{where} \quad \eta_{xy} = \text{diag}(\underbrace{-,-,-,-,-,-}_{6\ \text{times}}, \underbrace{+,+,\cdots,+}_{k\ \text{times}}). \tag{6.27}$$

The above matrix is well–defined *up to multiplication* on the left by $H = U(1) \times SO(6) \times SO(k)$ (our gauge invariance).

Applying the formulae of the previous subsection, we get

$$\mathfrak{F}^{-\,AB} = (\phi_1^* - \phi_2^*)^{-1} L^{AB}{}_x \, \eta^{xy} \, F^-_y, \tag{6.28}$$

so the immage of the central–charge map ζ (cfr. eqn.(6.23)) is the complex subspace of $\mathbb{C}^{6+k}$ spanned by the 6 vectors $L^{AB}{}_x \, E^x$ (where E^x is a canonical basis of $\mathbb{C}^{6+k}$), which — using the natural real structure of $\mathcal{N} = 4$ SUGRA — is naturally identified with a *real* 6–plane in $\mathbb{R}^{6+k}$.

We can be even more specific. We have the Euclidean space $\mathbb{R}^{6,k}$, endowed with an inner product, $(\cdot,\cdot)$, of signature $(k, 6)$ (given by η, cfr. eqn.(6.27)). The central charges define a *negative definite* 6–plane in $\mathbb{R}^{6,k}$, that is a linear subspace $W \subset \mathbb{R}^{6,k}$ of dimensions six such that $(\cdot,\cdot)|_W$ is *negative definite.* The space of all such negative definite planes is the non–compact Grassmannian [**86**]

$$\text{Gr}(6, \mathbb{R}^{6,k}) = \frac{SO(6,k)}{SO(6) \times SO(k)}, \tag{6.29}$$

and the central–charge map ζ

$$\zeta\colon \mathcal{M} \equiv \frac{SU(1,1)}{U(1)} \times \frac{SO(6,k)}{SO(6) \times SO(k)} \to \text{Gr}(6, \mathbb{R}^{6,k}) \equiv \frac{SO(6,k)}{SO(6) \times SO(k)}, \tag{6.30}$$

is — in the $\mathcal{N} = 4$ case — simply projection into the second factor.

This gives an alternative explanation of why, in the $\mathcal{N} = 4$ case, the target space $\mathcal{M}$ turns out to be that specific symmetric space.

EXERCISE 6.2. Work out the $\mathcal{N} = 3$ case.

Of course, the above is a triviality. However, in the case $\mathcal{N} = 2$ this construction will give us a much more interesting map, in fact some of the deepest structures of all algebraic as well as transcendental geometry (not to mention non–perturbative quantum QFT).

We end this section by summarizing the situation into a

GENERAL LESSON 6.1. *In $\mathcal{N}$–extended $D = 4$ SUGRA ($2 \le \mathcal{N} \le 4$), the central charges (or, equivalently, the graviphotons) define a map ζ from the scalars' manifold $\mathcal{M}$ to the Grassmannian describing the space of all $\mathcal{N}(\mathcal{N}-1)/2$ planes in some Euclidean space with given geometric properties (depending on $\mathcal{N}$).*

7.∗ U–duality and arithmetics

This section contains just a side comment.

We stated more than one time that, if our SUGRA model is just the low–energy limit of some more fundamental theory, possibly UV complete, like superstring or M–theory, only a discrete subgroup of the SUGRA U–duality group G is actually a symmetry of the full theory. In fact, this discrete subgroup should be of the *arithmetic* type.

For simplicity, let us consider the simplest situation: $G/H = SU(1,1)/U(1) \equiv Sp(2,\mathbb{R})/U(1) \simeq \mathfrak{H}$ (the upper half–plane). The discrete subgroups of $SL(2,\mathbb{R})$ are known as *Fuchsian groups* [**125, 126**]; they zoology is vast and intricate: for instance all compact Riemann surfaces of genus ≥ 2 can be written in the form $\mathfrak{H}/\Gamma$ for some Fuchsian group (the uniformization theorem). However only a zero–measure subset of the Fuchsian groups may realized as the U–symmetry of a physical theory: the congruence subgroups, namely finite groups Γ such that

$$\Gamma(N) \subset \Gamma \subset SL(2,\mathbb{Z}), \tag{7.1}$$

where $SL(2,\mathbb{Z})$ is the group of 2×2 integral matrices with unit determinant (called the *modular group),* and $\Gamma(N)$ is a *principal congruence subgroup of level* $N \in \mathbb{N}$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;:\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}. \tag{7.2}$$

These groups are not only ‘few’ (countably many) but have quite peculiar properties, very different, in particular, from those Γ’s which uniformize compact Riemann surfaces. For instance, if Γ is a congruence subgroup, $\mathfrak{H}/\Gamma$ is NONcompact, but has nevertheless finite volume. Distances from the boundary diverge in $\mathfrak{H}/\Gamma$ but in a mild way, *ect.* These are universal properties, which hold in general, as we shall try to argue. They also lead to a beautiful, powerful and deep theory of *automorphic forms and representations* which is useful in a countless number of physical problems, *e.g.* in black–hole physics, see Moore and coworkers [**127**].

Curmrun Vafa [**128**] has especially emphasized the above properties. He conjectures that they are among the landmarks of any low–energy theory which may arise out of supertring/M–theory vacuum.

The point is that G not only acts on the scalars’ manifold as an isometry, but also on the vectors field strengths

$$\begin{pmatrix} F^x_{\mu\nu} \\ G_{x\,\mu\nu} \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^x_{\mu\nu} \\ G_{x\,\mu\nu} \end{pmatrix}. \tag{7.3}$$

If our SUGRA is the low–energy limit of some fundamental theory, the vectors appearing in this formula are precisely the gauge fields of the parent theory which — in the vacuum of interest — are in their Coulomb phase. Typically, these vectors are part of some unified non–Abelian gauge connection in the UV theory, and hence have (in general) non–trivial electric and magnetic sources.

Integrating the vector of two–forms (7.3) on a sphere at infinity we get magnetic and electric charges m^x, e_x. On the space of charge vectors

$Q^t = (m^x, e_y)$ we have an *integral valued symplectic pairing* given by Dirac's quantization relation

$$[Q_1, Q_2] \equiv m_1^x \, e_{2\,x} - e_{1\,x} \, m_2^x \in \mathbb{Z}. \tag{7.4}$$

Assuming that all vectors have non–trivial electric and magnetic sources, this rule implies quantization of all charges, and hence Q takes value in some lattice $L \subset \mathbb{R}^{2n}$, equipped with the above integral Dirac symplectic form $[\cdot,\cdot]$. Only the discrete subgroup $Sp(2n,\mathbb{Z}) \subset Sp(2n,\mathbb{R})$ preserve this structure. Thus, under the above assumptions, only the elements $g \in G$ which get mapped in $Sp(2n,\mathbb{Z})$ under the embedding $G \to Sp(2n,\mathbb{R})$ may be a symmetry of the physical theory. Thus, at most, the symmetry group is

$$G_{\mathbb{Z}} \stackrel{\text{def}}{=} G \cap Sp(2n,\mathbb{R}). \tag{7.5}$$

However, often the actual symmetry group may be much smaller than $G_{\mathbb{Z}}$. In fact, a true symmetry should not only preserve the charge lattice L, but also (say) the points in L which can be realized as quantum numbers (m^x, e_y) of single–particle states.

Let $G_S \subset G_{\mathbb{Z}}$ be the true symmetry. The Vafa conjecture states that the non–compact space $G_S\backslash G/H$ has finite volume,

$$\mathrm{Vol}\,(G_S\backslash G/H) < \infty, \tag{7.6}$$

in a theory arising from superstrings.

The volume of the $G_{\mathbb{Z}}\backslash G/H$ can be computed in the form

$$\mathrm{Vol}(G_{\mathbb{Z}}\backslash G/H) = \sharp\mathbf{Z}(H)\,\frac{\mathrm{Vol}(G_{\mathbb{Z}}\backslash G)}{\mathrm{Vol}(H)}. \tag{7.7}$$

The hard factor, namely $\mathrm{Vol}(G_{\mathbb{Z}}\backslash G)$, was computed by Langlands in ref.[**129**], (see also [**130**]). It is given by the following very elegant formula

$$\mathrm{Vol}(G_{\mathbb{Z}}\backslash G) = \sharp(\pi_1(G))\,\prod_{k=1}^{l} \zeta(a_i), \tag{7.8}$$

where $\zeta(s)$ is Riemann's zeta function, $l = \mathrm{rank}\,G$, and $(a_1 = 2, a_2, \cdots, a_l)$ are the degrees of the basic Killing invariants of G (the exponents plus 1). Thus, the volume of $Sp(2n,\mathbb{Z})\backslash\mathfrak{H}_n$ is

$$\mathrm{Vol}(Sp(2n,\mathbb{Z})\backslash\mathfrak{H}_n) = n\,\frac{\zeta(2)\,\zeta(4)\ldots\zeta(2n)}{2^{(n+1)/2}\pi^{n(n+1)/2}}\prod_{k=1}^{n-1} k! \tag{7.9}$$

(see also [**131**]) where we used

$$\mathrm{Vol}(U(n)) = 2^{(n+1)/2}\pi^{n(n+1)/2}\prod_{k=1}^{n-1}\frac{1}{k!}.$$

Using Langland formula it is easy to compute the volume of $E_{7(7)\,\mathbb{Z}}\backslash E_{7(7)}/SU(8)$

$$\frac{\zeta(2)\,\zeta(6)\,\zeta(8)\,\zeta(10)\,\zeta(12)\,\zeta(14)\,\zeta(18)}{4c^{63}\pi^{35}}\prod_{k=1}^{7} k! \tag{7.10}$$

where c is the change of normalization in the generators of $SU(8)$

$$\mathrm{tr}_{\mathbf{28}}(t^a t^b) = \frac{c}{2}\,\mathrm{tr}_{\mathbf{8}}(t^a t^b).$$

Eqn.(7.10) is easily evaluated using

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!} B_k \, \pi^{2k} \tag{7.11}$$

where B_k are Bernoulli numbers ([**132**] **Proposition VII.7**).

CHAPTER 5

Symmetric spaces and σ–models

In order to be more concrete in our formulation of $\mathcal{N} \geq 3$ SUGRA's, we have to present explicit expression for the G–invariant metrics, bundles and connections over the symmetric space G/H. This requires to study in some detail the geometry of these remarkable Riemannian spaces.

From the gauge point of view introduced in the previous section, it is clear that we have to start from G–invariant geometric structures on the group manifold G itself and then 'gauge away' their spurious H part. So we start this chapter with the differential geometry of a Lie group.

1. Cartan connections on G

1.1. Left–invariant vector fields. By definition, a Lie group G is a group which has a differentiable structure so that the group operations

$$G \times G \to G \qquad (g,h) \mapsto gh, \tag{1.1}$$
$$G \to G \qquad g \mapsto g^{-1}, \tag{1.2}$$

are smooth maps. In particular, a Lie group G is a smooth manifold.

To be concrete, we see G as a group of matrices[1], $G \subset \mathbb{M}_N(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$), and take the differential structure induced on G by that of $\mathbb{M}_N(\mathbb{F}) \simeq \mathbb{F}^{N^2}$. As a matrix, an element of $g \in G$ has the form

$$g = \exp X \equiv \sum_{k=1}^{\infty} \frac{X^k}{k!}, \qquad \text{with } X \in \mathfrak{g} \subset \mathbb{M}_N(\mathbb{F}), \tag{1.3}$$

where $\mathfrak{g}$ is the Lie algebra of G (seen as an algebra of matrices[2]).

By eqn.(1.1), each element $h \in G$ defines a diffeomorphism

$$\begin{aligned} L_h \colon G &\to G \\ g &\mapsto hg \end{aligned} \tag{1.4}$$

called *left translation.* Let $X \in T_eG \simeq \mathfrak{g}$ be an element of the tangent space at the identity $e \in G$, and consider the vector $X(g) := (L_g)_* X \in T_gG$. It is a vector field on G. By construction

$$\begin{aligned} X(hg) &= (L_{hg})_* X = (L_h L_g)_* X = (L_h)_*\big((L_g)_* X\big) = \\ &= (L_h)_* X(g), \end{aligned} \tag{1.5}$$

so the vector field $X(g)$ is invariant under left translation or, how we shall say, *left–invariant.* Conversely, any left–invariant vector field arises in this

[1] By Ado's theorem [**227**], all finite dimensional Lie groups are groups of matrices, that is have a faithfull finite dimensional representation.

[2] The Lie bracket is given by the matrix commutator.

way. Hence the space of left–invariant vectors on G is naturally isomorphic to T_eG, that is isomorphic to the Lie algebra $\mathfrak{g}$.

An integral line of a left–invariant vector field[3], $X(g)$, is called a *one–parameter subgroup.* The commutator of two left–invariant vector fields, $X(g)$ and $Y(g)$, is again a left–invariant vector field. Hence if $X_a(g)$ ($a = 1, 2, \dots, \dim G$) is a basis of such vector fields[4]

$$[X_a(g), X_b(g)] = f_{ab}{}^c\, X_c(g), \tag{1.6}$$

where $f_{ab}{}^c$ are constants (the Lie group structure constants) satisfying the Jacobi identity.

Analogously, we can define *right–invariant* vector fields $\widetilde{X}_a$. The inversion map $i\colon g \mapsto g^{-1}$ interchanges the left– and right–invariant fields[5]

$$i_* X_a = \widetilde{X}_a. \tag{1.7}$$

1.2. Left–invariant and Cartan connections.

DEFINITION 1.1. A connection[6] D_i on a Lie group G is *left–invariant* if, given any two left–invariant vector fields, X and Y, the vector field

$$X^i D_i Y^j \partial_j \quad \equiv \quad D_X Y$$

is also left–invariant.

A left–invariant connection defines a multiplication on the Lie algebra $\mathfrak{g}$

$$\alpha\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \qquad\qquad \alpha(X, Y)^j = X^i D_i Y^j. \tag{1.8}$$

Given a connection D_i, we have a notion of parallel transport, and hence of geodesic curves[7]. It would be desirable that the geodesics (passing through

---

[3] That is the solution $g(t)$ to the differential equation

$$\frac{d}{dt} g(t) = X\big(g(t)\big), \qquad g(0) = e.$$

[4] The sign in the RHS of eqn.(1.6) is *tricky.* It depends on the way we interprete the action of the group. In a natural action there is an oveall sign minus in the commutator [**133**]; indeed, define the vector fields $\widehat{X}$ on G by the rule

$$\widehat{X} \cdot f(\phi) = \frac{d}{dt} f\Big(e^{tX} \cdot \phi\Big)\Big|_{t=0}$$

then $\widehat{Y}\widehat{X} \cdot f(\phi) = d_s d_t f(e^{tX} \cdot e^{sY} \cdot \phi)|_{t=s=0}$. Hence

$$[\widehat{X}, \widehat{Y}] = -\widehat{[X, Y]}.$$

The map $X \mapsto \widehat{X}$ may be seen as a lift of the Killing vector X from $\mathcal{M}$ to $L(\mathcal{M})$, the bundle of linear frames over $\mathcal{M}$, see [**190**]. We shall adhere to the viewpoint of ref.[**94**].

[5] Recall that we used the map $\mathcal{E} \mapsto \mathcal{E}^{-1}$ to interchange left/right actions.

[6] As common in the math literature, we use the words *'connection'* and *'covariant derivative with respect the (given) connections'* interchangeably.

[7] We stress that the definition of a geodesic curve $\gamma\colon \mathbb{R} \to \mathcal{M}$ depends only on the connection and it does not require the presence of a metric on $\mathcal{M}$. The geodesic equation reads

$$\dot{\gamma}^i D_i \dot{\gamma}^j = 0.$$

e) and the one–parameter subgroups correspond to the *same* curves; this is equivalent to the validity of the following natural equation

$$\boxed{\exp_e tX = \exp tX}$$

where the LHS is the exponential in the sense of differential geometry (cfr. chapter 3) and the RHS is the matrix exponential as in eqn.(1.3).

DEFINITION 1.2. A left–invariant connection on a Lie group G is called a *Cartan connection* if, for all vectors $X \in T_e G \simeq \mathfrak{g}$, the one–parameter subgroup $\exp(tX) \subset G$ and the geodesic $\gamma_{e,X}$ coincide. In particular,

$$\frac{d\gamma_{e,X}(t)}{dt} = X\big(\gamma_{e,X}(t)\big). \tag{1.9}$$

A consequence of this definition is that a Lie group G is always *geodesically complete* with respect to any Cartan connection.

LEMMA 1.1. *For a Cartan connection, the product $\alpha(\cdot,\cdot)$ defined in eqn.*(1.8) *is anti–commutative, that is,*

$$D_X Y = -D_Y X, \tag{1.10}$$

for all left–invariant vectors X, Y.

PROOF. Indee, the equation of the geodesic $\gamma_{e,X}$ reads

$$0 = \left(\frac{d\gamma_{e,X}}{dt}\right)^i D_i \left(\frac{d\gamma_{e,X}}{dt}\right)^j = X^i\big(\gamma_{e,X}(t)\big)\, D_i X^j\big((\gamma_{e,X}(t)\big) = \alpha(X,X)^j, \tag{1.11}$$

which, by linearity, implies $\alpha(X,Y) = -\alpha(Y,X)$. □

PROPOSITION 1.1. *On G there is a unique torsionless Cartan connection*

$$D_X Y = \frac{1}{2}[X,Y]. \tag{1.12}$$

PROOF. By the definition of torsion,

$$\begin{aligned} 0 = T(X,Y) &:= D_X Y - D_Y X - [X,Y] = \\ &= 2\, D_X Y - [X,Y]. \end{aligned}$$

□

REMARK. The general Cartan connection is given by $D_X Y = \lambda[X,Y]$, $\lambda \in \mathbb{R}$. Its torsion is given by $T(X,Y) = (2\lambda - 1)[X,Y]$.

1.3. Curvature of a Cartan connection. Let X, Y, Z be three left–invariant vector fields. Let us compute the curvature of a Cartan connection

$$\begin{aligned} R(X,Y)Z &:= D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = \\ &= \lambda^2\big([X,[Y,Z]] - [Y,[X,Z]]\big) - \lambda\,[[X,Y],Z] = \\ &= (\lambda^2 - \lambda)\,[[X,Y],Z] \end{aligned} \tag{1.13}$$

so we have two Cartan connections (with torsion) which are *flat*, namely $\lambda = 0, 1$. The corresponding parallel sections are, respectively, the left–invariant vector fields X, and the right–invariant ones $\widetilde{X}$. For the torsionless connection we get

THEOREM 1.1. *The curvature of the torsionless Cartan connection is*

$$R(X,Y) = -\frac{1}{4}[X,Y] \tag{1.14}$$

acting on $\mathfrak{g} \simeq T_eG$ *by the adjoint representation. In particular, the Riemann tensor is covariantly constant (and hence* G *is a symmetric space).*

Yes, gentlemen, the curvature it is minus one quarter the commutator of a natural, invariant, geometrical objects. Not only now we understand a possible origin of our tt^*–like formulae, but also we realize that these tt^*–like structures imply the existence of two related connections on the *same* bundle which are *flat* (albeit with torsion). This is a theme we will encounter many times in these lectures.

1.4. Invariant metrics. By the fundamental theorem of differential geometry[8], the above torsionless Cartan connection should be the Christoffel connection for any *left–invariant* metric on G. Since the left invariant fields span TG, a left–invariant metric is defined by its *constant* values on any basis $\{X_i\}$ of left–invariant vectors (that is a basis of $\mathfrak{g}$)

$$g(X_a, X_b) = g_{ab}. \tag{1.15}$$

A left–invariant metric $g(\cdot,\cdot)$ on G is so identified with a metric on $\mathfrak{g}$.

One has[9]

$$0 = D_c\, g(X_a, X_b) = \frac{1}{2}\, g([X_c, X_a], X_b) + \frac{1}{2}\, g(X_a, [X_c, X_b]),$$

so $g(\cdot,\cdot)$ should be an *invariant metric* on the Lie algebra $\mathfrak{g}$.

Unfortunately, not all groups have invariant metrics. By the Cartan–Killing criterion and the Schur's lemma, a simple Lie algebra $\mathfrak{g}_{\text{simple}}$ admits one and only one (up to normalization) invariant bilinear pairing

$$\mathfrak{g}_{\text{simple}} \times \mathfrak{g}_{\text{simple}} \to \mathbb{F},$$

namely the *Cartan–Killing form,* but this form is not always a real positive–definite inner–product. Its signature depends on the specific real form $\mathfrak{g}$ of the abstract complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$. See §. 3 below.

COROLLARY 1.1. *The left–invariant vectors* X_a *are Killing vectors for any left–invariant metric* g_{ab} *on the group manifold* G.

PROOF. Let Y be a left–invariant vector. Since the Lie derivative *is* a derivative

$$\begin{aligned}\Big(\mathcal{L}_Y g\Big)(X_a, X_b) &= \mathcal{L}_Y\Big(g(X_a, X_b)\Big) - g\Big(\mathcal{L}_Y X_a, X_b\Big) - g\Big(X_a, \mathcal{L}_Y X_b\Big)\\ &= Y^i \partial_i\, g_{ab} - g\big([Y, X_a], X_b\big) - g\big(X_a, [Y, X_b]\big) = 0.\end{aligned}$$

□

[8] It is just the statement that the Levi–Civita is the unique connection which is both *metric* and *torsionless.*

[9] Here D_c means D_{X_c}.

2. Maurier–Cartan forms

2.1. Left–invariant forms. Given a basis $\{X_a\}$ of left–invariant vector fields, we construct a dual basis $\{\omega^a\}$ of left–invariant 1–*forms* by defining

$$\omega^a(X_b) = \delta^a{}_b. \tag{2.1}$$

DEFINITION 2.1. The left–invariant forms ω^a are called *Maurier–Cartan forms.*

Just an in gauge theory[10], it is convenient to write ω as a 1–form taking values in the Lie algebra $\mathfrak{g}$. Let $\{T_a\}$ be a set of generators of $\mathfrak{g}$ corresponding to the same basis as the X_a's. We write

$$\omega = \omega^a \otimes T_a \;\in\; \Omega^1(G) \otimes \mathfrak{g}. \tag{2.2}$$

By construction, ω is the *unique* left–invariant element of $\Omega^1(G) \otimes \mathfrak{g}$ which, at the origin e, corresponds to the identity $\mathrm{Id}_\mathfrak{g}$ in the Lie algebra $\mathfrak{g}$ under the isomorphism

$$\Omega^1(G)\Big|_e \otimes \mathfrak{g} \equiv \mathfrak{g} \otimes T^*_e(G) \simeq \mathfrak{g} \otimes \mathfrak{g}^\vee \simeq \mathrm{End}(\mathfrak{g}).$$

From this characterization, we get a convenient formula for the Maurier–Cartan forms. Again, we consider $G \subset GL(N, \mathbb{F})$ as a group of matrices, and correspondingly $\mathfrak{g} \subset \mathbb{M}_N(\mathbb{F})$.

LEMMA 2.1. *Let ϕ^i be local coordinates on G. Write the generic group element as a matrix $g(\phi) \in GL(N, \mathbb{F})$.*

Then $\omega \in \Omega^1(G) \otimes \mathfrak{g} \subset \Omega^1(G) \otimes \mathbb{M}_N(\mathbb{F})$ is given by

$$\omega = g^{-1}\, dg. \tag{2.3}$$

Thinking of ω^a as a target space gauge field, it is 'pure gauge'.

PROOF. ω is invariant under $g(\phi) \to h\, g(\phi)$, so it is a left–invariant form. We have only to check its value at the identity e. Write $g(\phi) = \exp \phi^a T_a$. One has[11] $g^{-1}\, dg|_{\phi=0} = d\phi^a\, T_a \equiv \mathrm{Id}_\mathfrak{g} \in \mathrm{End}(\mathfrak{g})$. ☐

2.2. Maurier–Cartan equations. Since ω is 'pure gauge' it has a vanishing 'field strength'. This statement is known as the Maurier–Cartan equations.

PROPOSITION 2.1 (Maurier–Cartan). *One has*

$$d\omega + \frac{1}{2}[\omega, \omega] = 0, \tag{2.4}$$

or in components

$$d\omega^i + \frac{1}{2} f_{kl}{}^i\, \omega^k \wedge \omega^l = 0. \tag{2.5}$$

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[10] Think the ω^a as gauge fields in the target space G.

[11] Of course $X_a\big|_{\phi=0} = \partial/\partial\phi_a$ (cfr. footnote 4 on page 142). Then $\omega^a\big|_{\phi=0} = d\phi^a$.

The Maurier–Cartan equations are the dual relations, for the left–invariant 1–forms ω^a, to the Lie commutator relations for left–invariant vector fields X_a (cfr. eqn.(1.6)). Both sets of equations encode the same information, namely the Lie algebra structure constants $f_{ab}{}^c$, and both sets imply the same restrictions on these coefficients, namely the Jacobi identity

$$\big[[\omega,\omega],\omega\big]=0 \tag{2.6}$$

2.3. Haar invariant measure.

2.3.1. *Left Haar measure.* Let $\{\omega^a\}_{a=1}^n$ be a basis of Maurier–Cartan forms. The n–form

$$\Omega=\omega^1\wedge\omega^2\wedge\cdots\wedge\omega^n \tag{2.7}$$

is also left–invariant. Any other left–invariant n–form may be written as $f\,\Omega$ where f is a left–invariant *function*, hence a constant. Ω is a volume form on G. (Note that, if a left–invariant n–form is positive at e, is positive everywhere, for the given orientation).

Definition 2.2. The unique (up to normalization) left–invariant volume form on G is called the *left Haar measure* on G. Given a function $f\colon G\to\mathbb{C}$, we write

$$\int_G f(g)\,dg \tag{2.8}$$

for its integral with respect to the left Haar measure.

The left Haar measure is characterized by the property

$$\int_G f(hg)\,dg=\int_G f(g)\,dg \qquad \forall\, h\in G. \tag{2.9}$$

2.3.2. *Unimodular Lie groups.* On a group manifold G we have an action of $G\times G$ given, respectively, by left and right multiplications

$$L_{h_L}:\ g\mapsto h_L\,g \tag{2.10}$$

$$R_{h_R}:\ g\mapsto g\,h_R. \tag{2.11}$$

The two actions *commute* (they act on 'different indices'). By construction, Ω is invariant under left multiplication. Consider the n–form $R_g^*\Omega$. One has

$$L_h^*R_g^*\Omega=R_g^*L_h^*\Omega=R_g^*\Omega, \qquad \forall\, h,g\in G,$$

where we used the commutativity of the two actions. Hence $R_g^*\Omega$ is also a left–invariant n–form, and hence it should be a multiple of Ω,

$$R_g^*\Omega=\varrho(g)\,\Omega, \tag{2.12}$$

$$\text{with}\quad \varrho(gh)=\varrho(g)\,\varrho(h). \tag{2.13}$$

Definition 2.3. The quasicharacter $\varrho\colon G\to\mathbb{R}$ is called the *module* of the Lie group G. A Lie group with $\varrho(g)=1$ for all $g\in G$ is called *unimodular.*

Corollary 2.1. *On a Lie group there is a unique (up to normalization) right–invariant measure (the* right Haar measure*), given by* $\varrho(g)^{-1}dg$*. For a* unimodular *group the left and right Haar measure coincide.*

In particular, *any compact Lie group is unimodular.*

PROOF. If G is compact, Ω is the generator of $H^n(G, \mathbb{Z}) \simeq \mathbb{Z}$. Then

$$\varrho(h)\,\big[\Omega\big] = \big[R_h^* \Omega\big] \in H^n(G, \mathbb{Z}), \tag{2.14}$$

and $\varrho\colon G \to \mathbb{Z}$ is a smooth function hence a constant. □

3. Invariant metrics on a compact group

3.1. Bi–invariant metrics. We are now in a position to prove

PROPOSITION 3.1. *If G is compact, there exists a metric g_{ij} (unique up to normalization) which is both left and right invariant.*

PROOF. Let $(\cdot,\cdot)_h$ be any positive–definite smooth symmetric tensor. Given two vector fields $X, Y \in T_h G$, define a new inner product

$$\langle X, Y\rangle_h = \int_G (L_{g\,*}X, L_{g\,*}\eta)_{gh}\, dg.$$

One has

$$\langle L_{f\,*}X, L_{f\,*}Y\rangle_{fh} = \int_G (L_{gf\,*}X, L_{gf\,*}Y)_{gfh}\, dg = \int_G (L_{g\,*}X, L_{g\,*}Y)_{gh}\, dg = \langle X, Y\rangle_h$$

where we used $L_{g\,*}L_{f\,*} = L_{gf\,*}$ and the fact that, for a compact group, the left Haar measure is also right invariant. The above equation shows that $\langle\cdot,\cdot\rangle_h$ is left–invariant.

Then

$$g(X, Y)_h = \int_G \langle R_g^* X, R_g^* Y\rangle_{hg}\, dg$$

is right invariant for construction and left invariant because the two actions commute. □

In §. 1.4 we have seen that, for a *simple* G, a left–invariant metric — if it exists — should be proportional to the Cartan–Killing form on the left–invariant vector fields. Hence

COROLLARY 3.1. *G a semisimple compact Lie group. Then the Cartan–Killing form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$*

$$\mathrm{Kil}(X, Y) := \mathrm{Tr}\big(\mathrm{adj}\, X \circ \mathrm{adj}\, Y\big) \tag{3.1}$$

is negative definite.

If G is compact, we take as metric $g(X, Y) = -\mathrm{Kil}(X, Y)$. Being the unique left–invariant metric, $g(X, Y)$ should be also right–invariant, by proposition 3.1. Hence it has the full isometry group $G \times G$, associated to the two commuting actions of G on itself (left and right). The corresponding Killing vectors are the X_i's and the $\widetilde{X}_i$ (eqn.(1.7)),

$$[X_i, X_j] = f_{ij}{}^k X_k, \quad [X_i, \widetilde{X}_j] = 0, \quad [\widetilde{X}_i, \widetilde{X}_j] = -f_{ij}{}^k \widetilde{X}_k.$$

(but see footnote 4 on page 142).

3.2. Explicit formula for the invariant metric. In practice, we have our matrix representation for the group elements $g(\phi) \in GL(N, \mathbb{F})$ (cfr. lemma 2.1). Then we write the bi–invariant metric in the obvious form

$$g_{ij}(\phi)\, d\phi^i\, d\phi^j = -\lambda\, \mathrm{tr}\big[(g^{-1}\partial_i g)(g^{-1}\partial_j g)\big] d\phi^i\, d\phi^j, \tag{3.2}$$

that is as *minus the trace of the square of the Maurier–Cartan 'pure gauge field'* $g^{-1}\partial_i g$. λ is a normalization constant, which depends on the particular representation of G we use to define the trace in the RHS.

Now *everything is very explicit.* To prove eqn.(3.2), one has only to check that it is invariant under left/right translations, which is obvious, and the rest follows from the above uniqueness and existence results. Notice that this metric is positive definite if G is compact, but not in general.

We can rephrase the above result in a useful way. Take a basis of $\mathfrak{g}$ which is orthonormal with respect to the form $-$Kil, *i.e.* $-\mathrm{Kil}(T_i T_j) = \delta_{ij}$. Then the coefficient one–forms ω^i of $\omega \equiv \omega^i T_i$ are a *orthonormal frame* in T^*G with respect the invariant metric $(\cdot,\cdot)$

$$(\omega^i, \omega^j) = \delta^{ij}. \tag{3.3}$$

Therefore the ω^i can be identified with the vielbein form $e^i \equiv e^i{}_\alpha d\phi^\alpha$. Thus we can write

$$g^{-1}dg = e^i\, T_i \qquad (T_i \text{ an orthonormal basis in } \mathfrak{g}). \tag{3.4}$$

3.3. G is an Einstein space. In §. 1.3 we saw that the Riemann tensor of the invariant metric on G should be

$$R(X,Y) = -\frac{1}{4}[X,Y]. \tag{3.5}$$

Let now compute the Ricci tensor

$$\begin{aligned}\mathrm{Ric}(Y,Z) &= \omega^i\big(R(X_i,Y)Z\big) = -\frac{1}{4}\omega^i\big([[X_i,Y],Z]\big) \equiv \\ &\equiv -\frac{1}{4}\omega^i\Big(\mathrm{adj}\, Z \circ \mathrm{adj}\, Y(X_i)\Big) = -\frac{1}{4}\mathrm{Tr}\Big(\mathrm{adj}\, Z \circ \mathrm{adj}\, Y\Big) = \\ &= -\frac{1}{4}\,\mathrm{Kil}(Y,Z) = \frac{1}{4}g(Y,Z),\end{aligned} \tag{3.6}$$

hence *the Killing metric on a compact group G is Einstein with 'cosmological constant'* 1/4 (in the above normalization!).

4. Chiral models

A σ–model with target space a compact Lie group G is usually called a *chiral model.* It has a symmetry group $G_L \times G_R$, which is suggestive of the flavour group in massless QCD, $SU(N)_L \times SU(N)_R$. In facts, it is used to model the low–energy theory of hadrons — but to get the right physics you need to add other couplings, the Wess–Zumino–Witten terms [**3, 4, 5**], to the basic σ–model Lagrangian

$$\begin{aligned}\mathcal{L}_{\sigma\ \mathrm{model}} &= \frac{1}{2}\mathrm{tr}[(g^{-1}\partial_\mu g)(g^{-1}\partial^\mu g)] \equiv -\frac{1}{2}\mathrm{tr}[\partial_\mu g^{-1}\,\partial^\mu g] \\ &\equiv -\frac{1}{2}\mathrm{tr}[(\partial_i g^{-1}\,\partial_j g]\,\partial_\mu\phi^i\,\partial^\mu\phi^j.\end{aligned} \tag{4.1}$$

EXERCISE 4.1. Write the conserved $G_L \times G_R$ currents.

Due to the high symmetry, the chiral model in $D = 2$ is solvable both classically [**150**] and quantum mechanically [**151**]. Indeed the two differential operators

$$\partial_\xi + \frac{\lambda_0}{\lambda - \lambda_0} g^{-1}\partial_\xi g, \tag{4.2}$$

$$\partial_\eta - \frac{\lambda_0}{\lambda + \lambda_0} g^{-1}\partial_\eta g, \tag{4.3}$$

commute for all values of the spectral parameter $\lambda \in \mathbb{C}$.

5. Geometry of coset spaces G/H

To study the geometry of a coset space G/H *not necessarily summetric,* we adopt the physicists' strategy based on the gauge idea.

5.1. The gauge point of view. In chapter 4 we introduced the point of view that, instead of using G/H as target space for our σ–model, we may use the full group G provided we consider gauge–equivalent two field configurations $g_1, g_2 \colon \Sigma \to G$, if they differ by an arbitrary, space–time dependent, $h(x) \in H$,

$$g_1(x) = g_2(x)\, h(x). \tag{5.1}$$

Choose a (non–trivial) represention R of the simple compact group G, and let t_a ($a =, \dots, \dim H$) be the matrices representing the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in this representation, normalized so that

$$\operatorname{tr}_R(t^a t^b) = -\delta^{ab}. \tag{5.2}$$

Consider the following Lagrangian

$$\mathcal{L} = \frac{\lambda}{2} \operatorname{tr}_R\big[(g^{-1}\partial^\mu g - A^{a\,\mu}\, t_a)(g^{-1}\partial_\mu g - A^b_\mu\, t_b)\big]. \tag{5.3}$$

This Lagrangian is obiously invariant under (global) *left translation* $g \mapsto ag$, $a \in G$, and it is also invariant with respect to *local* right translation by the subgroup H *provided we also transform the gauge field* A_μ *in the appropriate way:*

$$g \mapsto g\, h(x), \qquad A_\mu \mapsto h(x)^{-1}\, A_\mu\, h(x) + h(x)^{-1}\partial_\mu h(x). \tag{5.4}$$

Integrate away the auxiliary gauge fields A^a_μ using their equations of motion

$$A^a_\mu = -\operatorname{tr}_R[t^a\, g^{-1}\partial_\mu g]. \tag{5.5}$$

The result is

$$\mathcal{L} = \frac{\lambda}{2} \operatorname{tr}_R\big[(g^{-1}\partial^\mu g)^\perp\, (g^{-1}\partial_\mu g)^\perp\big], \tag{5.6}$$

where $(\cdots)^\perp$ means the component in $\mathfrak{g}$ orthogonal to $\mathfrak{h}$ with repect to the Killing form (or, more generally, with respect the inner product $\operatorname{tr}_R[\cdot\,\cdot]$).

By construction,

$$g_{ij} := -\lambda \operatorname{tr}_R\big[(g^{-1}\partial_i g)^\perp (g^{-1}\partial_j g)^\perp\big], \tag{5.7}$$

is a 'metric' on G/H which is invariant under left translation by G. It is positive–definite if and only if the form $-\lambda\, \mathrm{Kill}(\cdot,\cdot)$ is positive–definite

definite on $\mathfrak{h}^\perp$. We have two possibilities: either $-\text{Kill}(\cdot,\cdot)$ is positive–definite and we take the overall factor $\lambda > 0$, or it is negative–definite and we take $\lambda < 0$. Both cases lead to physically sensible theories.

Let m^α be an orthonormal basis of $\mathfrak{m} \equiv \mathfrak{h}^\perp \subset \mathfrak{g}$ with respect the inner product $\pm\text{Kil}(\cdot,\cdot)$ (whichever sign gives a positive form). Then we can expand the $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ valued one–form $g^{-1}dg$ in the basis (t_a, m_α)

$$g^{-1}dg = e^\alpha m_\alpha + A^a t_a, \tag{5.8}$$

where the one–forms $A^a \in \Omega^1(G/H)$ are related to the previous gauge fields of the same name by[12] $A^a_{\text{gauge}} = \Phi^* A^a$. By construction, the one–forms e^α are an *orthonormal frame* (a vielbein) in T^*G/H with respect to the metric in eqn.(5.7) (for $\lambda = \pm 1$).

The differentials of the one–forms e^α, A^a satisfy a set of identities following from the Maurier–Cartan equations for G

$$\begin{aligned} &d\big(e^\alpha\, m_\alpha + A^a\, t_a\big) = \\ &\quad = -\frac{1}{2}\, e^\alpha \wedge e^\beta [m_\alpha, m_\beta] - e^\alpha \wedge A^a [m_\alpha, t_a] - \frac{1}{2}\, A^a \wedge A^b [t_a, t_b]. \end{aligned} \tag{5.9}$$

To write more explicit formulae, let us recall the structure of the Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ for an *arbitrary* Lie subgroup H of the simple[13] compact group G:

$$[t_a, t_b] = C_{abc}\, t_c \qquad (\mathfrak{h} \text{ is a Lie subalgebra of } \mathfrak{g}) \tag{5.10}$$

$$[t_a, m_\alpha] = M_{a\,\alpha\beta}\, m_\beta \qquad (\mathfrak{m} \text{ is a representation of } \mathfrak{h}) \tag{5.11}$$

$$[m_\alpha, m_\beta] = M_{c\,\alpha\beta}\, t_c + D_{\alpha\beta\gamma}\, m_\gamma. \tag{5.12}$$

Then eqn.(5.9) becomes

$$de^\alpha = -\frac{1}{2}\, D_{\beta\gamma\alpha}\, e^\beta \wedge e^\gamma - M_{a\beta\alpha}\, A^a \wedge e^\beta \tag{5.13}$$

$$dA^a = -\frac{1}{2}\, M_{a\,\alpha\beta}\, e^\alpha \wedge e^\beta - \frac{1}{2}\, C_{bca}\, A^b \wedge A^c. \tag{5.14}$$

These formulae are interesting[14], but their geometrical meaning is much more pregnant if the subgroup $H \subset G$ is such that G/H is actually a *symmetric* space in the sense of sect. 2 of chapt. 3. Thus, from now on we specialize to the symmetric case (which is the one relevant for SUGRA, see chapter. 4).

6. Symmetric spaces

6.1. Vielbeins, connections, and curvatures. In §. 2 of chapt. 3 we saw that a coset G/H is a Riemannian symmetric space if and only if $\mathfrak{g} =$

[12] As always, $\Phi\colon \Sigma \to \mathcal{M} \equiv G/H$ is the scalars' field configuration map.

[13] For a semisimple group the structure constants are totally antisymmetric, and hence (5.11) is a consequence of (5.10), while (5.12) follows from (5.11).

[14] Taking e^α as vielbeins and A^a as spin–connections, these are Cartan's structural equations showing that $-M_{a\,\alpha\beta}$ are the curvatures and -$D_{\beta\gamma\alpha}$ the torsions.

$\mathfrak{h} \oplus \mathfrak{m}$ with:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \tag{6.1}$$
$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \tag{6.2}$$
$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \tag{6.3}$$

Comapring with eqns.(5.10)–(5.12), we see that G/H is symmetric iff

$$D_{\alpha\beta\gamma} = 0.$$

We can rephrase this symmetry condition by saying that the Lie algebra $\mathfrak{g}$ has an involutive automorphism σ,

$$\sigma \colon \mathfrak{g} \to \mathfrak{g}, \qquad \sigma^2 = \mathrm{Id}_{\mathfrak{g}}, \tag{6.4}$$

defined by $\sigma|_{\mathfrak{h}} = +\mathrm{Id}_{\mathfrak{h}}$ and $\sigma|_{\mathfrak{m}} = -\mathrm{Id}_{\mathfrak{m}}$. Conversely, given a (real) Lie algebra $\mathfrak{g}$ with a non–trivial involutive auto–morphism σ, we can construct a pair (G, H) of Lie algebras so that G/H is symmetric by simply setting $\mathfrak{h}$ to be the eigenspace of σ corresponding to the eigenvalue $+1$, which is automatically a subalgebra since σ is an automorphism.

If the coset G/H is a *symmetric* space, the Maurier–Cartan identities (5.13)–(5.14) become

$$de^\alpha + M_{a\,\beta\alpha}\, A^a \wedge e^\beta = 0 \tag{6.5}$$
$$dA^a + \frac{1}{2}\, C_{bca}\, A^b \wedge A^c = -\frac{1}{2}\, M_{a\,\alpha\beta}\, e^\alpha \wedge e^\beta, \tag{6.6}$$

which now have a transparent differential–geometric meaning.

Define the $\mathfrak{h}$–valued one–form

$$\omega^\alpha{}_\beta := A^a\, M_{a\,\beta\alpha} \tag{6.7}$$

and call it the *connection form.* Eqn.(6.5) is then nothing else than the first Cartan's structural equation with vanishing torsion

$$\boxed{de^\alpha + \omega^\alpha{}_\beta\, e^\beta \equiv T^\alpha = 0} \tag{6.8}$$

so $(e^\alpha, \omega^\alpha{}_\beta)$ are the vielbein and the torsionless connection corresponding to the (unique up to scale) left–invariant metric g_{ij} we constructed in eqn.(5.7). Recall that there is a unique connection which is both metric and torsionless (the fundamental theorem of differential geometry), namely the Levi–Civita one; so we are guaranteed that the connection $\omega^\alpha{}_\beta$ is equivalent to the standard one given, in 'curved' indices, by the Christoffel symbols.

A^a is a $\mathfrak{h}$–valued gauge–field and the matrices $M_{a\,\beta}{}^\alpha$ are the matrices representing $\mathfrak{h}$ on $\mathfrak{m} \simeq TG/H$. Hence $\omega^\alpha{}_\beta$ is nothing else than the $\mathfrak{h}$–gauge–field A^a in the representation appropriate for T^*G/H. Thus, rewriting eqn.(6.6) in this representation, we get

$$R_{\alpha\beta} := \left(d\omega + \frac{1}{2}\omega \wedge \omega\right)_{\alpha\beta} = -\frac{1}{2}\, M_{a\,\alpha\beta}\, M_{a\,\gamma\delta}\, e^\gamma \wedge e^\delta, \tag{6.9}$$

which is the second Cartan's structural equation defining the curvature. (This shows, again, that $\mathrm{Hol}(G/H) \equiv H$).

Therefore, we can rewrite the $\mathfrak{h}$ curvature into *two* extremely illuminating forms. First, we can write it in the representation $\mathbb{H}$ol — which corresponds to the Riemann tensor in 'flat' indices — in the form

$$R_{\alpha\beta\,\gamma\delta} = -\frac{1}{2}\,M_{a\,\alpha\beta}M_{a\,\gamma\delta} = -\frac{1}{2}C_2(\mathbb{H}\mathrm{ol})\,\mathrm{Id}_{\mathfrak{h}} \tag{6.10}$$

where $C_2(V)$ stands for the second Casimir of $\mathfrak{h}$ in the V representation. The second form is obtained by writing the curvature, which is (of course) the field–strength of a $\mathfrak{h}$–valued YM field, in the adjoint representation:

$$\boxed{R = \left(dA + \frac{1}{2}A \wedge A\right)^a t_a = -\frac{1}{2}\big[e^\alpha\, m_\alpha, e^\beta\, m_\beta\big]} \tag{6.11}$$

that is our beloved *minus one quarter the commutator of natural geometric objects.* Now we also see what these 'natural' objects are: they are the projections on $\mathfrak{m}$ of the Maurier–Cartan forms of G. If, as it happens in $D = 3$ SUGRA, $\mathbb{H}$ol is a spinorial representation, these projections satisfy automatically the Clifford algebra property. In all other cases, they satisfy the corresponding (generalized) relations.

GENERAL LESSON 6.1. *Maybe we now understand where the tt^*–like structure (minus a quarter of a commutator) comes from.*

6.2. Explicit formulae for the Killing vectors. It will be useful to have explicit and compact formulae for the Killing vectors K_a corresponding to the (left) isometries of a symmetric space G/H. They may be written in many ways.

6.2.1. *Matrix form.* Identify G with a group of matrices *via* some representation R. From the left–action on G/H of the one parameter subgroup $\exp(t\,K) \subset G$,

$$(t, g) \longmapsto e^{t\,K} g, \tag{6.12}$$

we get

$$\pounds_{K_a}\, g = t_a\, g, \tag{6.13}$$

where t_a is matrix representing in $\mathrm{End}(R)$ the generator of $\mathfrak{g}$ corresponding to the left–invariant vector K_a.

6.2.2. *Gauge viewpoint.* To get a more convenient formula, we argue from the physical side, that is from the gauge point of view.

A simple way to extract the Killing vector associated to a one–parameter subgroup of the isometry group of a manifold $\mathcal{M}$ is to gauge that subgroup. From eqn.(1.13) of chapt. 1, we know that gauging a symmetry of $\mathcal{M}$ amounts to the replacement $\partial_\mu\phi^i \to \partial_\mu\phi^i - A^a_\mu\, K^i_a$,

$$\mathcal{L}_{\text{gauged}} = -\frac{1}{2}g_{ij}(\partial_\mu\phi^i - A^a_\mu K^i_a)(\partial^\mu\phi^j - A^{b\,\mu}K^j_b). \tag{6.14}$$

On the other hand, consider the symmetric space G/H where G acts on the left. The Lagrangian reads[15]

$$\mathcal{L}_{G/H} = \pm\frac{1}{2}\mathrm{Tr}\left[(g^{-1}\partial^\mu g - B^\mu)(g^{-1}\partial_\mu g - B_\mu)\right] = \tag{6.15}$$

$$= \pm\frac{1}{2}\mathrm{Tr}\left[(g\partial^\mu g^{-1} + gB^\mu g^{-1})(g\partial^\mu g^{-1} + gB^\mu g^{-1})\right] \equiv \tag{6.16}$$

$$\equiv \pm\frac{1}{2}\mathrm{Tr}\left[(gD^\mu g^{-1})(gD^\mu g^{-1})\right], \tag{6.17}$$

where B_μ is the $\mathfrak{h}$–valued gauge field gauging the local symmetry H acting on the right. Let us now gauge a subgroup $F \subset G$ acting on g on the left. The gauge symmetry reads

$$g(x) \mapsto f(x)\, g(x), \qquad f(x) \in F. \tag{6.18}$$

Under this transformation

$$gD_\mu g^{-1} \longmapsto f(gD_\mu g^{-1})f^{-1} + f\partial_\mu f^{-1}, \tag{6.19}$$

so, introducing a $\mathfrak{f}$–valued gauge connection $A_\mu \equiv A_\mu^a t_a$ transforming as

$$A_\mu \longmapsto fA_\mu f^{-1} + f\partial_\mu f^{-1}, \tag{6.20}$$

we can write the F–*gauge* invariant Lagrangian

$$\mathcal{L} = \pm\frac{1}{2}\mathrm{Tr}\left[(gD^\mu g^{-1} - A_\mu)(gD^\mu g^{-1} - A^\mu)\right]. \tag{6.21}$$

Comparing with eqn.(6.14), we see that the coefficient of A_μ^a is $g_{ij}K_a^i\partial^\mu\,\phi^j$, thus from eqn.(6.21) we get

$$\boxed{K_{i\ a} \equiv g_{ij}\,K_a^j = \mp\,\mathrm{Tr}\big[t_a\,(gD_i g^{-1})\big]} \tag{6.22}$$

where t_a is the generator of $\mathfrak{f}$ corresponding to the given Killing vector.

Since $g_{ij} = \mp\,\mathrm{Tr}[(gD_i g^{-1})(gD_j g^{-1})]$, eqn.(6.22) implies

$$K_a^j\,(gD_j g^{-1}) = t_a, \tag{6.23}$$

or[16],

$$\boxed{K_a^j\,D_j g = -t_a\,g} \tag{6.24}$$

If $\{t_a\}$ are generators of G normalized as $-\mathrm{Tr}[t_a\,t_b] = \delta_{ab}$, one has

$$(gD_i g^{-1}) = K_{i\,a}\,t_a. \tag{6.25}$$

which corresponds to eqn.(6.13). This formula may be intepreted by saying that the action of G bt multiplication on the left is accompanied by a compensating H–gauge transformation on the right given (at the infinitesimal level) by $K_a^i\,B_i \in \mathfrak{h}$.

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[15] The sign + is appropriate for G compact, − for G non–compact.

[16] $D_i g$ is defined by the rule

$$D_i g = -g(D_i g^{-1})g.$$

6.3. Relations with a construction in §§. 9.4.2, 10.2 of chapt. 2. To make contact with the discussion of $\mathcal{N}$–extended SUSY in chapt. 2, we take a different viewpoint. We consider the *flat* G–connection

$$\nabla := d + g^{-1}dg \qquad \nabla^2 = 0 \quad \text{(pure gauge configuration)}, \tag{6.26}$$

and decompose into the *even* and *odd* parts with respect the involutive automorphism σ. The even part is a new differential

$$D = d + g^{-1}dg\Big|_{\text{even}} = d + \omega \tag{6.27}$$

which is *metric* but no longer flat. The odd part is a tensor

$$g^{-1}dg|_{\text{odd}} = e \tag{6.28}$$

which is, in fact, the vielbein.

We used this point of view in §. **2.**10.2 (see also §. **2.**9.4.2 and elsewhere in these notes) in a (preliminary) discussion of special Kähler geometry. Here we see that that procedure is just another way of stating the main property of the Maurier–Cartan form for a symmetric G/H.

REMARK. However, in chapt. 2 we used the decompostion into even/odd parts *in full generality,* that is also for non symmetric manifolds, provided the relevant bundle has a *flat non–metric connection,* and there is a natural involution σ (as it happens in supersymmetry, see chapt.2).

7. Duality. Classification of symmetric manifolds

7.1. Duality. We have shown in the previous section that, given a simple *compact* Lie group G and an involutive automorphism of its algebra

$$\sigma\colon \mathfrak{g} \to \mathfrak{g}, \qquad \sigma^2 = 1,$$

we can construct a symmetric space G/H ($H = \exp[\mathfrak{h}]$, $\mathfrak{h} = \text{Fix}(\sigma)$) with a positive–definite left–invariant metric

$$-\text{tr}[(g^{-1}\partial_i g)_{\mathfrak{m}}(g^{-1}\partial_j g)], \tag{7.1}$$

where $(\cdots)_{\mathfrak{m}}$ is the projector on the -1 eigenspace of σ.

Consider now the following subspace of the complexification $\mathfrak{g}_{\mathbb{C}} = \mathbb{C}\otimes\mathfrak{g}$, namely

$$\mathfrak{g}' = \mathfrak{h} \oplus i\,\mathfrak{m} \qquad \Big(\subset \mathfrak{g}_{\mathbb{C}}\Big) \tag{7.2}$$

which corresponds to a different real form, G', of the complexified Lie group $G_{\mathbb{C}}$. $\mathfrak{g}'$ is again a Lie algebra with an involutive automorphism, and the Killing form $\text{Kil}(\cdot,\cdot)$ restricted to $\mathfrak{m}$ is now *positive*–definite. Then we can define a new symmetric space G'/H with positive definite metric

$$\text{tr}[(g^{-1}\partial_i g)_{\mathfrak{m}}(g^{-1}\partial_j g)]. \tag{7.3}$$

DEFINITION 7.1. The spaces G/H and G'/H are called *dual symmetric* manifolds.

Thus (non–flat) Riemannian symmetric spaces come in two dual pairs. Notice that the two spaces in a dual pair have the same holonomy group H and the same holonomy representation. As we already anticipated in chapt. 4, they are distinguished by the *sign* of their curvature. This is our next topic.

7.2. G/H **is Einstein.** In §. 3.3 we computed the Ricci curvature of a compact Lie group, showing that it is Einstein with 'cosmological constant'[17] $+1/4$. Let us redo that computation in the case of a general symmetric manifold G/H. We write X_α for the vector fields dual to the one–forms e^α. Let $X, Y \in \mathfrak{m} \simeq T_e G/H$. Then:

$$\begin{aligned}\mathrm{Ric}(Y,Z) = e^\alpha\big(R(X_\alpha, Y)Z\big) = -\frac{1}{4}e^i\big([[X_\alpha, Y], Z]\big) \equiv \\ \equiv -\frac{1}{4}e^\alpha\Big(\mathrm{adj}\, Z \circ \mathrm{adj}\, Y(X_\alpha)\Big) = -\frac{1}{4}\mathrm{Tr}\Big(\mathrm{adj}\, Z \circ \mathrm{adj}\, Y\Big|_{\mathfrak{m}}\Big).\end{aligned} \tag{7.4}$$

Now we claim that

$$\mathrm{Tr}\left(\mathrm{adj}\, X_\alpha \circ \mathrm{adj}\, X_\beta\Big|_{\mathfrak{m}}\right) = \frac{1}{2}\mathrm{Tr}\Big(\mathrm{adj}\, X_\alpha \circ \mathrm{adj}\, X_\beta\Big). \tag{7.5}$$

This is due to the special form of the commutation relations for a Lie algebra with symmetry σ. In fact eqn.(7.5) is equivalent to

$$\mathrm{Tr}\left(\mathrm{adj}\, X_\alpha \circ \mathrm{adj}\, X_\beta\Big|_{\mathfrak{m}}\right) = \mathrm{Tr}\left(\mathrm{adj}\, X_\alpha \circ \mathrm{adj}\, X_\beta\Big|_{\mathfrak{h}}\right) \tag{7.6}$$

and this last equation can be rewritten in terms of the structure constants of $\mathfrak{g}$ in the form

$$f_{\alpha\gamma a}\, f_{\beta a\gamma} = f_{\alpha a\gamma}\, f_{\beta\gamma a}, \tag{7.7}$$

which is trivially true. Then

PROPOSITION 7.1. *A symmetric manifold G/H, equipped with the left G–invariant metric (normalized as above) is Einstein with cosmological constant $+1/8$ if G is compact and $-1/8$ for its dual G'.*

Hence the pairs of dual symmetric spaces G/H and G'/H are, respectively, positively and negatively curved.

We recall a useful result:

THEOREM∗ (see [**86**]). *Let $\mathcal{M}$ be a homogenous[18] Einstein manifold with cosmological constant λ. Then:*

1. *if $\lambda > 0$ $\mathcal{M}$ is compact with finite π_1;*
2. *if $\lambda = 0$ $\mathcal{M}$ is flat;*
3. *if $\lambda < 0$ $\mathcal{M}$ is non–compact.*

(1) is Myers theorem. (2) is a theorem by D.V. Alekseevskii and B. N. Kimelfeld. (3) is elementary: we already proved it (see the footnote of page 14). In fact, if $\mathcal{M}$ is compact and has negative–definite Ricci tensor has no Killing vector, contrary to the assumption that it is homogeneous.

7.3. Classification. The irreducible symmetric spaces are usually classified in four types:

- **Type I:** spaces of the form G/H with G a real simple and compact Lie group;
- **Type II:** spaces of the form $(G \times G)/G \simeq G$, G a real simple and compact Lie group;

[17] In our normalization. In the textbooks one finds also other numbers (typically 1). But our are the natural normalizations, see Postnikov [**94**].

[18] That is $\mathrm{Iso}(\mathcal{M})$ is transitive.

Type III: the non–compact duals of Type I, *i.e.* G'/H.

Type IV: the non–compact duals of Type II, namely $H_{\mathbb{C}}/H$, where $H_{\mathbb{C}}$ is a complex simple simply–connected Lie group and H a maximal compact connected subgroup, and σ the complex conjugation of $H_{\mathbb{C}}$ whose fixed set is H.

From the discussion it follows

COROLLARY 7.1. *A non–compact irreducible symmetric space is a quotient G/H where G is a real simple non–compact Lie group with trivial center and H is a maximal compact subgroup of G.*

Thus we deduce the classification of the symmetric spaces from that of the real Lie groups. See the tables in refs. [**119, 120**].

8. Totally geodesic submanifolds. Rank

8.1. Totally geodesic submanifolds of G/H. In §. 2 of chapt. 4 we saw that the 'structure transporting maps' $\varrho_{D,d}$ (*i.e.* dimensional reduction and group disintegration) embed $\mathcal{M}_D$ into $\mathcal{M}_d$ as *a totally geodesic submanifold.* Hence we are, in particular, interested in the geometry of the totally geodesic submanifolds of Riemannian symmetric spaces. They are described by the following

THEOREM 8.1. *Let $\iota\colon \mathcal{W} \hookrightarrow G/H$ be a totally geodesic submanifold:*
1. *$\mathcal{W}$ is also a symmetric manifold;*
2. *the geodesic submanifolds of G/H passing trough a fixed point ϕ are in one–to–one correspondence with the subspaces $\mathfrak{s} \subset \mathfrak{m} \subset \mathfrak{g}$ such that*

$$[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subseteq \mathfrak{s}. \tag{8.1}$$

PROOF. **(1)**. Let $\phi \in \mathcal{W}$ be a point and s_ϕ the associated geodesic symmetry of G/H. s_ϕ sends the geodesics through ϕ to themselves (with the opposite orientation) and hence $\mathcal{W}$ to itself. The metric on $\mathcal{W}$ is $\iota^* g$; under s_ϕ this metric is mapped to $s_\phi^* \circ \iota^* g = \iota^* s_\phi^* g = \iota^* g$. Thus $s_\phi|_{\mathcal{W}}$ is an isometry. **(2)**. Let $\mathfrak{s} = T_\phi \mathcal{W} \subset T_\phi G/H \simeq \mathfrak{m}$. In view of the fact that the Riemann tensor of G/H is given by a double commutator, eqn.(8.1) is equivalent to the usual property of a totally geodesic submanifold $\mathcal{W}$, namely

$$R_{\mathcal{W}} T\mathcal{W} \subset \mathcal{W}. \tag{8.2}$$

So $\mathcal{W}$ is totally geodesic $\Rightarrow$ (8.1). Conversely, consider the submanifold of G/H defined by

$$\mathcal{W} := \exp_\phi \mathfrak{s} \tag{8.3}$$

it is easy to see (for instance with the help of normal coordinates) that it is totally geodesic only if (8.1) holds. □

COROLLARY 8.1 (Yet another argument!!). *In any space–dimension D, the scalars' manifold of a* SUGRA *with more than* 8 *supercharges is a symmetric space.*

PROOF. Apply the above theorem to $\varrho_{D,3}$. □

Moreover,

COROLLARY 8.2. *In $D = 4$ SUGRA the 'magnetic suscetibility' map*

$$\mu \colon \mathcal{M} \to \mathfrak{H}_V$$

(1) *is a totally geodesic embedding for $\mathcal{N} \geq 3$;*
(2) *is a holomorphic map for $\mathcal{N} = 1$;*
(3) *is a holomorphic embedding of the first space factor $SU(1,1)/U(1) \equiv Sp(2,\mathbb{R})/U(1)$ for $\mathcal{N} = 4$.*

8.2. Rank of G/H. From the curvature computations we already did too many times, it is clear that a totally geodesic submanifold $\mathcal{M} \hookrightarrow G/H$ is *flat* if and only if the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{s}$ is Abelian. This motivates the following

DEFINITION 8.1. The dimension of the maximal Abelian subalgebra of $\mathfrak{m}$ is called the RANK of the symmetric space G/H.

The rank is also the dimension of the maximal flat totally geodesic submanifolds of G/H. Two such submanifolds are related by left–translation by $H \subset G$. They are torii for G/H compact and Euclidean spaces otherwise.

A particular interesting class of symmetric spaces are those with *rank* 1.

PROPOSITION 8.1. *A symmetric space G/H has rank 1 if and only if the action of H on the unit sphere in T_eG/H is transitive. (That is if H is one of the Berger groups or $Spin(9)$ in dimension 16).*

PROOF. Absurt. Assume H is not transitive on the unit sphere in the tangent space.

Fix some unit $x \in \mathfrak{m} \simeq T_eG/H$, and let Hx be its H–orbit. The tanget space to Hx at x is $[\mathfrak{h}, x]$. Since H is not transitive, there is a unit vector $y \in \mathfrak{m}$, not proportional to x, which is orthogonal to the orbit Hx in x, and hence to its tangent space $[\mathfrak{h}, x]$. Thus

$$0 = ([\mathfrak{h}, x], y) = (\mathfrak{h}, [x, y]), \tag{8.4}$$

and $[x, y] \in \mathfrak{h}$ is orthogonal to all elements of $\mathfrak{h}$. Since the restriction of the Killing form to $\mathfrak{h}$ is non degenerate, $[x, y] = 0$. Then there are at least two non–proportional commuting vectors in $\mathfrak{m}$ and $\text{rank}(G/H) \geq 2$. degenerate □

9. Other techniques

As we mentioned before when the symmetric space G/H has a holonomy group $H = U(1) \times \widetilde{H}$, it is Kählerian. In these case much more powerful techniques exist to study its geometry. Of course we will discuss (if we have time) them after having developed the general theory of complex and Kähler space to a sufficient level.

In particular, the non–compact, Type III symmetric manifolds correspond to bounded domain, and in particular classical domains, in $\mathbb{C}^N$ [**134, 135, 136, 137, 138**]. We can use complex function theory to study them. We already saw a basic example. Our 'duality target space'

$$\frac{Sp(2V, \mathbb{R})}{U(1) \times SU(V)} \tag{9.1}$$

is the *Siegel upper–half space,* $\mathfrak{H}_V$, which is one of the most relevant domains in many complex variables theory. The Cayley transform maps it into the generalized disk $\mathfrak{D}_V$, opening the way to the standard methods in bounded domain theory, based on Hilbert space techniques [**139, 140**]. (These methods are reviewed in APPENDIX ...). The bounded domain viewpoint is the most convenient one for $\mathcal{N} = 1$ SUGRA in $D = 4$.

A most convenient approach to symmetric (or even homogeneous) non–compact Kähler space is the *thermodynamical analogy* which is discussed in sect...

10. An example: $E_{7(7)}/SU(8)$

As an illustration of the geometric techniques develloped in this chapter, we work out in some detail the case of $E_{7(7)}/SU(8)$ that is of $\mathcal{N} = 8$ $D = 4$ SUGRA. All other four–dimensional supergravities with $\mathcal{N} \geq 5$ can be obtained from this one by truncation, that is by restricting to the appropriate totally geodesic submanifold.

The reader may have a feeling that the exceptional Lie groups G_2, F_4, E_6, E_7 and E_8 are quite misterious objects, difficult to visualize and to understand[19]. In facts, in the literature [**142, 143**] there are a number of explicit constructions of these groups (we already mentioned the one based on the octonionic projective spaces [**122**]). It turns out that *each* of these mathematical constructions is word–for–word equivalent to the 'physical' realization given by a specific SUGRA model. The various constructions in [**142**] are related by morphisms which correspond to our $\varrho_{D,d}$ and to truncations to lower $\mathcal{N}$'s. Our physical discussion of E_7 below is, mathematically, nothing else than the *grading model* of the group (see *Example 1* on page 180 of ref. [**143**], or chapter 12 of ref. [**142**]). The analogous construction for maximal SUGRA in $D = 5$, based on the coset $E_{6(6)}/Sp(8)$, corresponds to the construction of E_6 in chapter 13 of ref. [**142**]. The same procedure applied to $\mathcal{N} = 16$, $\mathcal{N} = 12$, $\mathcal{N} = 10$, and $\mathcal{N} = 9$ in $D = 3$ gives the 'standard' construction of (respectively) E_8, E_7, E_6 and F_4. For E_8 this is, essentially, the same construction one gets fromthe $d = 2$ current algebra which leads, say, to the $E_8 \times E_8$ gauge symmetry in the heteoric string [**144**]. The 'magical' $\mathcal{N} = 2$ SUGRA's in $D = 5$ correspond to the 'magical square' construction of the exceptional Lie groups, see §. **5**.1.7 of ref. [**143**], and, from another viewpoint, to the construction of E_6 in page 181 of the last quoted review.

So, SUGRA is also a very powerful mathematical tool!

10.1. The exceptional group $E_{7(7)}$. From the physics of $\mathcal{N} = 8$ SUGRA we know a number of things about its scalars' space $\mathcal{M}$ which are indeed sufficient to construct the exceptional, 133–dimensional group $E_{7(7)}$ from scratch.

First of all, we know that is has a real symplectic **56** dimensional representation on the field–strengths $\mathcal{F}^+$'s. So we identify $E_{7(7)}$ with a group of $\mathbf{56} \times \mathbf{56}$ real symplectic matrices. We perform the Cayley transform in §. 6.1

[19] The group G_2 is easily understood as a subgroup of $Spin(7) \subset SO(8)$. G_2 is treated in great detail in APPENDIX C.

of chapt. 4 to rewrite it, more conveniently, as a group of complex symplectic matrices E

$$E^t \Omega E = \Omega \tag{10.1}$$

satisfying the reality condition in eqn.(6.8) of chapt.

$$E^* = \Sigma_1 E \Sigma_1. \tag{10.2}$$

Let $E = \exp L$ be an element of the group (so $L \in \mathfrak{e}_{7(7)}$). Write

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{10.3}$$

where A, B, C, D are 28×28 matrices. The conditions (10.1)(10.2) become, respectively

$$A^t = -D, \qquad B^t = C \tag{10.4}$$

$$A^* = D, \qquad B^* = C, \tag{10.5}$$

which imply $A^\dagger = -A$, and $B^\dagger = B$. Then the subalgebra of the block–diagonal elements of $e_{7(7)}$

$$\begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \tag{10.6}$$

consists of compact generators, whereas the complementary set

$$\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \tag{10.7}$$

of non–compact ones. Hence the subalgebra (10.6) is identified with the maximal compact subalgebra of $\mathfrak{e}_{7(7)}$.

The second thing we know from SUGRA is that this maximal compact subalgebra is $\mathfrak{aut}_R(8)$, namely $\mathfrak{su}(8)$, here in the **28**–dimensional representation. Writing an index of the **28** as an antisymmetric pair $[ab]$ of fundamental representation indices $(a, b = 1, 2, \dots, 8)$, one has

$$A^{[ab]}{}_{[cd]} = \Lambda^{[a}{}_{[c} \delta^{b]}{}_{d]}, \tag{10.8}$$

where $\Lambda^a{}_b$ are matrices in the defining representation of $\mathfrak{su}(8)$, that is: they are antiHermitean traceless 8×8 matrices.

Third information from SUGRA: writing $\mathfrak{e}_{7(7)} = \mathfrak{su}(8) \oplus \mathfrak{m}$, one has the isomorphisms

$$\mathfrak{m} \simeq T_\phi \mathcal{M} \simeq [[\wedge^4 \mathcal{S}_\phi]] \tag{10.9}$$

that is, the off–diagonal entries of L are in the the totally antisymmetric four indices $\mathfrak{su}(8)$ representation, subject to the reality condition that complex conjugation is equal to duality. Hence we write

$$B^{[ab][cd]} = B^{abcd} = \epsilon^{abcdefgh} B_{dfgh}, \tag{10.10}$$

where we adopt the convention which lowering/rising indices is the same thing as complex conjugation. Thus a general element of the Lie algebra of

$\mathfrak{e}_{7(7)}$ is given by

$$L = \begin{pmatrix} \Lambda^{[a}{}_{[e}\delta^{b]}{}_{f]} & B^{ab\,gh} \\ B_{cd\,ef} & \Lambda_{[c}{}^{[g}\delta_{d]}{}^{h]} \end{pmatrix} \tag{10.11}$$

$$\text{with} \quad \Lambda^a{}_b = -\Lambda_b{}^a, \quad \Lambda^a{}_a = 0, \tag{10.12}$$

$$\text{and} \qquad B^{ab\,cd} = \epsilon^{abcdefgh} B_{df\,gh}. \tag{10.13}$$

In total we have $63+70 = 133$ generators, so we have construct the full $\mathfrak{e}_{7(7)}$ Lie algebra.

To construct the compact version of $\mathfrak{e}_7$, we have only to use duality:

$$L_{\text{compact}} = \begin{pmatrix} \Lambda^{[a}{}_{[e}\delta^{b]}{}_{f]} & i\,B^{ab\,gh} \\ i\,B_{cd\,ef} & \Lambda_{[c}{}^{[g}\delta_{d]}{}^{h]} \end{pmatrix}. \tag{10.14}$$

10.2. $\mathcal{N} = 8$ couplings. We can insert the above explicit formulae in our construction of the SUGRA Lagrangians in chapt. 4. We write $\mathcal{E} = \exp L$ where L is as in eqn.(10.11), and compute the Maurier–Cartan form

$$(\partial_i \mathcal{E})\mathcal{E}^{-1} = \begin{pmatrix} Q_i^{[A}{}_{[E}\delta^{B]}{}_{F]} & P_i^{AB\,GH} \\ P_{i\,CD\,EF} & Q_{i\,[C}{}^{[G}\delta_{D]}{}^{H]} \end{pmatrix} \tag{10.15}$$

[Here we write $(\partial_i\mathcal{E})\mathcal{E}^{-1}$ instead than $\mathcal{E}^{-1}\partial_i\mathcal{E}$ because in SUGRA we think of the global group G as acting on the *right* of $\mathcal{E}$ and not on the *left* as we do in mathematics].

The indices A, B, C, D, E, F, G, H in eqn.(10.15) are *local* $SU(8)$ indices.

The target space one form with values in $\mathfrak{su}(8)_R$, $Q_i^A{}_B$ which appears in the RHS of eqn.(10.15), is by definition, the $\mathfrak{su}(8)_R$ gauge–connection. Its curvature is computed by the Maurier–Cartan equations to be *minus one quarter... blah... blah...* Can you see the paradox?

The basic ideas of this set of lectures is to get SUGRA by performing the *minimal number of computations.* But we discover that — nevertheless — we had done much *useless* work: there is no need to compute the Aut_R curvatures!! The mere knowledge of the field content of the theory, together with our smart 'target space equivalence principle' implies, *via* Berger's theorem, that (for $\mathcal{N} \geq 3$) $\mathcal{M}$ is a symmetric space and then the curvature computation was done by Maurier–Cartan for us[20]!

The off–diagonal terms, P_i^{ABCD}, are the vielbein, that is the tensors which represents the isomorphism $T\mathcal{M} \simeq [[\wedge^4\mathcal{S}]]$ with respect to standard basis. This vielbein defines the χ's SUSY transformation, as well as the scalars kinetic terms

$$-P_i^{ABCD} P_{j\,ABCD}\, \partial_\mu\phi^i\, \partial^\mu\phi^j. \tag{10.16}$$

The 4–fermions couplings are given by the curvature tensor, that we already computed too many times, up to some terms which originates either from the torsion part of the space–time connection (see the $D = 3$ case in chapt. 2)

[20] However, doing useless computations may be a good pedagogical strategy.

or from the supercovariantization of the supercurrent coupling. They are all described by $\varrho^*_{4,3}$ and are rather straightforward.

There is only one piece lacking in the picture. We can still gauge some subgroup of isometries, $\mathcal{G} \subset \mathrm{Iso}(\mathcal{M})$, getting a more general theory with all kinds of possible couplings[21]. Gaugings will be a major theme from now on.

11. * Symmetric and Iwasawa gauges

In this chapter we adopted the gauge point of view indescribing the cosets G/H: We added to the SUGRA model $\dim H$ unphysical scalars and as many gauge vectors to 'eat' them. For many applications it is useful to work in a *unitary* gauge. In such a gauge, all the scalar fields in $\mathcal{L}$ are physical, and no auxiliary gauge vector is present.

There are two natural unitary gauges, each with its own merits.

The first one is the *symmetric gauge.* One writes $G = \exp[\mathfrak{m}]$, that is (in the $E_{7(7)}$ example, say)

$$\mathcal{E} = \exp\begin{pmatrix} 0 & B^{ab\,gh} \\ B_{cd\,ef} & 0 \end{pmatrix}. \tag{11.1}$$

Now $\mathcal{E}$ is parametrized by $\dim G - \dim H$ physical scalars $B^{ab\,fg}$ (70, in the $E_{7(7)}$ example). To justify the choice, one has only to show that any configuration is gauge equivalent to one and only one of the above form. This follow from

LEMMA 11.1. *G/H a symmetric space. Let $\mathcal{E} \in G$. There is one and only one $\mathcal{E}'$ of the form* $\exp(\mathfrak{m})$ *with $\mathcal{E}^{-1}\mathcal{E}' \in H$.*

PROOF. Consider $\mathcal{P} := \mathcal{E}\sigma(\mathcal{E}^{-1}))$. $\sigma(\mathcal{P}) = \mathcal{P}^{-1}$. Write $\mathcal{P} = \exp p$. One has $\sigma(p) = -p$, hence $p \in \mathfrak{m}$. Set $\mathcal{E}' = \exp(p/2) \in \exp(\mathfrak{m})$. $(\mathcal{E}')^2 = \mathcal{E}\sigma(\mathcal{E}^{-1})$, so $\mathcal{E}^{-1}\mathcal{E}' = \sigma(\mathcal{E}^{-1})(\mathcal{E}')^{-1} = \sigma(\mathcal{E}^{-1}\mathcal{E}')$ thus $\mathcal{E}^{-1}\mathcal{E}'$ belongs to the fixed set in G of σ, that is to H.

If you feel the language too exoteric: embed $G \hookrightarrow Sp(2V, \mathbb{R})$. Then σ acts on $\mathcal{E}$ as $\sigma(\mathcal{E}) = (\mathcal{E}^t)^{-1}$. $\mathcal{P} = \mathcal{E}\mathcal{E}^t$ is a positive semi–definite symmetric matrix which has a *unique* square–root $\mathcal{E}'$ with all eigenvalues non–negative. □

There is another gauge, less symmetric but which has the merit of working well under the maps $\varrho_{D,d}$, in the sense that it makes the successive embeddings of totally geodesic submanifolds much more transparent. This is related to the fact that, in such a gauge, most of the couplings are *polynomial* in the scalar fields. This is the *Iwasawa gauge* [**11**]. One could argue in abstract terms, using the general theorem about Iwasawa decomposition of a Lie algebra [**120, 145**]. However, to be concrete, we look at G as a group of $2n \times 2n$ real symplectic matrices, $G \subset Sp(2n, \mathrm{R})$.

[21] At least for $D \leq 5$. In higher dimensions we may have couplings to higher form fields.

Working in the standard real realization[22] of $Sp(2n,\mathbb{R})$, we define the subgroup $\mathcal{B}_n \subset Sp(2n,\mathbb{R})$ consisting of the real matrices of the form

$$\mathcal{B}_n = \left\{ \begin{pmatrix} h & X(h^t)^{-1} \\ 0 & (h^t)^{-1} \end{pmatrix}, \begin{array}{c} h,\ X \in \mathbb{M}_n(\mathbb{R}),\quad X^t = X; \\ h \text{ upper } \textit{triangular} \text{ with} \\ \textit{positive} \text{ diagonal entries} \end{array} \right\}. \tag{11.2}$$

Notice that the change of basis

$$\begin{pmatrix} \delta_{a,\,b} & 0 \\ 0 & \delta_{n+1-a,\,b} \end{pmatrix}, \tag{11.3}$$

maps $\mathcal{B}_n$ into a group of *upper triangular* matrices, which can be uniquely parameterized as $\exp(D)\exp(N)$, with $D = \mathrm{diag}(d_i)$ diagonal and N *strictly* upper triangular. The group element $\exp(D)\exp(N) \in \mathcal{B}_n$ depends only polynomially on N, while the diagonal part is just $\exp(D) = \mathrm{diag}(e^{d_i})$. The triangular subgroup $\mathcal{B}_n$ has a very convenient parameterization. Luckily, each class in $Sp(2n,\mathbb{R})/U(n)$ admits one and only one triangular representative, so this convenient parameterization of $\mathcal{B}_n$ is, in fact, a very convenient parameterization of the coset space $Sp(2n,\mathbb{R})/U(n)$.

LEMMA 11.2. *An element $S \in Sp(2n,\mathbb{R})$ has a* UNIQUE *decomposition of the form*

$$S = b \cdot u, \qquad b \in \mathcal{B}_n, \quad u \in U(n). \tag{11.4}$$

PROOF. We have to show that all classes in $Sp(2n,\mathbb{R})/U(n)$ admit a representative of the form b. Equivalently, we have to show $\pi(\mathcal{B}_n) = \mathfrak{H}_n$, where $\pi\colon Sp(2n,\mathbb{R}) \to \mathfrak{H}_n$ is the map

$$\pi\colon \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto (i\,A + B)(i\,C + D)^{-1}. \tag{11.5}$$

Now

$$\begin{pmatrix} h & X(h^t)^{-1} \\ 0 & (h^t)^{-1} \end{pmatrix} \overset{\pi}{\longmapsto}\ i\,hh^t + X, \tag{11.6}$$

and the result follows from the well–known fact that any real symmetric positive definite matrix can be written in a *unique* way in the form hh^t with h an upper triangular matrix with positive diagonal entries (cfr. the Iwasawa decomposition for $GL(n,\mathbb{R})$, see ref.[**152**], **prop. 1.2.6**). The argument also implies the uniqueness of the Iwasawa decomposition. □

Let $G \subset GL(2n,\mathbb{R})$ be a subgroup which is stable under the involutive automorphism of $Sp(2n,\mathbb{R})$ (*i.e.* $g \in G \Rightarrow g^t \in G$), that is a totally geodesic symmetric submanifold. By the lemma, given an element $g \in G$ we have unique $b \in \mathbb{B}_n$ and $u \in U(n)$ such that $g = bu$. The general theorem of Iwasawa guarantee that $b \in G$ and then $u \in G \cap U(n) = H$. Thus, each point in G/H gets represented by a element of G of the form $\exp(D)\exp(N)$ where D is a diagonal matrix which belongs to an *Abelian* subalgebra $\mathfrak{a} \subset \mathfrak{g}$, with $\dim \mathfrak{a} = \mathrm{rank}(G/H)$, and N is an upper triangular matrix belonging

[22] That is the one associated to $\mathfrak{H}_n$ not the one related to $\mathfrak{D}_n$.

to a *nilpotent* subalgebra $\mathfrak{n} \subset \mathfrak{g}$, with $[\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n}$. If $D_i \equiv \mathrm{diag}(d_{i,a})$, and $N_\alpha \equiv \{n_{\alpha,ab} \mid b > a\}$ are basis of $\mathfrak{a}$ and $\mathfrak{n}$, respectively, we get[23]

$$\begin{aligned}&\exp(s^\alpha N_\alpha)\exp(t^i D_i) = \\ &= \begin{pmatrix} 1 & & P_{ab}(S^\alpha) \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} \exp(t^i d_{i,1}) & & 0 \\ & \ddots & \\ 0 & & \exp(t^i d_{i,n}) \end{pmatrix}\end{aligned} \tag{11.7}$$

with $P_{ab}(S^\alpha)$ polynomials. In this gauge the Maurier–Cartan take the form

$$g^{-1}dg = D_i\, dt^i + e^{-t^i D_i} N_\alpha e^{t^i D_i}\, F^\alpha{}_\beta(s)\, ds^\beta, \tag{11.8}$$

with $F^\alpha{}_\beta(s)$ polynomials in the s^γ's.

[23] For convenience we changed a little the parameterization of $\mathcal{B}_n$ by interchanging the order.

CHAPTER 6

Killing spinors and *AdS* Susy

The results of the previous chapters allow us to write down the Lagrangian $\mathcal{L}$ of any supergravity theory in any spacetime dimension D *but* for a few terms: the gauge couplings, the Yukawa ones, and the scalar potential. Pragmatically, the goal of the present chapter is to find out an *explicit and universal* formula for the scalar potential $V(\phi)$, valid for any D and $\mathcal{N}$. Once we have that formula, the problem of the scalar potential si solved once for all, and we may (and do) forget it for the rest of the lectures.

The universal formula for $V(\phi)$ is a Ward identity which can be deduced in just two lines [**163**]. Since we are perverted by geometry, we shall instead make a long and painful detour. I hope that what we learn along the route will justify the effort[1].

* * *

A basic idea in these lectures is that the geometrical structures one finds on the scalars' manifold $\mathcal{M}$ have the same general flavor as the ones in the physical spacetime Σ: metric, connections, gauge invariance, equivalence principle, *ect.* A field configuration, Φ, is then seen as a map which transports structure back and forward $\Sigma \leftrightarrow \mathcal{M}$. In this spirit, I prefer to introduce the next general geometrical feature from the point of view of physical space–time. Geometry will pave the way to the introduction of *AdS* supersymmetry ($\equiv$ conformal SUSY in one less dimension) and will lead (geometrically) to the general formula for the scalar potential of *any* supergravity theory.

Although we study *AdS* for rather 'technical' reasons, I would dare to say that no time spent in studying this particular space–time and its *super*symmetries is totally wasted.

* * *

This chapter is focused on the concept of *Killing spinor,* its geometrical meaning and physical implications. To motivate its introduction, we start in sect. 1 by reviewing the rôle of its bosonic counterpart — the Killing *vector* — in General Relativity. Given the relevance of anti–de Sitter space in the theory of Killing spinors, in sect. 2 we briefly review the geometry of this space in a language suited for our purposes.

In sect. 3 we introduce the Killing spinors, giving two (inequivalent) definitions: the *physical* and the *mathematical* ones. Their geometry is studied in sec. 4 along the lines of chapt. 3.

[1] Otherwise, jump ahead to the last equations of the chapter!

In sect. 5 we go back to physics, and write explicitly the spacetime charges associated to (asymptotic) Killing vectors (sec. 1) as flux integrals at infinity. This will allow us to construct the AdS/Poincaré *global* superalgebras of $D = 4$ SUGRA and, most importantly, to obtain the basic SUGRA Ward identity which provides the *universal formula for the scalar's potentials valid for any supergravity theory.* Technically, this is the main result of the present chapter.

REMARK. The material presented in this chapter may be seen as yet another manifestation of our *"minus one quarter the commutator of ⋯"* story. This is evident from sects. 3, 4 below. In particular, the geometric techniques for the symmetric spaces G/H, which we developed thinking of G/H as the *target* space, are here recycled with G/H seen as the physical space–time.

1. Spacetime charges in General Relativity

Warning: in this section $g_{\mu\nu}$ is the metric in physical spacetime!

In General Relativity one cannot, in general, define a conserved energy–momentum. The generators of space–time symmetries, such as momentum or angular momentum, can be defined only in space–time which have asymptotic Killing vectors, that is if the metric $g_{\mu\nu}$ approaches rapidly at infinity a reference metric $\bar{g}_{\mu\nu}$ which is invariant under the symmetry generated by the given conserved quantity.

Let us review how that works [**153**]. One starts with a metric $\bar{g}_{\mu\nu}$ which solves the Einstein equations of motion and is invariant under a group of isometries, $\mathrm{Iso}(\bar{g})$. The corresponding Lie group, $\mathfrak{iso}(\bar{g})$, is generated by Killing vectors K^m_μ with brackets[2]

$$[K^m, K^n] = -f^{mn}{}_p\, K^p. \tag{1.1}$$

To fix the ideas, we consider a vacuum–like configuration[3], with constant scalars and vanishing vector fields, which solves the Einstein equations

$$\overline{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\,\overline{R} - \overline{\Lambda}\,\bar{g}_{\mu\nu} = 0. \tag{1.2}$$

The effective cosmological constant $\overline{\Lambda}$ which receives contributions also from the 'vacuum' energy of the given field configuration.

Then one expands around this solution[4], setting $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, and $\Phi = \bar{\Phi} + \phi$. The Einstein equations are rewritten in the form

$$R^L_{\mu\nu} - \frac{1}{2}\,\bar{g}_{\mu\nu}R^L + \bar{\Lambda}\,h_{\mu\nu} = (T_{\mu\nu} + t_{\mu\nu}) \equiv \Theta_{\mu\nu}, \tag{1.3}$$

---

[2] This tricky sign again!

[3] This assumption is not needed. However it is the case we are interested in practice.

[4] A bar over a symbol denotes a quantity evaluated on the background configuration. Φ stands for all fields in the theory different from the metric.

where $R^L_{\mu\nu}$ is the Ricci tensor *linearized* around $\bar{g}_{\mu\nu}$, and $t_{\mu\nu}$ is minus the higher–order terms in $h_{\mu\nu}$ of the Einstein tensor $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. The background version of the Bianchi identity

$$\bar{D}^\mu \left(R^L_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^L + \overline{\Lambda}\, h_{\mu\nu} \right) \equiv 0 \tag{1.4}$$

holds, provided the background satisfies the equations of motion (1.2). Then the symmetric tensor $\Theta_{\mu\nu}$ is covariantly conserved with respect to the background connection

$$\bar{D}^\mu \Theta_{\mu\nu} = 0. \tag{1.5}$$

We stress that this is an *exact* result.

The K^m_μ are Killing vectors for the background metric $\bar{g}_{\mu\nu}$, hence

$$\bar{D}_\mu K^m_\nu + \bar{D}_\nu K^m_\mu = 0. \tag{1.6}$$

Combining eqns.(1.5)(1.6) we get and

$$\partial_\mu(\sqrt{-\bar{g}}\, \Theta^{\mu\nu}\, K^m_\nu) \equiv \bar{D}_\mu(\Theta^{\mu\nu}\, K^m_\nu) = \Theta^{\mu\nu} \bar{D}_\mu K^m_\nu = 0, \tag{1.7}$$

and the corresponding charge

$$M^m = \int_t d^3x\, \sqrt{-\bar{g}}\, \Theta^{0\nu} K^m_\nu, \tag{1.8}$$

is conserved provided the fluctuations $h_{\mu\nu}$ and ϕ vanish sufficiently rapidly at spatial infinity to justify all the formal manipulations we did.

The M^m's generate a Lie algebra isomorphic to $\mathfrak{iso}(\bar{g})$,

$$[M^m, M^n] = f^{mn}{}_p\, M^p, \tag{1.9}$$

as one easily checks by canonical manipulations.

Using the above procedure, if (say) the space–time geometry is (rapidly) asymptotic to flat Minkowski space, we construct the generators of the Poincaré group, P_μ, and $M_{\mu\nu}$, while if Σ is asymptotic to Anti–de–Sitter (AdS), *i.e.* the maximal symmetric solution with negative cosmological constant $\bar{\Lambda} < 0$, we get the generators M^{AB} of its isometry group,

$$\mathrm{Iso}(AdS_d) \simeq SO(d-1, 2).$$

For positive $\bar{\Lambda}$ the maximally symmetric solution is de Sitter space, and the same method gives us the generators of $\mathrm{Iso}(dS_d) \simeq SO(d, 1)$.

Most importantly, the spacetime charges M^m can be expressed as flux integrals at infinity [**153**][**82, 83**], as one would expect in a gauge theory. The actual expression is discussed in sect. 5 below.

2. ∗ *AdS* space

For convenience of the reader, I briefly review Anti–de Sitter space. The D dimensional anti–de Sitter space, AdS_D, corresponds to the hyperboloid

$$X_0^2 + X_D^2 - \sum_{i=1}^{D-1} X_i^2 = R^2, \tag{2.1}$$

in flat $\mathbb{R}^{2,D-1}$ space with metric

$$ds^2 = dX_0^2 + dX_D^2 - \sum_{i=1}^{D-1} dX_i^2. \tag{2.2}$$

By definition AdS_D has an isometry group $SO(2, D-1)$, and it is manifestly homogeneous and isotropic. Consider the point $(R, 0, \dots, 0) \in AdS_D$. It is invariant under the subgroup $SO(1, D-1) \subset SO(2, D-1)$ acting on the last D coordinates. Thus

$$AdS_D = SO(2, D-1)/SO(1, D-1) \tag{2.3}$$

which should be contrasted with the usual D–sphere, $S^D = SO(D+1)/SO(D)$. Thus anti–de Sitter and de Sitter ($= SO(1, D)/SO(1, D-1)$) are, in some sense, analytic continuations of the sphere.

Indeed, eqn.(2.1) can be solved as

$$X_0 = R\,\cosh\rho\,\cos\tau \tag{2.4}$$

$$X_D = R\,\cosh\rho\,\sin\tau \tag{2.5}$$

$$X_i = R\,\sinh\rho\,\Omega_i, \qquad i = 1, 2, \dots, D-1, \quad \sum\nolimits_i \Omega_i^2 = 1, \tag{2.6}$$

leading to the metric

$$ds^2 = R^2\big(\cosh^2\!\rho\, d\tau^2 - d\rho^2 - \sinh^2\!\rho\, d\Omega^2\big), \tag{2.7}$$

where $d\Omega^2$ is the usual 'round' metric on the sphere S^{D-2}. Here $\rho \geq 0$ and $0 \leq \tau \leq 2\pi$, if we wish to cover the hyperboloid once. However, the S^1 parameterized by τ is a closed time–like geodesic — which is not good for causality — and hence we *define* AdS_D to be the *universal cover* of the above hyperboloid, which just means that τ takes value in the full real line.

From eqn.(2.7) the relation $AdS_D \leftrightarrow S^D$ is manifest. You need only to take ρ imaginary to get the usual metric on the sphere.

Warning!! The analytical continuation of the AdS_D metric which gives us the sphere is <u>not</u> the Wick rotation which leads to the Euclidean version of AdS_D. Rather, the Euclidean AdS_D corresponds to the *non–compact dual* to the sphere $SO(D,1)/SO(D)$, see chapt. 5.

2.1. Spinorial representation. For later convenience, we write a coset representative of a point in AdS_D as a $Spin(2, D-1)$ element

$$\mathcal{E} \equiv \exp[\,t_{AB}\,\gamma^{AB}/4\,], \tag{2.8}$$

where γ^{AB} are the usual Dirac γ–matrices in signature $(2, D-1)$. We have a involutive authomorphism

$$\sigma\colon Spin(2, D-1) \to Spin(2, D-1)$$

given by

$$\sigma(\mathcal{E}) = \gamma_0\,\mathcal{E}\,\gamma_0. \tag{2.9}$$

The elements of $Spin(2, D-1)$ which are left fixed by σ are precisely the elements of $Spin(1, D-1)$. Then

$$AdS_D \equiv Spin(2, D-1)\big/Spin(1, D-1) = Spin(2, D-1)\big/\mathrm{Fix}(\sigma) \tag{2.10}$$

is a (Minkowskian) *symmetric space* to which we can apply the machinery introduced in chapter. 5 and write $(a, b = 1, 2, \dots, D)$

$$\mathcal{E}^{-1} d\mathcal{E} = \frac{1}{4}\omega^{ab}\,\gamma_{ab} + \frac{1}{2}e^{a}\,\gamma_0\gamma_a \tag{2.11}$$

where e^a and ω_{ab} are, respectively, the AdS_D vielbeins and spin–connection. The $SO(2, D-1)$ Killing vectors are given by eqn.(6.22) of chapt. 5

$$K_i^{AB} = \eta\,\mathrm{tr}\Big[\gamma^{AB}\,\mathcal{E} D_i \mathcal{E}^{-1}\Big], \tag{2.12}$$

where η is a normalization coefficient.

REMARK. From eqn.(2.11) we can obtain vielbeins and connections for other spaces by analytic continuation:

(1) $\gamma_0, \to i\,\gamma_0$, $\gamma_D \to i\,\gamma_D$ yields S^D (sphere);
(2) $\gamma_0, \to i\,\gamma_0$ yields dS_D (de Sitter);
(3) $\gamma_D \to i\,\gamma_D$ yields H_D, (hyperbolic space, the non–compact dual to the sphere).

Wick rotation relates AdS_D to H_D and dS_D to S^D.

3. Killing spinors

In supergravity one would like to extended the previous construction from the Poincaré (or AdS) algebra to the full superPoincaré (or super–AdS) algebra. *Alas!* In supergravity local space–time symmetries and local supersymmetries *ought to be unified!*

Our goal is to construct the supergenerators Q^A by mimiching the previous construction of the M^m's. From §. 1 it is clear that a central ingredient of M^m is the (asymptotic) Killing vector. We need a kind of fermionic counterpart to a Killing vector: such geometrical objects exist and are called (quite predictably) *Killing spinors.*

In facts, there are two different notions of what a Killing spinor is. One is the physical definition which, in a sense, is the most general (and deep) one. The other is the formal mathematical definition of Killing spinor, more handly to do geometry with. The two notions tend to coincide if the background is sufficiently nice (*i.e.* if it has enough symmetries).

3.1. The physical definition. Given a supergravity model, by a *supersymmetric background* we mean a solution to the (classical) equations of motion in which some supersymmetry is unbroken. If the parameter $\epsilon_\alpha^A(x)$ corresponds to an unbroken SUSY, it should leave invariant the given background

$$\delta\psi_\mu^A = \overline{\mathcal{D}}_\mu \epsilon^A = 0, \quad \delta\chi^i = \overline{\Xi}^i_A \epsilon^A = 0, \tag{3.1}$$

where in the RHS the bosonic fields are replaced by their background values (denoted by overbars). The variations of the bosonic fields vanish automatically, since our backgrounds are purely bosonic.

As the notation suggests (and we know from chapt2) the linear operator $\overline{\mathcal{D}}_\mu$ in the RHS of the first eqn.(3.1) is a first order differential operator[5]

$$\overline{\mathcal{D}}_\mu = \bar{D}_\mu + \begin{pmatrix} Q_i^A{}_C \partial_\mu \phi^i & \frac{i}{2}\,\overline{M}^{AD}\,\bar{\gamma}_\mu + \overline{\mathfrak{F}}^{AD}_{\rho\sigma}\,\bar{\gamma}^{\rho\sigma}\,\bar{\gamma}_\mu \\ \frac{i}{2}\overline{M}_{AC}\,\bar{\gamma}_\mu + \overline{\mathfrak{F}}_{AC\,\rho\sigma}\,\bar{\gamma}^{\rho\sigma}\,\bar{\gamma}_\mu & Q_{iB}{}^D \partial_\mu \phi^i \end{pmatrix}, \quad (3.2)$$

where $\bar{D}_\mu$ is the (background) $Spin(1, D-1)$ covariant derivative. We leave the indices implicit in the following.

In order to get nice and universal formulae, it is important to write the Lagrangian in a canonical form: in particular, the kinetic terms should be block–diagonalized between the gravitini and the spin–1/2 fields χ_a; terms of the form $\bar{\chi}\sigma^{\mu\nu} D_\mu \psi_\nu$ should be eliminated from $\mathcal{L}$ by a suitable redefinition of the gravitino fields, $\psi_\mu^A \mapsto \psi_\mu^A + A^{Aa}(\phi)\,\gamma_\mu \chi_a$. Then the only term in $\mathcal{L}$ containing *derivatives* of ψ_μ^A is the Rarita–Schwinger (RS) one, $\bar{\psi}_\mu \gamma^{\mu\nu\rho} \mathcal{D}_\nu \psi_\rho$, and the operator $\mathcal{D}_\mu$ appearing in the gravitino kinetic term is the same one entering in the SUSY transformations $\delta\psi_\mu = \mathcal{D}_\mu \epsilon$. As the RS term is Hermitean up to a total derivative,

$$(\overline{\mathcal{D}_\nu \epsilon})\gamma^{\mu\nu\rho}\psi_\rho + \bar{\epsilon}\,\gamma^{\gamma\nu\rho}(\mathcal{D}_\nu \psi_\rho) = D_\nu\big(\bar{\epsilon}\,\gamma^{\mu\nu\rho}\psi_\rho\big). \quad (3.3)$$

A non zero solution to eqns.(3.1) is called a *Killing spinor*.

The given background configuration is invariant under as many supersymmetries as there are linearly independent solutions to eqns.(3.1).

We linearize the gravitino equations of motion into the form

$$\bar{\gamma}^{\mu\nu\rho}\overline{\mathcal{D}}_\nu \psi = J^\mu, \quad (3.4)$$

by moving into the RHS of all the terms but the one explicitly written in the LHS. This equation defines what we mean by the effective supercurrent J^μ much as the linerized Einstein equations in §. 1 defined the effective energy–momentum tensor $\Theta^{\mu\nu}$.

Let ϵ be a Killing spinor. We now construct the $U(\mathcal{N})_R$–singlet vector which is the SUSY analogue to $\Theta^{\mu\nu} K_\nu^m$ of the gravitational case (cfr. §. 1)

$$\begin{aligned}(\bar{\epsilon} J^\mu) &\equiv \bar{\epsilon}\,\bar{\gamma}^{\mu\nu\rho}\overline{\mathcal{D}}_\nu \psi_\rho = \\ &= \bar{D}_\nu\big(\bar{\epsilon}\,\bar{\gamma}^{\mu\nu\rho}\psi_\rho\big),\end{aligned} \quad (3.5)$$

where we used eqn.(3.3). This expression, being the divergence of a two–form is automatically conserved. The supercharge is defined by

$$Q = \int_t \sqrt{-\bar{g}}\,(\bar{\epsilon} J^0) d^3x = \oint_\infty d\sigma_i\big(\bar{\epsilon}\bar{\gamma}^{0\,i\,\rho}\psi_\rho\big), \quad (3.6)$$

or, more covariantly,

$$\frac{1}{2}\oint \big(\bar{\epsilon}\,\bar{\gamma}^{\mu\nu\rho}\psi_\rho\big)\, d\sigma_{\mu\nu} \quad (3.7)$$

where $d\sigma_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma$ is the area element. See ref.[**82, 83**].

Remark. In order for the supercharge Q defined in eqn.(3.6) to be *fermionic* we must choose the Killing spinor ϵ to be *commuting.*

[5] For simplicity, we write the following equation in the metric $(+,-,-,\dots,-)$. Hence the γ–matrices satisfy $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$.

Thus the supercharges, just as the (asymptotic) isometry generators M^m, can be written as surface integrals at infinity. For the same reason, to define the supercharge we need only to have *asymptotic* Killing spinors. In fact, two spinors ϵ_1 and ϵ_2 which are asymptotic to the same Killing spinor give the same supercharge by effect of the SUSY Gauss' law. The simplest way to see this is to consider the canonical current associated to a SUSY variation of parameter ϵ (here Φ stands for all fields but the gravitino)

$$\begin{aligned} Q_\epsilon &= \int \left[\delta_\epsilon \overline{\psi}_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi}_\nu)} + \delta_\epsilon \Phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right] dv^\mu \\ &= \int \left[\overline{\mathcal{D}_\nu \epsilon} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi}_\nu)} + \delta_\epsilon \Phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right] dv^\mu \\ &= \int D_\nu \left(\bar{\epsilon} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi}_\nu)} \right) dv^\mu + \int \bar{\epsilon} \left(-\mathcal{D}_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi}_\nu)} + J^\mu \right) dv^\mu, \end{aligned} \tag{3.8}$$

where the precise form of differential operator $\mathcal{D}_\nu$ in the last line is defined by the above integration by parts and J^μ is the supercurrent. The supercharge is the sum of a surface term and a 'bulk' term which vanishes by the gravitino equations of motion or, more precisely, by the SUSY Gauss' law. This is required by consistency of the ψ_μ equations of motion with SUSY. The surface term is the one we obtained before. Thus any two smooth ϵ's asymptotic to the same spinor at infinity give the same supercharge (in backgrounds satisfying Gauss' constraint).

In the special case that the asymptotic Killing spinor can be continued in the 'bulk' to a *bona fide* Killing spinor, we have a supersymmetric configuration, and the associate generator Q is not only well defined but also identically zero, since it leaves the state unchanged. This is most easily seen writing Q as a volume integral, as in the second line of eqn.(3.8).

Eqn.(3.1) has an integrability condition, namely

$$[\overline{\mathcal{D}}_\mu, \overline{\mathcal{D}}_\nu] \epsilon^A = 0. \tag{3.9}$$

Computing the commutator, one gets an equation the generic form

$$\overline{R}_{\mu\nu\, ab}\, \gamma^{ab}\, \epsilon^A + \cdots = 0,$$

that is an *algebraic condition* on the Riemann tensor, and hence on the *holonomy group* of the supersymmetric background. This lead us back to the geometric structures studied in the previous chapters.

The Killing spinor equation, eqn.(3.1), simplifies for backgrounds which are *vacuum*–like: the scalars field ϕ^i have constant values, the vector fields A^x_μ vanish, and the metric $\bar{g}_{\mu\nu}$ is (typically) maximally symmetric.

Configurations which do NOT look as vacua — and hence are interpreted as some kind of object, typically solitonic, in *some* vacuum — but do have some (few) non–trivial Killing spinors, are called BPS OBJECTS. I believe that the reader is well aware of their physical relevance from M. Bertolini's course.

Let us look more closely at the Killing spinor equation in a 'vacuum–like' background. The terms containing the $U(\mathcal{N})_R$ connection, being proportional to $\partial_\mu \phi^i$ drop out. So do the terms containing the $U(\mathcal{N})_R$–covariant field strenght $\mathfrak{F}_{\rho\sigma}$. Changing notations to Majorana fermions (to write more compact formulae), we have

$$\delta\psi^a_\mu \equiv \left(\delta^{ab}\,\partial_\mu + \frac{1}{4}\delta^{ab}\,\omega^{mn}_\mu\gamma_{mn} - \frac{i}{2}\Big(M^{ab} + i\tilde{N}^{ab}\gamma_5\Big)\,\gamma_\mu\right)\epsilon^b = 0 \qquad (3.10)$$

$$\delta\chi^i \equiv (h^i{}_a + i\tilde{h}^i{}_a\gamma_5)\epsilon^a = 0, \qquad (3.11)$$

where $(M^{ab} \pm i\tilde{M}^{ab})$ and $(h^i{}_a \pm i\tilde{h}^i{}_a)$ are constant matrices (depending of the constant values taken by the scalars in the given background). The matrix $M^{ab} + i\tilde{M}^{ab})$ is Hermitean[6]; by a chiral redefinition of the gravitini, we can diagonalize it. In the new basis the first equation gets the form

$$D_\mu\epsilon^a + \frac{i}{2}m_a\gamma_\mu\epsilon^a = 0 \qquad \text{not summed over } a! \qquad (3.12)$$

The second equation, (3.11), is then an algebraic equation, requiring that $\epsilon_\pm$ is a zero eigenvalues of the matrix $(h^i{}_a \pm i\tilde{h}^i{}_a)$.

If the asymptotical background has maximal symmetry, namely:

- the Poincaré group for $\Lambda = 0$,
- $SO(D-1,2)$ for $\Lambda < 0$,
- $SO(D,1)$ for $\Lambda > 0$,

the conserved supercharges, if present, should organize themselves into spinorial representations of the (asymptotic) space–time symmetry group. Hence they form $\mathcal{N}_0$ copies of the basic spinorial representation appropriate for the given D and isometry group. These conserved supercharges generate an $\mathcal{N}_0$–extended SUSY algebra which is linearly realized on the physical states of the theory.

As with an ordinary bosonic symmetry, we say that SUSY is *unbroken* if $\mathcal{N}_0 = \mathcal{N}$. In the opposite case, $\mathcal{N}_0 = 0$, SUSY is *completely broken.* The gravitini become massive by the supersymmetric variant of the Higgs effect: each gravitino combines with the corresponding spin–1/2 goldstino into a massive spin–3/2 particle. The intermediate possibility, $0 < \mathcal{N}_0 < \mathcal{N}$ is called PARTIAL SUPERHIGGS. $\mathcal{N}_0$ gravitini remain massless, while the other $\mathcal{N} - \mathcal{N}_0$ get masses. The possibility of a *partial* breaking is peculiar of *local* supersymmetry: in the rigid case this cannot happen[7].

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[6] The fact that, in the canonical form of $\mathcal{L}$, the operator $\mathcal{D}_\mu$ in the SUSY variations is the same one which appears in the RS term implies, in particular, that the gravitino 'mass' term is proportional to

$$M^{ab}\bar{\psi}^a_\mu\gamma^{\mu\nu}\psi^b_\nu + i\widetilde{M}^{ab}\bar{\psi}^a_\mu\gamma^{\mu\nu}\gamma_5\psi^b_\nu,$$

with M^{ab}, $\widetilde{M}^{ab}$ the *same* matrices appearing in the SUSY transformation. The claim in the text that $M^{ab} + i\widetilde{M}^{ab}$ is Hermitean than follows from the fact that the gravitino 'masses' should be Hermitean.

[7] In the rigid case, SUSY is broken if and only if the energy of the vacuum is not zero. If the vacuum energy vanishes *all* supercharges are not broken, if it is non–zero *all* are broken.

From the integrability condition of eqn.(3.10)

$$\begin{aligned} 0 = & 4\left[D_\mu + \frac{i}{2} m_a \gamma_\mu, D_\nu + \frac{i}{2} m_a \gamma_\mu\right] \epsilon_a = \\ = & \left[R_{\mu\nu\alpha\beta}\gamma^{\alpha\beta} - 2\, m_a^2 \gamma_{\mu\nu}\right]\epsilon_a = \\ = & -\left[\frac{\Lambda}{D-1}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})\gamma^{\alpha\beta} + 2\, m_a^2 \gamma_{\mu\nu}\right]\epsilon_a, \end{aligned} \tag{3.13}$$

where in the last line we inserted the expression of the Riemann tensor for a *maximally symmetric* D–space (since we are assuming our asymptotic background to be vacuum–like), *i.e.*

$$R_{\mu\nu\alpha\beta} = \frac{K}{D-1}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \tag{3.14}$$

$$\text{where} \quad R_{\mu\nu} = K g_{\mu\nu} \;\Rightarrow\; K = -\Lambda. \tag{3.15}$$

Therefore, $\epsilon_a \neq 0$ implies

$$m_a^2 = -\frac{\Lambda}{D-1}. \tag{3.16}$$

Hence we must have $\Lambda \leq 0$, that is either *anti*–de Sitter or flat space (asymptotically) and the m_a's equal (up to a phase) to $\sqrt{-\Lambda/(D-1)}$ for all the gravitini $a = 1, 2, \ldots, \mathcal{N}_0$ corresponding to *unbroken* supersymmetries.

3.2. The mathematicians' definition. The mathematicians take the special case in eqn.(3.10) as their definition of Killing spinors. To make the dictionary with the maths' literature we must remember that they usually work in Euclidean signature, and that their gamma–matrices are *anti*Hermitean. So the formulae differ for some i's here and there.

Definition 3.1. A *Killing spinor* is a non–zero spinor such that

$$D_\mu\epsilon + m\,\gamma_\mu\epsilon = 0, \tag{3.17}$$

for some $m \in \mathbb{C}$. One says that the Killing spinor is *real* (resp. *imaginary*) if $m \in \mathbb{R}^*$ (resp. $i\,m \in \mathbb{R}^*$) and *parallel* if $m = 0$.

With this definition, the concept of Killing spinor is purely geometric, depending only on the Riemannian geometry of Σ.

As already mentioned, the geometers are mostly interested in Killing spinors for manifold Σ having Euclidean signature, whereas we, in physics, are interested into a variety of spaced–time signatures, depending on the particular application we are pursuing. For instance, the physical space–time, Σ, may be a generic manifold of signature $(D-1, 1)$, and we look for the Killing spinors of this Minkowskian space. Or we may have a Kaluza–Klein type situation, in which space time is taken of the form $\mathbb{R}^d \times \mathcal{K}$, with $\mathcal{K}$ Euclidean and compact. In this case, we look for Killing spinors of the compact space $\mathcal{K}$, which has positive signature.

However, conceptually, the main difference between the two definitions is that in physics it is fundamental that also the variation of the spin–1/2 fields, $\delta\chi^i$, vanishes. This requirement is not geometric, at least *a priori.*

3.3. First properties of the Killing spinors. Note that a Killing spinor on Σ is automatically a solution to the Dirac equation

$$\left(i\gamma^\mu D_\mu - \frac{D}{2}m\right)\epsilon = 0.$$

3.3.1. *Relation with the Killing vectors: The Euclidean case.* Let α, β be two *commuting* (real) Killing vectors (according to the above mathematical definition). We wish to show that the vector–bilinear

$$K_\mu(\alpha,\beta) = \alpha^\dagger\,\gamma_\mu\,\beta, \tag{3.18}$$

if non vanishing, is a Killing vector. Indeed, one has

$$D_\mu(\alpha^\dagger\gamma_\nu\beta) = -\frac{m}{2}\left(\alpha^\dagger\gamma_\mu^\dagger\gamma_\nu\,\beta + \alpha^\dagger\gamma_\nu\gamma_\mu\beta\right) = m(\alpha^\dagger\gamma_{\mu\nu}\beta) = -D_\nu(\alpha^\dagger\gamma_\mu\,\beta), \tag{3.19}$$

which is the Killing vector condition. Notice that the *pseudo*vector $\bar\alpha\gamma_\mu\gamma_5\beta$ lead to the equation

$$D_\mu(\alpha^\dagger\gamma_\nu\gamma_5\beta) + D_\nu(\alpha^\dagger\gamma_\mu\gamma_5\beta) = -\frac{m}{2}\,\alpha^\dagger(\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu)\gamma_5\beta = g_{\mu\nu}\,(m\alpha^\dagger\gamma_5\beta),$$

which is the equation for a *conformal* Killing vector.

3.3.2. *Minkowski signature.* The above holds for Euclidean spaces. In Minkowski signature with metric $(+,-,\dots,-)$ we have an i in the differential operator $D_\mu + \frac{i}{2}m\gamma_\mu$, and the bilinear reads

$$K_\mu(\alpha,\beta) = \bar\alpha\gamma_\mu\beta \equiv \alpha^\dagger\gamma_0\gamma_\mu\beta. \tag{3.20}$$

(If the spinors are Majorana or symplectic Majorana we can write this expression also in other forms). Now

$$D_\mu(\bar\alpha\gamma_\nu\beta) = -i\frac{m}{2}\alpha^\dagger(\gamma_\mu^\dagger\gamma_0\gamma_\nu - \gamma_0\gamma_\nu\gamma_\mu)\beta = im\,\bar\alpha\gamma_{\nu\mu}\beta = -D_\nu(\bar\alpha\gamma_\mu\beta)$$

which, again, is the Killing vector condition.

3.4. Killing spinors on AdS_D and S^m. We are interested, in particular, into the Killing spinors for the AdS_D space which is an analytic continuation of the sphere. We solve first the Killing spinor equation on the sphere S^D, and then continue the solution to AdS_D.

3.4.1. *Killing spinors for S^{2n}.* We work out the details for $D = 2n$, the odd dimensional case being essentially similar and left to the reader.

Write S^{2n} as $Spin(2n+1)/Spin(2n)$ and identify $Sp(2n+1)$ with the matrix group given by the (unique) irreducible spinorial representation. The generators of $Spin(2n+1)$ are then[8] $\frac{1}{2}\gamma_{ab}$ and $\frac{1}{2}\gamma_a$, with $a, b = 1, 2, \dots, 2n$.

Let $\mathcal{E} \in Spin(2n+1)$ be a coset representative. From chapt. 5 we know that the Maurier–Cartan form decomposes as

$$\mathcal{E}^{-1}d\mathcal{E} = \frac{1}{4}\omega^{ab}\gamma_{ab} + \frac{1}{2}e^a\,\gamma_a, \tag{3.21}$$

[8] As always when we work in Euclidean signature, we take the γ_a's to be antiHermitean.

where the one–forms ω^{ab} and e^a are, respectively, the connection and the metric vielbein of S^{2n}. The covariant derivative $\mathcal{D}_\mu = D_\mu + \frac{1}{2}\gamma_\mu$ is simply

$$\mathcal{D} := dx^\mu\, \mathcal{D}_\mu = d + \frac{1}{4}\omega^{ab}\gamma_{ab} + \frac{1}{2}dx^\mu\, \gamma_\mu = d + \mathcal{E}^{-1}d\mathcal{E}, \tag{3.22}$$

so the equation defining the Killing spinors, $\mathcal{D}\epsilon = 0$, has the general solution

$$\epsilon = \mathcal{E}^{-1}\epsilon_0, \tag{3.23}$$

with ϵ_0 an arbitrary *constant* spinor. Hence the number of linearly independent Killing spinors on the sphere S^{2n} is equal to the dimension of the Dirac spinor in $D = 2n+1$, namely 2^n. There are also 2^n solutions to the equation with the opposite sign, $(D_\mu - \frac{1}{2}\gamma_\mu)\epsilon = 0$, given by $\mathcal{E}^t\epsilon_0$.

From the above Killing spinors we can construct the Killing vectors of $\mathrm{Iso}(S^{2n})$. Equation (3.18), give the general formula

$$\epsilon^\dagger\gamma_\mu\,\epsilon = \epsilon_0^\dagger(\mathcal{E}^{-1})^\dagger\gamma_\mu\mathcal{E}^{-1}\epsilon = -2\,\epsilon_0^\dagger\big(\mathcal{E}D_\mu\mathcal{E}^{-1}\big)\epsilon_0$$

since, for the compact group $Spin(2n+1)$, the spinorial representation is unitary. Inserting eqn.(6.22) of chapt. 5 in the RHS, we get

$$\epsilon^\dagger\gamma_\mu\epsilon = \frac{1}{2^{n+1}}\,\epsilon_0^\dagger\gamma^{AB}\epsilon_0\; K_{\mu\,AB}, \qquad A,B = 1,2\dots,2n+1 \tag{3.24}$$

where

$$\gamma^{AB} = -\gamma^{BA} := \begin{cases}\gamma^{AB} & A,B = 1,2,\dots,2n\\ -\gamma^A & B = 2n+1.\end{cases} \tag{3.25}$$

and $K_{\mu\,AB}$ are the Killing vectors of the $Spin(2n+1)$ isometry of S^{2n}.

EXERCISE 3.1. Work out the details for S^{2n+1}.

3.4.2. *The AdS_D case.* From eqn.(2.11) we have

$$\mathcal{E}^{-1}d\mathcal{E} = \frac{1}{4}\omega^{ab}\,\gamma_{ab} + \frac{1}{2}e^a\,\gamma_0\gamma_a \tag{3.26}$$

where the conventions are such that γ_0, γ_D are Hermitean and the γ_i, $i = 1,2,\dots,D-1$ *anti*Hermitean. The linear map[9]

$$\varpi\colon \mathbb{C}l(1,D-1) \to \mathbb{C}l_0(2,D-1)$$

given by $\gamma_a \mapsto -i\gamma_0\gamma_a$ is an isomorphism since

$$(-i\gamma_0\gamma_a)(-i\gamma_0\gamma_b) + (-i\gamma_0\gamma_b)(-i\gamma_0\gamma_a) = \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}.$$

One has $\varpi(\gamma_{ab}) = (-i\gamma_0\gamma_{[a})(-i\gamma_0\gamma_{b]}) = \gamma_{ab}$. Then

$$\varpi(d + \mathcal{E}^{-1}d\mathcal{E}) = \mathcal{D} := d + \frac{1}{4}\,\omega^{ab}\gamma_{ab} + i\frac{1}{2}e^a\gamma_a. \tag{3.27}$$

An AdS_D Killing spinor is a (non–zero) solution to $\mathcal{D}\epsilon = 0$. The general solution is

$$\boxed{\epsilon = \varpi^{-1}\big(\mathcal{E}^{-1}\big)\epsilon_0} \tag{3.28}$$

with ϵ_0 any constant spinor.

[9] $\mathbb{C}l(p,q)$ denotes the Clifford algebra in signature (p,q); $\mathbb{C}l_0(p,q)$ is the even subalgebra. See APPENDIX C for the details, in particular for the isomorphism ϖ.

The Killing vectors are given by the bilinears $\bar{\epsilon}\gamma_\mu\epsilon' \equiv \epsilon^\dagger\gamma_D\gamma_\mu\epsilon'$,

$$\begin{aligned}
\epsilon^\dagger\gamma_D\gamma_\mu\epsilon' &= \epsilon_0^\dagger\varpi^{-1}\Big((\mathcal{E}^{-1})^\dagger\Big)\gamma_D\gamma_\mu\,\varpi^{-1}(\mathcal{E}^{-1})\epsilon_0' = \\
&= -\epsilon_0^\dagger\varpi^{-1}\Big((\mathcal{E}^{-1})^\dagger\gamma_0\gamma_D\gamma_0\gamma_\mu\mathcal{E}^{-1}\Big)\epsilon_0' = \\
&= -\epsilon_0^\dagger\varpi^{-1}\Big(\gamma_0\gamma_D(\mathcal{E}\gamma_0\gamma_\mu\mathcal{E}^{-1})\Big)\epsilon_0' = \\
&= 2\,\epsilon_0^\dagger\varpi^{-1}\Big(\gamma_0\gamma_D(\mathcal{E}D_\mu\mathcal{E}^{-1})\Big)\epsilon_0' = \\
&= 2\,\alpha\,\epsilon_0^\dagger\varpi^{-1}(\gamma_0\gamma_D\gamma_{AB})\epsilon_0'\;K_\mu^{AB} \\
&= 2\,\alpha\,\bar{\epsilon}_0\gamma_{AB}\epsilon_0'\;K_\mu^{AB}
\end{aligned}$$

where we used the explicit form of the $SO(2, D-1)$ Killing vectors, eqn.(3.28) and the identity

$$\gamma_D\gamma_0\mathcal{E}^{-1}\gamma_0\gamma_D = \mathcal{E}^\dagger.$$

α is a normalization constant. We take $\alpha = 1$ as a choice of normalization of the $SO(2, D-1)$ Killing vectors. Then we have the crucial identity

$$\boxed{\bar{\epsilon}\gamma_\mu\epsilon' = \;(\bar{\epsilon}_0\gamma_{AB}\epsilon_0')\;K_\mu^{AB}} \tag{3.29}$$

3.4.3. *de Sitter: no Killing spinors.* In the de Sitter case, we have an extra i around and the isomorphism argument does not work.

4. The geometry of Killing spinors

We recall that, mathematically, a Killing spinor ψ on a Riemannian (spin) manifold $\mathcal{M}$ is a solution to the equation[10]

$$(D_\mu + \alpha\,\gamma_\mu)\,\psi = 0,$$

with $\alpha \in \mathbb{C}$. If $\alpha = 0$, then a Killing spinor is the same thing as a parallel spinor. In theorem 5.1 of chapter 3 we already solved the problem of characterizing the manifolds $\mathcal{M}$ which have one or more such spinors. For convenience of the reader, we summarize the result in table 6.1.

Let us begin by generalizing the lemma **3.**5.1.

PROPOSITION 4.1. *Assume that on the Riemannian n–fold $\mathcal{M}$ there is a (non–vanishing) Killing spinor, ψ, satisfying $(D_i + \alpha\,\gamma_i)\psi = 0$. Then $\mathcal{M}$ is Einstein with $R_{ij} = 4(n-1)\alpha^2$.*

PROOF. Consider the integrability condition[11]

$$0 = 4\,[D_i + \alpha\,\gamma_i, D_j + \alpha\,\gamma_j]\,\psi = \Big(-R_{ijkl}\,\gamma^{kl} + 8\alpha^2\,\gamma_{ij}\Big)\psi$$

and contract it with γ^j. Using the identities $\gamma^j\gamma^{kl} = \gamma^{jkl} - \delta^{jk}\gamma^l + \delta^{jl}\gamma^k$, and $\gamma^j\,\gamma_{ij} = (n-1)\gamma_i$, we get

$$\Big[-R_{ijkl}\gamma^{jkl} - 2\Big(R_{ij} - 4(n-1)\alpha^2\,\delta_{ij}\Big)\gamma^j\Big]\psi = 0 \tag{4.1}$$

[10] Recall that, when doing geometry, we adhere to the mathematicians' convention $\gamma_i\gamma_j + \gamma_j\gamma_i = -2\,\delta_{ij}$, so the γ matrices are antiHermitean!

[11] The sign $-$ in front of the curvature arises because, in the present conventions the generators of $Spin(n)$ are $-\frac{1}{4}[\gamma_i, \gamma_j]$.

Manifold	Holonomy	dimension	(N_+, N_-)
flat	1	$2n$	$(2^{n-1}, 2^{n-1})$
flat	1	$2n+1$	2^{n-1}
Calabi–Yau	$SU(2m)$	$4m$	$(2,0)$
Calabi–Yau	$SU(2m+1)$	$4m+2$	$(1,1)$
hyper–Kähler	$Sp(2m)$	$4m$	$(m+1,0)$
G_2–manifold	G_2	7	1
$Spin(7)$–manifold	$Spin(7)$	8	$(0,1)$

TABLE 6.1. Riemannian manifolds with (N_+, N_-) parallel spinors of chirality ± 1.

The first term in the bracket vanishes by the first Bianchi identity. So

$$\Big(R_{ij} - 4(n-1)\alpha^2\,\delta_{ij}\Big)\gamma^j\,\psi = 0 \tag{4.2}$$

Let A_{ij} a symmetric tensor. One has

$$(A_{ij}\gamma^j)(A_{ik}\gamma^k) = -(A_{ij}A_{ij})\,\mathbf{1} = -\mathrm{tr}(A^\dagger A)\,\mathbf{1}, \tag{4.3}$$

so $A_{ij}\gamma^j\psi = 0$ and $\psi \neq 0$ implies $A_{ij} \equiv 0$. Apply this remark to eqn.(4.2). □

COROLLARY 4.1. *Killing spinors may exist only for α real (real Killing spinors) or α purely imaginary (imaginary Killing spinors).*

We are expecially interested in the real ones.

4.1. Real Killing spinors. The situation for $\alpha = 0$ is presented in table 6.1. It remains to discuss the case $\alpha \neq 0$. In fact, we can reduce the general case to the one we already solved. Let us start with a definition.

DEFINITION 4.1. Let $\mathcal{M}$ be a Riemann n–fold with metric $g_{\alpha\beta}\,(y)dy^\alpha dy^\beta$. The metric (or Riemannian) cone over $\mathcal{M}$, denoted $\mathcal{C}_\lambda(\mathcal{M})$, is the Riemannian $(n+1)$–fold $\mathbb{R}_+ \times \mathcal{M}$ endowed with the metric

$$ds^2 = dr^2 + \lambda^2 r^2\, g_{\alpha\beta}\,(y)dy^\alpha dy^\beta, \tag{4.4}$$

where $\lambda \in \mathbb{R}^*$.

REMARK. Respect to the standard definition, I took the liberty of introducing a free parameter, λ, for later convenience.

Now,

THEOREM 4.1 (Bär [**154**]). *$\epsilon(y)$ is a real Killing spinor on $\mathcal{M}$ for some $\alpha \in \mathbb{R} \iff \epsilon(y)$ is a parallel spinor on the cone $\mathcal{C}_{2\alpha}(\mathcal{M})$ (for suitable choices of the spin generators and spin connection).*

PROOF. Let e^a and ω^{ab} be the vielbein and connection one–forms for the manifold $\mathcal{M}$, satisfying the torsionless constraint $de^a + \omega^{ab} \wedge e^b = 0$. Let E^m, Ω^{mn} be the corresponding forms for the cone $\mathcal{C}_\lambda(\mathcal{M})$. We can take

$$E^0 = dr, \qquad E^a = \lambda\, r\, e^a \tag{4.5}$$
$$\Omega^{a0} = -\Omega^{0a} = \lambda\, e^a \qquad \Omega^{ab} = \omega^{ab}. \tag{4.6}$$

One checks that $dE^m + \Omega^{mn} \wedge E^n = 0$. Let

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\, \Omega_\mu^{mn}\, \gamma_{mn} \tag{4.7}$$

be the spin connection on the cone $\mathcal{C}_\lambda(\mathcal{M})$. Using the explicit connection

$$\nabla_0 = \partial_r \tag{4.8}$$
$$\nabla_\alpha = \partial_\alpha + \frac{1}{4}\, \omega_\alpha^{ab}\, \gamma_{ab} + \frac{\lambda}{2}\, e_\alpha^a\, \gamma_a \gamma_0. \tag{4.9}$$

As discussed in the APPENDIX C, there is an algebra isomorphism

$$\varpi \colon \mathbb{C}l_0(n+1) \to \mathbb{C}l(n)$$

given by

$$\varpi(\gamma_{ab}) \mapsto \gamma_{ab}, \qquad \varpi(\gamma_a\gamma_0) \mapsto \gamma_a.$$

Then

$$\varpi(\nabla_\alpha) = D_\alpha + \frac{\lambda}{2}\, \gamma_\alpha. \tag{4.10}$$

A parallel spinor on the cone is a spinor $\psi(y)$, independent of the coordinate r, which, after the change of the realization of the γ–matrices given by the isomorphism ϖ, is a solution on $\mathcal{M}$ to the equation

$$\left(D_\alpha + \frac{\lambda}{2} \gamma_\alpha \right) \psi = 0,$$

that is to the real Killing spinor equation for $\alpha = 2\lambda$. □

4.2. Cones: Sasaki manifolds and all that. Riemannian spaces $\mathcal{M}$ such that their metric cones $\mathcal{C}(\mathcal{M})$ have particular holonomy groups have got names (and deep theories) in the math literature. Here we limit to mention those names leaving for future lectures the analysis of their geometries. A complete reference is [**155**]. Definitions and results are summarized in table 6.2.

To complete the discussion, we state the following

PROPOSITION 4.2. *A (simply–connected) Riemannian cone is a product of* irreducible *Riemannian cones. A* SYMMETRIC *Riemannian cone is flat.*

PROOF. A good exercise for you. □

name manifold $\mathcal{M}$	$\dim \mathcal{M}$	$\mathcal{C}(\mathcal{M})$	$\mathrm{Hol}_0(\mathcal{C}(\mathcal{M}))$	(N_+, N_-)
Round sphere S^n	n	flat space $\mathbb{R}^{n+1}$	Id	$(2^{[n/2]}, 2^{[n/2]})$
Sasaki	$2n-1$	Kähler	$U(n)$	$(0,0)$
Sasaki–Einstein	$4m+1$	Calabi–Yau	$SU(2m+1)$	$(1,1)$
Sasaki–Einstein	$4m-1$	Calabi–Yau	$SU(2m)$	$(2,0)$
3–Sasaki	$4m-1$	hyperKähler	$Sp(2m)$	$(m+1,0)$
strict nearly–Kähler $6-$manifold	6	G_2–manifold	G_2	$(1,1)$
proper G_2–metric	7	$Spin(7)$–manifold	$Spin(7)$	$(1,0)$

TABLE 6.2. Manifolds having (N_+, N_-) real Killing spinors and their Riemannian cones $\mathcal{C}(\mathcal{M})$.

5. Nester form of the space–time charges

The space–time charges M^m associated with asymptotic isometries can be written as surface integrals at infinity in various ways (see refs.[**153**]). Here we shall introduce a particular form, due to Nester [**82, 83**], which is quite elegant and practical, and also well suited for applications to SUGRA.

We start from scratch[12]. We specialize to the case of a geometry which is asymptotic anti–de Sitter (AAdS) with cosmological constant Λ. The asymptotically flat case (AF) is recovered by taking $\Lambda = 0$ in the following formulae.

To keep the formulae simple, we work in $D = 4$.

We recall two identities:

$$\gamma_\sigma \gamma_{\mu\nu} + \gamma_{\mu\nu}\gamma_\sigma = 2\,\gamma_{\sigma\mu\nu} = 2\,\epsilon_{\sigma\mu\nu\tau}\,\gamma^\tau\,\gamma_5, \tag{5.1}$$

and

$$\begin{aligned} -\frac{1}{4}\,\delta^\alpha_\mu\,\epsilon^{\mu\nu\rho\sigma}\,\epsilon_{\alpha\beta\gamma\delta}\,R_{\rho\sigma}{}^{\gamma\delta} \;&=\; -\frac{1}{4}\delta^{\nu\rho\sigma}_{\beta\gamma\delta}\,R_{\rho\sigma}{}^{\gamma\delta} = \\ &\equiv -\frac{1}{4}\big[\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_\delta + \delta^\rho_\beta\delta^\sigma_\gamma\delta^\nu_\delta + \delta^\sigma_\beta\delta^\nu_\gamma\delta^\rho_\delta - \\ &\qquad - \delta^\nu_\beta\delta^\sigma_\gamma\delta^\rho_\delta - \delta^\rho_\beta\delta^\nu_\gamma\delta^\sigma_\delta - \delta^\sigma_\beta\delta^\rho_\gamma\delta^\nu_\delta\big]\,R_{\rho\sigma}{}^{\gamma\delta} \\ &= R_\beta{}^\nu - \frac{1}{2}\delta_\beta{}^\nu R. \end{aligned} \tag{5.2}$$

Consider the commutator

$$4\left[D_\mu + \frac{i}{2}m\,\gamma_\mu, D_\nu + \frac{i}{2}m\,\gamma_\nu\right] \equiv \Big(R_{\mu\nu}{}^{\alpha\beta}\,\gamma_{\alpha\beta} - 2\,m^2\,\gamma_{\mu\nu}\Big) \tag{5.3}$$

[12] Minkowski signature with metric $(+,-,\dots,-)$.

where $m = \sqrt{-\Lambda/3}$. In view of the above identities, one has

$$\begin{aligned}&\gamma_\sigma\Big(R_{\mu\nu}{}^{\alpha\beta}\,\gamma_{\alpha\beta} - 2\,m^2\,\gamma_{\mu\nu}\Big) + \Big(R_{\mu\nu}{}^{\alpha\beta}\,\gamma_{\alpha\beta} - 2\,m^2\,\gamma_{\mu\nu}\Big)\gamma_\sigma = \\ &= 2\Big(\epsilon_{\sigma\tau\alpha\beta}\,R_{\mu\nu}{}^{\alpha\beta} - 2\,m^2\epsilon_{\mu\nu\sigma\tau}\Big)\gamma^\tau\gamma^5.\end{aligned} \tag{5.4}$$

Multiply this expression by $\epsilon^{\sigma\rho\mu\nu}$ and use the identity eqn.(5.2)

$$\begin{aligned}2\epsilon^{\sigma\rho\mu\nu}&\left(\gamma_\sigma\left[D_\mu + \frac{i}{2}m\,\gamma_\mu, D_\nu + \frac{i}{2}m\,\gamma_\nu\right] + \left[D_\mu + \frac{i}{2}m\,\gamma_\mu, D_\nu + \frac{i}{2}m\,\gamma_\nu\right]\gamma_\sigma\right) = \\ &= \left[-4\left(R^{\rho\tau} - \frac{1}{2}g^{\rho\tau}\,R\right) - 2\cdot 3!\,m^2 g^{\rho\tau}\right]\gamma_\tau\gamma_5 = \\ &= -4\left(R^{\rho\tau} - \frac{1}{2}g^{\rho\tau}\,R - \Lambda g^{\rho\tau}\right)\gamma_\tau\gamma_5.\end{aligned} \tag{5.5}$$

Now write

$$\mathcal{D}_\mu := D_\mu + \frac{i}{2}m\,\gamma_\mu = \overline{\mathcal{D}}_\mu + \Omega_\mu + O(h^2) \tag{5.6}$$

where Ω_μ is the linear part in $h_{\mu\nu} = g_{\mu\nu} - \overline{g}_{\mu\nu}$. We assume Ω_μ to be of order $O(1/r^2)$, in order to have the right asymptotics. The omitted terms are of order $O(1/r^3)$. We have

$$\begin{aligned}\Theta^{\rho\lambda}\,\gamma_\lambda &= \left(R^{\rho\lambda} - \frac{1}{2}g^{\rho\lambda}\,R - \Lambda\,g^{\rho\lambda}\right)^{\text{Lin.}}\gamma_\lambda = \\ &= \epsilon^{\rho\sigma\mu\nu}(\gamma_\sigma\overline{\mathcal{D}}_\mu\Omega_\nu + \overline{\mathcal{D}}_\mu\Omega_\nu\gamma_\sigma)\gamma_5 = \big(\gamma^{\rho\mu\nu}\overline{\mathcal{D}}_\mu\Omega_\nu + \overline{\mathcal{D}}_\mu\Omega_\nu\gamma^{\rho\mu\nu}\big) = \\ &= \overline{\mathcal{D}}_\mu\big(\gamma^{\rho\mu\nu}\Omega_\nu + \Omega_\nu\gamma^{\rho\mu\nu}\big).\end{aligned} \tag{5.7}$$

Now let α, β be *commuting* spinors satisfying the equation

$$\overline{\mathcal{D}}_\mu\alpha = 0, \quad \Rightarrow \quad \Omega_\mu\alpha = \mathcal{D}_\mu\alpha + O(1/r^3), \tag{5.8}$$

that is α, β are Killing spinors for the Anti–de Sitter metric $\overline{g}_{\mu\nu}$. One has[13]

$$\begin{aligned}\Theta^{\rho\lambda}\,\bar\alpha\gamma_\lambda\beta &= \overline{D}_\mu\Big(\bar\alpha\big(\gamma^{\rho\mu\nu}\Omega_\nu + \Omega_\nu\gamma^{\rho\mu\nu}\big)\beta\Big) = \\ &= \overline{D}_\mu\Big(\bar\alpha\gamma^{\rho\mu\nu}\mathcal{D}_\nu\beta - (\overline{\mathcal{D}_\nu\alpha})\gamma^{\rho\mu\nu}\beta\Big).\end{aligned} \tag{5.9}$$

The LHS is the correct integrand for the space–time charge associated with the Killing vector $\bar\alpha\gamma_\mu\beta$. On the other hand, as we saw in §. 3.4.2, *all* $Spin(3,2)$ Killing vectors are of this form. More precisely, we have

$$\bar\alpha\gamma_\mu\beta = \frac{1}{2}(\bar\alpha_0\,\gamma_{AB}\,\beta_0)\;K^{AB}_\mu \tag{5.10}$$

[13] The overline outside the parenthesis means *background value,* whereas the one inside the parenthesis stands for *Dirac conjugate of the spinor.* I hope this is not too confusing.

(we have changed the normalization of the Killing vectors to adhere to the usual SUGRA conventions). Then

$$\begin{aligned}
&\frac{1}{2}(\bar{\alpha}_0\,\gamma_{AB}\,\beta_0)\,M^{AB} = \\
&\quad = \frac{1}{2}(\bar{\alpha}_0\,\gamma_{AB}\,\beta_0)\int_S \Theta^{\mu\nu}\,K^{AB}_\nu\,d\Sigma_\mu = \int_S \Theta^{\mu\nu}\,(\bar{\alpha}\gamma_\nu\beta)\,d\Sigma_\mu \\
&\quad = \int_S \overline{D}_\mu\Big(\bar{\alpha}\gamma^{\rho\mu\nu}\mathcal{D}_\nu\beta - (\overline{\mathcal{D}_\nu\alpha})\gamma^{\rho\mu\nu}\beta\Big)d\Sigma_\rho = \\
&\quad = \oint_{\partial S}\frac{1}{2}\Big(\bar{\alpha}\gamma^{\rho\mu\nu}\mathcal{D}_\nu\beta - (\overline{\mathcal{D}_\nu\alpha})\gamma^{\rho\mu\nu}\beta\Big)d\sigma_{\rho\mu},
\end{aligned} \tag{5.11}$$

where S is a space–like hypersurface in Σ. The surface integral in the RHS is Nester form of the space–time charges [**82, 83**].

6. The AdS/Poincaré Susy algebra

Armed with the explicit expressions of sec. 5 it is very easy to check the algebra generated by our supercharges [**156**]. We write the supercharge associated to a given (asymptotic) *commuting* Killing spinor α as

$$Q(\alpha) = \int_{\partial S} \bar{\alpha}\,\gamma^{\mu\nu\rho}\,\psi_\rho\,d\sigma_{\mu\nu}. \tag{6.1}$$

$Q(\alpha)$ generates the SUSY transformation $2\,\delta_Q(\alpha)$ of parameter α. Then

$$\begin{aligned}
\{Q(\alpha),Q(\beta)\} &= 2\,\delta_Q(\alpha)Q(\beta) = 2\int_{\partial S}\bar{\beta}\,\gamma^{\mu\nu\rho}\,\delta_Q(\alpha)\psi_\rho\,d\sigma_{\mu\nu} = \\
&= 2\int_{\partial S}\bar{\beta}\,\gamma^{\mu\nu\rho}\,\mathcal{D}_\rho\alpha\,d\sigma_{\mu\nu},
\end{aligned} \tag{6.2}$$

where now $\mathcal{D}_\rho$ is the operator appearing in the gravitino transformation, eqn. If $\mathcal{D}_\rho$ is simply $D_\rho + i\sqrt{-\Lambda/12}\,\gamma_\rho$, the RHS of eqn.(6.2) was already computed above to be equal to

$$(\bar{\alpha}_0\gamma_{AB}\beta_0)\,M^{AB} \tag{6.3}$$

where $M^{AB} = -M^{BA}$ are the $SO(2,D-1)$ generators. Therefore in AADS we get a SUSY algebra

$$\{Q_\alpha,\bar{Q}_\beta\} = (\gamma_{AB})_{\alpha\beta}\,M^{AB}, \tag{6.4}$$

which is the standard form of the AdS SUSY.

Taking $\Lambda \to 0$, the AdS isometry group, $SO(2,D-1)$, contracts to the Poincaré group and the above expression reduces to $(\gamma_\mu)_{\alpha\beta}\,P^\mu$, where now P^μ is the global energy–momentum of the asymptotically flat configuration. Indeed, eqn.(5.11) is exactly Nester original formula for P_μ [**82**]. This, of course, is Poincaré SUSY.

However, in general $\mathcal{D}_\rho$ contains other terms and we get a richer algebra. We shall be rather sketchy, looking for the general structures rather than the details of the computations (which you can find in the references). Since we are not going to write detailed formulae, it is better to revert to Majorana notation for the gravitini. Moreover, for the rest of this section we shall limit ourselves to $D=4$ SUGRA.

In $D=4$ SUGRA we have[14]

$$(\mathcal{D}_\rho)^{ab}=(D_\rho)^{ab}+\frac{i}{2}m(\phi)^{ab}\gamma_\mu-\frac{1}{4}\mathfrak{F}^{ab}_{\mu\nu}\gamma^{\mu\nu}\gamma_\rho \tag{6.5}$$

where $\mathfrak{F}^{ab}_{\mu\nu}$ is the local $U(\mathcal{N})_R$–covariant field–strenght defined in §. of chapt.

In order to have a well–defined SUSY algebra, the asymptotic regime should be reached rapidly enough. One must require that $h_{\mu\nu}=O(1/r)$, so the Ricci tensor deviates from the asymptotic value to order $O(1/r^3)$ and so does the matter energy–momentum tensor. This, in particular, requires $m(\phi)^{ab}=\sqrt{-\Lambda/3}\,\delta^{ab}+O(1/r^3)$ for $a,b=1,2,\dots,\mathcal{N}_0$, that is for the indices a corresponding to the *unbroken* SUSY subalgebra. Thus, in these particular backgrounds, there is no additional contribution to the surface integral from the second term in the RHS of eqn.(6.5). Instead there may be contributions from the field strength $\mathfrak{F}^{ab}_{\mu\nu}$. As we have already anticipated a couple of times, for AF backgrounds the flux at infinity of the $U(\mathcal{N})_R$–covariant field–strenght $\mathfrak{F}^{ab}_{\mu\nu}$ and its dual gives rise to the central charges Z^{ab}, which are linear combinations of the electric and magnetic charges of the basic vector fields with coefficients dependening on the values of the scalars at infinitiy, as we discussed, *e.g.* in §. of chapt. Thus, in the AF case, one gets a superalgebra with the general structure

$$\{Q^a,\bar{Q}^b\}=\delta^{ab}\,\gamma^\mu P_\mu+Z^{ab}+i\gamma_5\,\tilde{Z}^{ab}.$$

The *AdS* case is more interesting. First all all, what we expect to find?

In five dimensions we have the following Lie group isomorphisms[15]

$$Spin(5)\simeq Sp(4),\qquad Spin(4,1)\simeq Sp(2,2),\qquad Spin(3,2)\simeq Sp(4,\mathbb{R}),$$

so

$$\mathfrak{iso}(AdS_4)\simeq\mathfrak{sp}(4,\mathbb{R}), \tag{6.6}$$

which is, in particular, simple. The full bosonic symmetry realized on the supercharges should be $\mathfrak{sp}(4,\mathbb{R})\oplus\widetilde{\mathfrak{aut}}_R$. The SUSY automorphism algebra, $\widetilde{\mathfrak{aut}}_R$, is, however, smaller than in the Poincaré case where it is $\mathfrak{u}(\mathcal{N}_0)$. In fact the kinetic terms for the gravitini corresponding to the $\mathcal{N}_0$ unbroken supersymmetry is

$$-i\bar{\psi}^a_\mu\gamma^{\mu\nu\rho}\mathcal{D}_{\nu\,ab}\,\psi^b_\rho=-i\bar{\psi}^a_\mu\gamma^{\mu\nu\rho}D_\nu\psi^a_\rho-\sqrt{-\Lambda/3}\,\bar{\psi}^a_\mu\gamma^{\mu\nu}\psi^a_\nu+\cdots$$

In flat space, the second term in the RHS would be a mass for the spin–3/2 particle. In *AdS* provided the coefficient is precisely $\sqrt{-\Lambda/3}$, this 'mass term' combines with the curvature of the space in such a way that the net effect is that the gravitini propagate on the light–cone, that is are physically massless. However, the mass term breaks the chiral symmetry $U(\mathcal{N})$ down to

[14] Recall that the gravitino fields are redefined in such a way as to eliminate the cross–terms spin–3/2/spin–1/2 from the kinetic (derivative) terms. This also eliminates terms in the SUSY transformations different from those we write, up to higher terms in the fermions.

[15] Recall that, in our notations, $Sp(4,\mathbb{F})$ stands for the symplectic group having fundamental representation of dimension 4. It is often called $Sp(2,\mathbb{F})$, expecially in the math literature.

the vector–like subgroup $SO(\mathcal{N}_0)$, just as it will do in flat space. Therefore, we expect that the bosonic part of the AdS SUSY algebra is

$$SO(\mathcal{N}_0) \times Sp(4,\mathbb{R}). \tag{6.7}$$

This fits perfectly well into the classification theorems of superalgebras [**157**] which predicts the existence of a superalgebra, $Osp(\mathcal{N}_0\,|\,4)$, which has this bosonic subalgebra.

Let us check that this is precisely the result one obtain by inserting eqn.(6.5) into eqn.(6.2). Using the identity

$$\gamma^{\alpha\beta\sigma}\gamma^{\mu\nu}\gamma_\beta = 2i\gamma_5\,\epsilon^{\sigma\alpha\mu\nu} + 2g^{\sigma\mu}g^{\alpha\nu} - 2g^{\sigma\nu}g^{\alpha\mu}, \tag{6.8}$$

one finds [**158**]

$$-\bar{\epsilon}\gamma^{\mu\nu\rho}\mathcal{D}_\rho\epsilon = \cdots + \bar{\epsilon}\,(\mathfrak{F}^{\mu\nu} + i\gamma_5\widetilde{\mathfrak{F}}^{\mu\nu})\epsilon + (\cdots) \tag{6.9}$$

where $\cdots$ stands for the terms (we have already computed) which produce the expression $\bar{\epsilon}_0\gamma^{AB}\epsilon_0\,M_{AB}$ in the anticommutator of two Q's. Using the explicit formula for the (asymptotic) Killing spinors, eqn.(3.28), we get

$$\begin{aligned}
\bar{\epsilon}^a\epsilon^b &= (\epsilon^a)^\dagger\,\gamma_4\,\epsilon^b = (\epsilon_0^a)^\dagger\,\varpi^{-1}((\mathcal{E}^{-1})^t)\gamma_4\,\varpi^{-1}(\mathcal{E}^{-1})\epsilon_0^b \\
&= -i(\epsilon_0^a)^\dagger\,\varpi^{-1}((\mathcal{E}^{-1})^t\,\gamma_0\gamma_4\,\mathcal{E}^{-1})\epsilon_0^b = (\epsilon_0^a)^\dagger\,\varpi^{-1}(-i\gamma_0\gamma_4\,\mathcal{E}\mathcal{E}^{-1})\epsilon_0^b \\
&= (\epsilon_0^a)^\dagger\,\varpi^{-1}(-i\gamma_0\gamma_4\,\mathcal{E}\mathcal{E}^{-1})\epsilon_0^b = (\epsilon_0^a)^\dagger\,\gamma_4\,\epsilon_0^b = \\
&= \bar{\epsilon}_0^a\epsilon_0^b,
\end{aligned}$$

thus the first contribution linear in the field–strenght in the RHS of eqn.(6.9) reduces to $\mathfrak{F}^{\mu\nu}_{ab}\,(\bar{\epsilon}_0^a\epsilon_0^b)$ or, more elegantly[16], to

$$\mathfrak{F}^{\mu\nu}_{ab}\,(\bar{\epsilon}_0 L^{ab}\epsilon_0), \tag{6.10}$$

where $(L^{ab})_{cd} = \delta^{ab}_{cd}$ are the generators of $SO(\mathcal{N}_0)$ in the vector representation. Then this first term gives a contribution to the anticommutator of two Q's of the form

$$(\bar{\epsilon}\,L_{ab}\,\epsilon')\int\limits_{\partial S} *\mathfrak{F}^{ab} = (\bar{\epsilon}\,L_{ab}\,\epsilon')\;J^{ab}, \tag{6.11}$$

where the J^{ab} are *electric*–like charges which generates the global $SO(\mathcal{N}_0)$ symmetry. Very nice!

However the other term, $\bar{\epsilon}\,(*\mathfrak{F})\gamma_5\epsilon'$ (which is linear into the $\mathfrak{F}$–*magnetic* charges $m^{ab} = \int_{\partial S}\mathfrak{F}^{ab}$), is definitely <u>not</u> nice. The expression $\bar{\epsilon}^a\gamma_5\epsilon^b$ reduces to something like $\bar{\epsilon}_0^a(\mathcal{E}\gamma_5\mathcal{E}^{-1})\epsilon_0^b$, which is <u>not</u> a constant at infinity, nor an asymptotic Killing vector. So this contribution is ugly. It is not only ugly, it is also unexpected. The electric charges J^{ab} are enough to generate $SO(\mathcal{N}_0)$, and we do not expect any other bosonic generator of the global AdS SUSY algebra besides (M^{AB}, J^{ab}), since — by the arguments of [**157**] — we know it should correspond to $Osp(\mathcal{N}_0\,|\,4)$.

The magnetic term, anyhow, is there and it <u>should</u> be there. Why? because in the limit $\Lambda \to 0$, we should recover the Poincaré algebra, and both the magnetic and electric charges are needed to get the full SUSY algebra (with all possible central charges). This is related to the fact that, as $\Lambda \to 0$

[16] Recall that the ϵ_0^a are *commuting* spinors!

the automorphism group gets enhanced from $SO(\mathcal{N}_0)$ to $U(\mathcal{N}_0)$ and the $\mathfrak{F}$–fluxes should make a complete representation of this larger group.

How we solve the paradox? Well, Hawking already solved it for us [**159**].

From eqn.(2.7), we see that a massless photon moving radially satisfies the equation

$$d\tau = \frac{d\rho}{\cosh(\rho)} \equiv 2\frac{d\tanh(\rho/2)}{1+\tanh^2(\rho/2)}, \tag{6.12}$$

and the 'boundary' $\rho = \infty$ is reached in a *finite* time $\tau = \pi$. Thus, AdS is not a globally hyperbolic (*i.e.* there is no space–like Cauchy surface on which we can specify the initial data getting a unique time evolution). To get unique solutions, we need also to specify the boundary conditions at $\rho = \infty$ for the massless fields. For the gauge vectors, unitarity implies *reflective* B.C. There are two types: either the eletric fields $F_{0i} = O(1/r^2)$ and the magnetic fields $F_{ij} = O(1/r^3)$ or the other way around. Thus we can have either electric or magnetic fluxes at infinity, but not both. In the special case of $\mathfrak{F}_{\mu\nu}$, SUSY relates its boundary conditions to those of the graviton. The conclusion is that only the electric fluxes (charges) survives [**159**]. This is perfectly in agreement with our findings and solves (quite elegantly) our little paradox.

7. * Positive mass and BPS bounds

How we have already mentioned, the above machinery was introduced in order to prove a long standing conjecture in General Relativity, namely that the total mass of any asymptotically flat configuration is non–negative (and vanish if and only if the space is everywhere flat). This theorem also implies the stability of Minkowski space as a solution to the equations of Einstein.

In facts, R. Schoen and S.T. Yau did produce a proof of this crucial fact in refs.[**161**], but they proof is long and rather technical, and also it depends on details of the minimal submanifolds which are not true for $D \geq 8$. Then Witten presented in [**82**] a 'simple' proof of the positivity theorem, which was further simplified and generalized by many authors [**82, 83, 156, 160**].

7.1. Positive energy in Minkowski space. The simplest (and most basic) result states that, in all *physically sound* theories describing gravity coupled to matter, the total mass M of any asymptotically Minkowskian solution to the Einstein equation is *non–negative,* (and it vanishes if and only the space is globally Minkowski). The condition of being *physically sound* is reflected in the requirement that the matter energy–momentum tensor $T_{\mu\nu}$ satisfies the *dominant energy condition,* namely the matter energy density is required to be positive at all points and in all frames, or

$$T_{\mu\nu}\, U^\mu\, V^\nu \geq 0 \qquad \forall\, U^\mu,\ V^\nu \ \ \text{time–like.} \tag{7.1}$$

To prove positivity, one starts from eqn.(5.11), written for an asymptotically Minkowski background, for $\alpha = \beta$,

$$\begin{aligned}\frac{1}{2}(\bar{\alpha}_0\,\gamma^\mu\,\alpha_0)\,P_\mu &= \\ &= \oint_{\partial S}\frac{1}{2}\Big(\bar{\alpha}\gamma^{\rho\mu\nu}\mathcal{D}_\nu\alpha - (\overline{\mathcal{D}_\nu\alpha})\gamma^{\rho\mu\nu}\alpha\Big)d\sigma_{\rho\mu} = \\ &= \int_S \mathcal{D}_\mu\Big(\bar{\alpha}\gamma^{\rho\mu\nu}\mathcal{D}_\nu\alpha - (\overline{\mathcal{D}_\nu\alpha})\gamma^{\rho\mu\nu}\alpha\Big)\,dS_\rho,\end{aligned} \tag{7.2}$$

where $\mathcal{D}_\mu$ is the usual covariant derivative with respect the (spin lift of the) Levi–Civita connection, and we used the divergence theorem to rewrite the second line as a volume integral over the space–like 3–surface S (we choose local coordinates so that S is the surface $x^0 = \text{const.}$). Note that $\bar{\alpha}_0\,\gamma_\mu\,\alpha_0$ is a *time–like* vector.

Now the fundamental idea (due to Witten) is that in evaluating the RHS of eqn.(7.2) we can use *any* (commuting) spinor α, as long as it has the correct asymptotics

$$\alpha(x) = \alpha_0 + O\left(\tfrac{1}{r}\right), \tag{7.3}$$

and the idea is to make a smart choice of α which makes the RHS manifestly non–negative.

Using the identities (5.1) and (5.2), and the very same manipulations we used in sec. 5 to prove the Nester's formula, we get for the RHS of eqn.(7.2)

$$\int_S \Big\{T^\rho{}_\mu\,\bar{\alpha}\gamma^\mu\alpha + (\mathcal{D}_\mu\bar{\alpha})(\gamma^\rho\gamma^{\mu\nu} + \gamma^{\mu\nu}\gamma^\rho)(\mathcal{D}_\nu\alpha)\Big\}dS_\rho. \tag{7.4}$$

Since $\bar{\alpha}\gamma^\mu\alpha$ is non space–like, the first term in the brace, $T^\rho{}_\mu\,\bar{\alpha}\gamma^\mu\alpha$ is non–negative provided the matter satisfies the dominant energy condition. The second term is, explicitly, $(i,j = 1,2,3)$

$$\begin{aligned}(\mathcal{D}_i\bar{\alpha})(\gamma^0\gamma^{ij} + \gamma^{ij}\gamma^0)(\mathcal{D}_j\alpha) &= \\ &= 2(\mathcal{D}_i\alpha)^\dagger\gamma^{ij}(\mathcal{D}_j\alpha) = \\ &= -2g^{ij}(\mathcal{D}_i\alpha)^\dagger\,(\mathcal{D}_j\alpha) + (\mathcal{D}_i\alpha^\dagger\,\gamma^i)(\gamma^j\mathcal{D}_j\alpha).\end{aligned} \tag{7.5}$$

The first term is positive (metric $(+,-,-,-)$!). So the positivity theorem is proven provided we can choose our spinor α in such a way that $\gamma^i\mathcal{D}_i\alpha \equiv 0$. Spinors satisfying this equation and the boundary condition (7.3) are called *Witten spinors.* Witten spinors do exist, as one shows using standard linear analysis techniques. Then the proof of completed.

7.2. Stability of *AdS* space. In the proof of positive mass we have, on purpose, used for the covariant derivative the symbol $\mathcal{D}_\mu$ which, in the rest of the chapter, stands for the more complicated first–order differential operator appearing in the RS term (or in the gravitino SUSY transformations). The one used in the above proof corresponds to the SUSY transformation of *pure* $\mathcal{N} = 1$ supergravity with no cosmological constant. Taking $\mathcal{D}_\mu$ to be the SUSY transformation operator corresponding to various SUGRA theories we get other positive–energy theorems. For instance,

$$\mathcal{D}_\mu = D_\mu + i\sqrt{-\tfrac{\Lambda}{12}}\gamma_\mu \tag{7.6}$$

leads to the *AdS* space positive–energy theorem.

7.3. BPS bounds. In the same fashon, taking $\mathcal{D}_\mu$ to be the differential operator associated to an extended SUGRA

$$(\mathcal{D}_\mu\epsilon)^A = D_\mu\epsilon^A - \frac{1}{4}\mathfrak{F}^{AB}_{\rho\nu}\gamma^{\rho\nu}\epsilon_B \tag{7.7}$$

we get (in the Majorana notation)

$$\bar{\alpha}^A_0\left[\delta_{AB}\,P_\mu\gamma^\mu - (Z_{AB} + i\gamma_5\,\widetilde{Z}_{AB})\right]\alpha^B_0 = \int\limits_S \Big[\hat{T}^\rho{}_\mu\,\bar{\alpha}\gamma^\mu\alpha - g^{ij}(\mathcal{D}_i\alpha)^\dagger\,(\mathcal{D}_j\alpha)\Big]dS_\rho \geq 0 \tag{7.8}$$

where $\hat{T}_{\mu\nu}$ is the contribution to the energy–momentum from the matter *but* the gauge–vectors whose field–strengths enter in the differential operator (7.7). $\hat{T}_{\mu\nu}$ is assumed to satisfy the dominant energy condition.

Thus the matrix

$$\delta_{AB}\,P_\mu\gamma^\mu - (Z_{AB} + i\gamma_5\,\widetilde{Z}_{AB}) \geq 0, \tag{7.9}$$

which is equivalent to the BPS bound for the mass M

$$\boxed{|M| \geq |Z + i\widetilde{Z}|^2} \tag{7.10}$$

8. Sugra Ward identities

8.1. Susy Ward identities from positive mass. As we saw in sec. 7, Nester–type representations for the spacetime global charges, associated to asymptotic Killing vectors, was invented to prove *positivity of mass* in General Relativity, for *any* reasonable matter content, whether supersymmetric or not. Of course, if the particular model *is* locally supersymmetric, we get stronger results: the SUSY algebra, the BPS bounds, *etc.*

Now we shall apply this technology to a theory which *do have* $\mathcal{N}$–extended supersymmetry, but somehow forgetting this fact, and treating its just as gravity coupled to some *cleverly chosen* matter. In this context, we are free to add spurious fields producing an additional contribution to the energy–momentum tensor, $\theta_{\mu\nu}$, which acts as an *external source* for gravity

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,R = T_{\mu\nu} + \theta_{\mu\nu}, \tag{8.1}$$

where $T_{\mu\nu}$ is the energy–momentum tensor produced by the SUGRA fields.

Nester–like formulae hold on–shell, which means that the Einstein equations (8.1) of motion should be satisfied. In the manipulation below we shall use the analogue expressions for the supercharges, so we should require that the Rarita–Schwinger equations

$$R^\mu \equiv -\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho + J^\mu = 0 \tag{8.2}$$

are also satisfied. However, there is no need for the other fields to be on–shell, as long as the configuration is such that $D_\mu(T^{\mu\nu}+\theta^{\mu\nu}) = 0$, as required by the consistency of the Einstein equations (*via* the Bianchi identity). Thus, we can take as our background *any* constant value of the scalar fields

$\phi^i = \phi^i_0 =$ const. The scalars' contribution to $T^{\mu\nu}$ has the form $V(\phi_0)\, g^{\mu\nu}$, which is just another contribution to the cosmological constant, and hence it is consistent with the Bianchi identities for any constant value of $V(\phi_0)$. The vector fields are taken to vanish in our *gedanken* configuration. It is also convenient to choose the external sources to be $\theta_{\mu\nu} = -V(\phi_0)\, g_{\mu\nu}$, so that the effective cosmological constant is zero, and we can work in flat space-time and use Killing spinors (in the geometric sense!) which are strictly constant[17].

The Nester energy of such a configuration is zero

$$\begin{aligned} 0 &= \int\limits_{\partial S} d\sigma_{\mu\nu}\; \bar\epsilon\, \gamma^{\mu\nu\rho} D_\rho \epsilon = \int\limits_{\partial S} d\sigma_{\mu\nu} \left(\bar\epsilon\, \gamma^{\mu\nu\rho} \mathcal{D}_\rho \epsilon - \frac{i}{2}\, m\, \bar\epsilon\, \gamma^{\mu\nu} \epsilon \right) \\ &= \delta_Q(\epsilon) \int\limits_{\partial S} d\sigma_{\mu\nu}\, \bar\epsilon\, \gamma^{\mu\nu\rho}\, \psi_\rho - \frac{i}{2} \int\limits_{\partial S} d\sigma_{\mu\nu}\; m\, \bar\epsilon\, \gamma^{\mu\nu}\, \epsilon, \end{aligned} \tag{8.3}$$

where the $SU(\mathcal{N})_R$ indices are left implicit and ϵ is a *commuting* spinor. We evaluate the surface integrals in the RHS using the divergence theorem. The last term is covariantly constant, and hence has vanishing divergence; it drops out of the computation. We remain with

$$\begin{aligned} 0 &= \delta_Q(\epsilon) \int\limits_{\partial S} d\sigma_{\mu\nu}\, \bar\epsilon\, \gamma^{\mu\nu\rho}\, \psi_\rho = \\ &= \delta_Q(\epsilon) \int\limits_{S} d\sigma_\nu \Big(\overline{D_\mu \epsilon}\, \gamma^{\mu\nu\rho}\, \psi_\rho + \bar\epsilon\, \gamma^{\mu\nu\rho}\, D_\mu \psi_\rho \Big) = \\ &= \delta_Q(\epsilon) \int\limits_{S} d\sigma_\nu \Big(\overline{\mathcal{D}_\mu \epsilon}\, \gamma^{\mu\nu\rho}\, \psi_\rho + \bar\epsilon\, \gamma^{\mu\nu\rho}\, \mathcal{D}_\mu \psi_\rho \Big). \end{aligned} \tag{8.4}$$

In the last line we used a standard identity we already used to deduce the global SUSY algebra (and which should hold if the underlying theory is supersymmetric [**160**]). Next we recall a couple of useful relations

$$\gamma^{\mu\nu\rho} \mathcal{D}_\nu \psi_\rho - J^\mu := R^\mu \tag{8.5}$$

$$\delta_Q(\epsilon)\, \bar\epsilon R^\mu = -\frac{1}{2} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} - T^{\mu\nu} \right) \bar\epsilon \gamma_\nu \epsilon \equiv -\frac{1}{2} \theta^{\mu\nu}\, \bar\epsilon \gamma_\nu \epsilon. \tag{8.6}$$

The second equation follows from the first since the gravitino equations of motion, $R^\mu = 0$, get transformed into the Einstein equations, which, *in absence of external sources,* would read $\theta^{\mu\nu} = 0$; the overall coefficient, $-1/2$, is easily fixed by checking that the leading term in the curvatures,

$$\gamma^{\mu\nu\rho} D_\nu D_\rho \epsilon = \frac{1}{8} \gamma^{\mu\nu\rho} \gamma^{\alpha\beta} R_{\nu\rho\alpha\beta},$$

[17] Recall that we can choose the spinors in any way we like, provided they have the correct asymptotics.

is equal, for a maximally symmetric space, to $-\frac{1}{2}(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\gamma_\nu$. Then, the RHS of eqn.(8.4) becomes

$$\begin{aligned} 0 &= \delta_Q(\epsilon)\int_S d\sigma_\nu\Big(\overline{\mathcal{D}_\mu\epsilon}\,\gamma^{\mu\nu\rho}\,\psi_\rho + \bar\epsilon\,\gamma^{\mu\nu\rho}\,\mathcal{D}_\mu\psi_\rho\Big) = \\ &= \int_S d\sigma_\nu\,\big(\overline{\mathcal{D}_\mu\epsilon}\big)\gamma^{\mu\nu\rho}\big(\mathcal{D}_\rho\epsilon\big) - \delta_Q(\epsilon)\int_S d\sigma_\mu\,\bar\epsilon\,\Big(R^\mu - J^\mu\Big) = \\ &= \int_S d\sigma_\nu\,\big(\overline{\mathcal{D}_\mu\epsilon}\big)\gamma^{\mu\nu\rho}\big(\mathcal{D}_\rho\epsilon\big) + \frac{1}{2}\int_S d\sigma_\mu\,\theta^{\mu\nu}\,\bar\epsilon\gamma_\nu\epsilon + \delta_Q(\epsilon)\int_S d\sigma_\mu\,\bar\epsilon\,J^\mu \end{aligned} \tag{8.7}$$

One has $\bar\epsilon J^\mu = \overline{\big(\delta_Q(\epsilon)\chi^I\big)}Z(\phi)_I{}^J\gamma^\mu\chi_I$, where $Z(\phi)_I{}^J$ is the positive–definite matrix which enters the kinetic terms for the physical spin–$\frac{1}{2}$ fields

$$\frac{i}{2}\,e\,Z(\phi)_I{}^J\,\bar\chi_I\gamma^\mu D_\mu\chi_J. \tag{8.8}$$

Finally, eqn.(8.7) reduces to the identity

$$\begin{aligned} 0 = &\int_S d\sigma_\nu\,\big(\overline{\mathcal{D}_\mu\epsilon}\big)\gamma^{\mu\nu\rho}\big(\mathcal{D}_\rho\epsilon\big) + \frac{1}{2}\int_S d\sigma_\mu\,\theta^{\mu\nu}\,\bar\epsilon\gamma_\nu\epsilon + \\ &+ \int_S d\sigma_\mu\,\overline{\delta_Q(\epsilon)\chi^I}\,Z(\phi)_I{}^J\,\gamma^\mu\,\delta_Q(\epsilon)\chi^J. \end{aligned} \tag{8.9}$$

Defining the matrices $m(\phi)^{AB}$ and $C(\phi)^I{}_A$ by[18]

$$\delta\psi_\mu^A = D_\mu\epsilon^A + \frac{1}{2}\,m(\phi)^{AB}\,\gamma_\mu\epsilon_B \tag{8.10}$$

$$\delta\chi^I = \frac{1}{2}\,C(\phi)^I{}_A\,\epsilon^A, \tag{8.11}$$

and replacing them in the above identity we get the final formula

$$0 = \left(-\frac{3}{2}\,m(\phi)_{AC}\,m(\phi)^{CB} - \frac{1}{2}V(\phi)\,\delta_A{}^B + \frac{1}{4}C(\phi)_A{}^I Z(\phi)_I{}^J C(\phi)^B{}_J\right)\bar\epsilon^A\gamma_\mu\epsilon_B.$$

Since it holds for any (constant) value of ϕ, we can interpret it as a general formula for the scalar potential $V(\phi)$:

General lesson 8.1. *The scalar potential $V(\phi)$ of any $D = 4$ supersymmetric theory is given by the formula* [**162**]

$$\boxed{V(\phi)\,\delta_A{}^B = \frac{1}{2}C(\phi)_A{}^I Z(\phi)_I{}^J C(\phi)^B{}_J - 3\,m(\phi)_{AC}\;m(\phi)^{CB}} \tag{8.12}$$

Comparing with the integrability condition for the AdS_D Killing spinors, eqn.(3.13), we easily infer the general formula valid for any D:

$$\boxed{V(\phi)\,\delta_A{}^B = \frac{1}{2}C(\phi)_A{}^I Z(\phi)_I{}^J C(\phi)^B{}_J - (D-1)\,m(\phi)_{AC}\;m(\phi)^{CB}} \tag{8.13}$$

[18] We use the chiral convention; so ϵ^A has chirality $+1$ and ϵ_A chirality -1. Raising/lowering indices corresponds to complex conjugation.

REMARK. The first term in the RHS of eqn.(8.12) yields precisely the formula for $V(\phi)$ in *rigid* SUSY which corresponds to the operator statement[19]

$$H\,\delta^A{}_B = \{Q^A, (Q^B)^\dagger\} \geq 0 \qquad \text{in the sector } P_i = 0. \tag{8.14}$$

If only the first term was present, as it happens in global SUSY, we would had concluded that the scalar potential is ≥ 0 (unitary requires the matrix $Z_I{}^J$ to be positive definite); then $V(\phi)$ would vanish if and only if all $C_I^A = 0$, that is only if supersymmetry is *unbroken*; moreover, since the LHS is proportional to $\delta^A{}_B$, if *one* supersymmetry is unbroken, *all* supersymmetries must be unbroken.

Instead, in local SUSY, we have, in addition, the last term in eqn.(8.12), which is the contribution from the gravitini. It is *negative* semi–definite. One can understand it as the contribution from the *negative–norm* 'longitudinal' (helicity $\pm 1/2$) gravitini whose existence spoils naive arguments based on the positivity of the Hilbert space (just as it happens in any gauge theory, where we cannot have — simultaneously — a covariant formalism and a positive–definite Hilbert space). This new term changes everything with respect the global case. Now it is the matrix

$$V(\phi)\,\delta_A{}^B + 3\,m_{AC}\,m^{CA} \tag{8.15}$$

which is positive semi–definite. If this matrix has a zero eigenvalue, associated, say, to the eigenvector $\epsilon_A^{(0)}$, the supersymmetry generated by $\epsilon_A^{(0)}$ is *unbroken* in AdS/Minkowski space with $\Lambda = V(\phi) \leq 0$. Indeed, if (8.15) has a zero eigenvector $\epsilon_A^{(0)}$:

(1) the 'mass' m of the associated gravitino[20], $\psi_\mu^{(0)}$, is related to the cosmological constant Λ by the 'magical' relation

$$m = \sqrt{-\Lambda/3},$$

i.e. the gravitino is massless in the AdS sense (it propagates along *null* geodesics). This condition guarantees

$$\delta\psi_\mu^{(0)} = \overline{\mathcal{D}}_\mu \epsilon^{0)} = 0. \tag{8.16}$$

(2) from (8.12) $C_A{}^I\, Z_I{}^J C^B{}_J\, \epsilon_B^{(0)} = 0$, and we have automatically

$$\delta_\epsilon^{(0)} \chi^I = 0 \tag{8.17}$$

for all spin–$\frac{1}{2}$ fermions, since $Z_I{}^J$ is positive–definite.

However, now the non–negative matrix (8.15) is not proportional to $\delta_A{}^B$ any longer, and thus it may have $\mathcal{N}_0 < \mathcal{N}$ zero eigenvalues and $\mathcal{N} - \mathcal{N}_0$ non–zero ones. In this case, we have the spontaneous breaking from $\mathcal{N}$–SUSY to $\mathcal{N}_0$–SUSY. This possibility is called *partial super–Higgs* [**162**].

We may also have SUSY breaking, total or partial, at zero–vacumm energy, $V(\phi_{\min}) = 0$, if the negative gravitino contribution $-3m^2$ exactly cancels the positive spin–1/2 contribution $\frac{1}{2} C^\dagger Z C$. In some class of model this

[19] Assuming the Hilbert space has positive norm, as required by unitarity.

[20] 'Mass' between quotes stands for the coefficient of the bilinear $\frac{1}{2}\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu$ in the Lagrangian. It would coincide with the physical mass in the linearized theory around flat Minkowski space, but not otherwise.

cancellation is guaranteed (at the classical level!) by U–duality–like symmetries.

Finally, the breaking scales of the diverse supersymmetries are, *a priori,* unrelated.

8.2. A simpler technique. We can obtain the above result by simpler techniques. We assume to have a background in which the scalar fields ϕ^i are constant and the vector ones vanish. Then we consider the obvious identity

$$\delta_\epsilon \mathcal{L}\Big|_{\text{linear in } \psi_\mu} = 0. \tag{8.18}$$

and insert the definitions (8.10)(8.11),to recover eqn.(8.12) [**163**].

We preferred deduce the formula for $V(\phi)$ starting from positive–mass–like theorems in order to stress that the additional term one gets in local SUSY is strongly related to the geometry of AdS, the Killing spinors, and their integrability conditions, and that its physical motivation can be really *understood* precisely in these geometrical terms.

EXERCISE 8.1. Fill in the details in the direct derivation of eqn.(8.12).

CHAPTER 7

Parallel structures and isometries

The above geometrical results are sufficient to fully determine the Lagrangians of any *ungauged* supergravity. Our next task is to gauge our SUGRA models. In particular, we wish to understand which subgroups $\mathcal{G}$ of the global symmetry group G of a given $\mathcal{N}$–SUGRA may be consistently gauged while preserving the $\mathcal{N}$–extended supersymmetry.

The global symmetry group G acts on the scalars' by an isometry of the Riemannian manifold $\mathcal{M}$. Then our first task is to understand the peculiar properties of the isometry group, $\mathrm{Iso}(\mathcal{M})$, of the manifolds $\mathcal{M}$ which are compatible with a given *extended* supersymmetry. This purely geometric study is done in the present chapter. In the next one we shall use what we learn here to construct all possible supersymmetric gaugings. The hero here is the *momentum map*.

For definiteness, in this chapter we adopt the language of SUSY in $D = 3$ dimensions; the geometrical results are, of course, indepedent of the dimension of spacetime[1].

We recall the basic properties of the relevant geometries: on the Riemannian manifold $\mathcal{M}$ there is a vector bundle isomorphism

$$T^*\mathcal{M} \simeq S \otimes U \;\oplus\; \widetilde{S} \otimes \widetilde{U}$$

where the vector bundles S, $\widetilde{S}$ have structure group $Spin(\mathcal{N})_R$ (Aut_R for general D); they correspond to the irreducible Clifford–modules[2]. Moreover:

- **(rigid susy):** *the bundle S is flat.* This implies the existence of $\mathcal{N}(\mathcal{N}-1)/2$ parallel 2–forms, Σ^{AB}, trasforming in the adjoint of $Spin(\mathcal{N})_R$, and satisfying the algebraic identities inherited from the Clifford algebra $\mathbb{C}l(\mathcal{N}-1) \simeq \mathbb{C}l_0(\mathcal{N})$.
- **(sugra):** *S has a curvature given by $-\frac{1}{2}\Sigma^{AB}$,* where the Σ^{AB}'s are the two–forms representing the generators of $\mathfrak{spin}(\mathcal{N})_R$ under the above isomorphism.

We start with the geometry of $\mathrm{Iso}(\mathcal{M})$ in **(rigid susy):**

1. Momentum maps

1.1. Canonical symplectic structures on $\mathcal{M}$. From the analysis of chapt. 2, we know that $\mathcal{N}$–extended rigid SUSY implies the existence of

[1] And the maps $\varrho_{D,3}$ allow to relate the Lagrangian of any $D > 3$ SUGRA to the corresponding one in $D = 3$.

[2] The distinction between S and $\widetilde{S}$ applies only for $\mathcal{N} = 0 \mod 4$.

$\mathcal{N}(\mathcal{N}-1)/2$ canonical *parallel* 2–forms on $\mathcal{M}$,

$$(\Sigma^{AB})_{ij} = -(\Sigma^{BA})_{ij} := \begin{cases} g_{il}\,(f^B)^l{}_j, & \text{for } A=1 \\ (f^A)^k{}_i\, g_{kl}\,(f^B)^l{}_j & \text{for } A\neq 1, \end{cases} \tag{1.1}$$

where $(f^a)^i{}_j$, $a=2,3,\dots,\mathcal{N}$ are the parallel complex structures.

The 2–forms Σ^{AB} transform in the adjoint of $Spin(\mathcal{N})_R$, and — being parallel — are, in particular, *closed.* Fix the indices A,B and consider $(\Sigma^{AB})_{ij}$ as a $\dim\mathcal{M}\times\dim\mathcal{M}$ (skew–symmetric) matrix in the indices i,j. Then

$$\det\Sigma^{AB} = \det(f^A)\,\det(g)\,\det(f^B) = \det g \neq 0, \tag{1.2}$$

this means that *each* 2–form $\Sigma^{AB}=-\Sigma^{BA}$ satisfies[3]

$$d\Sigma^{AB}=0, \quad \text{and} \quad (\Sigma^{AB})^n = n!\,\mathrm{vol}\neq 0, \tag{1.3}$$

Eqns.(1.3) are the two conditions defining a symplectic 2–form. Thus

General lesson 1.1. *Let $\mathcal{M}$ be the target manifold of a rigid $\mathcal{N}$–extended*[4] *$D=3$ SUSY. On $\mathcal{M}$ there are* $\mathcal{N}(\mathcal{N}-1)/2$ symplectic structures, *Σ^{AB}, transforming in the adjoint representation of $Spin(\mathcal{N})_R$.*

The geometry of a symplectic manifold is equivalent to the Hamilton–Jacobi theory of classical mechanics[5]. In that case, the (single) symplectic structure is given by the closed two–form $dp_i\wedge dq^i$ in phase space. The theorem of Darboux states that any symplectic manifold has *local* coordinates such that the symplectic 2–form has this canonical expression.

In rigid SUSY we have the same geometric structure but replicated $\mathcal{N}(\mathcal{N}-1)/2$ times, and we also have a nice group action rotating the $\mathcal{N}(\mathcal{N}-1)$ symplectic structures. The 2–forms Σ^{AB} are canonically identified with endomorphisms of $T^*\mathcal{M}$, that is with $(\Sigma^{AB})_i^{\ j}\equiv(\Sigma^{AB})_{ik}\,g^{kj}$. These automorphisms generate an algebra with multiplication table

$$\begin{aligned}\Sigma^{AB}\Sigma^{CD} &= (\delta^{BC}\delta^{CD}-\delta^{AC}\delta^{BC})\mathbf{1}+\\ &+\delta^{AC}\Sigma^{BD}-\delta^{AD}\Sigma^{BC}-\delta^{BC}\Sigma^{AD}+\delta^{BD}\Sigma^{AC}+\Sigma^{ABCD},\end{aligned} \tag{1.4}$$

corresponding to the Clifford multiplication in $\mathbb{C}l_0(\mathcal{N})$.

Our next job is to construct the Hamiltonian functions generating the flows corresponding to the multi–symplectic isometries of $\mathcal{M}$.

1.2. Symplectic momentum maps. As discussed in chapt. 2, we cannot gauge in a supersymmetric way <u>all</u> the isometries of $\mathcal{M}$. In rigid SUSY, the symmetry $\mathrm{Aut}_R\equiv Spin(\mathcal{N})_R$ is necessarily rigid, and we can gauge only the isometries which *commutes* with the $Spin(\mathcal{N})_R$ group. These isometries have peculiar geometric properties.

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[3] Here $2n\equiv\dim\mathcal{M}$ which is even since we assume $\mathcal{N}\geq 2$ (otherwise there is no form Σ^{AB}), and vol stands for the canonically normalized volume $2n$–form associated to the metric g_{ij}.

[4] The number of supercharges is $\mathfrak{N}=2\mathcal{N}$. Stated in terms of the absolute number of supercharges, the result is true in *any* number of space–time dimensions D.

[5] General references for symplectic geometry and its relations to the Hamiltonian mechaninics are [**164, 165, 166, 167, 168, 169**].

Let K^i be a Killing vector generating an isometry commuting with $Spin(\mathcal{N})_R$. It should leave invariant the generators of the (global) $Spin(\mathcal{N})_R$ symmetry, namely the 2–forms Σ^{AB}. Thus

$$\pounds_K \Sigma^{AB} = 0. \tag{1.5}$$

A Killing vector satisfying this equation for all A, B will be called a *multi–symplectic* Killing vector. Only multi–symplectic isometries can be gauged while preserving the full $\mathcal{N}$–SUSY. (Of course, one may envisage the case in which this equation holds for a subset of the 2-forms, in such a way that the gauging still preserves an $\mathcal{N}_0 < \mathcal{N}$ SUSY subalgebra). Thus, without loss of generality (for our present purposes[6]) we can limit ourselves to such symmetries. We denote by Iso_0 the subgroup of $\mathrm{Iso}(\mathcal{M})$ of multi–symplectic isometries.

Explicitly, eqn.(1.5) reads

$$0 = \pounds_K \Sigma^{AB} = \big(i_K d + d i_K\big)\Sigma^{AB} = d\left(i_K \Sigma^{AB}\right). \tag{1.6}$$

By the Poincaré lemma, locally on $\mathcal{M}$ there exist functions $\mu^{AB}(K)$ such that

$$i_K \Sigma^{AB} = d\mu^{AB}(K) \qquad \text{(Hamiltonian functions)} \tag{1.7}$$

$\mu^{AB}(K)$ is clearly linear in K. We can interprete $\mu^{AB}(\cdot)$ as a linear map from the Lie algebra $\mathfrak{iso}_0$ to $\mathfrak{spin}(\mathcal{N})$ or

$$\mu \in (\mathfrak{iso}_0)^\vee \otimes \mathfrak{spin}(\mathcal{N}). \tag{1.8}$$

The Hamiltonian functions, seen as a map $\mathfrak{iso}_0 \to \mathfrak{spin}(\mathcal{N})$, is called the *momentum map.*

Momentum maps have a rich algebraic structure. To be concrete, we fix a basis $\{K^{i\,m}\}$ ($m = 1, 2 \ldots, \dim \mathrm{Iso}_0$) of the multi–symplectic Killing vectors, and write $\mu^{AB\,m} \equiv \mu^{AB}(K^m)$. Thus, in coordinates,

$$\partial_i \mu^{AB\,m} = (\Sigma^{AB})_{ij}\, K^{j\,m} \tag{1.9}$$

$$K^{i\,m} = -(\Sigma^{AB})^{ij}\, \partial_j \mu^{AB\,m} \qquad \text{NOT summed over } A\,B! \tag{1.10}$$

From the Clifford algebra (1.4), one has

$$\begin{aligned} D_k\big((\Sigma^{AB})_i^{\ k}\, \mu^{CD\,m}\big) &= (\Sigma^{AB})_i^{\ k} \partial_k \mu^{CD\,m} = (\Sigma^{AB}\Sigma^{CD})_{ik} K^{k\,m} = \\ &= -(\delta^{AC}\delta^{BD} - \delta^{AD}\delta^{BC}) K_i^m + (\Sigma^{ABCD})_{ij} K^{j\,m} + \\ &+ \partial_i\big(\delta^{AC}\,\mu^{BD\,m} - \delta^{AD}\,\mu^{BC\,m} - \delta^{BC}\,\mu^{AD\,m} + \delta^{BD}\,\mu^{AC\,m}\big). \end{aligned} \tag{1.11}$$

Notice that $\mu^{AB\,m}$ is defined up to an additive constant.

1.3. Action of $\mathfrak{iso}_0$ on the momentum maps. Let us compute $\pounds_{K^n}\mu^{AB\,m}$. We can evaluate it into two different ways. First

$$\begin{aligned} \pounds_{K^n}\mu^{AB\,m} &= K^{i\,n}\partial_i \mu^{AB,\,m} = (\Sigma^{AB})_{ij}\, K^{i\,n}\, K^{j\,m} \\ &= (\Sigma^{AB})^{ij}\, \partial_i \mu^{AB\,m}\, \partial_j \mu^{AB\,n} \qquad \text{(Poisson bracket)} \end{aligned} \tag{1.12}$$

[6] Non–multi–symplectic isometries have deep physical consequences which we shall address (if we have time) at the appropriate moment.

or,

$$\partial_i\Big(\pounds_{K^n}\mu^{AB\,m}\Big) = \pounds_{K^n}\partial_i\mu^{AB\,m} = (\Sigma^{AB})_{ij}\pounds_{K^n}K^{j\,m} = f^{mn}{}_p\partial_i(\mu^{AB\,p})$$

that is

$$\pounds_{K^n}\mu^{AB\,m} = f^{mn}{}_p\mu^{AB\,p} + \text{const.}, \tag{1.13}$$

We can reabsorbe the constant by a redefinition[7] of the $\mu^{AB\,m}$. Then the Poisson brachet of the momentum maps is equal to the Lie algebra commutator. (Of course, this is well–known from classical mechanics).

Therefore the momentum map $\mu^\bullet$ is a function

$$\mu^\bullet \colon \mathcal{M} \to \mathfrak{spin}(\mathcal{N}) \otimes \mathfrak{iso}_0$$

transforming according to the representation

$$\mathrm{Adj}\big(\mathfrak{spin}(\mathcal{N})\big) \otimes \mathrm{Adj}\big(\mathfrak{iso}_0\big).$$

1.4. 'Killing' maps. For $\mathcal{N} \geq 5$, the Clifford algebra of the *parallel* tensors contains also *symmetric* 2–tensors not proportional[8] to g_{ij}, *e.g.* Σ^{ABCD}. One may wonder what is the analogue of the momentum map for such *symmetric* parallel tensors. For any parallel 2–tensor T_{ij} one has[9]

$$\begin{aligned}\pounds_K T_{ij} &\equiv K^k D_k T_{ij} + T_{kj} D_i K^k + T_{ik} D_j K^k \\ &= D_i(K^k T_{kj}) + D_j(K^k T_{ik}),\end{aligned} \tag{1.14}$$

if, in addition, T_{ij} is K–invariant, $\pounds_K T_{ij} = 0$, we get

$$D_i(K^k T_{kj}) + D_j(K^k T_{ki}) = 0, \tag{1.15}$$

that is, $T_{ik}K^k$ is a *Killing vector.*

Thus, say, $(\Sigma^{ABCD})_{ij}$ can be seen as a map $\mathfrak{iso}_0 \to \mathfrak{iso}$.

1.5. ∗. Properties of $\mathfrak{iso}_0$ Killing vectors. The $\mathfrak{iso}_0$ condition

$$\pounds_K \Sigma^{AB} = 0 \tag{1.16}$$

may be written as in eqn.(1.14), *i.e.*

$$\begin{aligned}0 &= D_i K^k\,(\Sigma^{AB})_k{}^j + (\Sigma^{AB})_i{}^k\,D^j K_k = \\ &= D_i K^k\,(\Sigma^{AB})_k{}^j - (\Sigma^{AB})_i{}^k\,D_k K^j \quad \Rightarrow\end{aligned} \tag{1.17}$$

PROPOSITION 1.1. *The Killing vector K^i belongs to $\mathfrak{iso}_0$ iff the skew–symmetric endomorphism*

$$(A_K)^j{}_i := -D_i K^j \in \mathrm{End}(T\mathcal{M}) \tag{1.18}$$

commutes *with all the (skew–symmetric) endomorphisms* $(\Sigma^{AB})^i{}_j$.

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[7] This is NOT an elementary result. In general, there is a Lie algebra cocycle which may be an obstruction to a suitable redefinition of the momentum map. See the quoted references.

[8] Thus, by the corollary to Berger's theorem, for $\mathcal{N} \geq 5$ we must have $\mathcal{M}$ flat. We already know this result.

[9] Written in this way, the formula holds both for symmetric and antisymmetric tensors.

Under the isomorphism

$$T\mathcal{M}\otimes\mathbb{C}\simeq S\otimes U\ \oplus\ \widetilde{S}\otimes\widetilde{U}, \tag{1.19}$$

where S, $\widetilde{S}$ are irreducible $\mathbb{C}l_0(\mathcal{N})\otimes\mathbb{C}$–modules[10] (see §. **2.**4.1.3), the skew–symmetric endomorphism A_K is mapped to

$$A_K\mapsto \mathbf{1}\otimes\kappa\ \oplus\ \mathbf{1}\otimes\widetilde{\kappa},\qquad \kappa\in\begin{cases}\mathfrak{so}(U) & S \text{ has } \mathbb{R} \text{ structure}\\ \mathfrak{u}(U) & S \text{ has } \mathbb{C} \text{ structure}\\ \mathfrak{sp}(U) & S \text{ has } \mathbb{H} \text{ structure,}\end{cases} \tag{1.20}$$

and analogously for $\widetilde{\kappa}$, $\widetilde{U}$.

Example. For $\mathcal{N}=2$, with respect the unique complex structure, the Killing vectors in $\mathfrak{iso}_0$ are holomorphic.

Example. For $\mathcal{N}=3$, $\mathcal{M}$ is hyper–Kähler. Using the double index notation, eqn.(1.20) becomes $D_{\alpha a}K_{\beta b}=\epsilon_{\alpha\beta}\,\kappa_{ab}$. with $\kappa_{ab}\in\mathfrak{sp}(\dim\mathcal{M}/2)$. This is precisely eqn.(3.53) of Gaiotto–Witten [**170**].

Corollary 1.1. *The image of the Killing map Σ^{ABCD} is contained in $\mathfrak{iso}_0$ iff $\mathcal{N}\leq 4$.*

Proof. Under the isomorphism, the Σ^{ABCD} map reads

$$\mathbf{1}\otimes\kappa\longmapsto\Sigma^{ABCD}\otimes\kappa \tag{1.21}$$

and the immage is in $\mathfrak{iso}_0$ iff Σ^{ABCD} acts as $\pm\mathbf{1}$ on the irreducible Clifford–module. Notice that $\Sigma^{ABCD}(\mathfrak{iso}_0)\not\subset\mathfrak{iso}_0\ \Rightarrow\ \mathcal{M}$ is flat. □

1.5.1. *More properties of the momentum maps.* Consider the one–form

$$\begin{aligned}(\Sigma^{AB})_i^{\ k}\,\partial_k\mu^{CD\,m}&\in\mathfrak{spin}(\mathcal{N})\otimes\mathfrak{spin}(\mathcal{N})\otimes\mathfrak{iso}_0\otimes\Omega^1\\ &\in\Big(\odot^2\,\mathfrak{spin}(\mathcal{N})\otimes\mathfrak{iso}_0\otimes\Omega^1\Big)\oplus\Big(\wedge^2\,\mathfrak{spin}(\mathcal{N})\otimes\mathfrak{iso}_0\otimes\Omega^1\Big).\end{aligned} \tag{1.22}$$

Comparing with eqn.(1.11) and §. 1.4, the symmetric component (= the projection on the first direct summand in the RHS) is a Killing vector,

$$(\delta^{AD}\delta^{BC}-\delta^{AC}\delta^{BD})\,K^m_i+(\Sigma^{ABCD})_{ij}\,K^{j\,m},$$

while the skew–symmetric part (second summand) is an exact form

$$d(\delta^{AC}\,\mu^{BD\,m}-\delta^{AD}\,\mu^{BC\,m}-\delta^{BC}\,\mu^{AD\,m}+\delta^{BD}\,\mu^{AC\,m}).$$

Therefore, one has

$$\begin{aligned}D_i\big[(\Sigma^{AB})_j^{\ k}\,\partial_k\mu^{CD\,m}\pm(\Sigma^{CD})_j^{\ k}\,\partial_k\mu^{AB\,m}\big]\pm&\\ \pm D_j\big[(\Sigma^{AB})_i^{\ k}\,\partial_k\mu^{CD\,m}\pm(\Sigma^{CD})_i^{\ k}\,\partial_k\mu^{AB\,m}\big]&=0.\end{aligned} \tag{1.23}$$

and so $(\Sigma^{AB})_i^{\ k}D_j\partial_k\mu^{CD\,m}=-(\Sigma^{CD})_j^{\ k}D_i\partial_k\mu^{AB\,m}$. We write

$$\big(H[\mu^{AB\,m}]\big)^i_{\ j}:=D^i\partial_j\mu^{AB\,m}$$

[10] The summand $\widetilde{S}\otimes\widetilde{U}$ is present only for $\mathcal{N}$ even (this corresponds to the fact that in even dimensions we have two spinorial representations (chirality ±1) while in odd dimension there is only one.

for the (covariant) Hessian of the momentum map (seen as an element of $\mathrm{End}(T\mathcal{M})$). Then

LEMMA 1.1. $\boxed{\Sigma^{AB}\, H[\mu^{CD\, m}] = H[\mu^{AB\, m}]\, \Sigma^{CD}}$

1.6. Σ^{AB}–harmonic functions.

DEFINITION 1.1. A function $F(\phi)$ on $\mathcal{M}$ is Σ^{AB}*–harmonic* if there exists a function $G(\phi)$ such that

$$\partial_i F = -(\Sigma^{AB})_{ij}\, \partial^j G. \tag{1.24}$$

If $F(\phi)$ is Σ^{AB}–harmonic, so is $G(\phi)$ (because $(\Sigma^{AB})^2 = -1$). $G(\phi)$ is called *the conjugate function* to $F(\phi)$.

REMARK. A Σ^{AB}–harmonic function $F(\phi)$ is, in particular, harmonic in the usual sense:

$$D^i \partial_i F = -(\Sigma^{AB})_{ij}\, D^i \partial^j G = 0, \tag{1.25}$$

since $(\Sigma^{AB})_{ij} = -(\Sigma^{AB})_{ji}$.

The definition (1.24) is equivalent to

$$\Big(D_i - i(\Sigma^{AB})_i^{\ j} D_j\Big)(F + iG) = 0, \tag{1.26}$$

and we say that the complex function $F + iG$ is Σ^{AB}*–holomorphic.* A Σ^{AB}–harmonic function is the real part of a Σ^{AB}–holomorphic one. If the functions Φ_i are Σ^{AB}–holomorphic, any analytic function $F(\Phi_1, \Phi_2, \dots, \Phi_k)$ is also Σ^{AB}–holomorphic.

The condition of being Σ^{AB}–harmonic can be written as the condition of integrability of eqn.(1.24)

$$0 = D_{[j}\Big((\Sigma^{AB})_{i]}^{\ \ k} \partial_k F\Big) = (\Sigma^{AB})_{[i}^{\ \ k} D_{j]} \partial_k F. \tag{1.27}$$

Defining the Hessian (symmetric) endomorphism $H[F]^i{}_j = D^i \partial_j F$, the above condition reads

$$\Sigma^{AB}\, H[F] = -H[F]\, \Sigma^{AB}, \tag{1.28}$$

that is, F is Σ^{AB}–harmonic if and only if its Hessian $H[F]$ *anticommutes* with Σ^{AB}.

Next, we ask under which condition a function F can be simultaneously harmonic (or holomorphic) with respect to a *set* $\{\Sigma^{A_1,B_1}, \Sigma^{A_2,B_2}, \dots, \Sigma^{A_r,B_r}\}$ of parallel two–forms (with square -1). In particular, we are interested to holomorphic functions with respect to two particular sets: the set of *all* $\mathcal{N}(\mathcal{N}-1)/2$ parallel two forms Σ^{AB}, and the set of the $(\mathcal{N}-1)$ parallel forms of the form Σ^{1a}, *i.e.* the set of the generators of $\mathbb{C}l(\mathcal{N}-1)$. Then

COROLLARY 1.2. *A non–trivial harmonic/holomorphic function with respect to all* Σ^{AB}*'s exists only for* $\mathcal{N} \leq 2$*; a non–trivial harmonic/holomorphic function with respect to the* $\Sigma^{1,a}$*,* $(a = 2, 3, \dots, \mathcal{N})$ *exists only for* $\mathcal{N} \leq 3$.

Non–trivial means that the Hessian $D_i \partial_j F$ is not identically zero. Indeed the Hessian has to anticommute with all Σ^{AB}. But if it anticommutes with, say, Σ^{12} and Σ^{13}, it automatically *commutes* with $\Sigma^{23} = \Sigma^{12}\Sigma^{13}$.

REMARK. Of course, we can fix one parallel 2–form, say Σ^{AB}, use it to define a complex structure, prove that it is integrable, that is that there exist complex local coordinates x^i such that Σ^{AB} takes the special form:

$$(\Sigma^{AB})_{i\bar{j}} = i g_{i\bar{j}}, \quad (\Sigma^{AB})_{\bar{i}j} = -i g_{\bar{i}j}, \quad (\Sigma^{AB})_{ij} = (\Sigma^{AB})_{\bar{i}\bar{j}} = 0. \tag{1.29}$$

In these coordinates, a Σ^{AB}–harmonic function is just the real part of a holomorphic function of the x^i. Thus, for a *single* two–form, our notion is just the standard one. We are, however, also interested in the interplay between different choices of complex structures Σ^{AB}.

1.7. Σ–harmonicity of momentum maps. From eqn.(1.11) one has

$$\begin{aligned}(\Sigma^{AB})_i{}^j \partial_j \mu^{CD} &= \\ &= -2\,\delta^{A[C}\delta^{D]B}\, K_i - 4\,\delta^{[A[C}\,\partial_i \mu^{D]B]} + (\Sigma^{ABCD})_i{}^j\, K_j.\end{aligned} \tag{1.30}$$

Taking, say, $D = A$ with A, B and C all distinct

$$(\Sigma^{AB})_i{}^j \partial_j \mu^{CA} = -\partial_i \mu^{BC}, \tag{1.31}$$

and therefore

LEMMA 1.2. *The following momentum maps are Σ^{AB}–harmonic:*
(1) $\mu^{AC} = -\mu^{CA}$ *with* $C \neq B$;
(2) $\mu^{CB} = -\mu^{BC}$ *with* $C \neq A$.

Moreover

$$\boxed{\mu^{AC} + i\mu^{BC}, \quad C \neq A, B, \ \textit{is}\ \Sigma^{AB}\text{–holomorphic}} \tag{1.32}$$

2. T–tensors I

The momentum map $\mu^{\bullet}$ is a function from $\mathcal{M}$ to $\mathfrak{spin}(\mathcal{N}) \otimes \mathfrak{iso}_0$ which is not $\mathfrak{iso}_0$–gauge invariant; rather, it transforms in the adjoint representation. In order to construct a *gauge invariant* theory, we need to work with gauge–invariant quantities[11]. Therefore we need to study the gauge–invariant functions on $\mathcal{M}$.

The peculiar geometry of the isometries of a ‘supersymmetric’ manifold $\mathcal{M}$ gives us a set of natural functions directly related to the symmetries we wish to gauge, namely the momentum maps $\mu^{AB\,m}$. Then it is natural to construct the gauge–invariant expressions starting from these basic, God–given, functions.

Let $\mathfrak{g} \subset \mathfrak{iso}_0$ be the Lie subalgebra we wish to gauge. The simplest way to construct a $\mathfrak{g}$–invariant function out of the $\mu^{AB\,m}$’s is to take an *invariant symmetric tensor* in $\odot^2\mathfrak{g} \subset \odot^2\mathfrak{iso}_0$, l_{mn}, and consider the following tensor in $\odot^2\mathfrak{spin}(\mathcal{N})$

$$T^{AB,CD} = \mu^{AB\,m}\, l_{mn}\, \mu^{CD\,n}. \tag{2.1}$$

$T^{AB,CD}$ is $\mathfrak{g}$–invariant by construction. In the SUGRA jargon, it si called the *T–tensor associated with the invariant tensor* $l_{mn} \in \odot^2\mathfrak{g}$. $T^{AB,CD}$ is basically the invariant ‘square’ of the momentum map. This object has a

[11] Think, for instance to the superspace approach. The pre/super–potentials should be gauge–invariant functions on $\mathcal{M}$.

long and glorious history in both geometry and physics (for the particular case of just one symplectic structure, *i.e.* $\mathcal{N}=2$). Atiyah and Bott [**172, 32**] showed that (for $\mathcal{G}$ simple) the $T^{AB,AB}$ is a *$\mathcal{G}$–equivariant perfect Morse function*[12]. This should not be a surprise. Recall the relation between SUSY and Morse theory we discussed in chapt. 2. An informal proof of the fact that the diagonal entries of the T–tensor is a *perfect equivariant Morse function* can be obtained by applying the logic of chapt. 2 to the gauged SUSY models we shall construct in the next chapter out of $T^{AB,CD}$. Stated differently, to gauge an isometry of $\mathcal{M}$, we need gauge–invariant functions on $\mathcal{M}$ which *do encode* the $\mathcal{G}$–covariant topology of $\mathcal{M}$ in the correct way (SUSY vacua $\equiv$ equivariant cohomology classes). *The T–tensor has precisely these properties!!*

Witten and others used this function to get *exact quantum* expression for the path–integral (the non–Abelian localization formula [**173, 174, 175, 176**]), generalizing the $U(1)$ result of Duistermaat and Heckman [**177, 178**]. Again, these nice quantum properties, are expected on grounds of (extended) supersymmetry. Hence

General lesson 2.1. *The gauging of a supersymmetric theory is totally encoded in the corresponding T–tensor.*

Then we expect that the $\mathcal{N}$–SUSY completion of the minimal gauge coupling (*i.e.* the induced Yukawa terms and scalar potential) take a universal form in terms of $T^{AB,CD}$.

2.1. Properties of the T–tensor. Lemma 1.2 yields

Lemma 2.1. *Let $A\neq B$. Then the expressions*[13]

$$T^{AC,AD}-T^{BC,BD}, \qquad C\neq A,B,\ D\neq A,B \tag{2.2}$$

$$T^{AC,BD}+T^{BC,AD}, \qquad C\neq A,B,\ D\neq A,B \tag{2.3}$$

are Σ^{AB}–harmonic functions. In particular, the component

$$T^{AC,BC} \qquad C\neq A,B \tag{2.4}$$

is a Σ^{AB}–harmonic function. If A, B, C are all distinct

$$(\Sigma^{AB})_i^{\ j}\,\partial_j\big(T^{AC,AC}-T^{BC,BC}\big)=2\,\partial_i T^{AC,BC}. \tag{2.5}$$

The T–tensors satisfy a number of quite intricate differential identities. We write only the *very simplest.*

Lemma 2.2. *One has the identities:*

$$(\Sigma^{AB})_i^{\ k}\left\{K_k^m\,l_{mn}\,K_j^n+D_j\partial_k\left(\frac{1}{2}\,T^{AB,AB}\right)\right\}+(i\leftrightarrow j)=0. \tag{2.6}$$

[12] For a redable summary see Atiyah [**178**] §. 5.

[13] In the statement of the present lemma there is NO sum over the repeated indices!

NOT summed over A, B*!*

$$(\Sigma^{AB})_i{}^k\,(\Sigma^{AB})_j{}^h\,\left[K_k^m\,l_{mn}\,K_j^n + D_k\partial_h\left(\frac{1}{2}T^{AB,AB}\right)\right] =$$

$$= K_i^m\,l_{mn}\,K_j^n + D_i\partial_j\left(\frac{1}{2}T^{AB,AB}\right),$$

$$NOT\ summed\ over\ A,B! \tag{2.7}$$

Proof. The second identity is an immediate consequence of the first one: just contract it with $(\Sigma^{AB})_l{}^j$. For the second one, consider

$$(\Sigma^{AB})_i{}^k\,K_k^m\,l_{mn}\,K_j^n + (i \leftrightarrow j) = (\partial_i\mu^{AB\,m})\,l_{mn}\,K_j^n + (i \leftrightarrow j) =$$

$$= D_i\Big(\mu^{AB\,m}\,l_{mn}\,K_j^n\Big) + (i \leftrightarrow j) =$$

$$= -D_i\Big(\mu^{AB\,m}\,l_{mn}\,(\Sigma^{AB}\Sigma^{AB})_j{}^k K_k^n\Big) + (i \leftrightarrow j) =$$

$$= -(\Sigma^{AB})_j{}^h D_i\Big(\mu^{AB\,m}\,l_{mn}\,\partial_h\mu^{AB\,n}\Big) + (i \leftrightarrow j) +$$

$$= -(\Sigma^{AB})_j{}^h D_i\partial_h\left(\frac{1}{2}\mu^{AB\,m}\,l_{mn}\,\mu^{AB\,n}\right) + (i \leftrightarrow j) =$$

$$= -(\Sigma^{AB})_i{}^k D_j\partial_k\left(\frac{1}{2}T^{AB,AB}\right) + (i \leftrightarrow j).$$

□

3. Target space isometries in supergravity

In sects. 1, 2 we exploited the symplectic structures of the target space in rigid SUSY to construct natural $\mathcal{G}$–invariant functions on $\mathcal{M}$ which encode all the relevant aspects of the isometry group to be gauged. These functions, the components of the T–tensor, are perfect[14] geometric object with a crucial rôle in both the symplectic geometry and in the quantization procedure. This magics happens because $\mathcal{M}$ was a (multi–)symplectic manifold, and the symplectic forms Σ^{AB} have a peculiar relation to the way extended SUSY acts on $T\mathcal{M}$.

Our next problem is to gauge subgroups of Iso($\mathcal{M}$) in the *local* SUGRA case. After what we learned above in rigid supersymmetry, the obvious idea is to try to generalize the momentum maps $\mu^{AB\,m}$ and the T–tensors to the local situation. *But alas !!* In SUGRA, $\mathcal{M}$ is certainly not symplectic!

In the literature there are a few generalization of the momentum map idea (*e.g.* ref. [**179**]). But we are not looking for *any* generalization of $\mu^\bullet$, we are looking for a generalization intrinsically related to the geometry and physics of SUGRA.

Happily such a generalization exists! It appears that we can define naturally $\mu^\bullet$ and $T^{\bullet,\bullet}$ precisely for two broad classes of manifolds: *(i)* the (multi)symplectic ones (rigid SUSY); and *(ii)* those which we call 'SUGRA manifolds' (see the introduction to the present chapter for the specifics).

This remarkable fact is deeply rooted in the parallel structures over the SUGRA manifolds (in particular the universal, tt^*–like curvature of the Aut_R

[14] In the technical sense!!

bundles). In order to understand the issue geometrically, we have to consider the relations of the holonomy and isometry groups.

4. Holonomy *vs.* Isometries

In sect.1 we deduced the interplay between the holonomy of $\mathcal{M}$ (encoded in the system of parallel form, according to the FUNDAMENTAL PRINCIPLE 1.1 of chapt. 3) by hand using well–known results in symplectic geometry (*alias* Hamilton–Jacobi mechanics). In facts, there exists a more general theory relating the holonomy and isometry groups of a Riemannian manifold of which the symplectic case is just a special instance.

4.1. The derivation A_X. We begin by introducing[15] a new derivation acting on the tensor fields (in particular, forms) over a Riemannian manifold $\mathcal{M}$. Let X be any vector field. Set

$$\boxed{A_X = \pounds_X - D_X} \tag{4.1}$$

Since the difference of two derivations is again a derivation, so is the operator A_X. Let f be a smooth function: $A_X f = (\pounds_X - D_X)f = 0$, so A_X is a derivation which is trivial on functions and hence it is represented by a tensor (that is, it is an algebraic rather than a differential operator).

PROPOSITION 4.1 (see [**189, 191**]). *$\mathcal{M}$ is Riemannian. X, Y any vector fields on $\mathcal{M}$. Then*

$$A_X Y = -D_Y X. \tag{4.2}$$

Let K be a Killing vector. Then, for any vector field X

$$D_X(A_K) = R(K, X). \tag{4.3}$$

If K^1 and K^2 are both Killing vectors,

$$A_{[K^1,K^2]} = [A_{K^1}, A_{K^2}] + R(K^1, K^2) \tag{4.4}$$

(here $R(\cdot,\cdot)$ is the Riemann curvature).

PROOF. **(1)** One has

$$\begin{aligned} A_X Y \equiv \pounds_X Y - D_X Y &= && [\text{by definition of the torsion } T(X,Y)] \\ &= [X,Y] - D_X Y - \big(T(X,Y) - D_X Y + D_Y X + [X,Y]\big) \\ &= -D_Y X - T(X,Y) = && [\text{since the Levi–Civita connection is torsionless}] \\ &= -D_Y X. \end{aligned}$$

(2) By the definition of curvature:

$$\begin{aligned} R(K,X) &= [D_K, D_X] - D_{[K,X]} = \big[\pounds_K - A_K, D_X\big] - D_{[K,X]} \\ &= \big[\pounds_K, D_X\big] - D_{[K,X]} - [A_K, D_X]. \end{aligned} \tag{4.5}$$

For a Killing vector, however, one has also[16]

$$\big[\pounds_K, D_X\big] = D_{[K,X]} \tag{4.6}$$

---

[15] References for this section are: chapt. VI of ref. [**189**], expecially PROPOSITIONS 2.5 and 2.6 and §. 4; ref. [**191**], chapt. II; and ref. [**94**], chapt. 26 §. 4.

[16] The Levi–Civita connection D_X is constructed out ot the metric, and hence should transform covariantly under isometries.

Thus

$$R(K,X)=-[A_K,D_X]\equiv D_X(A_K). \tag{4.7}$$

For reference, we write this equation in coordinates

$$\boxed{D_iD_jK_k=-R_{jkil}K^l} \tag{4.8}$$

(3) Let K^1, K^2 be Killing fields

$$\begin{aligned} 0 &= \pounds_{K^1}\pounds_{K^2}-\pounds_{K^2}\pounds_{K^1}-\pounds_{[K^1,K^2]}= \\ &=(D_{K^1}+A_{K^1})(D_{K^2}+A_{K^2})-(D_{K^2}+A_{K^2})(D_{K^1}+A_{K^2})-(D_{[K^1,K^2]}+A_{[K^1,K^2]}) \\ &=R(K^1,K^2)+D_{K^1}A_{K^2}-D^2_KA_{K^1}+[A_{K^1},A_{K^2}]-A_{[K^1,K^2]} \end{aligned} \tag{4.9}$$

The identity (4.8) yields

$$D_{K^1}A_{K^2}=-R(K^1,K^2),\qquad D_{K^2}A_{K^1}=-R(K^2,K^1) \tag{4.10}$$

so (4.9) becomes

$$[A_{K^1},A_{K^2}]=A_{[K^1,K^2]}+R(K^1,K^2). \tag{4.11}$$

□

Example. In sect. 1.5 we studied the properties of the derivation A_K in the case of a (pluri)symplectic manifold. One may check that the properties of A_K we shall deduce in the present section hold, in particular, for the symplectic A_K.

Example. Let $\mathcal{M}=G$ be a group manifold (or, more generally, a symmetric space). Take the left–invariant vectors X^a as a basis of $T\mathcal{M}$, Then from proposition 1.1 of chapt.5 we have

$$A_{X^a}X^b=-D_{X^b}X^a=-\frac{1}{2}[X^b,X^a]\equiv\frac{1}{2}[X^a,X^b]. \tag{4.12}$$

So (up to normalization) A_X corresponds to the adjoint action of the Lie algebra $\mathfrak{g}$.

4.2. Kostant's theorem. The first general result is the following

Theorem 4.1 (Kostant [**186, 189**]). *Let $\mathfrak{hol}$ be the holonomy algebra of the Riemannian manifold $\mathcal{M}$ and $\mathfrak{n}(\mathfrak{hol})$ its normalizer in $\mathfrak{so}(\dim\mathcal{M})$. Let K be a Killing vector and let $A_K\in\mathrm{End}(T\mathcal{M})$ be the derivation $\pounds_K-D_K$. Let $\phi\in\mathcal{M}$ be any point. Then*[17]

$$A_K\big|_\phi\in\mathfrak{n}(\mathfrak{hol})\big|_\phi\subset\mathrm{End}(T_\phi\mathcal{M}). \tag{4.13}$$

Proof. By proposition (4.2), $(A_K)_{ij}\equiv-D_jK_i$ is a 2–form. Decompose the 2–form $(A_K)_{ij}|_\phi\in\mathfrak{so}(\dim\mathcal{M})$ as

$$A_K=B_K+E_K \tag{4.14}$$

where $B_K\in\mathfrak{hol}\subset\wedge^2T\mathcal{M}$ and $E_K\in\mathfrak{hol}^\perp\subset\wedge^2T\mathcal{M}$. We claim that the 2–form E_K is *parallel.* Indeed,for any vector X

$$D_XA_K=R(K,X)\in\mathfrak{hol}\subset\wedge^2T\mathcal{M}, \tag{4.15}$$

[17] $\mathfrak{n}(\mathfrak{hol})|_\phi\equiv\mathfrak{n}(\mathfrak{hol}|_\phi)$ where $\mathfrak{hol}|_\phi$ is the Lie algebra of the holonomy group viewed as the group of parallel transports along loops starting and ending at the point ϕ.

or, in coordinates,

$$D_k(A_K)_{ij} = D_k D_i K_j = -R_{ijkl} K^l \in \mathfrak{hol} \subset \wedge^2 T\mathcal{M}, \tag{4.16}$$

where we used the the Ambrose–Singer theorem. Eqn.(4.15) implies

$$D_k(E_K)_{ij} = 0, \tag{4.17}$$

i.e. E_K is *parallel.* By the FUNDAMENTAL PRINCIPLE 1.1 of chapt. 3, E_K commutes with the action of $\mathfrak{hol}$. Hence eqn.(4.13) is proven. □

4.3. First consequences of Kostant's theorem. Theorem 4.1 is the fundamental link between $\mathfrak{hol}$ and $\mathfrak{iso}$. Let us start to see what it implies for the isometry groups of the SUGRA manifolds.

Recall that, in a SUGRA manifold, $\mathfrak{hol} = \mathfrak{spin}(\mathcal{N}) \oplus \mathfrak{h}'$, so

COROLLARY 4.1. *$\mathcal{M}$ a SUGRA manifold. There exist functions A_K^{AB} such that*

$$-D_j K_i \equiv (A_K)_{ij} = \frac{1}{4}\, A_K^{AB}\, (\Sigma^{AB})_{ij} + (H_K)_{ij} \tag{4.18}$$

with $H_K \in \mathfrak{h}'$. In particular,

$$\big[H_K, \Sigma^{AB}\big] = 0. \tag{4.19}$$

COROLLARY 4.2. *There is a map $\sigma\colon \mathfrak{iso}(\mathcal{M}) \to \mathfrak{spin}(\mathcal{N})$ so that*[18]

$$\boxed{\pounds_{K^m}\Sigma = \mathrm{adj}_{\sigma(K^m)}\Sigma} \tag{4.20}$$

or, explicitly,

$$\pounds_{K^m}(\Sigma^{AB})ij = \sigma_{K^m}^{AC}\,(\Sigma^{CB})_{ij} - (\Sigma^{AC})_{ij}\,\sigma_{K^m}^{CB} \tag{4.21}$$

for certain $\sigma_{K^m}^{AB} = -\sigma_{K^m}^{BA} \equiv \sigma(K^m)$ (see the proof for their explicit expression). Moreover,

$$\boxed{\pounds_{K^m} Q_i^{AB} = -\mathcal{D}_i\,\sigma_{K^m}^{AB}} \tag{4.22}$$

PROOF. From the previous lemma, one has[19],

$$\begin{aligned}
\pounds_K(\Sigma^{AB})_i^{\ j} &= K^k D_k(\Sigma^{AB})_i^{\ j} + (D_i K^k)(\Sigma^{AB})_k^{\ j} + (\Sigma^{AB})_i^{\ k}(D_j K^k) = \\
&= -(K^k Q_k)^{AC}(\Sigma^{CB})_i^{\ j} + (\Sigma^{AC})_i^{\ j}(K^k Q_k)^{CB} + \\
&\qquad + \frac{1}{4}\, A_K^{CD}\Big(\Sigma^{CD}\Sigma^{AB} - \Sigma^{AB}\Sigma^{CD}\Big)_i^{\ j} = \\
&= \sigma_K^{AC}\,(\Sigma^{CB})_i^{\ j} - (\Sigma^{AC})_i^{\ j}\,\sigma_K^{CB},
\end{aligned} \tag{4.23}$$

where

$$\sigma_K^{AB} = A_K^{AB} - K^i Q_i^{AB}. \tag{4.24}$$

To get (4.22), recall that $-\frac{1}{2}\Sigma$ is the field–strenght of the *Spin*($\mathcal{N}$) connection Q_i^{AB}. We recognize in eqn.(4.21) the gauge transformation of the *Spin*($\mathcal{N}$) field strenght under an infinitesimal *Spin*($\mathcal{N}$) gauge transformation of parameter $-\sigma_{K^m}^{AB}$. Eqn.(4.22) is the usual gauge transformation of the connection. □

[18] Compare with eqn.(5.1) for the RIGID case!

[19] Recall that Q_i^{AB} is the canonical *Spin*($\mathcal{N}$) connection defined by SUGRA.

COROLLARY 4.3. *$\mathcal{M}$ an $\mathcal{N} \geq 3$* SUGRA *Riemannian manifold. Let $P(\cdot)$ a homogeneous* invariant *polynomial on $\mathfrak{spin}(\mathcal{N})$ of degree k (i.e. a generalized Casimir invariant),*

$$P(L) = P_{A_1B_1,A_2B_2\cdots A_kB_k}\, L^{A_1B_1}\, L^{A_2B_2}\, \cdots\, L^{A_kB_k}. \tag{4.25}$$

Consider the parallel *$2k$–form $P(\Sigma)$*

$$P(\Sigma) = P_{A_1B_1A_2B_2\cdots A_kB_k}\, \Sigma^{A_1B_1} \wedge \Sigma^{A_2B_2} \wedge \cdots \wedge \Sigma^{A_kB_k}. \tag{4.26}$$

Let K be ANY *Killing vector on $\mathcal{M}$. Then*

$$\pounds_K P(\Sigma) = 0. \tag{4.27}$$

REMARK. If $\mathcal{M}$ is compact, this result already follows from the theorem in the footnote of page 50. Indeed, a *parallel* form is *a fortiori* a harmonic form.

REMARK. Since $-\frac{1}{2}\Sigma$ is the $Spin(\mathcal{N})$ curvature, the closed $2k$–form

$$\left(-\frac{1}{2}\right)^k P(\Sigma) \tag{4.28}$$

represents a *characteristic class* [**84, 188, 189**] of the $Spin(\mathcal{N})$ bundle $M^{\oplus p} \oplus \widetilde{M}^{\oplus q}$, $(p + q = \dim \mathcal{M}/\mathbf{N}(\mathcal{N}))$, see chapt. 2.

5. The rigid case revisited. Superconformal gaugings

Let us reconsider the rigid SUSY manifolds in the light of Kostant's theorem. If $\mathcal{M}$ is a (pluri)symplectic manifold, the Killing vectors generating $\mathfrak{iso}_0$ are precisely those with $E_K = 0$. Indeed, from the definition of E_K in terms of $\pounds_K - D_K$, and the parallel property of the 2–forms Σ^{AB}, we get

$$\boxed{\pounds_K \Sigma^{AB} = [E_K, \Sigma^{AB}]} \tag{5.1}$$

Then, comparing with the theorem, we recover our proposition 1.1 of sect. 1.5.

The parallel tensors E_K correspond to the Lie subalgebra $\mathfrak{iso} \ominus \mathfrak{iso}_0$. To rule out degenerate situations, we consider manifolds $\mathcal{M}$ having $\mathfrak{hol}$ equal to the centralizer of $\mathfrak{spin}(\mathcal{N})$ in $\mathfrak{so}(\dim \mathcal{M})$; that is we require that the only parallel tensors are those in the canonical Clifford algebra $\mathbb{C}l(\mathcal{N} - 1)$. Comparing with eqn.(1.20), for a *general* Killing vector K,

$$A_K \mapsto \underbrace{\Big(\mathbf{1} \otimes \kappa \;\oplus\; \mathbf{1} \otimes \widetilde{\kappa}\Big)}_{B_K} + \underbrace{\Big(\sigma \otimes \mathbf{1} \;\oplus\; \widetilde{\sigma} \otimes \mathbf{1}\Big)}_{E_K} \tag{5.2}$$

$$\text{with } \sigma,\, \widetilde{\sigma} \in \mathfrak{spin}(\mathcal{N}) \text{ and } \tau,\, \widetilde{\tau} \text{ as in eqn.(1.20).} \tag{5.3}$$

E_K can be seen as a map

$$E_\bullet \colon \mathfrak{Lie}\Big(\mathrm{Iso}(\mathcal{M})/\mathrm{Iso}_0(\mathcal{M})\Big) \to \mathfrak{spin}(\mathcal{N}). \tag{5.4}$$

Since $\mathfrak{iso}_0$ is the kernel of the map $E_\bullet$ in eqn.(5.4), we have:

COROLLARY 5.1. *$\mathcal{M}$ a rigid $\mathcal{N}$–*SUSY *manifold. Assume that the only parallel tensors are those in the Clifford algebra. Then*

$$\mathfrak{iso} \subseteq \mathfrak{iso}_0 \oplus \mathfrak{spin}(\mathcal{N}). \tag{5.5}$$

In facts we have a stronger result

Proposition 5.1. *$\mathcal{M}$ a rigid $\mathcal{N}$–SUSY manifold. Assume*

$$\mathfrak{iso} \supseteq \mathfrak{iso}_0 \oplus \mathfrak{spin}(\mathcal{N}). \tag{5.6}$$

then $\mathcal{M}$ is a metric cone. Conversely: If a rigid $\mathcal{N}$–SUSY manifold is a metric cone then its isometry algebra satisfies eqn.(5.6)*.*

Proof. Let K be a Killing vector whose $\mathfrak{iso}_0$ component vanishes (*i.e.* $B_K \equiv 0$). We decompose A_K into a basis of $\mathfrak{spin}(\mathcal{N})$:

$$D_i K_j = \frac{1}{2}(E_K)^{AB}\,(\Sigma^{AB})_{ij}. \tag{5.7}$$

Since both $D_i K_j$ and Σ^{AB} are parallel tensors, the coefficients $(E_K)^{AB}$ are numerical constants. Then, by taking linear combinations with *constant* coefficients, we may construct a basis of $(\mathfrak{iso} \ominus \mathfrak{iso}_0)$ Killing vectors $\{K^{AB}\}$ such that

$$D_i K^{AB}_j = (\Sigma^{AB})_{ij} \equiv (E_{K^{AB}})_{ij} \tag{5.8}$$

$$\Rightarrow \quad \pounds_{K^{AB}} \Sigma^{CD} = [\Sigma^{AB}, \Sigma^{CD}] = f^{AB\;CD}{}_{EF}\,\Sigma^{EF}. \tag{5.9}$$

Let

$$\begin{aligned} V_i &= \frac{1}{\mathcal{N}(\mathcal{N}-1)}(\Sigma^{AB})_i{}^k\, K^{AB}_k = \frac{1}{\mathcal{N}(\mathcal{N}-1)} D_i(K^{AB\,k})\,K^{AB}_k = \\ &= \partial_i\left(\frac{1}{2\mathcal{N}(\mathcal{N}-1)}K^{AB\,k}\,K^{AB}_k\right) \equiv \partial_i V. \end{aligned} \tag{5.10}$$

From the Clifford algebra, one gets

$$g_{ij} = D_i V_j = D_i \partial_j V, \tag{5.11}$$

so V_i is a *conformal* Killing vector, $\pounds_V g_{ij} = 2 g_{ij} \Rightarrow \pounds_V V = 2V$ *i.e.* $g^{ij}\partial_i V \partial_j V = 2V$.

A vector field V_i such that $g_{ij} = D_i V_j$ is called *concurrent.* A Riemannian manifold is a metric cone if and only if it has a concurrent vector field, cfr. **theorem III.5.4** and **theorem III.5.5** in ref. [**46**]. The converse it obvious. If $\mathcal{M}$ is a cone, it has a concurrent vector V_i. Then consider the vectors $K^{AB}_i = (\Sigma^{AB})_{ij} V^j$. Then

$$D_i K^{AB}_j = (\Sigma^{AB})_{jk}\, D_i V^k = (\Sigma^{AB})_{ji} \quad \text{antisymmetric in } i \leftrightarrow j, \tag{5.12}$$

so the K^{AB}_i are $\mathfrak{spin}(\mathcal{N})$ Killing vectors. □

Corollary 5.2 (for the *cognoscenti*). *From eqn.5.11 we see that in the Kähler case ($\mathcal{N} = 2$), V is a* preferred *Kähler potential. In the hyperKähler case ($\mathcal{N} = 3, 4$), V is the* hyperKähler potential *which exists if and only if $\mathcal{M}$ is a cone* [**180**]*. Thus,* inter alia *we have a formula for the Kähler potential of any $\mathcal{N} = 2, 3, 4$ conical model*

$$\boxed{V = \frac{1}{2\mathcal{N}(\mathcal{N}-1)} K^{AB\,k} K^{AB}_k} \tag{5.13}$$

Remark. The special case $\mathcal{N} = 3$ in the above proposition is a central result in the theory of 3–Sasakian manifolds: *$\mathcal{M}$ hyperKähler with a $Spin(3)$ isometry rotating the 3 complex structures $\Leftrightarrow$ $\mathcal{M}$ is a metric cone over a 3–Sasakian manifold.* See [**180**] and [**181**] (expecially **proposition 1.6** and

theorem A). In fact our result is a bit stronger than those in the math literature.

5.1. The *Spin*($\mathcal{N}$) 'momentum map'. From eqn.(5.9) we see that $\pounds_{K^{AB}}\Sigma^{CD}+([AB]\leftrightarrow[CD])=0$. Writing explicitly the Lie derivative, this is

$$d\Big(i_{K^{AB}}\Sigma^{CD}+i_{K^{CD}}\Sigma^{AB}\Big)=0. \tag{5.14}$$

This allows us to define a new $\mathfrak{spin}(\mathcal{N})$ 'momentum map' as the function $M^{AB\;CD}$ such that

$$i_{K^{AB}}\Sigma^{CD}+i_{K^{CD}}\Sigma^{AB}=2\,dM^{AB\;CD}. \tag{5.15}$$

PROPOSITION 5.2. *The Spin*($\mathcal{N}$) *momentum map is given by*

$$\boxed{M^{AB\;CD}=-\frac{1}{2}\,K^{AB\,i}K_i^{CD}} \tag{5.16}$$

PROOF. From eqn.(5.8) one has

$$(i_{K^{AB}}\Sigma^{CD})_j\equiv K^{AB\,i}(\Sigma^{CD})_{ij}=-K^{AB\,i}D_jK_i^{CD}. \tag{5.17}$$

$\square$

5.2. The $\mathfrak{iso}_0(\mathcal{N})$ momentum map for conical susy–manifolds.

PROPOSITION 5.3. *Let $\mathcal{M}$ a* conical *rigid $\mathcal{N}$–*SUSY *manifold. Let K^m be a Killing vector belonging to $\mathfrak{iso}_0$. Its momentum map is given by*

$$\boxed{\mu^{AB\,m}=-\frac{1}{2}K^{AB\,j}\,K_j^m} \tag{5.18}$$

PROOF.

$$\begin{aligned}\partial_i(K^{AB\,j}\,K_j^m)&=(\Sigma^{AB})_i^{\;j}\,K_j^m+K^{AB\,j}\,D_iK_j^m=\\&=(\Sigma^{AB})_i^{\;j}\,K_j^m-K^{AB\,j}\,D_jK_i^m=\\&=(\Sigma^{AB})_i^{\;j}\,K_j^m-(K^{AB\,j}\,D_iK_j^m-K^{m\,j}\,D_iK_j^{AB})-K^{m\,j}\,D_iK_j^{AB}\\&=(\Sigma^{AB})_i^{\;j}\,K_j^m-K^{m\,j}\,D_iK_j^{AB}=\\&=(\Sigma^{AB})_i^{\;j}\,K_j^m-K^{m\,j}\,(\Sigma^{AB})_{ij}=2(\Sigma^{AB})_i^{\;j}\,K_j^m,\end{aligned} \tag{5.19}$$

where we used $\pounds_{K^{AB}}K^m=0$. $\square$

5.3. Physical meaning of E_K: Superconformal invariance. The subgroup of $\mathrm{Iso}(\mathcal{M})/\mathrm{Iso}_0(\mathcal{M})$ acts on the fermions (infinitesimally) trough $E_K\in\mathfrak{spin}(\mathcal{N})$, *i.e.* trough an automorphism of the SUSY algebra. This corresponds to the situation in which the $Spin(\mathcal{N})_R$ rotations by themselves are not symmetries of the theory, but there is another symmetry group acting as an isometry of $\mathcal{M}$ plus a $Spin(\mathcal{N})_R$ transformation, which *is* a true R–symmetry under which the supercharges transform non–trivially. This is certainly the case if the rigid model is *superconformal.* In this case we have a full $Spin(\mathcal{N})$ R–symmetry group, under which the supercharges transform in the vector representation. Thus the hypothesis of proposition 5.1 are fulfilled, and $\mathcal{M}$ is a cone:

THEOREM 5.1. *In rigid* SUSY *the theory is superconformal* $\Rightarrow \mathcal{M}$ *is a metric cone.*

In chapt. 2 we saw that, conversely, if $\mathcal{M}$ is a cone the model is superconformal.

5.4. *. Gauging rigid susy *vs.* gauging sugra. From the above theorem we can understand a *major* difference between rigid and local SUSY. In the rigid case we have typically[20]

$$\mathfrak{n}(\mathfrak{hol}) \simeq \mathfrak{spin}(\mathcal{N}) \oplus \mathfrak{hol} \tag{5.20}$$

and, *a priori* A_K can have a non–vanishing second–projection $E_K \in \mathfrak{spin}(\mathcal{N})$. If $E_K \neq 0$, necessarily $\pounds_K \Sigma \neq 0$ and $K \notin \mathfrak{iso}_0$. So (in general) only a subgroup of the isometry group of $\mathcal{M}$ acts 'holomorphically' with respect to all the complex structures f^a, and only this particular subgroup can be gauged while preserving SUSY. On the contrary, in local SUSY (SUGRA) we have

$$\mathfrak{n}(\mathfrak{hol}) \simeq \mathfrak{hol}, \tag{5.21}$$

hence for ALL Killing vectors $E_K \equiv 0$; thus all the isometry group may (*a priori*) be gauged.

In ref.[**50**] (see discussion after their eqn.(2.31) or their §. 4.2) it is stated that the $\mathcal{N} = 2$ case is exceptional in what only holomorphic isometries can be gauged. This statement is correct, but geometrically quite subtle.

A Kähler manifold may have — in general — both holomorphic and non–holomorphic isometries and, surely enough, gauging the non–holomorphic ones cannot be consistent with SUSY (either rigid or local). So the claim is obvious from this point of view, except that our arguments above apply to all $\mathcal{N} \geq 2$. How we solve the paradox?

The point is that our arguments apply, without further specification, to the *strict* $\mathcal{N} = 2$ SUGRA manifolds which are: *(i)* Hodge (see sec. **2.**6) so the Kähler form is the Chern–class of some line bundle $\mathcal{L}$, and *(ii)* $\mathcal{L}$ is a power of $\mathcal{K}$, the canonical line bundle of $\mathcal{M}$. The two condition together imply that the *strict* $\mathcal{M}$'s are Kähler–Einstein (in other words: for the strict $\mathcal{N} = 3$ manifolds holds the theorem we proven for $\mathcal{N} = 3, 4$ in page 124). The Kähler–Einstein metrics are quite rigid. In the compact case there is *at most one* such metric in each topological class of manifolds (they correspond to algebraic varieties with ample canonical bundle equipped with the unique Calabi–Yau metric). In the non compact case, the same rigidity holds under some mild assumptions about regularity at infinity. Now a small miracle happens: for Einstein–Kähler manifolds (at least in the compact case) *all isometries* do are holomorphic, see theorem **III.5.1** in ref. [**191**]. No paradox.

However, since $Spin(2)_R$ is Abelian, there is no problem in a further twisting of the line bundle $\mathcal{L}$ which does not need to be a power of $\mathcal{K}$. In this more general case, we get a Hodge manifold which is not Kähler–Einstein and in which we are not guaranteed that all isometries are complex automorphisms. This 'twisting luxury' is special to $\mathcal{N} = 2$.

[20] This is true for a *generic* $\mathcal{N} \leq 4$ theory. For $\mathcal{N} \geq 5$ $\mathcal{M}$ is flat and $\mathfrak{hol}$ is trivial.

6. The Cartan–Kostant isomorphism

In ref.[**50**] the authors construct, by clever physicists'–style manipulations, a remarkable Lie algebra homomorphism, which we like to present in the original version due to E. Cartan (for the symmetric spaces) and B. Kostant (in general).

6.1. General theorems. Let $\mathcal{M}$ be *any* Riemannian manifold and $\phi \in \mathcal{M}$ an arbitrary point. $\wedge^2 T_\phi\mathcal{M}$ (viewed as a subspace of $\mathrm{End}(T_\phi\mathcal{M})$ *via* $\Lambda_{ij} \leftrightarrow \Lambda^i{}_j \equiv g^{ik}\Lambda_{kj}$) is a Lie algebra $\mathfrak{s} \simeq \mathfrak{so}(\dim \mathcal{M})$. Set

$$\mathfrak{S} = \mathfrak{s} \oplus T_\phi\mathcal{M}. \tag{6.1}$$

On $\mathfrak{S}$ we introduce a bracket $[\cdot,\cdot]$

$$[s_1, s_2] = s_1 s_2 - s_2 s_1 \in \mathfrak{s}, \qquad \text{for } s_1, s_2 \in \mathfrak{s} \tag{6.2}$$

$$[s, v] = -[v, s] = s(v) \in T_\phi\mathcal{M}, \qquad \text{for } s \in \mathfrak{s},\ v \in T_\phi\mathcal{M} \tag{6.3}$$

$$[v_1, v_2] = R_\phi(v_2, v_1) \in \mathfrak{s} \qquad \text{for } v_1, v_2 \in T_\phi\mathcal{M}. \tag{6.4}$$

where $R_\phi(\cdot,\cdot)$ is the Riemann curvature endomorphism at the given (arbitrary) point $\phi \in \mathcal{M}$.

BEWARE!! $\mathfrak{S}$ is <u>not</u> a Lie algebra, in general (Jacobi does not hold!)

We define a map

$$\theta_\phi \colon \mathfrak{iso}(\mathcal{M}) \to \mathfrak{S} \tag{6.5}$$

as

$$\theta_\phi(K) = A_K\big|_\phi + K\big|_\phi \in \mathfrak{s} \oplus T_\phi\mathcal{M}. \tag{6.6}$$

Although $\mathcal{S}$ is not (in general) a Lie algebra, the bracket $[\cdot,\cdot]$ defines a Lie algebra structure when restricted on the image $\theta_\phi(\mathfrak{iso}(\mathcal{M})) \subset \mathfrak{S}$; *thus* $\mathfrak{g}_\phi := \theta_\phi(\mathfrak{iso}(\mathcal{M}))$ is *a Lie algebra!!* Indeed, we have a stronger result:

THEOREM 6.1 (Kostant [**186**]). *Let* $\mathfrak{g}_\phi = \theta_\phi(\mathfrak{iso}(\mathcal{M})) \subset \mathfrak{S}$ *equipped with the bracket* $[\cdot,\cdot]$ *in eqns.*(6.2)–(6.4). *The map* $\theta_\phi \colon \mathfrak{iso}(\mathcal{M}) \to \mathfrak{g}_\phi$ *is a Lie algebra* ISOMORPHISM.

PROOF. Let K^1, K^2 be Killing vectors. From eqns.(4.2) and (4.2) we have

$$[K^1, K^2] = D_{K^1}K^2 - D_{K^2}K^1 = A_{K^1}K^2 - A_{K^2}K^1, \tag{6.7}$$

$$[A_{K^1}, A_{K^2}] = A_{[K^1,K^2]} + R(K^1, K^2). \tag{6.8}$$

Evaluating both sides at the point ϕ, we get

$$\big[\,\theta_\phi(K^1), \theta_\phi(K^2)\,\big] = \theta_\phi(\,[K^1, K^2]\,). \tag{6.9}$$

(In facts

$$\begin{aligned}\big[\,\theta_\phi(K^1), \theta_\phi(K^2)\,\big] &= [A_{K^1}, A_{K^2}] + A_{K^1}K^2 - A_{K^2}K^1 + R(K^2, K^1)\\ &= [K^1, K^2] + \big[A_{K^1}, A_{K^2}\big] = \theta_\phi\big([K^1, K^2]\big)\ !!)\end{aligned}$$

□

Let $\varrho_\phi\colon \mathfrak{iso}(\mathcal{M}) \to \mathfrak{s}$ be the map $K \mapsto A_K\big|_\phi$, corresponding to the first summand[21] of θ_ϕ. The image $\varrho_\phi(\mathfrak{iso}) \subset \mathfrak{g}_\phi$ is a Lie subalgebra. Then

PROPOSITION 6.1 (Cartan/Kostant [**186**]). *Let $\mathcal{M} = G/H$ be irreducible symmetric. We have the isomorphism*

$$\varrho_\phi(\mathfrak{iso}(\mathcal{M})) \simeq \mathfrak{hol}_\phi \simeq \mathfrak{h}. \tag{6.10}$$

In fact this is just another way of stating that a symmetric space has the form G/H and the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with $\mathfrak{m} \simeq T_\phi\mathcal{M}$, and $\mathfrak{h} \simeq \mathfrak{hol}$ (cfr. sect. **3.**2). Historically, things went in the reverse order: Cartan arrived to understand the symmetric spaces as coset manifolds G/H, with H the fixed points of an involutive homomorphism $\sigma\colon G \to G$, precisely following the above construction.

6.2. Explicit formulae for G/H. Again, we identify G as a group of matrices acting in some representation R. The Lie algebra $\mathfrak{g}$ is identified with the matrices representing it in the R representation (so $\mathfrak{g} \subset \mathfrak{gl}(R)$).

Let t^m be a basis of the matrices representing $\mathfrak{g} \simeq \mathfrak{iso}(G/H)$, $[t^m, t^n] = f^{mn}{}_p\, t^p$, and let K^m be the left–invariant vector fields ($\equiv \mathfrak{g}$ Killing vectors) with $K^m|_e = t^m$. For concreteness, we also assume that $\sigma(g) = (g^{-1})^t$, as in the application we have in mind.

Consider the map

$$\Theta\colon (G/H, \mathfrak{g}) \to \mathfrak{gl}(R) \tag{6.11}$$

given by

$$(\phi, K^m) \mapsto \theta_\phi(K^m) \in \mathfrak{gl}(R) \tag{6.12}$$

Let

$$\mathfrak{h}_\phi \equiv \{K \in \mathfrak{g} \mid \sigma(\theta_\phi(K)) = \theta_\phi(K)\} \tag{6.13}$$

By the above theorems, $\mathfrak{h}_\phi$ is equal to the Lie algebra of the isotropy group at ϕ, and $K \in \mathfrak{h}_\phi \;\Rightarrow\; A_K \equiv \theta_\phi(K)$.

Let $g \in G$ be a representaive of the point $\phi \in G/H$. Let H_ϕ be the isotropy group at ϕ. $k \in H_\phi$ if and only if

$$k\, g = g\, h \qquad \text{for some } h \in H. \tag{6.14}$$

Passing to the Lie algebras, $\tau \in \mathfrak{h}_\phi$ if and only if $(g^{-1}\tau g) \in \mathfrak{h}$. Hence the Lie algebra isomorphism $\theta_\phi\colon \mathfrak{s}_\phi \to \mathfrak{h}$ is given simply by

$$\theta_\phi(K^m) = (g^{-1} t^m g). \tag{6.15}$$

Notice that this isomorphism is unique up to 'H–gauge transformations'.

Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ ($T_eG/H \simeq \mathfrak{m}$); let h^I (resp. m^a) be a basis for the matrices representing $\mathfrak{h}$ (resp. $\mathfrak{m}$). (So $\{t^m\} \equiv \{h^I, m^a\}$). Then we get the fundamental formulae

$$\boxed{g^{-1} t^m g = L^m_I\, h^I + L^m_a\, m^a} \tag{6.16}$$

$$\boxed{A_{K^m} = L^m_I\, h^I} \tag{6.17}$$

We specialize to the case of a SUGRA manifold, as in corollary 4.1. Thus $\mathfrak{h} = \mathfrak{spin}(\mathcal{N}) \oplus \mathfrak{h}'$, and we split the basis $\{h^I\}$ as $\{\gamma^{AB}\}$, $\{h^\alpha\}$. Comparing with corl. 4.1, we have

[21] That is $\varrho_\phi := \pi_1 \circ \theta_\phi$.

Corollary 6.1. *$G/[Spin(\mathcal{N})\otimes H']$ a Riemannian symmetric manifold. One has*

$$\boxed{g^{-1}\,t^m\,g=\frac{1}{4}A^{AB}_{K^m}\,\gamma^{AB}+H^{\alpha}_{K^m}\,h^{\alpha}+L^m_a\,m^a} \tag{6.18}$$

To get the formula as in corl. 4.1, one has only to convert the 'flat' (matrix) indices into 'curved' ones using the vielbeins (extracted from the Maurier–Cartan forms, as in chapt. 5).

7. The covariant momentum map

In this section we define the *covariant momentum map* for a SUGRA manifold $\mathcal{M}$ in perfect analogy with the symplectic one in the rigid case, just making everything $Spin(\mathcal{N})$–covariant. The following construction reduces in the $\mathcal{N}=3,4$ cases (*i.e.* for $\mathcal{M}$ Quaternionic–Kähler) to the one given[22] by Galicki and Lawson [**184, 185**].

7.1. Definition and first properties.

Let $\mathcal{Q}^{\vee}\subset\wedge^2T^*\mathcal{M}$ be the subbundle spanned by the 2–forms Σ^{AB}. One would like to state a definition of the following sort:

Definition 7.1. Let $\mathcal{M}$ be a SUGRA manifold. The *covariant momentum map* $\boldsymbol{\mu}^{\bullet}\in\Gamma(\mathcal{Q}\otimes\mathfrak{iso}^{\vee})$ is the unique (smooth) section such that

$$\mathcal{D}\boldsymbol{\mu}^{AB\,m}=i_{K^m}\Sigma^{AB} \tag{7.1}$$

where $\mathcal{D}=d+Q$ is the $Spin(\mathcal{N})$–covariant exterior derivative.

But, is this definition well–posed? That is, does eqn.(7.1) have always a solution $\boldsymbol{\mu}^{AB\,m}$? Is it really unique?

The answer is *yes* to both questions. The uniqueness of $\boldsymbol{\mu}^{\bullet}$ has to be understood (of course) as a section of $\Gamma(\mathcal{Q}\otimes\mathfrak{iso}^{\vee})$, that is:

Claim 7.1. *Given any $Spin(\mathcal{N})$–invariant k–linear map*

$$P\colon\ \odot^k\mathfrak{spin}(\mathcal{N})\to\mathbb{C} \tag{7.2}$$

(i.e. a generalized Casimir invariant of corl. 4.3) the $2(k-l)$–form

$$\begin{aligned}&P_l(\boldsymbol{\mu}^{m_1},\cdots,\boldsymbol{\mu}^{m_l},\Sigma)\equiv\\&\equiv P_{A_1B_1A_2B_2\cdots A_kB_k}\,\underbrace{\boldsymbol{\mu}^{A_1B_1\,m_1}\,\cdots\,\boldsymbol{\mu}^{A_lB_l\,m_l}}_{l\ \text{times}}\,\Sigma^{A_{l+1}B_{l+1}}\wedge\cdots\wedge\Sigma^{A_kB_k}\end{aligned} \tag{7.3}$$

is unique.

[22] Note that the $\mathcal{N}\geq3$ SUGRA manifolds are either Quaternionic–Kähler or symmetric G/H. Since Galicki–Lawson have constructed the momentum map in the Quaternionic–Kähler case, while in the symmetric case one may work algebraically, one could wonder if constructing the theory of covariant momentum maps from scratch is a good idea or not. However, somebody has said that "understanding is unifying", so a uniform treatment, instead than a case–by–case one may be more appropriate didatically (and not only). Ironically, the most intricate case is precisely $\mathcal{N}=2$, which is the less understood in the literature (and less studied since it is considered — erroneously — trivial).

$\boldsymbol{\mu}^{AB\,m}$ itself is defined only *up to a Spin*($\mathcal{N}$) *(local) gauge transformation* (*i.e.* up to changes of trivialization of the bundle $\mathcal{Q}$). One has[23]

$$dP_l^{m_1\cdots m_l} = \frac{1}{k+1-l}\sum_{r=1}^{l} i_{k^{m_r}} P_{l-1}^{m_1\cdots\widehat{m}_r\cdots m_l}. \tag{7.4}$$

We assume $\mathcal{N} \geq 3$, so that $Spin(\mathcal{N})$ is semi–simple. In the case $\mathcal{N} = 2$ $Spin(2)$ is Abelian, and hence the connection is trivial in the adjoint representation; thus $D = d$ acting on $\boldsymbol{\mu}^{\bullet}$. Hence, for $\mathcal{N} = 2$, the covariant momentum map is just the ordinary (symplectic) momentum map. In this special case unicity holds up to an additive constant.

We shall need the self–evident

LEMMA 7.1. *One has the identity*

$$\pounds_K = i_K d + d i_K = i_K(\mathcal{D}-Q) + (\mathcal{D}-Q)i_K = i_K\,\mathcal{D} + \mathcal{D}\,i_K - (i_K Q)$$

where (again) $\mathcal{D}$ is the Spin($\mathcal{N}$)–covariant exterior derivative and the one–form $Q \equiv d\phi^i\, Q_i^{AB}$ is the Spin($\mathcal{N}$)–connection.

Then, as claimed,

PROPOSITION 7.1. *$\mathcal{M}$ is a $\mathcal{N} \geq 3$* SUGRA *manifold, K^m any Killing vector of $\mathcal{M}$. There* EXISTS *a* UNIQUE[24] *function $\boldsymbol{\mu}^{AB\,m}$ such that*

$$\mathcal{D}\boldsymbol{\mu}^{AB\,m} = i_{K^m}\Sigma^{AB}, \tag{7.5}$$

where $\mathcal{D}$ is Spin($\mathcal{N}$)–covariant. Explicitly,

$$\boxed{\boldsymbol{\mu}^{AB\,m} = 2\,A_{K^m}^{AB}} \tag{7.6}$$

where $A_{K^m}^{AB}$ are the coefficient of A_{K^M} (cfr. corollary 4.1).

PROOF. Take the $\mathcal{D}$ of both sides of eqn.(7.5). $\mathcal{D}^2$ is the curvature that we have computed too many times, *i.e.* $-\frac{1}{2}\Sigma$. So,

$$\begin{aligned}-\frac{1}{2}\mathrm{adj}(\Sigma)\,\boldsymbol{\mu}^{\bullet\,m} &= \mathcal{D}(i_{K^m}\Sigma) = \pounds_{K^m}\Sigma - i_{K^m}\mathcal{D}\Sigma + (i_{K^m}Q) = \\ &= \pounds_{K^m}\Sigma + (i_{K^m}Q)\Sigma = -\mathrm{adj}(\Sigma)\Big((i_{K^m}Q) + \sigma(K^m)\Big),\end{aligned} \tag{7.7}$$

where we used lemma 7.1 and corollary 4.2. Then

$$\boldsymbol{\mu}^{AB\,m} = 2\Big(K^{i\,m}\,Q_i^{AB} + \sigma^{AB}(K^m)\Big) \equiv 2A_{K^m}^{AB}. \tag{7.8}$$

□

Our definition is well–posed! Notice that $\boldsymbol{\mu}^{\bullet}$ is (up to normalization) just the $\mathfrak{spin}(\mathcal{N})$–projection of the basic endomorphism A_K defined by [**186, 189**]

$$\pounds_K = D_K + A_K, \tag{7.9}$$

[23] We set $P_l(\boldsymbol{\mu}^{m_1}, \cdots, \boldsymbol{\mu}^{m_l}, \Sigma) = P_l^{m_1\cdots m_l}$.

[24] In the above sense!

whose geometrical meaning is explained in §. VI.4 of [**189**]. Compare also with [**184, 185**], or eqn.(12.4.5) of ref.[**155**]. Explicitly, one has

$$\boxed{A_K = \frac{1}{8}\boldsymbol{\mu}^{AB}\,\Sigma^{AB} + H_K} \tag{7.10}$$

i.e. $\frac{1}{2}\boldsymbol{\mu}^\bullet$ is the $\mathfrak{spin}(\mathcal{N})$ projection of the Cartan–Kostant morphism[25]

$$\theta_\phi \colon \mathfrak{iso} \to \mathfrak{spin}(\mathcal{N}) \oplus \mathfrak{h}' \oplus T_\phi\mathcal{M}. \tag{7.11}$$

It is geometrically natural (and often physically useful, see *e.g.* ref.[**50**]) to generalize the momentum map to the full space in the RHS. Then

DEFINITION 7.2. $\mathcal{M}$ a SUGRA manifold. The *generalized momentum map* is the map

$$\widetilde{\boldsymbol{\mu}} \colon \mathcal{M} \times \mathfrak{iso} \to \mathfrak{spin}(\mathcal{N}) \oplus \mathfrak{h}' \oplus T_\phi\mathcal{M} \tag{7.12}$$

$$(\phi, K^m) \mapsto \left(2\,\theta_\phi(K^m)\Big|_{\mathfrak{spin}(\mathcal{N})},\; 2\,\theta_\phi(K^m)\Big|_{\mathfrak{h}'},\; 2\,\theta_\phi(K^m)\Big|_{T\mathcal{M}} \right) \tag{7.13}$$

$$\mapsto \left(\tfrac{1}{4}\boldsymbol{\mu}^{AB\,m}\,\Sigma^{AB},\; 2\,H_{K^m},\; 2\,K^{i\,m}\right). \tag{7.14}$$

The fact that the Cartan–Kostant map is an isomorphism onto its image implies that the generalized momentum map behaves well under $\mathrm{Iso}(\mathcal{M})$.

REMARK. In the case $\mathcal{N} = 3$, the construction in PROPOSITION 7.1 reduces to the one given by Galicki and Lawson [**184, 185**] for Quaternionic–Kähler manifolds. The authors of ref.[**50**] write some of the above formulae for the covariant momentum map for general $\mathcal{N}$'s (cfr. eqns.(2.27)–(2.31) of [**50**]), although they arrive at them from a different route. Their $\mathcal{S}^{IJ}$ correspond to our σ^{AB}.

7.2. The map $\boldsymbol{\mu}^\bullet$ and the isometry algebra $\mathfrak{iso}(\mathcal{M})$. Uniqueness, in particular, implies

$$\boldsymbol{\mu}^\bullet(\varphi_* K) = \varphi^*\boldsymbol{\mu}^\bullet(K) \quad \forall\,\varphi \in \mathrm{Iso}(\mathcal{M}) \text{ and } K \in \mathfrak{iso}(\mathcal{M}). \tag{7.15}$$

where φ, in general, induces also a change of trivialization of $\mathcal{Q}$ ($\equiv$ a $Spin(\mathcal{N})$ gauge transformation). Let us check this at the infinitesimal level. Recall that eqn.(4.3) gives

$$[A_{K^m}, A_{K^n}] = A_{[K^m,K^n]} + R(K^m, K^n). \tag{7.16}$$

Then

$$\begin{aligned}
&\pounds_{K^m}\big(\boldsymbol{\mu}^{AB}(K^n)\big)\,\Sigma^{AB} = \\
&= \pounds_{K^m}\Big(\boldsymbol{\mu}^{AB\,n}\,\Sigma^{AB}\Big) - \boldsymbol{\mu}^{AB\,n}\,\pounds_{K^m}\Sigma^{AB} = \\
&= 8\Big\{(D_{K^m} + A_{K^m})A_{K^n} - A_{K^n}(D_{K^m} + A_{K^m})\Big\}_{\mathfrak{spin}(\mathcal{N})} - \boldsymbol{\mu}^{AB\,n}\,[\sigma_{K^m}, \Sigma]^{AB} \\
&= 8\Big\{-R(K^m, K^n) + [A_{K^m}, A_{K^n}]\Big\}_{\mathfrak{spin}(\mathcal{N})} - [\sigma_{K^m}, \boldsymbol{\mu}^{\bullet\,n}]^{AB}\,\Sigma^{AB} \\
&= 8\Big\{-A_{[K^m,K^n]}\Big\}_{\mathfrak{spin}(\mathcal{N})} - [\sigma_{K^m}, \boldsymbol{\mu}^{\bullet\,n}]^{AB}\,\Sigma^{AB} = \\
&= -f^{mn}{}_p\,\boldsymbol{\mu}^{AB\,p}\,\Sigma^{AB} - [\sigma_{K^m},\, \boldsymbol{\mu}^{\bullet,n}]^{AB}\,\Sigma^{AB},
\end{aligned}$$

[25] Recall that $\mathfrak{hol} \simeq \mathfrak{spin}(\mathcal{N}) \oplus \mathfrak{h}'$.

that is the $\boldsymbol{\mu}^{\bullet}$'s transform according to the adjoint of $\mathfrak{iso}$ up to the usual $Spin(\mathcal{N})$ gauge transformation of parameter $(\sigma_{K^m})^{AB}$.

Projecting eqn.(7.16) on $\mathfrak{spin}(\mathcal{N})$ we get the nice identity

$$\left[\frac{1}{2}\boldsymbol{\mu}(K^m), \frac{1}{2}\boldsymbol{\mu}(K^n)\right]^{AB} = -f^{mn}{}_l \, \frac{1}{2}\boldsymbol{\mu}^{AB\,l} - \frac{1}{2}\, K^{i\,m} K^{i\,n}\, \Sigma^{AB}_{ij}, \tag{7.17}$$

where $\boldsymbol{\mu}^{AB}(K^m)$ are seen as matrices in the $Spin(\mathcal{N})$ indices A, B.

7.3. Other properties of the covariant momentum map. Let $A \neq B$, and $A, B \neq C$. Then

$$\left[(\Sigma^{AB})_i{}^j + i\,\delta_i{}^j\right]\mathcal{D}_j\big(\boldsymbol{\mu}^{AD\,m} + i\boldsymbol{\mu}^{BD\,m}\big) = 0, \tag{7.18}$$

which is the covariant counterpart to Σ^{AB}–holomorphicity. In general, all rigid case identities with only one derivative take the same form in the local case. Identities with two derivatives get new terms from the curvatures. This terms can be always re–expressed in terms of commutators of $\boldsymbol{\mu}^{\bullet}$, as we did above. We leave to the reader to play with such gymnastics. We shall try to go on in the theoy without introducing (too) heavy identities.

7.4. Symmetric spaces $G/[Spin(\mathcal{N}) \times H']$. The case of a symmetric space, the *covariant momentum map* is explicitly computed by eqn.(6.18). One has

$$\boldsymbol{\mu}^{AB\,m} = -\eta\,\mathrm{tr}_R\Big[\gamma^{AB}\, g^{-1}\, t^m\, g\Big] \tag{7.19}$$

where γ^{AB} are the matrices representing the subalgebra $\mathfrak{spin}(\mathcal{N}) \subset \mathfrak{g}$ in the representation R and η is the obvious normalization coefficient (depending on the representation R we use to write the representative group elements as matrices).

We can also easily write down the *generalized momentum map.* The generators of $\mathfrak{g}$ can be decomposed as $\{t^m\} \equiv \{\gamma^{AB}, h^\alpha, m^i\}$ where $\{h^\alpha\}$ span $\mathfrak{h}' \equiv \mathfrak{hol} \ominus \mathfrak{spin}(\mathcal{N})$ and $\{m^i\}$ generate $\mathfrak{m} = \mathfrak{g} \ominus \mathfrak{hol}$. Then we can formally extend our definition of the $Spin(\mathcal{N})$–covariant map to a G–covariant *generalized momentum map*[26]

$$\boldsymbol{\mu}^{AB\,m} = -\eta\,\mathrm{tr}_R\Big[\gamma^{AB}\, g^{-1}\, t^m\, g\Big] \tag{7.20}$$

$$\boldsymbol{\mu}^{\alpha\,m} = -\eta\,\mathrm{tr}_R\Big[h^{\alpha}\, g^{-1}\, t^m\, g\Big] \tag{7.21}$$

$$\boldsymbol{\mu}^{i\,m} = -\eta\,\mathrm{tr}_R\Big[m^{i}\, g^{-1}\, t^m\, g\Big]. \tag{7.22}$$

The properties of the momentum map follows from the statement that the Cartan–Kostant map is an *isomorphism.*

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[26] Comparing with the Cartan–Kostant isomorphism, eqn.(7.22) should give (up to normalization) the Killing vector $K^{i\,m}$. Is this consistent with the formula for the Killing vectors we got in chapt. 5? *Yes.* You have to remember that (7.22) is the Killing vector in 'flat' indices. To convert it to 'curved' indices you have to use the vielbein which is given by the $\mathfrak{m}$–projection of the Maurer–Cartan form. Thus, assuming (7.22), we get

$$\begin{aligned} K^m_i d\phi^i \propto (g^{-1}dg)_i\,\mathrm{tr}[m^i\, g^{-1}t^m g] &= \mathrm{tr}[(g^{-1}dg)_{\mathfrak{m}}\, g^{-1}t^m g] = \\ &= -[(Dg^{-1})g\, g^{-1}t^m g] = -\mathrm{tr}[t^m(gDg^{-1})] \end{aligned}$$

which is our formula in chapt. 5.

8. T–tensors II

$\mathcal{M}$ a SUGRA manifold. Let $\mathcal{G} \subset \mathrm{Iso}(\mathcal{M})$ be a subgroup, and $\mathfrak{L}$ its lie algebra. Let $l_{mn} \in \odot^2 \mathfrak{L}^\vee$ be a symmetric $\mathrm{Adj}_{\mathcal{G}}$–invariant pairing $\mathfrak{L} \times \mathfrak{L} \to \mathbb{R}$. In perfect analogy to sect. 2, we define the *covariant* T–tensor as

$$T^{AB,CD} \equiv \frac{1}{4}\,\boldsymbol{\mu}^{AB\,m}\,l_{mn}\,\boldsymbol{\mu}^{CD\,n}. \tag{8.1}$$

In view of the Cartan–Kostant isomorphism, it is convenient to extend this definition to a symmetric tensor $\mathcal{T} \in \odot^2 \mathfrak{iso}(\mathcal{M})$, namely

$$\mathcal{T}(\phi) = \theta_\phi(K^m)\,l_{mn}\,\theta_\phi(K^n). \tag{8.2}$$

Notice that, say,

$$\mathcal{T}\Big|_{\mathfrak{hol}^\perp \times \mathfrak{hol}^\perp} = K_i^m\,l_{mn}\,K_j^n \tag{8.3}$$

and similar identifications hold for the other components.

The GENERAL LESSON 2.1 has a local counterpart.

GENERAL LESSON 8.1. *In* SUGRA *a gauging* $(\mathcal{G}, l_{mn})$ *is completely described by the corresponding covariant* T*–tensor.*

The covariant T*–tensor can be generalized to an* $\mathrm{Iso}(\mathcal{M})$*–covariant object (the generalized momentum map) trasforming in the representation* $\odot^2\mathfrak{iso}$ *of* $\mathrm{Iso}(\mathcal{M})$*.*

REMARK. We have fixed our attention to the $D = 3$ gaugings. However, the deep relations between the parallel geometric structures of SUSY/SUGRA and the isometry algebra are (obviously) universal, and in fact in the next chapter we shall see that the gauging in any $D > 3$ will require only small modifications (conceptually, at least).

CHAPTER 8

Gauging and potential terms

1. Gaugings in rigid Susy

In this chapter we study (geometrically) the problem of gauging (in an $\mathcal{N}$–supersymmetric fashion) the symmetries of the ungauged SUSY/SUGRA models we costructed above. Doing this, we complete (at least in principle) our program of constructing the most general $\mathcal{N}$–supersymmetric field theory in D space–time dimensions. In particular, we wish to know which gauge groups and which matter representations are compatible with an $\mathcal{N}$–extended supersymmetry, and understand the physics and the geometry of these couplings.

Just as we did in chapt. 2, we start with $D = 3$. Again, this is the 'universal' dimension: any theory in $D \geq 3$ can be dimensionally reduced to $D = 3$, while in $D = 3$ we have a larger span of possible values of $\mathcal{N}$, and hence more general geometrical structures. The conclusion will hold in any dimension (in which the theory makes sense) up to straightforward *mutatis mutandis* which we shall discuss in (sufficient) detail.

We adopt the *diet* formulation of $D = 3$ SUSY/SUGRA, that is we dualize all vectors *á la* dWHS [**50**] in such a way that the vectors' derivatives enter in the Lagrangian $\mathcal{L}$ only trough the Chern–Simons (CS) terms

$$\frac{k^{mn}}{4\pi}\left(A_m \wedge dA_n + \frac{2}{3} A_m \wedge [A, A]_n\right).$$

Notice that gauge invariance requires k_{mn} to be a constant (*i.e.* independent of the scalars ϕ^i). The matrix k^{mn} may be assumed to be symmetric.

2. CS coupled to $\mathcal{N} = 1$ gauged σ–models

As it is well–known, the pure CS theory is a TFT (topological field theory) without any local degree of freedom. We can write an $\mathcal{N} = 1$ supersymmetric version of the CS TFT, namely

$$\frac{k^{mn}}{4\pi}\left(A_m \wedge dA_n + \frac{2}{3}\, A_m \wedge [A, A]_n - \lambda^{\alpha}_{m} \lambda_{\alpha\, n}\right).$$

where the Majorana fermion $\lambda_{\alpha\, m}$ is the gaugino associated to the vector $A_{\mu\, m}$ (conventions as in [**170**]). This TFT is invariant under the SUSY transformations

$$\delta A_{\mu\, m} = \epsilon^{\alpha}\, (\sigma_\mu)_{\alpha\beta}\, \lambda^{\beta}_{m}, \tag{2.1}$$

$$\delta \lambda_{\alpha\, m} = -(\sigma_\mu)_{\alpha\beta} \epsilon^{\beta}\, F^{\mu}_{m}, \tag{2.2}$$

($F^{\mu}_{m} = \frac{1}{2} \epsilon_{\mu\nu\rho}\, F_{\nu\rho\, m}$ is the field–strength) as it is easily checked since the variation of both terms are proportional to $k^{mn}\, F^{\mu}_{m} \epsilon^{\alpha} (\sigma_\mu)_{\alpha\beta} \lambda^{\beta}_{n}$.

Next we couple this supersymmetric TFT to some matter. The vectors enter in the kinetic terms of the scalars and fermions *via* the covariant derivatives

$$D_\mu \phi^i = \partial_\mu \phi^i - A_{\mu\, m}\, K^{i\, m} \tag{2.3}$$

$$D_\mu \chi^i = \partial_\mu \chi^i + \partial_\mu \phi^j\, \Gamma^i_{jk} \chi^k - A_{\mu\, m}\, D_j K^i\, \chi^j \tag{2.4}$$

while the gaugini λ_m may enter the Lagrangian only trough the Yukawa couplings

$$Y^m_i(\phi)\, \lambda^\alpha_m\, \chi^i_\alpha.$$

$\mathcal{N} = 1$ supersymmetry requires this coupling to have a special form. The terms linear in λ in the variation of $\mathcal{L}$ are:

$$\delta A^\mu_m\, J^m_\mu + Y^m_i\, \lambda^\alpha_m\, \delta\chi^i, \tag{2.5}$$

where J^m_μ is the matter part of the gauge current

$$J^m_\mu = -g_{ij} K^{i\, m} \partial_\mu \phi^j + \cdots , \tag{2.6}$$

and hence the Yukawa coupling is identified with the Killing vector K^m_i,

$$\lambda^\alpha_m \chi^i_\alpha\, K^m_i, \tag{2.7}$$

where K^i_m is normalized so that the scalars' covariant derivative reads as in eqn.(2.3) (thus the coupling constant is absorbed in the normalization of K^m_i).

Notice that the gaugini enter the full Lagrangian only algebraically, and thus they can be integrated away. Eliminating the auxiliary fermions λ's, we get a physical Yukawa coupling

$$l_{mn}\, K^m_i\, K^n_j\, \chi^i \chi^j \tag{2.8}$$

where l_{mn} is the inverse of the matrix $k^{mn}/4\pi$.

As we saw in chapt. 2, in the $\mathcal{N} = 1$ theory, in addition to the Yukawa coupling (2.8) induced by the gauge interactions, we may have Yukawa terms coming from a (real) superpotential $W(\phi)$. $W(\phi)$ should be a gauge invariant function on $\mathcal{M}$, namely

$$\mathcal{L}_{K^m} W = 0. \tag{2.9}$$

The complete (rigid) $\mathcal{N} = 1$ Yukawa couplings are then given by

$$\big(l_{mn}\, K^m_i\, K^n_j\, + D_i \partial_j W\big)\, \chi^i \chi^j, \tag{2.10}$$

while the scalars' potential $V(\phi)$ is fixed by the general Ward identity we discussed in chapter 6 to be

$$V(\phi) = \frac{1}{2}\, G^{ij}\, \partial_i W\, \partial_j W. \tag{2.11}$$

3. $\mathcal{N}$–extended (rigid) CS gauge theories

3.1. Consistency conditions. Let our $D = 3$ model (with vectors dualized in the CS form) have (rigid) $\mathcal{N}$–supersymmetry. Forgetting about the invariance under $Q^2, Q^3, \cdots, Q^{\mathcal{N}}$, we may consider it simply as an $\mathcal{N} = 1$ model. Hence its Yukawa couplings and potential should have the form (2.10)(2.11) for some real function $W(\phi)$. However we may forget $\mathcal{N} - 1$ supercharges in various ways; the Lagrangian $\mathcal{L}$ should be independent of the choice we make. As we discussed in detail in chapt. 2, replacing Q^1

with Q^a ($a = 2, 3, \dots, \mathcal{N}$) as the unforgotten supercharge, amounts to the replacement

$$\chi^i \longmapsto (f^a)^i{}_j \, \chi^j \equiv (\Sigma^{1a})^i{}_j \, \chi^j, \tag{3.1}$$

where $f^a \in \mathrm{End}(T\mathcal{M})$ are the basic parallel complex structures. Written in terms of the new fermionic fields $\chi^{ai} = (\Sigma^{1a})^i{}_j \, \chi^j$, the Yukawa coupling and potential should also be of the general form in eqn.(2.10)(2.11). Therefore, in order for Q^a to generate a supersymmetry of the Lagrangian $\mathcal{L}$, there must be a superpotential $W^a(\phi)$ such that

$$(\Sigma^{1a})^t \, (K^m \, l_{mn} \, K^n + D\partial W^a) \, \Sigma^{1a} = (K^m \, l_{mn} \, K^n + D\partial W) \tag{3.2}$$

$$G^{ij} \, \partial_i W^a \, \partial_j W^a = G^{ij} \, \partial_i W \, \partial_j W \tag{3.3}$$

(NOT summed over a!)

If such a W^a exists, we are guaranteed that the supercharge Q^a generates a supersymmetry of $\mathcal{L}$. Thus $\mathcal{N}$–SUSY requires the existence of a complete set

$$W \equiv W^1, \ W^2, \ W^3 \ \cdots, \ W^{\mathcal{N}}$$

of (real) superpotentials satisfying the equations (3.2)(3.3).

Notice that $Spin(\mathcal{N})$ itself is not necessarily a symmetry; $\mathcal{L}$ is invariant under $Spin(\mathcal{N})$ only if the functional forms of the various $W^a(\phi)$'s are related by suitable isometries $\varphi^a \colon \mathcal{M} \to \mathcal{M}$

$$W^a(\phi) = W\big(\varphi^a(\phi)\big). \tag{3.4}$$

In general $Spin(\mathcal{N})$ is only an automorphism of the formalism.

The equations (3.2) for $a = 2, 3, \dots, \mathcal{N}$ have NO solution in general. However we are interested precisely in the *special* conditions under which a solution does exist: we wish to solve the following

PROBLEM. *Given a scalars' manifold $\mathcal{M}$, compatible with $\mathcal{N}$–SUSY, determine the subgroups $\mathcal{G} \subset \mathrm{Iso}_0(\mathcal{M})$ which may be gauged in an supersymmetric way as well as the associated gauge couplings l_{mn} which preserve the full $\mathcal{N}$–SUSY.*

In view of eqns.(3.2), we can formalize our problem in this form:

PROBLEM. *For which data $(\mathcal{M}, \mathcal{G}, l_{mn})$ there exist functions*

$$\big\{W^1(\phi) \equiv W(\phi), \ W^2(\phi), \ \cdots, \ W^{\mathcal{N}}(\phi)\big\} \tag{3.5}$$

which are gauge invariant

$$\pounds_{K^m} W^a = 0, \tag{3.6}$$

and solve eqns.(3.2)(3.3)*?*

As stated, this is a geometrical problem about the isometries of the Riemannian manifolds $\mathcal{M}$ having a special holonomy, $\mathfrak{hol}(\mathcal{M}) \subseteq \mathcal{C}(\mathfrak{spin}(\mathcal{N}))$, allowing for $\mathcal{N}(\mathcal{N}-1)/2$ parallel 2–forms Σ^{AB}. We discussed such isometry groups in the previous chapter. The relevant notions were the *momentum map* and the *T–tensor*. By GENERAL LESSON **7**.2.1, the possible solutions to the above problem should correspond to conditions on these natural objects. Indeed, the supersymmetric gaugings take a universal form when written in terms of the T–tensor. Stated differently, the data $(\mathcal{M}, \mathcal{G}, l_{mn})$ are encoded in the T–tensor

$$T^{AB,CD} = \mu^{AB\,m}\, l_{mn}\, \mu^{CD\,n}, \tag{3.7}$$

and, as we shall see momentarily, the equations for W^a depend on the basic data $(\mathcal{M}, \mathcal{G}, l_{mn})$ only trough the T–tensor. Thus, at the end of the day, the solution to the our problem will boil down to a geometric constraint on the T–tensor.

3.2. The consistency conditions for $\mathcal{N} \leq 3$. Let us see the implications of the above constraints for increasing $\mathcal{N}$'s. For $\mathcal{N} = 1$ there is no condition. In the $\mathcal{N} = 2$ case we have a single complex structure $f = \Sigma \equiv \Sigma^{12}$. From eqn.(3.2) we have

$$(\Sigma^{12})_i{}^k\, (\Sigma^{12})_j{}^h \Big\{ K_k^m l_{mn} K_h^n + D_h \partial_k W \Big\} = \Big\{ K_i^m l_{mn} K_j^n + D_i \partial_j W^2 \Big\} \tag{3.8}$$

while from lemma (2.7) of chapt. 7

$$\begin{aligned}(\Sigma^{12})_i{}^k\, (\Sigma^{12})_j{}^h \Big\{ K_k^m l_{mn} K_h^n + D_h \partial_k \left(\frac{1}{2} T^{12,12}\right) \Big\} &= \\ = \Big\{ K_i^m l_{mn} K_j^n + D_i \partial_j \left(\frac{1}{2} T^{12,12}\right) \Big\}&.\end{aligned} \tag{3.9}$$

Subtracting the two equations, we get

$$(\Sigma)_i{}^k\, (\Sigma)_j{}^h D_h \partial_k \left(W - \frac{1}{2} T^{12,12} \right) = D_i \partial_j \left(W^2 - \frac{1}{2}\, T^{12,12} \right). \tag{3.10}$$

In view of our discussions in chapt. 7, this equation has a simple interpretation: *The function* $\left(W - \frac{1}{2} T^{12,12}\right)$ *is* Σ*–harmonic* and[1]

$$W^2 - \frac{1}{2} T^{12,12} = -\left(W - \frac{1}{2} T^{12,12} \right). \tag{3.11}$$

Using holomorphic coordinates adapted to the complex structure Σ, this is equivalent to the existence of a holomorphic function $\mathcal{F}(\phi)$ such that

$$W = \frac{1}{2} T^{12,12} + \mathcal{F} + \overline{\mathcal{F}}. \tag{3.12}$$

Of course, $\mathcal{F}$ is just the holomorphic superpotential of the $\mathcal{N} = 2$ superspace formalism. The $\mathcal{N} = 2$ Chern–Simons vector superfield contains, beside A_μ and two auxiliary spin–1/2 fermions, also two auxiliary (real) scalars D and σ (see *e.g.* [**171**]). The additional auxiliary fermion produces new Yukawa terms, while the elimination of the auxiliary scalars leads to a gauge contribution to the scalars' potential of the form $g^{ij}(\partial_i T)(\partial_j T)$ where $T \equiv T^{12,12}$. On top of these gauge contributions there are the usual Yukawa and potential terms arising from the superpotential $\mathcal{F}$ (in the $\mathcal{N} = 2$ sense). Thus

$$V(\phi) = \begin{cases} \frac{1}{2} g^{ij} \partial_i W\, \partial_j W & \mathcal{N} = 1 \text{ formalism} \\ \frac{1}{8} g^{ij}\, \partial_i T\, \partial_j T + g^{i\bar{j}}\, \partial_i \mathcal{F}\, \partial_{\bar{j}} \overline{\mathcal{F}} & \mathcal{N} = 2 \text{ formalism} \end{cases}$$

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[1] Recall from §. 1.6 of chapt. 7 that, for a Σ–harmonic function,

$$D_i \partial_j F = -\Sigma_i{}^k\, \Sigma_j{}^h D_k \partial_h F.$$

A priori the equality holds up to functions having vanishing Hessian. But the condition on the scalars' potential eliminates this residual ambiguity.

The $\mathcal{N}=1$ and $\mathcal{N}=2$ expressions differ by cross–terms of the form

$$g^{ij}\,\partial_i T\,\partial_j \mathrm{Re}(\mathcal{F}) \tag{3.13}$$

but these vanish identically since

$$g^{ij}\partial_i T\,\partial_j\mathcal{F} = 2\mu^m\, l_{mn}\, K^i\partial_i\mathcal{F} = \mu^m\, l_{mn}\, \pounds_{K^n}\mathcal{F} = 0, \tag{3.14}$$

by gauge invariance of the $\mathcal{N}=2$ superpotential. Therefore the $\mathcal{N}=1$ and $\mathcal{N}=2$ formalisms lead to the same Lagrangian.

In conclusion, we have proven the following

LEMMA 3.1. *In an $\mathcal{N}\geq 2$ gauged* SUSY *model, we have the relation*

$$\boxed{W(\phi) = \frac{1}{2}\,T^{1\,a,1\,a} + F^a} \tag{3.15}$$

($a=2,3,\dots,\mathcal{N}$) where F^a is the real part of a f^a–holomorphic function satisfying

$$\pounds_{K^m} F^a = 0. \tag{3.16}$$

Moreover,

$$\boxed{W^a = T^{1\,a,1\,a} - W} \tag{3.17}$$

We interpret eqns.(3.15) (or, equivalently, eqns.(3.10)) as a set of *linear* equations defining the unknown function $W(\phi)$ in terms of the geometric gauging data $T^{AB,CD}$.

The general solution, $W(\phi)$, is the sum of a particular solution, $W^{(0)}(\phi)$, and the general solution of the homogeneous equation (*i.e.* the equations with $T^{1\,a,1\,a}$ set to zero). But a solution $W_h(\phi)$ of the homogeneous equation is precisely a function which is f^a–harmonic for *all* complex structures f^a. We already know that, for $\mathcal{N}\geq 4$ (three complex structures) such a function is trivial, that is $D_i\partial_j W=0$, while for $\mathcal{N}=3$ $D_i\partial_j W$ is at most a constant. Thus we learn the

GENERAL LESSON 3.1. *For a given gauging, $(\mathcal{G}, l_{mn})$, the Yukawa couplings and scalar potential which complete the Lagrangian to a $\mathcal{N}\geq 3$ model — if they exist at all — are essentially*[2] *unique.*

3.2.1. *$\mathcal{N}=3$ CS models.* For $\mathcal{N}=3$, $W(\phi)$ is (essentially) uniquely determined from the gauging data $(\mathcal{G}, l_{mn})$. Let us show explicitly that, in this case, *there is* a solution for each $\mathcal{G}\subset \mathrm{Iso}_0(\mathcal{M})$ and each symmetric invariant tensor l_{mn}. Indeed, the function

$$W(\phi) = \frac{1}{2}\Big(T^{1\,2,1\,2} + T^{1\,3,1\,3} - T^{2\,3,2\,3}\Big) \tag{3.18}$$

is a solution to eqn.(3.15) (while $\pounds_{K^m} W = 0$ holds by construction). To prove this, we have only to check that $T^{1\,3,1\,3} - T^{2\,3,2\,3}$ is a $\Sigma^{1\,2}$–harmonic function, and correspondingly that $T^{1\,2,1\,2} - T^{2\,3,2\,3}$ is $\Sigma^{1\,3}$–harmonic. Both facts are guaranteed by Lemma 2.1 of chapt. 7.

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[2] The residual non–uniqueness is related, in particular, to the fact that the momentum maps themselves are defined up to an additive constant (say for Abelian groups) and hence the T–tensors may be shifted by a function which, in the flat case, is at most quadratic.

REMARK. Notice that the gauged model is invariant under three supercharges, Q^1, Q^2 abnd Q^3, but not under the fourth one which, in the ungauged model, is automatically conserved, namely the one associated to the complex structure $f^3 \equiv f^1 f^2$. Indeed, in general, $W - \frac{1}{2}T^{23,23} \equiv \frac{1}{2}(T^{12,12} + T^{13,13} - 2\,T^{23,23})$ is <u>not</u> Σ^{23}–harmonic.

Thus *all gaugings* $(\mathcal{G}, l_{mn})$ *are admissible in* $\mathcal{N} = 3$ SUSY. However the gauge invariant Lagrangian is <u>unique</u> up to the small freedom in the definition of the T–tensor for the Abelian part of $\mathcal{G}$.

3.2.2. *A* no–go *theorem for* $\mathcal{N} \geq 4$. This is not true for larger $\mathcal{N}$. Gaiotto and Witten [**170, 182**] give a physical argument why generic gaugings are not compatible with $\mathcal{N} \geq 4$ SUSY. Consider the subclass[3] of $\mathcal{N}$–supersymmetric models wich are obtained, *via* dWHS duality, from $D = 3$ models whose vectors have both F^2 canonical kinetic terms and Chern–Simons interactions. Then the gauge vectors get massive [**183**] and have (say) helicity +1. The $\mathcal{N}$–SUSY algebra has $\mathcal{N}$ helicity lowering operators, so a massive vector multiplet should contain states with helicity λ

$$\lambda \;=\; 1,\; \frac{1}{2},\; 0 \cdots,\; 1 - \frac{\mathcal{N}}{2}.$$

In particular, for $\mathcal{N} \geq 4$, we have states with helicity -1 which are also massive vectors. Since (rigid) SUSY commutes with the gauge symmetry, all the above states transform in the same way under $\mathcal{G}$, that is in the adjoint representation (which is the representation for the gauge vectors $\lambda = +1$). For $\mathcal{N} \geq 4$, we have also $\lambda = -1$ vectors, always in the adjoint. Thus the gauge vectors transforms according to several copies of the adjoint representation, but this is forbidden in non–Abelian gauge theories.

$\mathcal{N} \geq 4$ gaugings *do* exist, but they are somehow exceptional, being possible only for very specific gauge groups, matter representations, and couplings. Given that the $\mathcal{N} = 3$ gauged model is essentially unique (for given $(\mathcal{G}, l_{mn})$), and that any $\mathcal{N} \geq 4$ model is, in particular, an $\mathcal{N} = 3$ model, *all* $\mathcal{N} \geq 4$ *gauged models should be defined by the* $\mathcal{N} = 3$ *superpotential* W *in eqn.*(3.18), which — for suitable T–tensors — magically happens to satisfy the conditions (3.15) for all a's. For a generic T–tensor this is impossible, as the previous 'no–go' remark implies.

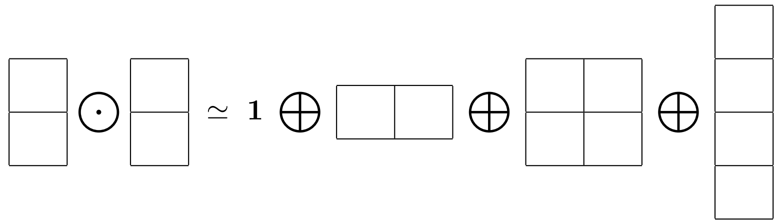
Our next task is to describe the circumstances which make the miracle to happen.

3.3. General $\mathcal{N}$'s: The main theorem. To solve the consistency conditions (3.15) it is natural to decompose the 'source' $T^{AB,CD}$, which transforms according to the *reducible* $Spin(\mathcal{N})$ representation[4] $\odot^2\text{Adj} \simeq (\wedge^2 V) \odot (\wedge^2 V)$, into *irreducible* representations.

[3] This class is somewhat 'generic', that is 'dense' in coupling costant space. Roughly speaking, the *no–go* theorem forbidding $\mathcal{N} \geq 4$ gaugings may be evaded only in a 'zero–measure set' of the coupling space. Then we expect that the solutions to our problem for $\mathcal{N} \geq 4$ constitute, at most, a zero–measure subset of all gaugings.

[4] V stands for the $\mathcal{N}$–dimensional vector representation of $Spin(\mathcal{N})$.

In terms of Young tableaux the decomposition of the T–tensor read



(3.19)

The main result of this section is the following

THEOREM 3.1. *A solution W to the consistency condition* (3.15) *exists iff the $\boxplus$ component of the T–tensor vanishes.*

PROOF. Assume $T^{AB,CD}\Big|_{\boxplus} = 0$. Then we have

$$T^{AB,CD} = \delta^{AC}\, T^{BD} - \delta^{AD}\, T^{BC} - \delta^{BC}\, T^{AD} + \delta^{BD}\, T^{AC} + T^{[ABCD]} \tag{3.20}$$

with $T^{AB} = T^{BA}$. Then, for $a \neq 1$

$$T^{1\,a,1\,a} = T^{1\,1} + T^{a\,a}. \tag{3.21}$$

Therefore the consistency conditions (3.15) become

$$W - \frac{1}{2}\left(T^{1\,1} + T^{a\,a}\right) \text{ is } \Sigma^{1\,a} - \text{harmonic for } a = 2, \dots, \mathcal{N}.$$

Let $c, d \neq 1$ or a (thus $\mathcal{N} \geq 3$). One has

$$T^{1\,c,1\,d} - T^{a\,c,a\,d} = (T^{cd} + \delta^{cd}\, T^{1\,1}) - (T^{cd} + \delta^{cd}\, T^{aa}) = \delta^{cd}(T^{1\,1} - T^{aa}).$$

By lemma 2.1 of chapt. 7 the LHS is $\Sigma^{1\,a}$–harmonic. So is the RHS. Thus $(T^{1\,1} - T^{a\,a})$ is $\Sigma^{1\,a}$–harmonic.

Take $W = T^{1\,1}$. Then

$$\begin{aligned} W - \frac{1}{2}\left(T^{1\,1} + T^{a\,a}\right) &= \frac{1}{2}\left(T^{1\,1} - T^{a\,a}\right) \\ &\equiv \Sigma^{1\,a} - \text{harmonic } \forall\, a = 2, \dots, \mathcal{N}. \end{aligned} \tag{3.22}$$

So a solution to eqns.(3.15) exists. We already know that it is unique. On the other hand, assume $T\Big|_{\boxplus} \neq 0$. This can happen only for $\mathcal{N} \geq 4$, since $T\Big|_{\boxplus} \equiv 0$ for $\mathcal{N} \leq 3$. Then the model is, in particular, an $\mathcal{N} = 4$ theory. But, in this special case, we already noticed that the consistency conditions have no solution. (See also sect. 4 below). □

REMARK. One advantage of the above formulation of the gauging problem is that the statement for local SUGRA will be *exactly* the same.

COROLLARY 3.1. *Set $W^1 = W$. Then for $A = 1, 2, \dots \mathcal{N}$ one has*

$$\boxed{W^A = T^{AA}} \tag{3.23}$$

PROOF. For $A = 1$ this is definition. For $A = a$ from eqns.(3.17)(3.21)

$$W^a = T^{1\,a,1\,a} - W = T^{1\,1} + T^{a\,a} - T^{1\,1} = T^{a\,a}. \tag{3.24}$$

Notice that

$$W^A = \frac{1}{2}\Big(T^{AB,AB} + T^{AC,AC} - T^{BC,BC}\Big) \qquad \forall\, B, C:\ A, B, C \text{ all distinct.} \tag{3.25}$$

□

REMARK. The result encodes the gaugings for all space–time dimensions D. We shall discuss this issue at the end of the chapter in the context of SUGRA. The reader may fill in the detail for the rigid case.

3.4. Fermionic shifts and potentials. By *fermionic shifts* we mean the SUSY transformation of the χ^i's evaluated in a constant bosonic background, namely the matrices $C^i_a(\phi)$ such that

$$\delta\chi^i = \cdots\cdots + C^i_a(\phi)\,\epsilon^a. \tag{3.26}$$

In the $\mathcal{N}=1$ formalism the SUSY transformation of fermions read

$$\delta\chi^i = \not{D}\phi^i\epsilon - (\delta_Q\phi^j)\,\Gamma^i_{jk}\chi^k - g^{ij}\,\partial_j W\,\epsilon. \tag{3.27}$$

If $\mathcal{N}>1$, we have a similar formula for each supercharge Q^a

$$\delta(f^a\chi)^i = \cdots\cdots - g^{ij}\,\partial_j W^a\,\epsilon^a, \qquad \text{(not summed over } a!) \tag{3.28}$$

so ($f^1=1$)

$$C^i_A = (f^A)^i_k\,g^{kj}\,\partial_j W^A. \tag{3.29}$$

From the basic Ward identity of chapt. 6, we know that

$$\frac{1}{2}\,g_{ij}\,C^i_A\,C^j_B = \delta_{AB}\,V(\phi). \tag{3.30}$$

Let us check this. From eqn.(3.25), for $A\neq B$ we can write ($C\neq A,B$)

$$W^A = \frac{1}{2}\Big(T^{AB,AB} + T^{AC,AC} - T^{BC,BC}\Big) \tag{3.31}$$

$$W^B = \frac{1}{2}\Big(T^{AB,AB} - T^{AC,AC} + T^{BC,BC}\Big). \tag{3.32}$$

Then (taking $A=a$ and $B=1$)

$$C^i_1 \equiv \partial^i W^1 = \mu^{1a\,m}\,l_{mn}\,(f^a)^{ij}\,K^n_j - \frac{1}{2}\partial^i\Big(T^{a\,C,a\,C} - T^{1\,C,1\,C}\Big)$$

$$C^i_a \equiv (f^a)^{ij}\,\partial_j W^a = -\mu^{1a\,m}\,l_{mn}\,K^{i\,n} + \frac{1}{2}\,(f^a)^{ij}\,\partial_j\Big(T^{a\,C,a\,C} - T^{1\,C,1\,C}\Big)$$

In view of eqn.(2.5) we can rewrite this as in the form [check]

$$\boxed{C^a_i = \mu^{a\,1\,m}\,l_{mn}\,K^n_i + \partial_i T^{aC,1C}} \tag{3.33}$$

where $C\neq 1,a$. From these we have

$$C^a_i g^{ij} C^a_j = \mu^{a\,1\,m} l_{mn}\mu^{a\,1\,n} + (\partial^i T^{aC,1C})(\partial_i T^{aC,1C}) = C^1_i g^{ij} C^1_j = 2V(\phi)$$
$$C^a_i g^{ij} C^1_j = 0.$$

and more generally that

$$C^A_i G^{ij} C^B_j = 2\delta^{AB}\,V(\phi). \tag{3.34}$$

3.5. Covariant expressions. For future reference we define

$$\boxed{C^{AB}_i = \mu^{AB\,m} l_{mn} K^n_i + \partial_i T^{AB}} \tag{3.35}$$

whose diagonal entries are just $\partial_i W^A$ and $C^{A1}_i = C^A_i$. By $Spin(\mathcal{N})$ covariance, we have

$$\delta(f^A\chi)^i = \cdots + g^{ij}\,C^{AB}_j\epsilon^B + \cdots. \tag{3.36}$$

4. Example: $\mathcal{N}=4$

In ref. [**170**] Gaiotto and Witten gave a very elegant interpretation of the constraint on the T–tensor in $\mathcal{N}=4$ CS SUSY. Let us recall that situation: The group $Spin(4)$ decomposes into the product $Spin(3)_1 \times Spin(3)_2$; correspondingly, the scalars' manifold $\mathcal{M}$ splits in the product of two hyperKähler spaces, $\mathcal{M}_1 \times \mathcal{M}_2$. Each factor manifold $\mathcal{M}_i$ ($i=1,2$) has three parallel 2–forms[5] ω_i^a ($a=1,2,3$). The ω_1^a's transform in the adjoint of $Spin(3)_1$ and are inert under $Spin(3)_2$, while the opposite holds for the ω_2^a's. The $\mathfrak{spin}(4)$–momentum map decomposes into two *independent* $\mathfrak{spin}(3)$–momentum maps on the factor manifolds, $\mu^{a\,m}$ and $\widetilde{\mu}^{a\,m}$, respectively in the representations $(\mathbf{3},\mathbf{1})$ and $(\mathbf{1},\mathbf{3})$ of $Spin(3)_1 \times Spin(3)_2$. Then the T–tensor belongs to the following representation

$$T \in \odot^2\Big((\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})\Big) \simeq \underbrace{(\mathbf{1},\mathbf{1})}_{\mathbf{1}} \oplus \underbrace{(\mathbf{1},\mathbf{1})}_{T^{[ABCD]}} \oplus \underbrace{(\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5})}_{\boxplus} \oplus \underbrace{(\mathbf{3},\mathbf{3})}_{\square\square}. \quad (4.1)$$

The $(\mathbf{3},\mathbf{3})$ component of the T–tensor couples the momentum map of $\mathcal{M}_1$ to the momentum map of $\mathcal{M}_2$

$$T\Big|_{(\mathbf{3},\mathbf{3})} = \mu^{a\,m}\, l_{mn}\, \widetilde{\mu}^{b\,n}, \quad (4.2)$$

while the other components are bilinear in the momentum map of a sigle factor manifold

$$T\Big|_{(\mathbf{1},\mathbf{1})\oplus(\mathbf{5},\mathbf{1})} = \mu^{a\,m}\, l_{mn}\, \mu^{b\,n}, \qquad T\Big|_{(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{5})} = \widetilde{\mu}^{a\,m}\, l_{mn}\, \widetilde{\mu}^{b\,n}. \quad (4.3)$$

The consistency constraint requires the components $(\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5})$ to vanish. These are components of the T–tensor which pertain to a single space. The (gauge) coupling between the two hyperKähler manifolds is not constrained by $\mathcal{N}=4$ SUSY, and the condition splits into two independent conditions for the factor spaces. Thus we can assume that we have just one manifold, say, $\mathcal{M}_1$. Now one has

$$T^{ab\,cd} = \epsilon^{abe}\epsilon^{cdf}\, T^{ef} \quad (4.4)$$

(all indices taking the values $1,2,3$), and the constraint is

$$T^{ab} \equiv \mu^{a\,m}\, l_{mn}\, \mu^{b\,n} = \delta^{ab}\, T. \quad (4.5)$$

Comparing with the remark on page 220, we see that this condition guarantees $\mathcal{N}=4$ SUSY, since, in this case, $T^{12,12} + T^{13,13} - 2\,T^{23,23}$ vanishes.

If $\mathcal{M}$ is symmetric[6], the Cartan–Kostant story allows to rewrite the condition (4.5) in terms of the algebra of $\mathfrak{iso}(\mathcal{M})$. Unfortunately, a symmetric hyperKähler is flat. $\mathcal{M}$, seen as a vector space, has a quaternionic structure, so we write the coordinates in the form q^I_α ($\alpha = 1,2$ and $I = 1,2,\dots,2m$), subjected to the usual symplectic reality condition

$$(q^I_\alpha)^* \stackrel{\text{def}}{=} q^\alpha_I = \epsilon^{\alpha\beta}\, \Omega_{IJ}\, q^J_\beta. \quad (4.6)$$

[5] $\omega_1^a = \eta^{aAB}\Sigma^{AB}$ where η^{aAB} is the 't Hooft tensor.

[6] The statement holds also for $\mathcal{M}$ simply *homogeneous*, provided $\mathcal{M}$ is either *compact* or *non–Ricci–flat*. The second case is impossible for $\mathcal{M}$ hyperKähler, while the first implies that $\mathcal{M}$ is a flat torus (for a proof, see footnote on page 14).

In this notation, the momentum map reads

$$\mu^{m}_{\alpha\beta} \equiv (\sigma^a)_{\alpha\beta}\, \mu^{a\,m} = \kappa^{m}_{IJ}\, q^I_\alpha\, q^j_\beta, \tag{4.7}$$

where the symmetric matrices κ^m_{IJ} generate the symplectic representation of $\mathfrak{iso}_0$ defined by[7]

$$A_{K^m} \;\longleftrightarrow\; \mathbf{1}\otimes\kappa^m. \tag{4.8}$$

Eqn.(4.5) becomes

$$0 = l_{mn}\, \mu^m_{(\alpha\beta}\, \mu^n_{\gamma)\delta} = \big(l_{mn}\, \kappa^m_{IJ}\, \kappa^n_{KL}\big)\, q^I_{(\alpha} q^J_\beta q^K_{\gamma)} q^L_\delta, \tag{4.9}$$

or [**170**]

$$l_{mn}\, \kappa^m_{(IJ}\, \kappa^n_{K)L} = 0. \tag{4.10}$$

A gauging of the model is defined by the following data: a group $\mathcal{G}$, with a symplectic representation $\{\kappa^m_{IJ}\}$, and a CS matrix l_{mn}. Our goal is to characterize the gaugings $(\mathcal{G}, \kappa^m_{IJ}, l_{mn})$ which preserve $\mathcal{N}=4$–SUSY.

THEOREM 4.1 (Gaiotto–Witten [**170**]). *The $\mathcal{N}=4$ gaugings $(\mathcal{G}, \tau^m_{IJ}, l_{mn})$ are in one–to–one correspondence with the Lie superalgebras whose fermionic generators form a quaternionic representation of the bosonic subalgebra and having an invariant non–degenerate quadratic form (l_{mn}, Ω_{IJ}). The Chern–Simons couplings are determined by the restriction of the quadratic form to the bosonic subalgebra.*

PROOF. Introduce bosonic M^m and fermionic λ_I generators of the Lie subalgebra $(m = 1, 2\ldots, \dim\mathcal{G},\; I = 1, 2, \ldots, \dim\mathcal{M}/2))$. Consider the brackets

$$[M^m, M^n] = f^{mn}{}_p M^p \tag{4.11}$$

$$[M^m, \lambda_I] = \kappa^m_{IJ}\Omega^{JK}\lambda_K \tag{4.12}$$

$$\{\lambda_I, \lambda_j\} = \kappa^m_{IJ}\, l_{mn}\, M^n. \tag{4.13}$$

These brackets define a Lie superalgebra if and only if the super–Jacobi identity holds. The only case which is not automatic, is the Jacobi identity with three λ's

$$[\lambda_I, \{\lambda_J, \lambda_K\}] + [\lambda_J, \{\lambda_K, \lambda_I\}] + [\lambda_K, \{\lambda_I, \lambda_J\}] = 0 \tag{4.14}$$

which corresponds precisely to eqn.(4.10). □

The Lie supergroup are classified in [**157**]. See table 8.1 for the list of those relevant for the $\mathcal{N}=4$ gaugings.

REMARK. A rigid $\mathcal{N}\geq 5$ model is, in particular, an $\mathcal{N}=4$ model. Since in this case $\mathcal{M}$ is necessarily flat, these gaugings should be described by the Gaiotto–Witten theorem. In the table we added a column with max $\mathcal{N}$, meaning the maximal supersymmetry we can construct with that Lie superalgebra. For the details see refs. [**193, 194, 192, 195, 196**]. There you can find many other interesting results about superconformal gaugings in three dimensions (for all $\mathcal{N}$'s).

[7] Recall from the Cartan–Constant isomorphism that the endomorphisms A_K do represent $\mathfrak{iso}_0$ if $\mathcal{M}$ is flat.

Lie superalgebra	gauge group	max $\mathcal{N}$
$U(2\|2)$	$SU(2) \times SU(2) \times U(1)$	8
$U(m\|n)$	$SU(m) \times SU(n) \times U(1)$	6
$Osp(2\|n)$	$SO(2) \times Sp(2n)$	6
$Osp(m\|n)$	$SO(m) \times Sp(2n)$	5
$F(4)$	$SO(7) \times SU(2)$ 5	5
$G(3)$	$G_2 \times SU(2)$	5
$D(2\|1;\alpha)$	$SO(4) \times Sp(2)$	5

TABLE 8.1. Lie supergroups with a symplectic action of the bosonic subgroup on the fermionic generators having a non–degenerate quadratic form. (Cfr. [**192**]).

5. Gauged supergravities

In passing from rigid SUSY to SUGRA, many things happens. First of all, there are $\mathcal{N}$ gravitini, ψ^A_μ, in the vector representation of $Spin(\mathcal{N})$. Second, the $Spin(\mathcal{N})$ symmetry is gauged: From the point of view of the χ^i's the $Spin(\mathcal{N})$ connection is identified with a part of the the Christoffel connection of $\mathcal{M}$, in view of the isomorphism[8]

$$T\mathcal{M} \simeq S \otimes U.$$

As in the rigid case, the constraints on the possible gaugings stem from the consistency of the Yukawa couplings. In supergravity there are *three* classes of such couplings

$$e\, g\, \left\{ \frac{1}{2} A_1^{AB}\, \overline{\psi}^A_\mu \gamma^{\mu\nu} \psi^B_\nu + A^A_{2\,i}\, \overline{\psi}^A_\mu \gamma^\mu \chi^i + \frac{1}{2}\, A_{3\,ij}\, \overline{\chi}^i \chi^j \right\}$$

where $A_1^{AB} = A_1^{BA}$ and $A_{3\,ij} = A_{3\,ji}$. From the GENERAL LESSONS of chapter 7 we know that the three Yukawa tensors A_1, A_2 and A_3 should have a universal form in terms of the T–tensor. The constraints on the T–tensor express exactly the requirement that the induced Yukawa tensors have the correct algebraic properties.

5.1. Gauging supergravity. The gauge group, $\mathcal{G}$ is a subgroup of $\mathrm{Iso}(\mathcal{M})$. In SUGRA it is usual to write the embedding $\mathfrak{g} \hookrightarrow \mathfrak{iso}(\mathcal{M})$ in terms of an *embedding tensor* [**197, 198, 50**]. In the present (*diet*) situation it is written as an element of $\mathfrak{iso} \otimes \mathfrak{iso}$, which we write as l_{mn}. The (infinitesimal) gauge transformations then have the form

$$\delta\Phi = \Lambda(x)^m\, l_{mn}\, \pounds_{K^n}\, \Phi \qquad (5.1)$$

where the $\Lambda(x)^m$'s are (spacetime dependent) parameters. If $\mathcal{G}$ is semi–simple, l_{mn} is a multiple of the Cartan–Killing form on each simple group factor, and hence it is symmetric, $l_{mn} = l_{nm}$. This property holds in general

[8] There are subtleties in the $\mathcal{N} = 2$ case, see §. **7**.5.4.

(in facts $\mathfrak{iso}(\mathcal{M}) \simeq \ker l \oplus \operatorname{im} l$ should be an orthogonal decomposition). The requirement that the image of l is a Lie subalgebra reads

$$l_{mp}\, l_{nq}\, f^{pq}{}_k = c_{mn}{}^h\, l_{hk} \tag{5.2}$$

with $c_{mn}{}^h$ the structure constants of $\mathfrak{g} \equiv \mathfrak{Lie}(\mathcal{G})$.

The scalars' covariant derivative is

$$\mathcal{D}_\mu \phi^i = \partial_\mu \phi^i - g\, l_{mn}\, A^m_\mu\, K^{i\,n} \tag{5.3}$$

where we inserted the coupling g as an order–counting device. This derivative transforms covariantly

$$\mathcal{D}_\mu \phi \to (\delta^i{}_j + g\, l_{mn}\, \Lambda^m \partial_j K^{i\,n})\, \mathcal{D}_\mu \phi^j, \tag{5.4}$$

provided the gauge fields transform as

$$l_{mn}\, \delta A^n_\mu = l_{mn}\Big(\partial_\mu \Lambda^n - g\, c_{pq}{}^n A^p_\mu\, \Lambda^q\Big). \tag{5.5}$$

Consider the gauge variation of the gravitino ψ^A_μ. One has

$$\delta_K \psi^A_\mu = \pounds_K \psi^A_\mu = (A_K + D_K)\psi^A_\mu = \tag{5.6}$$

$$= A^{AB}_K \psi^B_\mu + K^i Q^{AB}_i \psi^B_\mu \tag{5.7}$$

and the new covariant derivative is

$$\mathcal{D}_\mu \psi^A_\nu = \nabla_\mu \psi^A_\nu + \partial_\mu \phi^i\, Q^{AB}_i \psi^B_\nu + g\, l_{mn}\, A^m_\mu\, \mathcal{A}^{AB\,n}\, \psi^B_\nu \tag{5.8}$$

where ∇_μ is the covariant derivative with respect the curved spacetime connection, and we wrote $\mathcal{A}^{AB\,n} \equiv A^{AB}_{K^n}$ for the A_K–derivation (= one–half the covariant momentum map!). Covariance requires also

$$\mathcal{D}_\mu \epsilon^A = \nabla_\mu \epsilon^A + \partial_\mu \phi^i\, Q^{AB}_i \epsilon^B + g\, l_{mn}\, A^m_\mu\, \mathcal{A}^{AB\,n}\, \epsilon^B. \tag{5.9}$$

Now the SUSY variation of the Rarita–Schwinger kinetic term, $-\frac{i}{2}\varepsilon^{\mu\nu\rho}\, \overline{\psi}^A_\mu \mathcal{D}_\nu \psi^A_\rho$, has an additional contribution steming from the fact that

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = \cdots + g\, l_{mn}\, F^m_{\mu\nu}\, \pounds_{K^n}, \tag{5.10}$$

where $\cdots$ stands for terms already present in the ungauged theory (*i.e.* for $g = 0$). A similar term appears in the variation of the χ's kinetic terms, since $(f^A)^i{}_j \delta\chi^j = \gamma^\mu \mathcal{D}_\mu \phi^i \epsilon^A$+2–fermions. The SUSY variation of the covariantized original terms then read

$$\delta\mathcal{L}_0 = \frac{i}{2} g\, l_{mn}\, \epsilon^{\mu\nu\rho}\, F^m_{\mu\nu} \left(\mathcal{A}^{AB\,n}\, \overline{\psi}^A_\mu \epsilon^B + \frac{1}{2} K^n_i\, (f^A)^i{}_j\, \overline{\chi}^j \gamma_\mu \epsilon^A \right). \tag{5.11}$$

in order to cancel this variation, one introduces the Chern–Simons term

$$\mathcal{L}_{CS} = \frac{i}{4} g\, \varepsilon^{\mu\nu\rho}\, A^m_\mu\, l_{mn}\, \big(\partial_\nu A^n_\rho - \frac{1}{3} g\, c_{pq}{}^n A^p_\nu A^q_\rho\big), \tag{5.12}$$

and the SUSY variation of the vector fields

$$l_{mn}\, \delta A^m_\mu = l_{mn} \Big[2\, \mathcal{A}^{AB\,m}\, \overline{\psi}^A_\mu \epsilon^B + K^m_i (f^A)^i{}_j \overline{\chi}^j \gamma_\mu \epsilon^A \Big] \tag{5.13}$$

However, this variation of the vectors produces additional SUSY variations at order $O(g)$

$$\begin{aligned} &\delta(\mathcal{L}_0 + \mathcal{L}_{CS}) = \\ &\quad = e\, g\, l_{mn} \big(2\, \mathcal{A}^{AB\,m}\, \overline{\psi}^A_\mu \epsilon^B + K^m_i (f^A)^i{}_j \overline{\chi}^j \gamma_\mu \epsilon^A \big)\, K^n_k\, \mathcal{D}^\mu \phi^k + \cdots \end{aligned} \tag{5.14}$$

where we wrote only the terms linear in the fermions. This variation is cancelled by introducing Yukawa couplings

$$e\left\{\frac{1}{2}A_1^{AB}\,\overline{\psi}_\mu^A\gamma^{\mu\nu}\psi_\nu^B+A_{2\,i}^{A}\,\overline{\psi}_\mu^A\gamma^\mu\chi^i+\frac{1}{2}\,A_{3\,ij}\,\overline{\chi}^i\chi^j\right\} \tag{5.15}$$

and fermionic shifts in the SUSY transformations

$$\delta\psi_\mu^A=\cdots+g\,A_1^{AB}\gamma_\mu\epsilon^B,\qquad \delta\chi^i=\cdots-g\,A_2^{iA}\epsilon^A \tag{5.16}$$

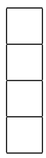
where $\cdots$ stands for $O(g^0)$ terms already present in the ungauged theory. Finally, to cancel the variations at order $O(g^2)$, we should add a scalars' potential, which we already know is given by the universal Ward identity of chapter 6.

5.2. Constraints on the covariant T–tensor. Let us look at the conditions under which the term in $\delta\mathcal{L}$ proportional to $g\,\psi_\mu^A\psi_\nu^B\psi_\rho^C$ vanishes. There are two sources of such terms: the variation of A_μ inside the gravitino kinetic term, and the variation of e in front of the Yukawa bilinear in ψ_μ^A.

$$\begin{aligned}
&-\frac{i}{2}g\,\overline{\psi}_\mu^A\gamma^{\mu\nu\rho}\psi_\rho^B\,l_{mn}\,\mathcal{A}^{AB\,n}\,\delta A_\nu^M\Big|_\psi+\frac{g}{2}\,\delta e\,A_1^{AB}\,\overline{\psi}_\mu^A\gamma^{\mu\nu}\psi_\nu^B=\\
&=-\frac{1}{2}g\,\overline{\psi}_\mu^A\gamma^{\mu\nu\rho}\psi_\rho^B\,l_{mn}\,\mathcal{A}^{AB\,n}\,\mathcal{A}^{CD\,m}\,\overline{\psi}_\nu^C\epsilon^D+\frac{1}{4}g(\overline{\epsilon}^C\gamma^\rho\psi_\rho^C)\,A_1^{AB}\,\overline{\psi}_\mu^A\gamma^{\mu\nu}\psi_\nu^B=\\
&=-\frac{g}{2}T^{AB,CD}\,(\overline{\psi}_\mu^A\gamma^{\mu\nu\rho}\psi_\rho^B\;\overline{\psi}_\nu^C\,\epsilon^D)+\frac{g}{4}(A_1^{AB}\delta^{CD})(\overline{\psi}_\mu^A\gamma^{\mu\nu}\psi_\nu^B\;\overline{\psi}_\rho^C\gamma^\rho\epsilon^D)
\end{aligned}$$

(the formulae are meant to be schematic).

Notice that the totally antisymmetric part of the T–tensor, $T^{[AB,CD]}$, corresponding to the vertical tableux



decouples from the first term in the last line. Indeed, since in $3D$ the Majorana fermions have only two real components $\psi_{[\mu}^{[A}\psi_\nu^B\psi_{\rho]}^{C]}\equiv 0$, by Fermi statistics. Therefore, the condition $\delta\mathcal{L}\big|_{\psi\psi\psi}=0$ gives:

GENERAL LESSON 5.1. *In order a given* SUGRA *gauging to be allowed the associated T–tensor should satisfy the algebraic condition*

$$T^{AB,CD}=\frac{1}{4}\Big(\delta^{AC}A_1^{BD}-\delta^{BC}A_1^{AD}-\delta^{AD}A_1^{BC}+\delta^{BD}A_1^{AC}\Big)+T^{[AB,CD]}, \tag{5.17}$$

that is

$$T^{AB,CD}\Big|_{\boxplus}=0. \tag{5.18}$$

As anticipated, this is precisely the same condition as in rigid SUSY.

Comparing with the definition of T^{AB}, eqn.(3.20), we get[9]:

[9] Recall that there is a factor 1/4 in the definition of the covariant T–tensor with respect to the rigid one. Here we are rather cavalier with numerical constants. They are not really computed, rather they are fixed at the end by 'geometrical arguing' (as we advocated many times). The final formulae are surely correct, the intermediate ones are meant to be just schematic.

COROLLARY 5.1. *For $\mathcal{N} \geq 3$ we have:*

$$\boxed{A_1^{AB} = T^{AB}} \tag{5.19}$$

The cases $\mathcal{N} = 1, 2$ are special:

(1) *For $\mathcal{N} = 1$, both sides of eqn.*(5.17) *are identically zero. This corresponds to the fact that the Yukawa couplings contain terms coming from a real superpotential (as in the rigid case) and hence may be not zero even if* no *gauging is present. We can parameterize all Yukawa/potential couplings in terms of the gravitino mass, A_1, (which is essentially the real superpotential) together with the symmetric tensor $T_{ij} \equiv K_i^m \, l_{mn} \, K_j^n$ which fully encodes the gauging of the given $\mathcal{N} = 1$ model.*

(2) *For $\mathcal{N} = 2$, the* LHS *of eqn.*(5.17) *is a singlet of $Spin(2)$. Hence only the singlet part of A_1^{AB} (that is its trace $\delta_{AB} \, A_1^{AB}$) is determined in terms of the T–tensor. The components corresponding to $Spin(2)$ charge ± 2*

$$\tfrac{1}{2}(A_1^{22} - A_1^{11}) \pm i \, A_1^{12} \tag{5.20}$$

are not *determined by the gauging. Again, as in the rigid case, this corresponds to the fact that we may add a non–trivial complex superpotential (a general solution to the homogeneous consistency condition). Again we can parameterize all the Yukawa and potential couplings in terms of the T–tensor plus the complex quantity $\frac{1}{2}(A_1^{22} - A_1^{11}) + i \, A_1^{12}$. (See chapter* **??** *for further details).*

We have still to determine the Yukawa tensors A_2 and A_3, the fermionic shifts and potential, and check that everything works fine. We do this in the next subsection, exploiting the Cartan–Kostant isomorphism.

5.3. Fermionic shifts, Yukawa tensors and potential. The direct computation of the Yukawas and fermionic shifts, for the general case, is quite involved (see ref. [**50**] for some details). Thus, instead of computing, let us try to argue on general grounds on their possible forms.

First of all, the fermionic shifts can be read directly from the SUSY current, namely from the terms in the Lagrangian $\mathcal{L}$ linear in the gravitino fields ψ_μ^A. Hence

$$\delta\psi_\mu^A = \cdots + g \, A_1^{AB} \, \gamma_\mu \epsilon^B \tag{5.21}$$

$$\delta\chi^i = \cdots - g \, g^{ij} \, A_{2\,j}^{A} \, \epsilon^A \tag{5.22}$$

so it is enough to determine the Yukawa tensors (the scalars' potential is then predicted by the universal formula of chapt. 7). We already know that they have a universal expression in terms of the T–tensor.

The first tensor, A_1^{AB}, was already computed in eqn.(5.17). To determine $A_{2\,i}^{A}$ and A_{3ij}, we return to the scaling argument we used in chapt. 2 to predict the 4–Fermi couplings. Rescaling the volume of $\mathcal{M}$ (or, more correctly, the Planck scale) we can 'switch off' the supergravity couplings, ending up with a rigid $\mathcal{N}$–SUSY model for which we know both the fermionic shift $A_{2\,i}^{A}$ and the Yukawa matrix A_{3ij}. Moreover, we have a scaling property with respect to the gauge coupling g. Therefore the correct formula for the SUGRA case should be the one in the rigid case (minimally covariantized, that is with derivatives replaced by $Spin(\mathcal{N})$–covariant derivatives) *plus* corrections which vanish in the above rigid limit. As in chapt. 2, the possible corrections should have (at least) one more factor of g_{ij} than the ones present in the rigid limit. Thus $A_2{}^A{}_i$ should not be corrected (except for covariantization), while

$$A_{3\,ij} = A_{3\,ij}^{\text{rigid}}\Big|_{\text{covariantized}} + g_{ij}\,F\big(T^{AB,CD}\big) \tag{5.23}$$

where, by scaling in g, F is some *linear* function of the T–tensor.

To determine F, we use a clever trick of ref. [**50**]. In order to make the action of the $Spin(\mathcal{N})$ group more manifest, one introduces an overcomplete system of spin–1/2 fermions, $\chi^{A\,i}$, $A = 1, 2, \ldots, \mathcal{N}$, $i = 1, 2, \ldots, \dim\mathcal{M}$ by

$$\chi^{A\,i} = (f^A)^i{}_j\,\chi^j. \tag{5.24}$$

The fact that these fermions are not independent is written as the constraint

$$\chi^{A\,i} = \mathbb{P}^{A\,i}_{B\,j}\,\chi^{B\,j}, \tag{5.25}$$

where $\mathbb{P}^{A\,i}_{B\,j}$ is the projector

$$\mathbb{P}^{A\,i}_{B\,j} := \frac{1}{\mathcal{N}}\Big(\delta^{AB}\,\delta^i_j - (\Sigma^{AB})^i{}_j\Big). \tag{5.26}$$

Now one writes the fermionic shift in the form

$$\delta\chi^{A\,i} = \cdots - g^{ij} A_2^{AB}{}_j\,\epsilon^B \tag{5.27}$$

and the $\chi\chi$ Yukawa couplings as

$$\frac{1}{2}\,e\,g\,A_{3\,ij}^{AB}\;\overline{\chi}^{A\,i}\chi^{B\,j}. \tag{5.28}$$

These matrices should satisfy the algebraic conditions

$$A_2^{AC}{}_k\,\mathbb{P}^{C\,k}_{B\,i} = A_2^{AB}{}_i \tag{5.29}$$

$$A_3^{AC}{}_{ik}\,\mathbb{P}^{C\,k}_{B\,j} = A_3^{AB}{}_{ij} \tag{5.30}$$

$$A_3^{AB}{}_{ij} = A_3^{BA}{}_{ji}. \tag{5.31}$$

As we shall see momentarily, these relations are sufficient to uniquely fix our unknown function F.

The χ–shift is read directly from eqn.(3.35),

$$A_2^{AB}{}_i = \boldsymbol{\mu}^{AB\,m}\,l_{mn}\,K_i^n + \mathcal{D}_i T^{AB} \tag{5.32}$$

where we used the fact that our T–tensor satisfies the constraint $T\Big|_{\boxplus} = 0$. In fact, as written, eqn.(5.32) is correct for $\mathcal{N} \geq 3$, for reasons explained in corollary 5.1. The general expression, valid for all $\mathcal{N}$'s, is

$$\boxed{A_2^{AB}{}_i = \boldsymbol{\mu}^{AB\,m}\,l_{mn}\,K_i^n + \mathcal{D}_i A_1^{AB}} \tag{5.33}$$

which has the correct rigid limit. Notice that the first term in the RHS is proportional to

$$\theta(K^m)\,l_{mn}\,\theta(K^n)\Big|_{\mathfrak{spin}(\mathcal{N})\otimes\mathfrak{m}} \tag{5.34}$$

so it is again a component of the T–tensor as generalized by the Cartan–Kostant isomorphism. For this reason we denote is as T_i^{AB}

$$T_i^{AB} \equiv -T_i^{BA} := \boldsymbol{\mu}^{AB\,m}\, l_{mn}\, K_i^n. \tag{5.35}$$

Notice that

$$\mathcal{D}_{(i}T_{j)}^{AB} = (\Sigma^{AB})_{(i}{}^k K_k^m\, l_{mn}\, K_{j)}^n. \tag{5.36}$$

One shows that the $A_2^{AB}{}_i$ in eqn.(5.33) satisfies eqn.(5.29) as a consequence of the fact that $T\big|_{\boxplus} = 0$. *Indeed, we already know this:* as long as we have expressions with only one (covariant) derivative the identities of the rigid case apply to the local one too. Instead, expressions which contain $\mathcal{D}_i\mathcal{D}_jT^{AB,CD}$ are different in the two case due to the presence, in the SUGRA case of a antisymmetric term in $i \leftrightarrow j$ due to the non–trivial $Spin(\mathcal{N})$ curvature. But $A_2^{AB}{}_i$ (which is, morally speaking, like the auxiliary field F in $D = 4$, $\mathcal{N} = 1$ SUSY) contains only first derivatives, and the algebraic proof of $Spin(\mathcal{N})$ invariance, under the condition $T\big|_{\boxplus} = 0$ in sect. 3.3 applies word–for–word.

On the contrary, the Yukawa couplings contain the second derivative and hence corrections are necessary in order to maintain the necessary symmetry properties. The relation between the curvature terms, arising from the commutation of covariant derivatives, and the symmetry condition forces a term proportional to g_{ij} to be present, as we shall see momentarily. No other correction are needed (nor possible), in agrement with the general arguments.

In the rigid case one has for the diagonal entries[10] (no sum over A!)

$$-\frac{\mathcal{N}}{2}\, A_{3\,ij}^{AA} = \partial_i A_{2\,j}^{AA} + K_i^m\, l_{mn}\, K_j^n, \tag{5.37}$$

which we can upgrade to a fully $Spin(\mathcal{N})$ covariant expression with the help of the projector $\mathbb{P}_{B\,j}^{A\,i}$. Therefore, in the local case we must have

$$(-1/2)\mathcal{N}^2 A_{3\,ij}^{AB} = \mathcal{N}\mathcal{D}_i A_{2\,j}^{AB} - K_i^m\, l_{mn}\, K_k^n\, \mathcal{N}\mathbb{P}_{B\,j}^{A\,k} + g_{ik}F^{AC}\,\mathcal{N}\mathbb{P}_{B\,j}^{C\,k}. \tag{5.38}$$

where F^{AB} is the covariant counterpart to the function F in eqn.(5.23), which should be linear in the components of the T–tensor. The first term in the RHS already satisfies the projection constraint, as a consequence of $T\big|_{\boxplus} = 0$.

Now,

$$\begin{aligned}\mathcal{D}_i A_{2\,j}^{AB} &= \mathcal{D}_{(i}T_{j)}^{AB} + \mathcal{D}_{[i}T_{j]}^{AB} + \mathcal{D}_{(i}\mathcal{D}_{j)}T^{AB} + \mathcal{D}_{[i}\mathcal{D}_{j]}T^{AB} = \\ &= \mathcal{D}_{(i}\mathcal{D}_{j)}T^{AB} + \mathcal{D}_{[i}T_{j]}^{AB} + \\ &\quad + (\Sigma^{AB})_{(i}{}^k K_k^m l_{mn} K_{j)}^n - \frac{1}{2}(\Sigma^{AC})_{ij}T^{CB} - \frac{1}{2}(\Sigma^{BC})_{ij}T^{AC},\end{aligned} \tag{5.39}$$

where we used eqn.(5.36) and the explicit form of the $Spin(\mathcal{N})$ curvature. In the RHS, the first line has already the correct symmetry under the interchange $(A, i) \leftrightarrow (B, j)$; the other terms should give symmetric terms after

[10] The normalization is different with respect to the one we used in the rigid case to get formulae simuilar to the ones one finds in the SUGRA literature. Nothing in the argument depends on the specific numeric coefficients.

adding the other two terms in the RHS of eqn.(5.38). One has

$$\begin{aligned}
\text{RHS of eqn.(5.38)} &= \mathcal{D}_{(i}\mathcal{D}_{j)}T^{AB} + \mathcal{D}_{[i}T^{AB}_{j]}+\\
&+ (\Sigma^{AB})_{(i}{}^{k} K^m_k l_{mn} K^n_{j)} - \frac{1}{2}(\Sigma^{AC})_{ij}T^{CB} - \frac{1}{2}(\Sigma^{BC})_{ij}T^{AC}+\\
&- \delta^{AB}K^m_i l_{mn}K^n_j + K^m_i l_{mn} K^n_k (\Sigma^{AB})^k{}_j + g_{ij}F^{AB} - F^{AC}(\Sigma^{CB})_{ij} =\\
&= \mathcal{D}_{(i}\mathcal{D}_{j)}T^{AB} + \mathcal{D}_{[i}T^{AB}_{j]} + \delta^{AB}K^m_i l_{mn}K^n_j + K^m_{[i} l_{mn}K^n_k(\Sigma^{AB})^k{}_{j]}+\\
&+ g_{ij}F^{AB} - (\Sigma^{BC})_{ij}\left(\frac{1}{2}T^{AC} - F^{AC}\right) - \frac{1}{2}(\Sigma^{AC})_{ij}T^{CB}
\end{aligned} \tag{5.40}$$

The first line in the RHS has the required symmetry. The second line has the correct symmetry iff

$$F^{AB} = F^{BA} \tag{5.41}$$

$$T^{AC} - 2F^{AC} = -T^{AC} \tag{5.42}$$

that is

$$\boxed{F^{AB} = T^{AB}} \tag{5.43}$$

Then the Yukawa couplings are completely determined in terms of the T–tensor.

Once determined the fermionic shifts, the scalars' potential is also determined by the Ward identity. The validity of the Ward identity requires the fermionic shifts to satisfy a number of subtle algebraic identities which we shall not verify here. The diligent reader may check them by himself/herself (and the curious, but not so diligent, can trust reference [**50**]).

5.4. ∗. Identities for future reference. The tensor $A^{AB}_{2\,i}$ satisfies two equations. Firstly

$$A^{AB}_{2\,i} = \underset{\text{antisymm.}}{T^{AB}_i} + \underset{\text{symm.}}{\mathcal{D}_i A^{AB}_1} \tag{5.44}$$

and then the projection (5.29) which we write as

$$(\mathcal{N}-1)\, A^{AB}_{2\,i} = A^{AC}_{2\,k}(\Sigma^{CB})^k{}_i. \tag{5.45}$$

Taking the symmetric and antisymmetric parts in A, B, we get

$$2(\mathcal{N}-1)\mathcal{D}_i A^{AB}_1 = \Big(\mathcal{D}_k A^{AC}_1 (\Sigma^{CB})^k{}_i + (A \leftrightarrow B)\Big) + 4\mathcal{D}_i T^{AC,CD} \tag{5.46}$$

$$2(\mathcal{N}-1)T^{AB}_i = (T^{AC\;k} + \mathcal{D}^k A^{AC}_1)(\Sigma^{CB})_{ki} - (A \leftrightarrow B)\Big) \tag{5.47}$$

where we used the definition of the covariant momentum map ad T–tensor

$$T^{AC\;k}\,(\Sigma^{CB})_{ki} = \mathcal{D}_i(\boldsymbol{\mu}^{AC\,m}\, l_{mn}\, \boldsymbol{\mu}^{CB\,n}) = 4\mathcal{D}_i T^{AC,CD}. \tag{5.48}$$

6. Symmetric target spaces

Things simplify a lot if the scalar manifold $\mathcal{M}$ is *symmetric* (or just homogeneous). This is automatically true for $\mathcal{N} \geq 5$ (or, if one wishes a D–independent statement, for a number of supercharges $\mathfrak{N} \geq 9$), but even for $\mathcal{N} = 2, 3$ and 4 there are many interesting[11] Kähler (resp. Quaternionic–Kähler) manifolds who do are symmetric (or homogeneous). In the Kähler case they are the Hermitean symmetric manifolds; in the Quaternionic–Kähler one, we have the (symmetric) Wolf spaces, see ref. [**199**] and tables below. (The homogeneous, non–symmetric, Quaternionic–Kähler spaces are known as the Alekseevskiǐ manifolds: there are three infinite families, ref. [**200**]). The following formulae and results will apply to all such cases.

We may assume the symmetric space $\mathcal{M}$ to be *irreducible* without loss of generality. Thus, in this section, we write G/H for $\mathcal{M}$ with G simple and $H \equiv Spin(\mathcal{N}) \times H'$ its maximal compact subgroup. We also recall from chapts. 3, 5 that the Lie algebra $\mathfrak{g}$ of G decomposes into irreducible representations[12] of $Spin(\mathcal{N})$ as

$$\mathfrak{g} = \mathfrak{spin}(\mathcal{N}) \oplus \mathbf{1}^{\oplus \dim \mathfrak{h}'} \oplus S^{\oplus m} \tag{6.1}$$

where S is an irreducible spin representation, and $m = \dim \mathcal{M}/\mathbf{N}(\mathcal{N})$.

6.1. Lifting the consistency constraint [**50**]**.** We saw in sect. 5.2 that the consistency requirements on the gauging data $(\mathcal{G}, l_{mn})$ reduce to the single equation $T\big|_{\boxplus} = 0$. The (generalized) T–tensor is easily computed with the help of eqn.(6.18), which we rewrite for the convenience of the reader (as always, $g \in G$ stands a representative of the coset G/H)

$$g^{-1}\, t^m\, g = \frac{1}{4} A^{AB}_{K^m}\, \gamma^{AB} + H^{\alpha}_{K^m}\, h^{\alpha} + L^m_a\, m^a. \tag{6.2}$$

In particular,

$$T^{AB,CD}(g) = \eta\ \mathrm{tr}\Big[\gamma^{AB}\, g^{-1}\, t^m\, g\Big]\, l_{mn}\ \mathrm{tr}\Big[\gamma^{CD}\, g^{-1}\, t^n\, g\Big], \tag{6.3}$$

where η is a suitable normalization constant. Now we must require

$$T^{AB,BC}(g)\Big|_{\boxplus} = 0 \quad \forall\, g \in G. \tag{6.4}$$

6.1.1. *The generalized T–tensor.* It is convenient to consider the *generalized* T–tensor defined as (cfr. eqns.(7.20)–(7.22) of chapt. 7)

$$T^{k,h}(g) = \eta\ \mathrm{tr}\Big[t^k\, g^{-1}\, t^m\, g\Big]\, l_{mn}\ \mathrm{tr}\Big[t^h\, g^{-1}\, t^n\, g\Big] \tag{6.5}$$

$$T^{k,h}(g) \in \odot^2 \mathfrak{g} \simeq \odot^2 \big(\mathfrak{spin}(\mathcal{N}) \oplus \mathfrak{h}' \oplus \mathfrak{m}\big). \tag{6.6}$$

or, more abstractly,

$$\mathcal{T}(g) := \sum_{m,n} l_{mn}\ (g^{-1}\, t^m\, g) \otimes (g^{-1}\, t^m\, g) \in \mathfrak{g} \odot \mathfrak{g}. \tag{6.7}$$

$\mathfrak{g} \odot \mathfrak{g}$ is a G–module (a representation) with respect to the tensor product of the adjoint representation $\varrho \equiv \mathrm{Adj}_G \otimes \mathrm{Adj}_G$

$$\varrho(h)(x \odot y) = hxh^{-1} \odot hyh^{-1}, \qquad h \in G,\ x, y \in \mathfrak{g},$$

so

$$\varrho(g^{-1})\mathcal{T}(e) \equiv \mathcal{T}(g). \tag{6.8}$$

[11] Interesting, in particular, for the 'phenomenological' applications.

[12] This is true for $\mathcal{N} \neq 4$. For $\mathcal{N} = 4$ the adjoint rep. is not irreducible. The modifications are obvious.

Let us decompose the G–module $\mathfrak{g}\odot\mathfrak{g}$ into *irreducible* G–modules R_i, where $R_0 \equiv \mathbf{1}$ stands for the *unique* trivial representation corresponding to the Cartan–Killing form of the isometry group G,

$$\mathfrak{g}\odot\mathfrak{g} = \mathbf{1}\oplus\Big[\bigoplus\nolimits_{i\geq 1} R_i\Big]. \tag{6.9}$$

Obviously,

$$\mathcal{T} = \lambda_{\mathcal{T}}\, K + \sum_{i\geq 1} \mathcal{T}\Big|_{R_i} \tag{6.10}$$

(K is the Cartan–Killing form). Comparing with eqn.(6.8), we see that the full functional dependendece of $\mathcal{T}(g)$ on the scalars is encoded in the G–action ϱ, that is in the decomposition of $\mathcal{T}(g)$ into G–irreducible components.

Each irreducible G–representation R_i can be further decomposed into irreducible $Spin(\mathcal{N})\subset G$ representations

$$R_i = \bigoplus_j R_{i,j}. \tag{6.11}$$

Now, as $Spin(\mathcal{N})$ representations,

$$\begin{aligned}\bigoplus_{i,j} R_{i,j} &= \mathfrak{g}\odot\mathfrak{g} \equiv \odot^2\big(\mathfrak{spin}(\mathcal{N})\oplus \mathbf{1}^{\oplus k}\oplus S^{\oplus m}\big) = \\ &= \odot^2\mathfrak{spin}(\mathcal{N}) \;\oplus\; \mathbf{1}^{\oplus k(k+1)/2} \;\oplus \mathfrak{spin}(\mathcal{N})^{\oplus k} \;\oplus \\ &\quad\oplus\; \big(S\odot S\big)^{k(k+1)/2} \;\oplus\; \big(S\wedge S\big)^{k(k-1)/2} \;\oplus \\ &\quad\oplus\; S^{\oplus km} \;\oplus\; \big(\mathfrak{spin}(\mathcal{N})\otimes S\big)^{\oplus m}.\end{aligned} \tag{6.12}$$

In the third line of the RHS we have only *spinorial* representations. In the second line enter only irrepresentations which are contained in $S\otimes S$; all such irrepresentations have the form $\wedge^l V$, where V is the vector representation of $Spin(\mathcal{N})$ (see ref...). In the first line, we have our old friend $\odot^2\mathfrak{spin}(\mathcal{N})$ up to copies of $\mathbf{1}$ and $\mathfrak{spin}(\mathcal{N})$ (the adjoint irrepr.). The decomposition of $\odot^2\mathfrak{spin}(\mathcal{N})$ was given in eqn.(3.19) in terms of $SO(\mathcal{N})$ Young tableaux. Thus eqn.(6.12) proves the following:

LEMMA 6.1. *In the $Spin(\mathcal{N})$ decomposition $\odot^2\mathfrak{g}\simeq\bigoplus_{i,j} R_{i,j}$ there is* only *one* copy *of the irreducible $Spin(\mathcal{N})$ representation $\boxplus$. Therefore, in the G decompostion $\odot^2\mathfrak{g}\simeq\bigoplus_i R_i$, there is* one and only one *irreducible G–representation, written $R_{\boxplus}$, such that the irreducible $Spin(\mathcal{N})$ representation $\boxplus$ appears (with multiplicity 1) in its $Spin(\mathcal{N})$ decomposition (cfr. eqn.(6.11))*

$$R_{\boxplus} = \bigoplus_j R_{\boxplus,j} = \boxplus \;\oplus\; \cdots \tag{6.13}$$

COROLLARY 6.1. *The condition $T^{AB,CD}(g)\big|_{\boxplus}=0$, identically on G/H,* IS EQUIVALENT *to the condition*

$$\mathcal{T}\Big|_{R_{\boxplus}} = 0 \tag{6.14}$$

where $\mathcal{T}$ is the G–representation ϱ in eqn.(6.8).

$\mathcal{N}$	G/H	dim.	Adj_G	$\odot^2(\mathrm{Adj}_G)$
5	$\frac{Sp(4,2k)}{Sp(4)\times Sp(2k)}$	$8k$	$(2,0,\dots)$	$(0,\dots)\oplus(0,1,\dots)\oplus(0,2,\dots)\oplus\underline{(4,0,\dots)}$
6	$\frac{SU(4,k)}{SU(4)\times U(k)}$	$8k$	$(1,0,\dots,1)$	$(0,\dots)\oplus(1,\dots,1)\oplus(0,1,\dots,1,0)\oplus\underline{(2,0,\dots,2)}$
8	$\frac{SO(8,k)}{SO(8)\times SO(k)}$	$8k$	$(0,1,\dots)$	$(0,\dots)\oplus(0,0,0,1,\dots)\oplus(2,\dots)\oplus\underline{(0,2,\dots)}$
9	$\frac{F_{4(-20)}}{SO(9)}$	16	$\mathbf{52}$	$\mathbf{1}\oplus\mathbf{324}\oplus\underline{\mathbf{1053}}$
10	$\frac{E_{6(2)}}{SO(10)\times U(1)}$	32	$\mathbf{78}$	$\mathbf{1}\oplus\mathbf{650}\oplus\underline{\mathbf{2430}}$
12	$\frac{E_{7(-5)}}{SO(12)\times Sp(2)}$	64	$\mathbf{133}$	$\mathbf{1}\oplus\mathbf{1539}\oplus\underline{\mathbf{7371}}$
16	$\frac{E_{8(8)}}{SO(16)}$	128	$\mathbf{248}$	$\mathbf{1}\oplus\mathbf{3875}\oplus\underline{\mathbf{27000}}$

TABLE 8.2. Symmetric spaces for $\mathcal{N}\geq 5$ supergravity in $D=3$. The representation $R_{\boxplus}$ of G is underlined in the decompostion of $\odot^2(\mathrm{Adj}_G)$. Dots $\dots$ represent zero weights. Taken from [**50**].

Thus, the consistency condition is reduced to a simple group–theoretical criterion.

REMARK. By Kostant theorem, the above result holds even if G/H is not symmetric, provided that it is *compact* or *not Ricci–flat*.

COROLLARY 6.2. $\mathcal{N}\geq 5$*: the full isometry group G is always an adimissible gauge group $\mathcal{G}$.*

Indeed, G is simple. The G–invariant tensor l_{mn} should be proportional to the Cartan–Killing form K of $\mathfrak{g}$, and hence $\mathcal{T}=\lambda\, K$, which is always a solution to eqn.(6.14).

In table 8.2 (taken from ref. [**50**]) we write the representation $R_{\boxplus}$ for all symmetric spaces arising in $\mathcal{N}\geq 5$ $D=3$ SUGRA. The analogous results $\mathcal{N}=4$ symmetric spaces (Wolf spaces) are presented in table 8.3 (always taken from ref. [**50**]). Obviously I do not derive here these table (they were obtained from computer calculations using the LiE package [**201**]). You can easily check all the results using the on–line version of LiE at

`http://www-math.univ-poitiers.fr/ maavl/LiE/`.

The checking takes just a few seconds. The group theory tables of ref. [**202**] are also quite helpful.

Notice that the gauging criterion depends essentially only on the isometry group G. Two distinct SUGRA's with the same G have *the same gaugings:*

G/H	dim.	Adj_G	$\odot^2(\mathrm{Adj}_G)$
$\frac{Sp(2m,2)}{Sp(2)\times Sp(2m)}$	$4m$	$(2,0,\dots)$	$(0,\dots)\oplus(0,1,\dots)\oplus(0,2,\dots)\oplus\underline{(4,0,\dots)}$
$\frac{SU(m,2)}{SU(2)\times U(m)}$	$4m$	$(1,0,\dots,1)$	$(0,\dots)\oplus(1,\dots,1)\oplus(0,1,\dots,1,0)\oplus\underline{(2,0,\dots,2)}$
$\frac{SO(4,m)}{SO(4)\times SO(m)}$	$4m$	$(0,1,\dots)$	$(0,\dots)\oplus(0,0,0,1,\dots)\oplus(2,\dots)\oplus\underline{(0,2,\dots)}$
$\frac{G_{2(2)}}{SO(4)}$	8	**14**	$\mathbf{1}\oplus\mathbf{27}\oplus\underline{\mathbf{77}}$
$\frac{F_{4(4)}}{Sp(6)\times Sp(2))}$	28	**52**	$\mathbf{1}\oplus\mathbf{324}\oplus\underline{\mathbf{1053}}$
$\frac{E_{6(2)}}{SU(6)\times Sp(2)}$	40	**78**	$\mathbf{1}\oplus\mathbf{650}\oplus\underline{\mathbf{2430}}$
$\frac{E_{7(-5)}}{SO(12)\times Sp(2)}$	64	**133**	$\mathbf{1}\oplus\mathbf{1539}\oplus\underline{\mathbf{7371}}$
$\frac{E_{8(-24)}}{E_7\times Sp(2)}$	112	**248**	$\mathbf{1}\oplus\mathbf{3875}\oplus\underline{\mathbf{27000}}$

TABLE 8.3. Symmetric spaces for $\mathcal{N}=4$ supergravity in $D=3$. The representation $R_{\boxplus}$ of G is underlined in the decompostion of $\odot^2(\mathrm{Adj}_G)$. Dots ... represent zero weights. Taken from [**50**].

e.g. $\mathcal{N}=12$ SUGRA and $\mathcal{N}=4$ SUGRA with

$$\mathcal{M}=E_{7(-5)}/[SO(12)\times Sp(2)], \tag{6.15}$$

having the same isometry group G, have the same possible gaugings $(\mathcal{G},l_{mn})$ since in both cases the unwanted representation is the **7371** of $E_{7(-5)}$.

6.2. Summary of $D=3$ and some general lessons. Before going to higher spacetime dimensions D, let us pause a while to review what we have done in $D=3$, restate it in a way suited for further generalization, and list the lessons we have learned:

(1) The gauge group is embedded in the isometry group of the scalars' manifold, $\mathcal{G}\hookrightarrow G\equiv \mathrm{Iso}(\mathcal{M})$. At the Lie algebra level this corresponds to a Lie algebra monomorphism

$$\mathfrak{Lie}(\mathcal{G})\equiv\mathfrak{l}\rightarrow\mathfrak{g}\equiv\mathfrak{iso}(\mathcal{M}), \tag{6.16}$$

which is expressed by a tensor[13]

$$l \in \mathfrak{l} \otimes \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g} \qquad \text{('embedding tensor')}. \tag{6.17}$$

Gauge invariance requires l to be invariant under the adjoint action of $\mathfrak{l}$.

(2) We use the invariant tensor l to make the derivatives in the Lagrangian to be covariant with respect to the gauge group $\mathcal{G}$. The SUSY variation of $\mathcal{L}$ acquires new terms, proportional to the $\mathcal{G}$ field–strengths, from the commutator of the covariantized derivatives. In order to cancel them, in $D = 3$, we must add a Chern–Simons term, whose structure requires the tensor l to be symmetric, namely

$$l \in \odot^2 \mathfrak{g}. \tag{6.18}$$

(3) To cancel the residual terms in the SUSY variation $\delta\mathcal{L}$, we must add to $\mathcal{L}$ the Yukawa terms (5.15)

$$e \left\{ \frac{1}{2} A_1^{AB}\, \overline{\psi}_\mu^A \gamma^{\mu\nu} \psi_\nu^B + A_{3\,i}^A\, \overline{\psi}_\mu^A \gamma^\mu \chi^i + \frac{1}{2}\, A_{3\,ij}\, \overline{\chi}^i \chi^j \right\}, \tag{6.19}$$

and a potential V, while we modify the Fermi SUSY variations by some scalar shifts. These corrections are expressed in terms of the Yukawa tensors A_1^{AB}, $A_{2\,i}^A$ and $A_{3\,ij}$ *via* the universal (algebraic) Ward identities of SUGRA (cfr. chapt. 6).

(4) The interplay between the isometries and the parallel structures of $\mathcal{M}$ implies that the three Yukawa tensors are linear in the components of the generalized T–tensor

$$\mathcal{T}(\phi) := \theta_\phi(K^m)\, l_{mn}\, \theta_\phi(K^n), \tag{6.20}$$

where

$$\theta_\phi \colon \mathfrak{iso}(\mathcal{M}) \longrightarrow \mathrm{End}(T_\phi\mathcal{M}) \oplus T_\phi\mathcal{M}, \tag{6.21}$$

is the Cartan–Kostant monomorphism[14] (Cfr. the GENERAL LESSONS of chapt. 7).

(5) The above linear map can be inverted, namely we can write the T–tensor $\mathcal{T}$ as a linear combination of the Yukawa tensors A_1^{AB}, $A_{2\,i}^A$ and $A_{3\,ij}$.

(6) Since $\mathfrak{hol}(\mathcal{M}) \simeq \mathfrak{spin}(\mathcal{N}) \oplus \mathfrak{h}'$, and the action of $\mathfrak{spin}(\mathcal{N}) \subset \mathfrak{hol}$ on $T\mathcal{M}$ is trough a *spinorial* representation, one has

$$\mathcal{T} \in \odot^2 \mathfrak{spin}(\mathcal{N}) \oplus \mathfrak{spin}(\mathcal{N})^{\oplus k_1} \oplus \mathbf{1}^{\oplus k_2} \oplus \text{(spinorial)}, \tag{6.22}$$

in terms of $Spin(\mathcal{N})$–representations[15].

[13] For simplicity, we assume $\mathcal{G}$ and G to be semi–simple. In this case the adjoint representation coincides with its dual, and we omit the dual mark $\vee$ in our formulae. We also identify $\mathfrak{l}$ and its image in $\mathfrak{g}$. The modifications for the non–semisimple case are obvious, although somehow tricky. See ref. [**203**].

[14] *I.e.* a linear map which is an isomorphism onto its image. This isomorphism is a Lie algebra isomorphism!

[15] $\mathfrak{spin}(\mathcal{N})$ stands for the adjoint representation.

(7) On the other hand, by item 5, the tensor $\mathcal{T}$ cannot contain any irreducible $Spin(\mathcal{N})$–representation which is not present in the tensors A_1^{AB}, $A_{2\,i}^{A}$ and $A_{3\,ij}$, namely

$$A_1^{AB} \longrightarrow V \odot V \tag{6.23}$$

$$A_{2\,i}^{A} \longrightarrow V \otimes S \tag{6.24}$$

$$A_{3\,ij} \longrightarrow S \odot S, \tag{6.25}$$

where V is the $\mathcal{N}$–dimensional vector representation and S the spinorial one. Since $\odot^2 S$ is the direct sum of totally skewsymmetric representations $\wedge^r V$, we learn that $\mathcal{T}$ should satisfy the constraint

$$\boxed{\mathcal{T}\Big|_{\odot^2 \mathfrak{spin}(\mathcal{N})} \in \odot^2 V \oplus \wedge^4 V} \tag{6.26}$$

Of course, this is the same constraint on the T–tensor we obtained by a direct computation. *Here we see that we could have deduced it from symmetry considerations alone,* just group theory!

(8) If the isometry group G is transitive (and $\mathcal{M}$ is irreducible), the Cartan–Kostant *isomorphism,* together with the tt^*–structure of the Aut_R curvatures *('minus one quarter...'),* allow us to identify the $\mathrm{Adj}^{\odot^2}G$–orbits of the 'embedding tensor' l with the generalized T–tensor

$$\mathcal{T}(gH) \stackrel{!}{=} l(g) \equiv \mathrm{Adj}^{\odot^2}(g^{-1})(l), \tag{6.27}$$

and the constraint (6.26) becomes

$$l(g) \in \oplus_k R_k \subset \odot^2 \mathfrak{g} \qquad \text{as } G-\text{representations!} \tag{6.28}$$

where the irreducible representations R_k may contain only the following representations of the subgroup $Spin(\mathcal{N}) \subset G$:

$$\begin{aligned}&\mathbf{1},\quad (\odot^2 V)_{\text{traceless}},\quad \wedge^2 V,\quad \wedge^4 V,\\ &\text{as well as the spinorial irrepr. contained}\\ &\text{in}\quad (V\otimes S)\cap \odot^2\mathfrak{g}\quad\text{and}\quad S\cap\odot^2\mathfrak{g}.\end{aligned} \tag{6.29}$$

Again this statement is equivalent to the one given before.

In the next section we rephrase these items into a GENERAL LESSON valid for all space–time dimensions D.

7. $D \geq 4$ *gauged* supergravity

Our last task is to extend the gaugings to different dimensions D. Our analysis below holds for any D, although, in presence of higher form–fields $A_{\mu_1\mu_2\cdots\mu_k}$ ($k \geq 2$), there are some additional subtleties (that are outside the scope of this course) [**204, 205, 206**]. For $D \leq 5$ such fields may be dualized away, and we may formulate any SUGRA theory with a bosonic sector consisting just of scalars and vectors besides the graviton.

7.1. Review of the ungauged theory. There are a few physical differences between $D \geq 4$ and $D = 3$. First of all, in $D = 3$ (in the dWHS dual version) the vector fields do not propagate physical states; they are 'auxiliary' fields, and we can add to the ungauged Lagrangian $\mathcal{L}_0$ as many of them as we wish or need. On the contrary, in $D \geq 4$, A_μ propagates physical states, which should belong to SUSY representations, so their *number* and *quantum numbers* are uniquely determined by the SUSY content of the ungauged theory. More concretely, on the target space $\mathcal{M}$ we have a flat vector bundle $\mathcal{V} \to \mathcal{M}$, such that the field strenghts 2–forms F, and their $(D-2)$–form duals $G = \frac{\partial \mathcal{L}_0}{\partial F}$, satisfy[16]

$$F \in \wedge^2 T^*\Sigma \otimes \Phi^*\mathcal{V}, \qquad G \in \wedge^{(D-2)} T^*\Sigma \otimes \Phi^*\mathcal{V}^\vee. \tag{7.1}$$

The situation in $D = 4$ is slightly different, since both F and G are two–forms. As explained in chapt. 1, in this case, we should combine the F's and the G's into a double–size vector $\mathcal{F}$ with

$$\mathcal{F}_+ \in \wedge^2_+ T^*_{\mathbb{C}}\Sigma \otimes \Phi^*\mathcal{V}, \qquad \mathcal{F}_- \in \wedge^2_- T^*_{\mathbb{C}}\Sigma \otimes \Phi^*\overline{\mathcal{V}}. \tag{7.2}$$

On $\mathcal{M}$ we have two other natural vector bundles. *(i)* $\Psi \to \mathcal{M}$ is the vector bundle corresponding to the gravitino fields, ψ^A_μ. It has structure group Aut_R. *(ii)* $\Upsilon \to \mathcal{M}$ is the bundle associated with the spin–1/2 fields.

Υ decomposes into the Whitney sum of fiber bundles of the form $S_R \otimes U_\rho$, where S is the Aut_R–bundle associated to a suitable representation R, and U is the H'–bundle[17] associated to some representation ρ . For $D = 4$, the relations between these fermionic bundles and $T\mathcal{M}$ is given in the tables at the end of chapt. 2, where the representation pairs (R, ρ) are also listed.

The subgroup $G \subset \mathrm{Iso}(\mathcal{M})$, which is actually a symmetry of the ungauged theory, acts on the bundles $\mathcal{V}$, Ψ and Υ, to guarantee invariance of the vectors' and fermions' couplings (which have a non–trivial dependence on the scalars ϕ). In, say $D = 4$, the action of G on $\mathcal{V}$ gives a group morphism

$$\mu^\sharp \colon G \to Sp(2n, \mathbb{R}), \tag{7.3}$$

induced by the 'suscetibility map'

$$\mu \colon \mathcal{M} \to Sp(2n, \mathbb{R}). \tag{7.4}$$

More generally, G acts on the field–strengths according to a linear representation ρ. On the other hand, the actions on Ψ and Υ are obtained by suitable projections of the Cartan–Kostant map θ, that is, by the generalized momentum maps.

In this context, our problem is to determine the possible gaugings which are compatible with $\mathcal{N}$–extended local SUSY in D dimensions.

7.2. General principles. What we have learned about SUSY and, in particular, the above discussion of the $D = 3$ gaugings, together with symmetry considerations, suggest the following general statement:

[16] Recall that, by abuse of notation, when we write $F \in E$, with E a vector bundle, we really mean $F \in C^\infty(E)$, that is F is a smooth section of E.

[17] Recall that the Lie group H' is defined by the condition $\mathfrak{hol}(\mathcal{M}) \simeq \mathfrak{aut}_R \oplus \mathfrak{h}'$, where $\mathfrak{h}'$ is the Lie algebra of H'.

GENERAL LESSON 7.1. *Let $\mathcal{V}$, Ψ and Υ be the target space bundles associated, respectively, with the vector, spin–3/2 and spin–1/2 fields. Let $\mathcal{G} \subset \mathrm{Iso}(\mathcal{M})$ be the symmetry of $\mathcal{L}_0$ to be gauged. Let*

$$l\colon \mathcal{V} \to \mathfrak{iso}(\mathcal{M}), \tag{7.5}$$

be a linear map whose image is a Lie subalgebra $\mathfrak{l} \subset \mathfrak{iso}(\mathcal{M})$ isomorphic to $\mathfrak{Lie}(\mathcal{G})$. Then the SUSY *completion of the minimal gauge coupling*[18]

$$D_\mu \Phi^a \to D_\mu \Phi^a - A_\mu^M \, l_{Mm} \pounds_{K^m} \Phi^a,$$

is completely encoded in the three Yukawa tensors

$$A_1 \in \widehat{\odot}^2 \Psi^\vee, \quad A_2 \in \Psi^\vee \otimes \Upsilon^\vee, \quad A_3 \in \widehat{\odot}^2 \Upsilon^\vee, \tag{7.6}$$

where $\widehat{\odot}$ stands for the non–symmetrized, symmetrized or anti–symmetrized tensor product depending on the peculiar symmetry and reality properties of the spinors and their scalar bilinears in the given space–time dimension D (see sect. 2.1.1 for the list). The three Yukawa tensors can be linearly combined into a single tensor $\mathcal{T}(\phi)$

$$\mathcal{T}(\phi) := \rho(\widetilde{\mu}(\phi))^{-1} l \cdot \theta_\phi \tag{7.7}$$

where $\cdot$ stands for the Cartan–Killing inner product in $\mathfrak{g}$,

$$\theta_\phi \in \big(\mathrm{End}(T\mathcal{M}) \oplus T\mathcal{M}\big)_\phi \otimes \mathfrak{g} \tag{7.8}$$

is the Cartan–Kostant morphism, and $\widetilde{\mu}$ is any lift of μ,

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \widetilde{\mu}\ } & \begin{cases} Sp(2n,\mathbb{R}) & D=4 \\ GL(n,\mathbb{R}) & D\geq 5 \end{cases} \\ {\scriptstyle \mathrm{Id}}\Big\downarrow & & \Big\downarrow \\ \mathcal{M} & \xrightarrow{\ \mu\ } & \begin{cases} Sp(2n,\mathbb{R})/U(n) & D=4 \\ GL(n,\mathbb{R})/O(n) & D\geq 5 \end{cases} \end{array} \tag{7.9}$$

with μ the usual 'susceptibility' map[19].

A gauging l is allowed if and only if the decompostion of $\mathcal{T}(\phi)$ into irreducible representations of $\mathrm{Aut}_R \times H'$ contains only the representations associated with the three vector bundles in eqn.(7.6).

The $D = 4$ case is special, since the bundle $\mathcal{V}$ contains both the 'electirc' field strenghts $F = dA$ (corresponding, say, to the upper block of the $2n$–vector $\mathcal{F}$) as well as their 'magnetic' duals, $*\partial\mathcal{L}/\partial F$. In a local Lagrangian, you cannot have both A_μ^M and its magnetic counterpart $A_{\mu\, M}$; hence, in

[18] Φ^a is a shorthand for all the fields in the theory *but* the vectors A_μ.

[19] We have defined μ only in the tricky dimension $D = 4$. Here we fill the gap. In $D \geq 5$ the vectors' kinetics terms have the form $f_{xy}(\phi)\, F^x_{\mu\nu}\, F^{y\,\mu\nu}$ for some real, symmetric, positive–definite matrix f_{xy}. The space of all the symmetric positive–definite metrics is identified with the coset $GL(n,\mathbb{R})/O(n)$ by the map

$$\mathcal{E} \mapsto \mathcal{E}^t\, \mathcal{E} \equiv f$$

($\mathcal{E}$ is the vielbein in the standard sense). The map $\mu\colon \phi \mapsto \mathcal{E}(\phi)$. The group $GL(n,\mathbb{R})$ (automorphisms of the formalisms) acts on the vectors as

$$F \mapsto g\,F, \qquad \mathcal{E} \mapsto \mathcal{E}\, g^{-1}.$$

a meaningful theory, only the 'electric' vectors may be used to gauge the global symmetries. Of course, what we mean by 'electric' depends on the duality frame we use. The invariant physical requirement is that the vector entering into the Lagrangian are mutually local fields. *In view of the Dirac quantization rule, this is equivalent to requiring that gauging l satisfies* [**207**]

$$l_{Mm}\, l_{Nn}\, \Omega^{MN} = 0, \tag{7.10}$$

where Ω^{MN} is the $Sp(2n,\mathbb{R})$ symplectic matrix. This physical condition is automatically satisfied for a gauging l whose T–tensor $\mathcal{T}(\phi)$ has the correct decomposition into $\mathrm{Aut}_R \times H'$ *representations. (This is a nice consistency check on the full picture!)*

If $\mathrm{Iso}(\mathcal{M}) = G$ *is transitive,*

$$\mathcal{T}_M{}^{\alpha}\, t_\alpha = \rho(g^{-1})_M{}^N\, l_{Mn}\, (g\, t^m\, g^{-1}) \tag{7.11}$$

(see e.g. [**207, 208**]*) and the gauging constraints may be lifted to a problem about the representations of G as we did in sect. 6.1 for the $D = 3$ case.*

Let us argue that the above GENERAL LESSON is correct. Going trough the logical steps that we did in $D = 3$, as summarized in §. 6.2, we easily convince ourselves that the above conditions on the 'embedding tensor' $l_{M\,m}$ are at least necessary in order to have supersymmetry. On the other hand, we may think of relating a SUGRA model in D dimensions to one in three dimension by 'dimensional reduction'. However, after the gauging (or the addition of a superpotential) the scalars' potential (and hence the effective cosmological constant Λ) is not longer zero. Then, in general, we have no solution of the Einstein equations with a manifold of the form $T^{D-3} \times \Sigma_3$, and the dimensional reduction is not straightforward. However, $\Lambda = O(g^2)$, so, the dimensional reduction argument still works at order $O(g)$, so we certainly get the correct *linear* constraints on $l_{M\,m}$. Moreover, the condition we get from SUSY at level $O(g^2)$ should be equivalent, on general grounds, to the Ward identity of chapt. 6, which has an universal form in terms of the tensors A_1^{AB} and $A_{2\,i}^{A}$, which, again, depends only on their *representation content.* So one does not expect any new independent constraint at the $O(g^2)$ level.

In my view, the most convincing evidence for the above GENERAL LESSON is the fact that its statement is *very* geometric, as it should.

We shall present a few checks of the above results. In $D = 4$, for $\mathcal{N} \geq 3$ $\mathcal{M}$ is a symmetric space, and hence we need only to verify that the various representations match with those predicted above. For $\mathcal{N} = 1, 2$ the situation is more 'geometric' and we shall discuss the relevant topics in Part 3. (See ref.[**209**] for a nice treatment of the general gauged $\mathcal{N} = 2$ supergravity in a language similar to the one used here).

7.3. ∗. Gaugings and Peccei–Quinn symmetries. In $D = 4$, the ungauged Lagrangian $\mathcal{L}_0$ is not invariant under the full U–duality group G which (for $\mathcal{N} \geq 3$, at least) is identified with the isometry group $\mathrm{Iso}(\mathcal{M})$. Only the equations of motion are invariant. Gauging a continuos symmetry $\mathcal{G}$ of the ungauged theory which is <u>not</u> a symmetry of the the ungauged

Lagrangian is rather tricky. But it can be done, under certain circumstances, is a supersymmetric fashion [**208**].

We have already stated that the vectors A_μ^M gauging $\mathcal{G}$ should be *mutually local.* Without loss of generality[20], we can assume that the first n–components of the field–streght $2n$–vector $\mathcal{F}_{\mu\nu}$ corresponds to the curvatures of the mutually local connection fields A_μ^M. Then an element of $Sp(2n,\mathbb{R})$ acts as

$$\begin{pmatrix} F \\ G \end{pmatrix} \to \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} AF + CG \\ DG + BF \end{pmatrix} \tag{7.12}$$

so preserving the mutual locality implies $C = 0$. Then the symplectic condition gives

$$D = (A^t)^{-1}, \qquad B = (A^t)^{-1} S \quad \text{with } S \text{ symmetric.} \tag{7.13}$$

Let $\mathcal{P}(2n) \subset Sp(2n,\mathbb{R})$ be the 'parabolic' subgroup of matrices of the form

$$\begin{pmatrix} A & 0 \\ (A^t)^{-1} S & (A^t)^{-1} \end{pmatrix}. \tag{7.14}$$

From lemma **5.**11.2 we know that all coset elements $\xi \in Sp(2n,\mathbb{R})/U(n)$ have a representative in $\mathcal{P}(2n)$; in fact, we may even choose A to be lower–triangular with positive diagonal entries. Thus

$$\mathcal{P}(2n)/O(n) \simeq Sp(2n,\mathbb{R})/U(n). \tag{7.15}$$

The isomorphism 7.15 gives us as a reduced 'susceptibility' map

$$\mu^\flat \colon \mathcal{M} \to \mathcal{P}(2n)/O(n). \tag{7.16}$$

Now we can restate our GENERAL LESSON 7.1 with $\mathcal{G} \subseteq \mathcal{P}(2n)$ and μ replaced by $\mu^\flat$, forgetting about the locality condition which is now *automatic.*

Using the results of chapt. 1, we see that the ungauged Lagrangian changes under the transformation (7.14) by

$$\delta \mathcal{L}_0 = \frac{1}{4} S_{MN}\, F^M_{\mu\nu}\, \widetilde{F}^{N\,\mu\nu}, \tag{7.17}$$

which is a total derivative (and hence does not affect the equation of motion). A symmetry under which the Lagrangian transforms as in eqn.(7.17) is called *a Peccei–Quinn symmetry.*

Gauging a generic subgroup $\mathcal{G} \subseteq \mathcal{P}(2n)$, we may also gauge some Peccei–Quinn symmetry[21]. Now the group parameter, $S_{MN}(x)$, is not a constant,

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[20] In refs. [**63, 208**], it is discussed how different choices lead to *inequivalent* gauged theories. However, in our 'abstract' geometric formulation, a different choice for the embedding of the curvatures of the local fields in the $2n$–vector $\mathcal{F}$ can be compensated by a corresponding modification of the 'susceptibility' map μ, which needs not to be the canonical embedding $G/H \to Sp(2n,\mathbb{R})/U(n)$. However the map μ is still a totally geodesic embedding: by arguments presented in chapt. 5, μ encodes precisely the constant matrix $\mathsf{E} \in Sp(2n,\mathbb{R})$ of refs. [**63, 208**]. The inequivalent choices of E are labelled by the double coset $G\backslash Sp(2n,\mathbb{R})/GL(n,\mathbb{R})$.

[21] By Cartan's criterion, the *gauged* Peccei–Quinn symmetries correspond to $\mathfrak{r}(\mathfrak{Lie}(\mathcal{G}))$, the *radical* of the Lie algebra of $\mathcal{G}$ (cfr. ref. [**226**] §. I.5). Hence they cannot be present in *semi–simple* gaugings.

so (7.17) is not a total derivative any longer. Then we must cancel this variation by adding a Chern–Simons–like term (much as we did in $D = 3\,!$)

$$\mathcal{L}_{CS} \propto \varepsilon^{\mu\nu\rho\sigma}\, l_{M\,NP}\, A^M_\mu\, A^N_\nu \left(\partial_\rho A^P_\sigma - \frac{1}{2} f_{QR}{}^P\, A^Q_\rho\, A^R_\sigma\right) \tag{7.18}$$

where $l_{M\,NP}$ is the projection of l on[22]

$$\mathfrak{r}\big(\mathfrak{Lie}(\mathcal{G})\big) \otimes \mathcal{V}^\vee \subset \mathfrak{Lie}(\mathcal{G}) \otimes \mathcal{V}^\vee. \tag{7.19}$$

The resulting Lagrangian is supersymmetric. See ref. [**208**] for further details.

8. An example: $\mathcal{N} = 3$ supergravity in $D = 4$

To illustrate the above results, I discuss in some detail a simple example, namely gauged $\mathcal{N} = 3$ supergravity (in four space–time dimensions) which is paradigmatical, as the title and abstract of ref.[**210**] imply.

As we saw in chapt. 4, the scalars' manifold is

$$SU(3,k)\Big/\big(SU(3) \times SU(k) \times U(1)\big), \tag{8.1}$$

where k is the number of matter gauge multiplets coupled to SUGRA. Again, we write $g \in SU(3,k)$ for a coset representative, with the global group $SU(3,k)$ acting on the left, and the local one, $H = SU(3) \times SU(k) \times U(1)$, acting on the right. To be explicit, let

$$J = \begin{pmatrix} \mathbf{1}_{3\times 3} & 0 \\ 0 & -\mathbf{1}_{k\times k} \end{pmatrix}. \tag{8.2}$$

The elements $g \in SU(3,k)$ are identified with the $(3+k)\times(3+k)$ unimodular complex matrices such that

$$g^\dagger\, J\, g = J. \tag{8.3}$$

The Lagrangian itself is invariant only under the subgroup $SO(3,k) \subset G$, which do not mix 'electric' and 'magnetic' fields.

We have $(3+k)$ vector fields, so we can gauge at most a subgroup of G of dimension $(3+k)$. Following ref. [**211**], we shall limit ourselves to semi–simple subgroups $K \subset SO(3,k)$ (so, no Peccei–Quinn symmetries).

The subgroup K to be gauged should fulfill the following requirements. *(i)* the fundamental representation D of $SU(3,k)$ must split as

$$D \underset{K}{\rightarrow} \mathrm{adj} \oplus \mathrm{adj} \tag{8.4}$$

(one summand for the 'electric' fields and one for the 'magnetic' ones). Furthermore K must preserve the metric J, which therefore can be identified with the Cartan–Killing metric of K. Then the structure constants $f_{ab}{}^c$ of K become fully antisymmetric by lowering the upper index with the metric J. The Lie algebra of K, $\mathfrak{K}$, must be a real subalgebra of $\mathfrak{so}(3,k)$ of dimension $(3+k)$ and with a bilinear invariant of signature $(3,k)$. Thus we have at most three non–compact generators, and the only possibilities are [**211**]

$$K = SO(3) \times K_n \tag{8.5}$$

$$K = SO(3,1) \times K_{n-3}, \tag{8.6}$$

[22] See previous footnote for the definition of $\mathfrak{r}(\cdot)$.

where K_n stands for any compact Lie group of dimension n [recall that the adjoint representation embeds any compact Lie algebra $\mathfrak{L}$ of dimension n into the Lie algebra $\mathfrak{so}(n)$].

In both cases, using our eqn.(7.11), the $\mathcal{T}$ tensor is easily computed. It corresponds to the 'boosted structure constants' of the gauged subgroup K [**210, 211**],

$$\mathcal{T}^L{}_{MN}(g) = (g^{-1})^L{}_R \, f^R{}_{PQ} \, (g)^P{}_M \, (g)^Q{}_N. \tag{8.7}$$

Our GENERAL PRINCIPLE 7.1 predicts that the fermionic shifts (and hence the Yukawa couplings) are linear in this $\mathcal{T}$ tensor. Explicitly, one finds [**210, 211**]

$$\delta\psi^A_\mu = \cdots - \frac{i}{8} g \Big(\mathcal{T}^A_{PQ} \, \epsilon^{BPQ} + (A \leftrightarrow B) \Big) \gamma_\mu \epsilon_B \qquad \text{(gravitino)} \tag{8.8}$$

$$\delta\chi = \cdots - \frac{1}{4} g \, \mathcal{T}^B_{BA} \, \epsilon^A \qquad \text{(dilatino)} \tag{8.9}$$

$$\delta\lambda^i = \cdots - \frac{1}{2} g \, \mathcal{T}^i_{BC} \, \epsilon^{ABC} \, \epsilon_A \qquad \text{(gluino singlets)} \tag{8.10}$$

$$\delta\lambda_{A\,i} = \cdots - \left(\mathcal{T}^B{}_{iA} - \frac{1}{2} \mathcal{T}^B{}_{iB} \, \delta^B_A \right) \epsilon_B \qquad \text{(gluino triplets).} \tag{8.11}$$

in full agreement with the general principles. One checks [**210, 211**] that these fermionic shifts do satisfy the algebraic conditions following from the universal Ward identity relating the scalar potential to the fermionic shifts.

The important lesson is that the fermionic shifts and Yukawa tensors are precisely the projections of $\mathcal{T}$ on the respective representations of the $\mathrm{Aut}_R \times H'$ local symmetry associated with the given Fermi bilinear. The numerical coefficients of these projection are universal, as we saw in $D = 3$.

9. Gauging maximal supergravity in D dimensions

We apply the previous GENERAL LESSON to the case of *maximal* supergravity in D space–time dimensions, that is SUGRA with $\mathfrak{N} = 32$ 'conserved supercharges'. All SUGRA's with $\mathfrak{N} \geq 18$ are truncations of these maximal theories.

The scalars' manifold is symmetric; the cosets G/H corresponding to maximal SUGRA in $D \geq 3$ were found in chapt. 4. For convenience of the reader, we have listed the groups G an H in table 8.4. In the same table you find the decompostion of the G–representation $\mathcal{V} \otimes \mathrm{Adj}_G \ni \mathcal{T}$ into irreducible representations (following [**207, 208**]). The bottom line is the $D = 3$ case that we already studied in detail. The table is easily checked using the LiE package or the tables in [**202**].

9.1. Supersymmetric gaugings. The basic criterion for a supersymmetry preserving gauging, l_{Mm}, is that the associated $\mathcal{T}$–tensor,

$$\mathcal{T}(g) = \varrho_{\mathcal{V}\otimes\mathrm{Adj}}(g) \, l,$$

D	G	H	decomposition of the G–module $\mathcal{V} \otimes \mathrm{Adj}_G$
7	$SL(5)$	$Sp(4)$	$\mathbf{10} \otimes \mathbf{24} = \mathbf{10} \oplus \boxed{\mathbf{15}} \oplus \mathbf{40} \oplus \mathbf{175}$
6	$SO(5,5)$	$Sp(4) \times Sp(4)$	$\mathbf{16} \otimes \mathbf{45} = \mathbf{16} \oplus \boxed{\mathbf{144}} \oplus \mathbf{560}$
5	$E_{6(6)}$	$Sp(8)$	$\mathbf{27} \otimes \mathbf{78} = \mathbf{27} \oplus \boxed{\mathbf{351}} \oplus \mathbf{1728}$
4	$E_{7(7)}$	$SU(8)$	$\mathbf{56} \otimes \mathbf{133} = \mathbf{56} \oplus \boxed{\mathbf{912}} \oplus \mathbf{6480}$
3	$E_{8(8)}$	$SO(16)$	$\mathbf{248} \odot \mathbf{248} = \boxed{\mathbf{1}} \oplus \boxed{\mathbf{3875}} \oplus \mathbf{27000}$

TABLE 8.4. Decompostion of $\mathcal{V} \otimes \mathrm{Adj}_G$ into irreducible G–representations. The (extended) $\mathcal{T}$–tensor belongs to the $\mathcal{V} \otimes \mathrm{Adj}_G$. SUSY requires that only its components in the *boxed* irreducible representaions do not vanish.

D	$H \equiv \mathrm{Aut}_R$	$\widehat{\odot}^2 \Psi^\vee$	$\Psi^\vee \otimes \Upsilon^\vee$	$\widehat{\odot}^2 \Upsilon^\vee$
7	$Sp(4)$	$\mathbf{1} \oplus \mathbf{5}$	$\mathbf{5} \oplus \mathbf{10} \oplus \mathbf{15} \oplus \mathbf{35}$	$\mathbf{1} \oplus \mathbf{5} \oplus \mathbf{14} \oplus$ $\oplus\, \mathbf{30} \oplus \mathbf{35} \oplus \mathbf{35}'$
6	$Sp(4) \times Sp(4)$	$(\mathbf{4}, \mathbf{4})$	$(\mathbf{4}, \mathbf{4}) \oplus (\mathbf{4}, \mathbf{4}) \oplus$ $\oplus\, (\mathbf{4}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{4})$	$(\mathbf{4}, \mathbf{4}) \oplus (\mathbf{4}, \mathbf{16}) \oplus$ $\oplus (\mathbf{16}, \mathbf{4}) \oplus (\mathbf{16}, \mathbf{16})$
5	$Sp(8)$	$\mathbf{36}$	$\mathbf{27} \oplus \mathbf{42} \oplus \mathbf{315}$	$\mathbf{1} \oplus \mathbf{27} \oplus \mathbf{36} \oplus \mathbf{308} \oplus$ $\oplus\, \mathbf{315} \oplus \mathbf{792} \oplus \mathbf{825}$
4	$SU(8)$	$\mathbf{36} \oplus \overline{\mathbf{36}}$	$\mathbf{28} \oplus \overline{\mathbf{28}} \oplus$ $\oplus \mathbf{420} \oplus \overline{\mathbf{420}}$	$\mathbf{420} \oplus \overline{\mathbf{420}} \oplus$ $\oplus\, \mathbf{1176} \oplus \overline{\mathbf{1176}}$
3	$SO(16)$	$\mathbf{1} \oplus \mathbf{135}$	$\mathbf{128} \oplus \overline{\mathbf{1920}}$	$\mathbf{1} \oplus \mathbf{1820} \oplus \overline{\mathbf{6435}}$

TABLE 8.5. Possible Yukawa couplings in various dimensions for maximal SUGRA and the corresponding Aut_R–representations.

has non–vanishing components *only* in those irreducible G–representations $\subset \mathcal{V} \otimes \mathrm{Adj}_G$ which, when decomposed into representations of the holonomy subgroup H, contain only the H–irrepresentations appearing into the Yukawa tensors A_1, A_2 and A_3.

The H–representations of the Yukawa couplings are listed in table 8.5, (again following [**207, 208**]). Note that in $D = 7$, the mass–matrices A_1 and

A_3 should be *anti*symmetric in the fermionic indices as the scalar bilinear in the $D = 7$ pseudoMajorana spinors.

9.2. Example: $D = 4$**.** Consider, for instance, $\mathcal{N} = 8$ supergravity in four dimensions. One has $G = E_{7(7)}$ and $H = SU(8)$. The three $E_{7(7)}$ representations appearing in $\mathcal{V} \otimes \mathrm{Adj}$ decompose under $SU(8)$ as follows:

$$\mathbf{56} \xrightarrow{SU(8)} \mathbf{28} \oplus \overline{\mathbf{28}} \tag{9.1}$$

$$\mathbf{912} \xrightarrow{SU(8)} \mathbf{36} \oplus \overline{\mathbf{36}} \oplus \mathbf{420} \oplus \overline{\mathbf{420}} \tag{9.2}$$

$$\mathbf{6480} \xrightarrow{SU(8)} \mathbf{28} \oplus \mathbf{420} \oplus \mathbf{1280} \oplus \mathbf{1512} \oplus \text{(conjugate reprs.)} \tag{9.3}$$

Comparing with table 8.5, we see that the representation **6480** cannot be present.

Also the **28** cannot be present. The reason is that ${A_2}_A{}^{BCD}$ satisfies the — less obvious — identity ${A_2}_A{}^{ACD} = 0$, and hence it is a section of a *proper* sub–bundle of $\Psi^\vee \otimes \Upsilon^\vee$, corresponding to the representation $\mathbf{420} \oplus \overline{\mathbf{420}}$ of $SU(8)$. Indeed, from our GENERAL LESSONS, valid in any dimension, we know there exists an equation of the form[23]

$$c_1\, {A_2}_A{}^{BCD} + c_2\, A_1^{B[C}\, \delta_a^{D]} = \mathcal{T}^{BCD}{}_A \equiv (g^{-1})^{BC}{}_{EF}\, l^{EF}{}_m\, (\boldsymbol{\mu}^m)^B{}_A, \tag{9.4}$$

where the 8×8 matrix $(\boldsymbol{\mu}^\bullet)^B{}_A$ is the component of the covariant momentum map in the Lie algebra $\mathfrak{su}(8)$, which is traceless by definition. Taking the trace of both sides of eqn.(9.4), we get ${A_2}_A{}^{ACD} = 0$, *i.e.* $A_2 \in \mathbf{420} \oplus \overline{\mathbf{420}}$.

Since the T–tensor is an $E_{7(7)}$–covariant object, it is enough to impose the representation constraint at one point of the G orbit, say at the origin. Then we get a condition in terms of the embedding tensor l only

$$\boxed{\mathbb{P}_{\mathbf{912}}\, l = l} \tag{9.5}$$

where $\mathbb{P}_{\mathbf{912}}$ is the projector on the **912** representation of $E_{7(7)}$. The interested reader may find in ref. [**208**] the explicit form of this projector, as well as the projector for the other relevant representations for maximal SUGRA in $D = 4$ and $D = 5$.

At this point, the complete classification of all possible SUSY–preserving gaugings of $\mathcal{N} = 8$ SUGRA is reduced to a problem in group theory, albeit *not* a trivial one. The full list of gaugings with gauge group $\mathcal{G} \subset SL(8, \mathbb{R})$ (no Peccei–Quinn gauge symmetries or other fancy mechanisms) can be found in refs. [**212, 213**]. There are other, trickier gaugings, see refs. [**214, 215**] as well as refs. [**208, 207**] for examples.

Assuming $\mathcal{G} \subset SL(8, \mathbb{R})$, one gets all gauge groups of the form

$$SO(8-p, p), \qquad p = 0, 1, \dots 4,$$

as well as some non–semisimple groups — called $CO(p, q, 8-p-q)$ — whose Lie algebra is the semi–direct sum of $\mathfrak{so}(p, q)$ with a solvable Lie algebra.

[23] c_1 and c_2 are some non–vanishing numerical constants whose precise value is unmaterial for our argument.

$CO(p,q,8-p-q)$ is obtained from the group $SO(8-q,q)$ by a suitable contraction.

For the similar analysis of gauged maximal SUGRA in $D>4$ dimensions, see *e.g.* refs.[**216, 217, 218**].

10. ADDENDUM: A *puzzle* and its resolution

Some of the results of this chapter may be puzzling. I refer to the case of gauged *rigid* SUSY in $D=3$: we have found that, for $\mathcal{N}\geq 4$, only certain gauge groups are allowed. In particular, in a Chern–Simons theory with maximal (rigid) SUSY, $\mathcal{N}=8$, very few gauge groups $\mathcal{G}$ are permitted (only $\mathcal{G}=SO(4)$ according to the analysis of ref.[**192**]).

On the other hand, we know that, in $D=3$, we must have the $\mathcal{N}=8$ super–Yang Mills (SYM) theory for *any* (reductive) Lie group G: Just dimensionally reduce the $D=10$ one ! We also know that $\mathcal{N}=8$ SYM can be formulated as a Chern–Simons theory, thanks to the dWHS duality.

It seems like we got a *paradox* here.

In fact, there are a few subtleties. First of all, the gauge group of the dual CS theory is not the same one of the original SYM. If G is the SYM gauge group, (which we assume to be *semi–simple*) one has

$$\mathcal{G}=G\ltimes A \tag{10.1}$$

where A is an Abelian Lie algebra whose generators $P^{\hat{a}}$ transform in the adjoint of G

$$[t^a,t^b]=f^{ab}{}_c\,t^c \qquad [t^a,P^{\hat{b}}]=f^{ab}{}_c\,P^{\hat{c}} \qquad [P^{\hat{a}},P^{\hat{b}}]=0. \tag{10.2}$$

The invariant tensor l_{mn} for the SYM *á la* CS is associated to the Casimir invariant $t^aP^{\hat{a}}$.

We know that the target space $\mathcal{M}$ for $\mathcal{N}=8$ is flat *i.e.* $\mathbb{R}^{8m}$, $m=\dim G$. The isometry group is

$$O(8m)\ltimes\mathbb{R}^{8m}, \tag{10.3}$$

while the subgroup leaving invariant the symplectic forms Σ^a is

$$\mathrm{Iso}_0(\mathcal{M})=O(m)\ltimes\mathbb{R}^{8m}. \tag{10.4}$$

The semi–simple part of the gauge group, G, should embed in $SO(m)$ while A should embed in $\mathbb{R}^{8m}$. In fact we know that the scalars transform in the adjoint of G *i.e.*

$$X^{\alpha\,a}\mapsto\lambda_b\,f^{abc}\,X^{\alpha\,c} \qquad (\mathfrak{Lie}(G)) \tag{10.5}$$

$$X^{\alpha\,a}\mapsto X^{\alpha\,a}+\delta^{\alpha\,8}\,\lambda^{\hat{a}} \qquad (\mathfrak{Lie}(A)), \tag{10.6}$$

breaking $Spin(8)_R$ symmetry (and hence the conformal invariance).

The momentum map of $\mathfrak{Lie}(A)$ is *linear* in the scalar fields $X^{\alpha\,a}$

$$\mu^{AB\,\hat{a}}=(\Sigma^{AB})_{8\,\alpha}\,X^{\alpha\,a}, \tag{10.7}$$

whereas that of $\mathfrak{Lie}(G)$ is *quadratic*

$$\mu^{AB\,a}=-\frac{1}{2}(\Sigma^{AB})_{\beta\,\gamma}\,f^{abc}\,X^{\beta\,b}\,X^{\gamma\,c}. \tag{10.8}$$

Since the invariant tensor $l_{mn} \in \mathfrak{Lie}(G) \otimes \mathfrak{Lie}(A)$, the T–tensor is cubic in the $X^{\alpha\, a}$'s

$$T^{AB,CD} = g f_{abd}\, X^{\alpha\, a}\, X^{\beta\, b}\, X^{\gamma\, c} (\Sigma^{AB})_{\alpha\,\beta}\, (\Sigma^{CD})_{\gamma\, 8} + \Big([AB] \leftrightarrow [CD]\Big) \quad (10.9)$$

Now this T–tensor *does* satisfy the consistency constraints (although the $Spin(8)$ symmetry is explicitly broken). To check explicitly is painful, but we know that we can formulate the model in the $\mathcal{N} = 1$ formalism, and hence there exists a superpotential W (cubic in the fields) with the right properties. Alternatively, consider the $\mathcal{N} = 8$ model as a special instance of an $\mathcal{N} = 4$ model, and go to the next subsection.

However, a T–tensor cubic in the scalars is an operator of (canonical) dimension 3/2, while conformal invariance requires dimension 2. Hence this 'trick' cannot give us new superconformal theories. Therefore the results of ref.[**192**] (which refers to the superconformal case) are correct.

10.1. $\mathcal{N} = 4$. We have, of course, also $\mathcal{N} = 4$ SYM. The gauge group is again $G \ltimes A$, and l_{mn} is as before. Calling M^a the generators of G and P^a those of A, and adding fermionic generators Q_I ($I = 1, 2, \dots, 2 \dim G$), the Gaiotto–Witten superalgebra reads

$$[M^a, M^b] = f^{ab}{}_c\, M^c \quad (10.10)$$

$$[M^a, P^b] = f^{ab}{}_c\, P^c \quad (10.11)$$

$$[P^a, P^b] = 0 \quad (10.12)$$

$$[M^a, Q_I] = (\tau^a)_{IJ}\, \Omega^{JK}\, Q_K \quad (10.13)$$

$$[P^a, Q_I] = 0 \quad (10.14)$$

$$\{Q_I, Q_J\} = (\tau^a)_{IJ}\, k_{ab}\, P^b \quad (10.15)$$

where the matrices $(\tau^a)_{IJ}$ give a symplectic representation of G, in the SYM case

$$\tau^a = \begin{pmatrix} f^{a\,b}{}_c & 0 \\ 0 & -f^{a\,b}{}_c \end{pmatrix}, \quad (10.16)$$

and k_{ab} is (minus) the Killing form of G. In the Gaiotto–Witten case, the only non–trivial Jacobi identity is the one with three odd generators

$$[Q_I, \{Q_J, Q_K\}] + \text{cyclic permutations} = 0. \quad (10.17)$$

In the present case, this also is trivial since $\{Q_I, Q_J\}$ gives P^a which commutes with Q_K.

Notice that the above superalgebra is (a generalization of) Poincaré supersymmetry.

APPENDICES

APPENDIX A

Elements of Differential Geometry

STILL TO BE WRITTEN

APPENDIX B

Real, Complex and Quaternionic Structures

(Hyper–)complex structures are a unifying theme of SUSY geometry. Roughly speaking the pattern is:

- 2 supercharges $\Rightarrow$ $\mathbb{R}$–structure;
- 4 supercharges $\Rightarrow$ $\mathbb{C}$–structure;
- 8 supercharges $\Rightarrow$ $\mathbb{H}$–structure;
- 16, 32 supercharges $\Rightarrow$ 'Magical'–structure.

In this appendix we review $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ structures mostly to fix conventions. The *connoscenti* may skip it without loss.

Hamilton's Quaternions $\mathbb{H}$. We recall that the quaternions, $\mathbb{H}$, is the real algebra of numbers of the form $a+bi+cj+dk$, $(a,b,c,d\in\mathbb{R})$ with the three imaginary units i,j,k satisfying the relations

$$i^2=j^2=k^2=-1,\quad ij=-ji=k,\quad jk=-kj=i,\quad ki=-ik=j.$$

$\mathbb{H}$ is associative but NOT commutative. The conjugate of the quaternion $\xi=a+bi+cj+dk$ is the quaternion $\bar{\xi}=a-bi-cj-dk$. Conjugation is an *antiautomorphism:* $\overline{\xi\eta}=\bar{\eta}\,\bar{\xi}$. One has $\xi\bar{\xi}=\xi\bar{\xi}=|\xi|^2=a^2+b^2+c^2+d^2$, the Euclidean $(\text{metric})^2$ of $\mathbb{H}$ identified with the vector space $\mathbb{R}^4$; the inner product of two quaternions, ξ,η, — viewed as real 4–vectors — is simply $\langle\xi|\eta\rangle=(\xi\bar{\eta}+\eta\bar{\xi})/2$.

The metric $\xi\mapsto|\xi|$ is a *norm*, that is $|\xi\eta|=|\xi|\cdot|\eta|$. Indeed: $|\xi\eta|^2=(\xi\eta)\overline{(\xi\eta)}=\xi\eta\cdot\bar{\eta}\bar{\xi}=\xi|\eta|^2\bar{\xi}=|\eta|^2\cdot|\xi|^2$, by associativity. Therefore

Proposition. *Quaternions ξ of unit norm $|\xi|=1$ form a group under multiplication. This group is identified with the sphere S^3.*

In fact $|\xi|=1$ means $a^2+b^2+c^2+d^2=1$ which is the unit sphere in $\mathbb{R}^4$.

$\mathbb{H}$ can be realized in terms of 2×2 matrices as

$$a+bi+cj+dk\longmapsto a+ib\sigma_1+ic\sigma_2+id\sigma_3,$$

where σ_k are the usual Pauli matrices. Then an element $\xi\in\mathbb{H}$ is of the form $|\xi|e^{i\vec{\lambda}\cdot\vec{\sigma}}$ and the group of unit quaternions is $SU(2)$.

$\mathbb{H}$ is a *division algebra:* any element $\xi\neq 0$ has an inverse. In fact if $\xi=|\xi|e^{i\vec{\lambda}\cdot\vec{\sigma}}$, $\xi^{-1}=|\xi|^{-1}\,e^{-i\vec{\lambda}\cdot\vec{\sigma}}$.

1. $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ Structures on Vectors Spaces

1.1. Complex Structures. A *complex structure* on a real space $\mathcal{V}$ is a $\mathbb{R}$–linear operator I with $I^2=-1$. We can define the product of a vector $v\in\mathcal{V}$ by the complex scalar $(a+bi)$ as $(a+bI)v$. This makes $\mathcal{V}$ into a

$\mathbb{C}$–space. A real space $\mathcal{V}$ with a complex structure has necessarily an even dimension, $\dim_{\mathbb{R}} \mathcal{V} = 2n$; in suitable bases I takes the block form

$$I = \begin{pmatrix} 0 & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & 0 \end{pmatrix}. \tag{1.1}$$

It is convenient to identify $\mathcal{V} \simeq \mathbb{R}^{2n}$ with $\mathbb{R}^2 \otimes \mathbb{R}^n$ writing $I = (i\sigma_2) \otimes \mathbf{1}$. I has eigenvalues $\pm i$. One denotes the corresponding eigenspaces $\mathcal{V}_\pm$. $\mathcal{V}_+$ is a complex vector space of (complex) dimension n on which $(a + bi)$ acts by ordinary complex multiplication. One has $\mathcal{V}_+ \oplus \mathcal{V}_- \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathcal{V}$, the complexification of $\mathcal{V}$. As *real* vector spaces $\mathcal{V}_+ \simeq \mathcal{V}$.

As a matter of notation a basis of $\mathcal{V}_+$ will be denote as $v_1, v_2, \dots v_n$ and the conjugate base of $\mathcal{V}_-$ as $v_{\bar{1}}, v_{\bar{2}} \dots v_{\bar{n}}$. Then $v_k + v_{\bar{k}}$ and $i(v_k - v_{\bar{k}})$ make a real basis for $\mathcal{V}$.

1.2. Real Structures. Conversely let $\mathcal{W}$ be a complex vector space of (complex) dimension n. A *real structure* on $\mathcal{W}$ is *anti*linear map R with $R^2 = 1$. The eigenvalues of R are ± 1. Let $\mathcal{W}_\pm$ be the corresponding eigenspaces. Vectors $v \in \mathcal{W}_+$ are called *real*, while vectors $w \in \mathcal{W}_-$ are purely *imaginary.* The definition is coherent, since $Ri = -iR$, so multiplication by i in $\mathcal{W}$ transforms real vectors into imaginary and viceversa. One has $\mathcal{W} = \mathcal{W}_+ \otimes_{\mathbb{R}} \mathbb{C}$.

Let our complex space $\mathcal{W}$ be obtained from a real space $\mathcal{V}$ as in §.**1.1**. $R^2 = 1$ and $RI = -IR$, so we can always choose the basis so that $I = (i\sigma_2) \otimes \mathbf{1}$ and $R = \sigma_3 \otimes \mathbf{1}$. Then $\mathcal{V} = \mathcal{W}_+ \oplus \mathcal{W}_-$, as real spaces, and $I\mathcal{W}_\pm = \mathcal{W}_\mp$. The relation between the basis vectors $v_k, v_{\bar{k}}$ and $v_k + v_{\bar{k}}$, $i(v_k - v_{\bar{k}})$ corresponds to that between eigenvectors of σ_2 and σ_3.

1.3. Quaternionic Structures. Let $\mathcal{W}$ be a *complex* vector space. A *quaternionic structure* on $\mathcal{W}$ is an *anti*linear map J such that $J^2 = -1$. Note the similarity with the real structure in §.**1.2**: the only difference is the sign of the square of the anti–isomorphism: quaternionic structures are also called *pseudoreal.*

A complex space $\mathcal{W}$ with a quaternionic structure is made into a left $\mathbb{H}$–module by defining multiplication (on the left) by the quaternion $(a + bi + cj + dk)$ as the action of the $\mathbb{R}$–linear operator $(a + bi + cJ + diJ)$. Such a space has always even (complex) dimension.

Let $\mathcal{W}$ be obtained from a real space $\mathcal{V}$ of (real) dimension $4n$. Then $\mathcal{W}$ has both a real and a quaternionic structure. Without loss of generality, we can choose the basis so that

$$I = (i\sigma_2) \otimes 1_{2\times 2} \otimes \mathbf{1}_{n\times n}, \qquad J = \sigma_3 \otimes (i\sigma_2) \otimes \mathbf{1}_{n\times n}, \tag{1.2}$$

and then $K = IJ = -\sigma_1 \otimes (i\sigma_2) \otimes \mathbf{1}_{n\times n}$ (note that the matrices are real). Consider the *auxiliary* complex structure $L = 1_{2\times 2} \otimes (i\sigma_2) \otimes \mathbf{1}_{n\times n}$. It has eigenvalues $\pm i$; let $\mathcal{V}^\pm$ be the corresponding eigenspaces. The base vectors of $\mathcal{V}^+$ will be denoted as $v_{\alpha i}$, the index $\alpha = 1, 2$ corresponding to the first factor in the tensor product in the RHS of eqn.(1.2) and the index $i = 1, \dots, n$ to the last. On $\mathcal{V}^+$ the matrices I, J, K act as

$$I = i\sigma_2 \otimes \mathbf{1}_{n\times n}, \qquad J = i\sigma_3 \otimes \mathbf{1}_{n\times n}, \qquad K = -i\sigma_1 \otimes \mathbf{1}_{n\times n}, \tag{1.3}$$

and in $\mathcal{V}^-$ by the conjugate matrices. We recover the Pauli matrices realization of $\mathbb{H}$.

As a basis of $\mathcal{V}_\mathbb{C}$ we can consider $\{v_{\alpha i}, \epsilon_{\alpha\beta}(v_{\alpha i})^*\}$. In this basis

$$I = (i\sigma_2) \otimes 1_{2\times 2} \otimes \mathbf{1}_{n\times n}, \qquad J = i\sigma_3 \otimes 1_{2\times 2} \otimes \mathbf{1}_{n\times n} \tag{1.4}$$

The real structure associated with L, R_L, becomes in this basis:

$$R_L = -(i\sigma_2) \otimes (i\sigma_2) \otimes \mathbf{1}_{n\times n}. \tag{1.5}$$

Eqns.(1.4)(1.5) allows us to introduce the formalism more useful (and used) in the context of supersymmetry. We consider the space $\mathcal{V}_\mathbb{C} = \mathcal{V} \otimes_\mathbb{R} \mathbb{C} \simeq \mathcal{V}^+ \oplus \mathcal{V}^-$ and identify $\mathcal{V}_\mathbb{C} \simeq \mathbb{C}^2 \otimes \mathbb{C}^{2n}$; a basis element is written as $v_{\alpha a}$, $\alpha = 1, 2$, $a = 1, \dots, 2n$. This space is viewed as a (left) $\mathbb{H}$–module by $(a + bi + cj + dk) \mapsto (a + ib\sigma_2 + ic\sigma_3 - id\sigma_1)$ acting on the index α. We have a real antisymmetric $2n \times 2n$ matrix Ω, with $\Omega^2 = -1$, such that the (auxiliary) real structure $R_L = (i\sigma_2) \otimes \Omega$ (cfr. eqn.(1.5)). Then a vector is *real* (and so belongs to the original real space $\mathcal{V}$) if $R_L v = v$ or in components[1]

$$x^{\alpha a} = \epsilon^{\alpha\beta}\, \Omega^{ab}\, (x^{b\beta})^*. \tag{1.6}$$

1.4. Symmetry Groups. Assume the real vector space $\mathcal{V}$ of dimension $2n$ (resp. $4n$) has a complex (resp. quaternionic) structure and a (positive–definite) inner product such that the imaginary units I, J, K are antisymmetric — as in eqns.(1.1) and (1.2) — and hence orthogonal. The linear maps preserving both the inner product and the complex structure are

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad A, B \in \mathbb{R}(n), \quad AA^T + BB^T = \mathbf{1}, \quad AB^T = BA^T, \tag{1.7}$$

i.e. $U \equiv A + iB$ is unitary, $UU^\dagger = \mathbf{1}$. Thus the group of orthogonal transformations leaving invariant the complex structure is $U(n)$.

In the quaternionic case the matrices $A, B \in \mathbb{R}(2n)$ are further constrained by the condition

$$A\Omega = \Omega A, \qquad B\Omega = -\Omega B \tag{1.8}$$

i.e. $U^* = (A - iB) = \Omega U \Omega^{-1}$. Then

$$UU^\dagger = 1, \quad U^T \Omega U = \Omega, \tag{1.9}$$

and the relevant group is $Sp(n) \equiv U(2n) \cap Sp(n, \mathbb{C})$. In the basis $v_{\alpha a}$ introduced at the end of §.**1.3**, $Sp(n)$ acts on the index a.

This is not the whole story, however. We have still the possibility of an orthogonal transformation which, while not leaving invariant the imaginary units, acts as an automorphism of $\mathbb{H}$. This is the $Sp(1) \simeq SU(2)$ group of unit quaternions. In the basis $v_{\alpha a}$ it acts on the index α. The actions of $Sp(1)$ and $Sp(n)$ obviously commute, so we can consider $Sp(1) \otimes Sp(n)$ as a natural group preserving both the $\mathbb{H}$–structure and the inner product.

[1] Recall that R_L is an anti–linear map.

2. Transitive Actions on Spheres

We have seen in §.**1.2** that the groups $SO(n)$ (REAL case), $U(n)$ and $SU(n)$ (COMPLEX case), and $Sp(1)\otimes Sp(n)$ and $Sp(n)$ (QUATERNIONIC case) act, respectively, on $\mathbb{R}^n$, $\mathbb{R}^{2n}$ and $\mathbb{R}^{4n}$ by orthogonal matrices. Hence they preserve the unit sphere S^{m-1} ($m = n$, $2n$, or $4n$). The crucial point, common to all cases, is that this action is *transitive.*

PROPOSITION 2.1. *The groups*

(1) *$SO(n)$,*
(2) *$U(n)$ and $SU(n)$,*
(3) *$Sp(1)\otimes Sp(n)$, $U(1)\otimes Sp(n)$ and $Sp(n)$,*

act TRANSITIVELY *on the sphere S^{n-1},respectively S^{2n-1} and S^{4n-1}.*

REMARK. Indeed one has

$$\begin{gathered} S^{n-1} = SO(n)/SO(n-1), \quad S^{2n-1} = SU(n)/SU(n-1)???, \\ S^{4n-1} = Sp(n)/Sp(n-1). \end{gathered} \tag{2.1}$$

To show the proposition, one starts from the projective spaces $P^{n-1}(\mathbb{K})$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. These spaces are defined as $(X^1, \dots, X^n)$, $X^k \in \mathbb{K}$, not all vanishing, with $(X^1, \dots, X^n) \approx (\lambda X^1, \dots \lambda X^n)$, $\forall \lambda \in \mathbb{K}^*$. Choose $\lambda = (\sum_k |X^k|^2)^{-1/2}$. The new homogeneous coordinates X^k satisfy $\sum_k |X^k|^2 = 1$, *i.e.* they represent a point in the unit sphere. Thus we get surjective maps $S^{n-1} \to P^{n-1}(\mathbb{R})$, $S^{n-1} \to P^{2n-1}(\mathbb{C})$, and $S^{4n-1} \to P^{n-1}(\mathbb{H})$, (Hopf bundles). The automorphism group of the projective space, $SL(n, \mathbb{K})$, obviously acts transitively on $\mathbb{K}^n$; hence its compact subgroup preserving the norm $\sum_k |X^k|^2$ acts transitively on $S^{n \dim \mathbb{K}-1}$: this is $SU(n)$ in the $\mathbb{C}$ case and $Sp(n)$ for $\mathbb{H}$. The isotropy subgroup of a point in $\mathbb{K}^n$ is $SL(n-1, \mathbb{K})$; its compact subgroup is the isotropy of a point in $S^{n \dim \mathbb{K}-1}$. This implies proposition 2.1 .

The projective space $P^{n-1}(\mathbb{K})$ is obtained from $S^{n \dim \mathbb{K}-1}$ by taking the quotient with respect the group of elements in $\mathbb{K}$ of unit norm (so the fiber is $S^{\dim \mathbb{K}-1}$). Then

$$P^{n-1}(\mathbb{C}) = SU(n)/U(1)\otimes SU(n-1), \quad P^{n-1}(\mathbb{H}) = Sp(n)/Sp(1)\otimes Sp(n-1).$$

In the next appendix we shall generalize these constructions to a fourth division algebra besides $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, the Cayley numbers (octonions) $\mathbb{O}$. In this way we will found other three groups acting transitively on spheres namely:

$$G_2, \qquad Spin(7), \qquad Spin(9). \tag{2.2}$$

APPENDIX C

Clifford algebras, Octonions, Triality, and G_2

Fermions are spinors or, in a fancier language, elements of a *Clifford module.* As argued in the Introduction, the geometry of supersymmetry reflects the algebraic structure of fermions. We have also motivated the necessity of working in different space–time dimensions. Therefore we begin by a careful analysis of the Clifford algebras in arbitrary dimensions (and signature of the metric). The main theme is to elucidate the relation between the Dirac matrices and the four classical real *division algebras:* the reals numbers, $\mathbb{R}$, the complex, $\mathbb{C}$, the quaternions, $\mathbb{H}$, and the octonions, $\mathbb{O}$. *SUSY* geometric structures will be *real* (2 supercharges), *complex* (4 supercharges), and *quaternionic* or *symplectic* (8 supercharges), and the Clifford algebras is a first manifestation of this pattern. The extension to $\mathbb{O}$ allow us to understand the peculiarities of the spin groups *spin(7)* and *spin(8)*, as well their friend the exceptional Lie group G_2, which are relevant for the geometry of superstring and M-theory compactifications.

1. Real Clifford Algebras

Clifford algebras is the mathematical name for the Dirac matrices. We adopt the mathematical language for two reasons: (1) we need to work in arbitrary spacetime dimension; (2) we are primarily interested in geometric structures which are more naturally described in that language. Of course everything can be restated in physic language, as we do whenever useful.

1.1. The Clifford Algebra of a Quadratic Form. Let $\mathcal{V}$ be a finite–dimensional real vector space and $Q\colon \mathcal{V} \to \mathbb{R}$ a quadratic form. The form Q defines in $\mathcal{V}$ a (symmetric) inner product, $Q(x, y)$, by the formula

$$2\,Q(x, y) = Q(x + y) - Q(x) - Q(y) \tag{1.1}$$

A real algebra with unit, $\mathcal{A}$, is a Clifford algebra for $(\mathcal{V}, Q)$ if there is an injective linear map[1] $i\colon \mathcal{V} \to \mathcal{A}$ and[2]

$$x^2 = Q(x). \tag{1.2}$$

The first result is

THEOREM 1.1. *For each pair $(\mathcal{V}, Q)$ there is a universal Clifford algebra $\mathbb{C}l(Q)$ unique up to isomorphism. $\mathbb{C}l(Q)$ is $\mathbb{Z}_2$–graded.*

[1]The injectivity condition is superfluous, we add it for convenience.

[2] Here and in the sequel we identify an element $x \in \mathcal{V}$ with its image in $\mathcal{A}$ (cfr. the previous note). In the same way we identify real numbers λ and elements $\lambda \cdot 1 \in \mathcal{A}$ (this applies, in particular to the RHS of (1.2)).

We sketch the proof. Let $\mathcal{V}_k = \mathcal{V} \otimes \mathcal{V} \otimes \cdots \otimes \mathcal{V}$ (k terms) and consider the space

$$\mathcal{T} = \mathbb{R} \oplus \mathcal{V} \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \cdots \tag{1.3}$$

$\mathcal{T}$ is an algebra with respect the product $(x, y) \mapsto x \otimes y$. The vector space $\mathcal{V}$ is naturally identified with a subspace of $\mathcal{T}$. Consider the ideal $\mathcal{I} \subset \mathcal{T}$ generated by all the expressions of the form $x \otimes x - Q(x)$. The algebra

$$\mathbb{C}l(Q) = \mathcal{T}/\mathcal{I} \tag{1.4}$$

is the universal Clifford algebra for the quadratic form Q. Its universality follows from the analogue property of the tensor product (see *e.g.* [**228**]). Consider the map[3] $\alpha\colon \mathcal{V} \to \mathbb{C}l$ given by $\alpha(x) = -x$. Since $\alpha(x)^2 = Q(x)$, it defines a Clifford algebra for Q which, by universality, should be isomorphic to $\mathbb{C}l$. Let $\alpha\colon \mathbb{C}l \to \mathbb{C}l$ be the corresponding automorphism. One has $\alpha^2 = 1$. Then $\mathbb{C}l$ decomposes into the direct sum of two subspaces on which α acts as multiplication by ± 1. We call them, respectively, the spaces of *even* and *odd* elements

$$\mathbb{C}l = \mathbb{C}l^0 \oplus \mathbb{C}l^1,$$

moreover

$$\mathbb{C}l^a \cdot \mathbb{C}l^b \subset \mathbb{C}l^{a+b \bmod 2}$$

so $\mathbb{C}l$ is a $\mathbb{Z}_2$–graded algebra.

Working with $\mathbb{Z}_2$–graded algebras, it is convenient to introduce a modified tensor product called the *super–tensor product* (or *super–product*).

Definition 1.1. Let $\mathcal{A}$, $\mathcal{B}$ be two $\mathbb{Z}_2$–graded algebras. Their *super–tensor* product, written $\mathcal{A} \otimes_s \mathcal{B}$, is the $\mathbb{Z}_2$–graded algebra having the underlying vector space $\mathcal{A} \otimes \mathcal{B}$ and product

$$(a \otimes b)\,(a' \otimes b') = (-1)^{\deg(b)\,\deg(a')}\, aa' \otimes bb'. \tag{1.5}$$

The reader will immediately recognize in this definition the standard product for the graded algebra of bosonic/fermionic operators in a physical theory: one has first to (anti)commute b past a'.

If we have two spaces with quadratic forms, $(\mathcal{V}_1, Q_1)$ and $(\mathcal{V}_2, Q_2)$, we can construct the *direct sum quadratic form,* denoted $Q_1 \oplus Q_2$, *i.e.* the quadratic functional on the space $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ given by

$$Q(x_1 + x_2) = Q_1(x_1) + Q_2(x_2) \qquad x_1 \in \mathcal{V}_1,\ x_2 \in \mathcal{V}_2.$$

Proposition 1.1. *One has the isomorphism*

$$\mathbb{C}l(Q_1 \oplus Q_2) \approx \mathbb{C}l(Q_1) \otimes_s \mathbb{C}l(Q_2) \tag{1.6}$$

Proof. Let $\alpha\colon \mathcal{V} \to \mathbb{C}l(Q_1) \otimes_s \mathbb{C}l(Q_2)$ be the linear map

$$\alpha(x_1 + x_2) = x_1 \otimes 1 + 1 \otimes x_2.$$

By the definition of the *super–product* $(x_1 \otimes 1)(1 \otimes x_2) = x_1 \otimes x_2$ and $(1 \otimes x_2)(x_1 \otimes 1) = -x_1 \otimes x_2$. Then

$$\begin{aligned}\alpha(x_1 + x_2)^2 = (x_1 \otimes 1 + 1 \otimes x_2)^2 &= x_1^2 \otimes 1 + 1 \otimes x_2^2 \\ &= Q_1(x_1) + Q_2(x_2) \equiv Q(x_1 + x_2).\end{aligned}$$

[3] To save print we omit reference to the quadratic form Q if there is no ambiguity.

By the universality property of the theorem, α extends to a morphism of Clifford algebras, $\alpha^\sharp\colon \mathbb{C}l(Q) \to \mathbb{C}l(Q_1) \otimes_s \mathbb{C}l(Q_2)$ which is easily seen to be an isomorphism. □

Let $\dim \mathcal{V} = n$. All real quadratic functional Q on $\mathcal{V}$ can be put in the canonical form

$$x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2. \tag{1.7}$$

In eqn.(1.7) Q is written as the direct sum of one dimensional forms. In one dimension we have only three inequivalent quadratic forms: Q_1, Q_{-1} and Q_0, according if $Q(e) = \pm 1$ or 0 on a base vector e; the corresponding Clifford algebra, $\mathbb{C}l(Q_\varepsilon)$ is isomorphic to the algebra $\mathbb{R}[e]/(e^2 - \varepsilon)$ with $\varepsilon = \pm 1, 0$. For $\varepsilon = -1$ this is the algebra $\mathbb{C}$ (*complex* numbers); for $\varepsilon = 1$ it is the algebra $\mathbb{D}$ of *double* numbers ($a + be$ where $a, b \in \mathbb{R}$ and $e^2 = 1$); for $\varepsilon = 0$ it is the algebra $\mathbb{G}$ of *dual* numbers ($a + \xi$ with $a \in \mathbb{R}$ and ξ a real Grassmanian element). So proposition 1.1 gives

THEOREM 1.2. *For a quadratic form Q on a vector space of dimension n, rank r and signature $(p, r-p)$*

$$\mathbb{C}l(Q) = \underbrace{\mathbb{D} \otimes_s \cdots \otimes_s \mathbb{D}}_{p\ \text{times}} \otimes_s \underbrace{\mathbb{C} \otimes_s \cdots \otimes_s \mathbb{C}}_{(r-p)\ \text{times}} \otimes_s \underbrace{\mathbb{G} \otimes_s \cdots \otimes_s \mathbb{G}}_{(n-r)\ \text{times}}. \tag{1.8}$$

In particular $\dim \mathbb{C}l(Q) = 2^n$.

REMARK. The choice between a metric of signature (p, q) and one of signature (q, p) is a matter of convention. However the corresponding Clifford algebras are NOT isomorphic over the reals (their Gamma–matrices are related by $\Gamma' \leftrightarrow \sqrt{-1}\,\Gamma$). Therefore issues like Hermiticity of the Dirac matrices or reality of Majorana–like spinors look different in the two cases. Of course this simply means that when we change our convention for the signature of the metric we have also to change our conventions for spinors. The theory of real Clifford algebras specifies in which way.

For the rest of the chapter we assume the form Q to be definite of signature (p, q). We write $\mathbb{C}l(p, q)$ for its universal Clifford algebra.

EXAMPLE. In two–dimensions we have three algebras: $\mathbb{C}l(0, 2)$, $\mathbb{C}l(1, 1)$, and $\mathbb{C}l(2, 0)$. In the first case the identification

$$1 = 1 \otimes_s 1, \quad i = i \otimes_s 1, \quad j = 1 \otimes_s i, \quad k = i \otimes_s i, \tag{1.9}$$

gives

$$\mathbb{C}l(0, 2) \approx \mathbb{H} \qquad \text{(Hamilton's quaternions)}. \tag{1.10}$$

In the second case the identifications

$$1 = 1 \otimes_s 1, \quad \sigma_3 = e \otimes_s 1, \quad i\sigma_2 = 1 \otimes_s i, \quad \sigma_1 = e \otimes_s i, \tag{1.11}$$

and in the third case

$$1 = 1 \otimes_s 1, \quad \sigma_3 = e \otimes_s 1, \quad \sigma_1 = 1 \otimes_s e, \quad i\sigma_2 = e \otimes_s e, \tag{1.12}$$

lead to[4]

$$\mathbb{C}l(1,1) \approx \mathbb{R}(2) \qquad \mathbb{C}l(2,0) \approx \mathbb{R}(2). \tag{1.13}$$

Note, however, that $\mathbb{C}l(1,1) \not\approx \mathbb{C}l(2,0)$ as $\mathbb{Z}_2$*–graded algebras* since the grading is different in the two cases: the even elements are 1 and σ_1 in $\mathbb{C}l(1,1)$ and 1 and $i\sigma_2$ in $\mathbb{C}l(2,0)$.

1.2. First Properties. Let $e_1, \ldots, e_n$ be a basis of $\mathcal{V}$, orthonormal with respect to the product $Q(\cdot,\cdot)$ in eqn.(1.1). We write Γ_i for the image of e_i in $\mathbb{C}l$ and set $\eta_{ij} = Q(e_i, e_j) \equiv \delta_{ij}\varepsilon_i$. In terms of the Γ_i's, eqn.(1.2) reads

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2\,\eta_{ij}. \tag{1.14}$$

It is convenient to specify an explicit basis for $\mathbb{C}l$. There is a basis element for each subset $I \subset \{1, 2, \ldots, n\}$. Writing I as $\{i_1 < i_2 < \cdots < i_m\}$, it is

$$\Gamma_I = \Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_m}. \tag{1.15}$$

(with $\Gamma_\emptyset = 1$). One has

$$\Gamma_I\Gamma_J = (-1)^{\tau(I,J)}\left(\prod_{k\in I\cap J}\varepsilon_k\right)\Gamma_{I\Delta J}$$

where $I\Delta J = (I\cup J)\setminus(I\cap J)$ and $\tau(I,J) = \sharp\{(i,j)\in I\times J|\ \ i>j\}$. In particular

$$\Gamma_I^2 = (-1)^{m(m-1)/2}\left(\prod_{k\in I}\varepsilon_k\right).$$

Proposition 1.2. *The center of the algebra* $\mathbb{C}l(p,q)$ *is given by* $\mathbb{R}\cdot 1$ *if* n *is even and by* $\mathbb{R}\cdot 1 + \mathbb{R}\cdot\Gamma_{[n]}$ *where* $[n]$ *denotes the full set* $\{1,\ldots,n\}$.

$\Gamma_{[n]}$ is the chirality operator generalizing to arbitrary dimensions the usual Dirac matrix γ_5.

Proof. Let x be an element of $\mathbb{C}l$. Since the Dirac matrices Γ_i generates the algebra, x belongs to the center if and only if $\Gamma_i\,x = x\,\Gamma_i$ for all i. This is equivalent to $\Gamma_i\,x\,\Gamma_i = \varepsilon_i\,x$. Write x in the above basis as $\sum x_I\,\Gamma_I$. Then

$$\Gamma_i x\Gamma_i = \sum x_I\Gamma_i\Gamma_I\Gamma_i = \sum_{i\in I}(-1)^{m(I)-1}\,\varepsilon_i\;\,x_I\Gamma_I + \sum_{i\notin I}(-1)^{m(I)}\,\varepsilon_i\;\,x_I\Gamma_I.$$

Since the two terms in the RHS have different signs, this expression may be equal to $\varepsilon_i\,x$ only if one of them vanishes for $\forall i$. This leaves two possibilities $I=\emptyset$ or $[n]$. In the first case we get 1 which is a central element. In the second, we have $\Gamma_i\Gamma_{[n]}\Gamma_i = (-1)^{(n-1)}\varepsilon_i\,\Gamma_I$; and $\Gamma_{[n]}$ is central for n odd. □

Note that

$$\Gamma_{[n]}^2 = (-1)^{n(n-1)/2}\prod_k\varepsilon_k. \tag{1.16}$$

In an irreducible Clifford module $\Gamma_{[n]}$ is represented, for odd n, as multiplication by a number. This number ±1 if the RHS of eqn.(1.16) is $+1$, and $\pm i$ otherwise.

[4] Here and in the sequel $\mathbb{A}(n)$ denotes the algebra of the $n\times n$ matrices whose entries are elements of the algebra $\mathbb{A}$.

2. Bott Periodicity and Majorana Spinors

For supersymmetry is crucial to know which kinds of fermions exist in a Minkowski space of signature (p, q). It is our next subject.

2.1. Matrix Form of $\mathbb{C}l(p,q)$. We wish to construct explicit matrix realizations of the universal algebras $\mathbb{C}l$. We start by comparing the Clifford algebras for spacetime dimensions n and $n+2$. Note that in the isomorphisms below we use *ordinary* tensor–products not super.

PROPOSITION 2.1. *One has*

$$\mathbb{C}l(p+1,q+1) \simeq \mathbb{C}l(p,q) \otimes \mathbb{C}l(1,1) \tag{2.1}$$

$$\mathbb{C}l(p+2,q) \simeq \mathbb{C}l(q,p) \otimes \mathbb{C}l(2,0) \tag{2.2}$$

$$\mathbb{C}l(p,q+2) \simeq \mathbb{C}l(q,p) \otimes \mathbb{C}l(0,2). \tag{2.3}$$

Notice the inversion $p \leftrightarrow q$ in the RHS of the last two equations.

PROOF. Explicitly the isomorphism is

$$\begin{aligned} \Gamma_i &= \Gamma_i \otimes \Gamma_{[2]} \quad \text{for} \quad i = 1, \dots, p+q \\ \Gamma_i &= 1 \otimes \Gamma_i \quad \text{for} \quad i > p+q. \end{aligned}$$

The inversion is due to a minus sign in the square of $\Gamma_{[2]}$ in the definite case. □

Using eqns.(1.10)(1.13), the above isomorphisms read

$$\mathbb{C}l(p+1,q+1) \simeq \mathbb{C}l(p,q) \otimes \mathbb{R}(2) \tag{2.4}$$

$$\mathbb{C}l(p+2,q) \simeq \mathbb{C}l(q,p) \otimes \mathbb{R}(2) \tag{2.5}$$

$$\mathbb{C}l(p,q+2) \simeq \mathbb{C}l(q,p) \otimes \mathbb{H}. \tag{2.6}$$

COROLLARY 2.1. *We have*

$$\begin{cases} \mathbb{C}l(p,q) \simeq \mathbb{C}l(p-q,0) \otimes \mathbb{R}(2^q) & \text{for } p \geq q \\ \mathbb{C}l(p,q) \simeq \mathbb{C}l(0,q-p) \otimes \mathbb{R}(2^p) & \text{for } p \leq q. \end{cases} \tag{2.7}$$

Thus we need to study only the Clifford algebras of definite *quadratic forms.*

COROLLARY 2.2. *We have: (1)* $\mathbb{C} \otimes \mathbb{H} \simeq \mathbb{C}(2)$*; (2)* $\mathbb{H} \otimes \mathbb{H} \simeq \mathbb{R}(4)$*. Combing with* $\mathbb{A} \otimes \mathbb{R}(n) = \mathbb{A}(n)$*, valid for all real algebras* $\mathbb{A}$*, one gets*

$$\mathbb{C}(m) \otimes \mathbb{H}(n) \simeq \mathbb{C}(2nm), \qquad \mathbb{H}(n) \otimes \mathbb{H}(m) \simeq \mathbb{R}(4nm). \tag{2.8}$$

PROOF. (1) From proposition 2.1 we get two expressions for $\mathbb{C}l(1,2)$, namely $\mathbb{C}l(0,1)\otimes\mathbb{C}l(1,1) \simeq \mathbb{C}\otimes\mathbb{R}(2) \simeq \mathbb{C}(2)$ and $\mathbb{C}l(0,1)\otimes\mathbb{C}l(0,2) \simeq \mathbb{C}\otimes\mathbb{H}$. (2) Compute $\mathbb{C}l(2,2)$ in two ways: as $\mathbb{C}l(0,2)^{\otimes 2}$ and as $\mathbb{C}l(2,0)^{\otimes 2}$. □

THEOREM 2.1 (Periodicity mod. 8). *One has*

$$\mathbb{C}l(p+8k,q+8l) \simeq \mathbb{C}l(p,q) \otimes \mathbb{R}(2^{4(k+l)}). \tag{2.9}$$

I.e. — a part for the dimension of the matrices — the properties of the Clifford algebra are periodic in p, q *mod. 8.*

In fact, combing with corollary 2.1, we see that the properties of the Clifford algebra depend — except for matrix dimensions — only on $(p-q)$ mod. 8. This result is the prototype of many (related) mod. 8 periodicity theorems. The general phenomenon is called *Bott periodicity.*

PROOF. Using eqns.(2.4)-(2.6) twice we get

$$\mathbb{C}l(p+4,q) \simeq \mathbb{C}l(p,q+4) \simeq \mathbb{C}l(p,q) \otimes \mathbb{H}(2), \tag{2.10}$$

and iterating

$$\begin{aligned}\mathbb{C}l(p+8,q) &\simeq \mathbb{C}l(p+4,q+4) \simeq \mathbb{C}l(p,q+8)\\ &\simeq \mathbb{C}l(p,q) \otimes \mathbb{H}(2) \otimes \mathbb{H}(2) \simeq \mathbb{C}l(p,q) \otimes \mathbb{R}(16).\end{aligned}$$

In the last step we used an identity from eqn.(2.8). □

Corollary 2.1 and the theorem reduce the general case to the definite case in dimension ≤ 8. Starting with the know results in dimensions one and two, using eqns.(2.4)-(2.10) and the identities (2.8), it is easy to write down the complete isomorphism table.

PROPOSITION 2.2. *We have*[5]

$$\begin{array}{cc}\mathbb{C}l(1,0) \simeq \mathbb{R} \oplus \mathbb{R} & \mathbb{C}l(0,1) \simeq \mathbb{C}\\ \mathbb{C}l(2,0) \simeq \mathbb{R}(2) & \mathbb{C}l(0,2) \simeq \mathbb{H}\\ \mathbb{C}l(3,0) \simeq \mathbb{C}(2) & \mathbb{C}l(0,3) \simeq \mathbb{H} \oplus \mathbb{H}\\ \mathbb{C}l(4,0) \simeq \mathbb{H}(2) & \mathbb{C}l(0,4) \simeq \mathbb{H}(2)\\ \mathbb{C}l(5,0) \simeq \mathbb{H}(2) \oplus \mathbb{H}(2) & \mathbb{C}l(0,5) \simeq \mathbb{C}(4)\\ \mathbb{C}l(6,0) \simeq \mathbb{H}(4) & \mathbb{C}l(0,6) \simeq \mathbb{R}(8)\\ \mathbb{C}l(7,0) \simeq \mathbb{C}(8) & \mathbb{C}l(0,7) \simeq \mathbb{R}(8) \oplus \mathbb{R}(8)\\ \mathbb{C}l(8,0) \simeq \mathbb{R}(16) & \mathbb{C}l(0,8) \simeq \mathbb{R}(16).\end{array} \tag{2.11}$$

From eqns.(2.7)(2.9)(2.11) we see that the various universal Clifford algebras $\mathbb{C}l(p,q)$ are matrix algebras with entries in the classical division algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ (the last one, the octonions $\mathbb{O}$, will also enter the game, see sects. 4–8 below). Thus we have an explicit matrix realization as required. We refer to the elements $\Gamma_i \in \mathbb{C}l$ as Γ–*matrices.*

REMARK. **(Hermitean properties)**. The concrete Γ_k matrices constructed above have a special form. Going trough the construction we see that they have the structure

$$i^l\, \sigma_{k_1} \otimes \sigma_{k_2} \otimes \cdots \otimes \sigma_{k_m} \tag{2.12}$$

which implies that $\Gamma_k \in U(2^m)$. (More precisely, a part for an overall power of i, the Γ_k are *monomial matrices*: all elements in each row and in each column vanish, except for a single one equal to ± 1). If $\mathbb{C}l(p,q)$ is a *real* (resp. *quaternionic*) matrix algebra, $\Gamma_k \in O(2^m) \subset U(2^m)$, (resp. $\Gamma_k \in Sp(2^{m-1}) \subset U(2^m)$. Comparing the relations $\Gamma_k\Gamma_k^\dagger = 1$ and $(\Gamma_k)^2 = \varepsilon_k$ we get $\Gamma_k^\dagger = \varepsilon_k\, \Gamma_k$.

REMARK. (**Complex Clifford algebras**). The story becomes much simpler if one considers the Clifford algebras over the field $\mathbb{C}$. The complexified algebras $\mathbb{C}l(p,q) \otimes_{\mathbb{R}} \mathbb{C}$ depend, up to isomorphism, only on the dimension $n = p+q$; indeed the map $\Gamma_k \to (\sqrt{-1})^{\varepsilon_k}\, \Gamma_k$ transforms any

[5] Note the symmetry between $\mathbb{C}l(p,0)$and $\mathbb{C}l(0,8-p)$. They differ only in dimension as predicted by the mod 8 periodicity.

$\mathbb{C}l(p,q) \otimes_{\mathbb{R}} \mathbb{C}$ into $\mathbb{C}l(p+q,0) \otimes_{\mathbb{R}} \mathbb{C}$. We write $\mathbb{C}l(n,\mathbb{C})$ for this complex algebra. Tensoring the table in eqn.(2.11) with $\mathbb{C}$, we get[6]

$$\mathbb{C}l(n,\mathbb{C}) \simeq \begin{cases} \mathbb{C}(2^m) & \text{for } n=2m \\ \mathbb{C}(2^m)\oplus\mathbb{C}(2^m) & \text{for } n=2m+1. \end{cases} \tag{2.13}$$

The direct sum for *odd* dimension is related to the fact, already discussed, that in this case the center is two–dimensional. The direct summands correspond to the subalgebras in which $\Gamma_{[n]}$ takes, respectively, the values $+(-1)^{n(n-1)/4}$ and $-(-1)^{n(n-1)/4}$. In the physical applications, one usually considers only one irreducible summand.

2.2. Zoology of Majorana and Weyl Spinors. We are interested in the reality properties of spinors in space–times of different dimension and signature. The explicit matrix realization of the algebras $\mathbb{C}l(p,q)$ defines a basic module which we denote as $\mathcal{S}(p,q)$; we will refer to its elements as *spinors*. We can read their reality properties of the elements directly from the matrix form of the Clifford algebras, §.2.1.

From a complex point of view, things are simple. (Irreducible) complex spinors — called *Dirac spinors* — exist for all dimensions and signatures. Their dimension can be read in eqn.(2.13):

$$\dim \mathcal{S}(n,\mathbb{C}) = 2^{[n/2]}. \tag{2.14}$$

Over the reals, we have to distinguish three basic kinds of spinors: *real*, *complex*, and *quaternionic* (or symplectic) according if the corresponding $\mathbb{C}l(p,q)$ is a matrix algebra over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. They are known, respectively, as *Majorana*, *Dirac*, and *symplectic–Majorana spinors.* The Majorana fermions, in the present setting, are easy: they just have real components.

The *symplectic–Majorana* spinors satisfy the reality condition appropriate for a vector space with a $\mathbb{H}$–structure. In the two indices notation the symplectic–Majorana condition reads

$$(\lambda_{ia})^* =_{\text{def}} \lambda^{ia} = \epsilon^{ij}\Omega^{ab}\lambda_{jb}, \tag{2.15}$$

where, as a matter of notation, we write λ^{ia} for $(\lambda_{ia})^*$. Thus

Definition 2.1. A *symplectic–Majorana* spinor is an element of a representation space for a Clifford algebra isomorphic to a matrix algebra over $\mathbb{H}$. It is a double–index vector obeying the reality condition in eqn.(2.15).

Example. Consider five dimensional Minkowski space with signature $(+,-,-,-,-)$. One has $\mathbb{C}l(1,4) \simeq \mathbb{H}(2)\oplus\mathbb{H}(2)$, so — with this choice of signature — it is natural to work with symplectic–Majorana spinors λ_{ia} ($i=1,2, a=1,\dots,4$). In the same way, it is natural to write $6D$ fermions (for both signatures) in terms of symplectic–Majoranas. Although the number of degrees of freedom in both cases are equal to that of a Dirac fermion, the quaternionic structure is relevant. [Anticipation for the *conoscenti*: *SUSY* theories obtained by dimensional reduction of six (five) dimensional models

[6] Recall that $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} \simeq \mathbb{C}\oplus\mathbb{C}$ and $\mathbb{A}(n)\otimes\mathbb{B}(n) \simeq (\mathbb{A}\otimes\mathbb{B})(nm)$ for all real algebras $\mathbb{A},\mathbb{B}$. The first isomorphism is easily proved by comparing the tensors products with $\mathbb{C}$ of the two columns in eqn.(2.11) which must be equal.

have a built–in quaternionic structure which enforces a hyperKahler (or quaternionic–Kahler) geometry on the scalar's manifold].

The above results are summarized in:

PROPOSITION 2.3. *Majorana and symplectic–Majorana spinors exist for the following signatures* (p, q)*:*

Majorana spinors	for $p-q=0,1,2 \mod 8$
symplectic–Majorana	for $p-q=4,5,6 \mod 8$

REMARK. From a physical point of view, its evident that if (symplectic)Majorana spinors exist for signature (p, q) they should exist also for (q, p) since the difference between the two is just a matter of convention. Consider the 'physical' situation, four dimension with Minkowski signature. One has $\mathbb{C}l(3,1) \simeq \mathbb{R}(4)$ and $\mathbb{C}l(1,3) \simeq \mathbb{H}(2)$, so it seems that in one convention ones gets Majorana and in the other symplectic–Majorana. In fact, both possibilities are open in both case. In signature $(+,-,-,-)$ Dirac's equation reads $(i\Gamma\cdot\partial - m)\psi = 0$, which requires — for real ψ — $i\Gamma_k$ to be real matrices; since the map $\Gamma_k \leftrightarrow i\Gamma_k$ interchanges $\mathbb{C}l(p,q) \leftrightarrow \mathbb{C}l(q,p)$ we get the Γ–matrices of the Clifford algebra $\mathbb{C}l(3,1) \simeq \mathbb{R}(4)$ which are real. In the opposite convention, $(-,+,+,+)$, a non–tachyonic fermion has equation $(i\Gamma\cdot\partial - im)\psi = 0$, and again the Γ–matrices are those of $\mathbb{C}l(3,1)$. So the physical condition is independent of the conventions. On the contrary, a tachyonic fermion (imaginary mass) could be chosen to be symplectic–Majorana.

Weyl Spinors. In addition to reality properties, a spinor can be restricted by a *chirality* condition. A chiral spinor is an element of $\mathbb{C}l(p,q)$ which is an eigenvector of $\Gamma_{[n]}$. For odd n, $\Gamma_{[n]}$ is a central element and hence it acts as a scalar in any irreducible module; in this sense in odd dimension a spinor is by definition 'chiral'.

We consider the case $p+q = 2m$. Recall that $\Gamma_{[n]}^2 = (-1)^{(m-q)} \equiv (-1)^{(p-q)/2}$. The projectors

$$P_\pm = \frac{1}{2}\left(1 \pm (-1)^{(p-q)/4}\,\Gamma_{[n]}\right) \tag{2.16}$$

decompose a spinor into components of definite chirality, $\psi = \psi_+ + \psi_-$, where $\psi_\pm \in \mathcal{S}_\pm(p,q) \equiv P_\pm \mathcal{S}(p,q)$. The spaces $\mathcal{S}_\pm(p,q)$ do not support a representation of the Clifford algebra $\mathbb{C}l$; however they do carry a representation of the subalgebra $\mathbb{C}l^0$ of *even* elements. As we shall see in §.3.1, this suffices to guarantee covariance with respect to the Lorentz group (although discrete symmetries, like parity, may be lost). Spinors in $\mathcal{S}_\pm(p,q)$ are called *Weyl* spinors or a spinors *of definite chirality*, *right*, ψ_+, and *left*, ψ_-. A Weyl field has one half the components of a Dirac one.

One can ask whether the chirality condition can be imposed together with a reality one. In this case we would get a *Majorana–Weyl* spinor having one quarter the degree of freedom of a Dirac field. For even n, the projectors $P_\pm$ in eqn.(2.16) belong to the algebra $\mathbb{C}l(p,q)$ precisely when $p-q = 0 \mod 4$. Comparing with proposition 2.3, we have

PROPOSITION 2.4. *Majorana–Weyl (resp. symplectic–Majorana–Weyl) spinors exist for the following signatures* (p, q)*:*

Majorana–Weyl	for $p - q = 0 \mod 8$
symplectic–Majorana–Weyl	for $p - q = 4 \mod 8$

REMARK. In the remaining two cases, $p - q = 2 \mod 8$ and $p - q = 6 \mod 8$, the chiral components of a (symplectic)Majorana spinors are each other conjugate $\psi_\pm = \psi^*_\mp$. In these dimensions the content of a Majorana spinor is equivalent to that of a Weyl spinor and we can switch from one description to the other.

3. Spin Groups

The spinor spaces $\mathcal{S}(p, q)$ are physically relevant because they support a representation of the Lorentz group $SO(p, q)$. In fact on $\mathcal{S}(p, q)$ act larger groups known as $\text{pin}(p, q)$ and $\text{Spin}(p, q)$[7].

3.1. The Groups $\text{pin}(p, q)$ **and** $\text{Spin}(p, q)$**.** We return to the setting of §.1.1: $\mathcal{V}$ is a real vector space with a non–degenerate quadratic form Q of signature (p, q). Let $\mathcal{U}_\pm \subset \mathcal{V}$ be the hypersurfaces $Q(x) = \pm 1$; $\mathcal{U}_\pm$ are two hyperboloids except for $(p, 0)$ where $\mathcal{U}_+ = S^{p-1}$, $\mathcal{U}_- = \emptyset$, and $(0, q)$ (the other way around). As always, we identify $\mathcal{U}_\pm$ with their image in $\mathbb{C}l(p, q)$. Since $x^2 = Q(x) = \pm 1$, x is invertible in $\mathbb{C}l(p, q)$: $x^{-1} = \pm x$. If $x_k \in \mathcal{U}_\pm$ then $y = x_1 x_2 \cdots x_n$ is also invertible. Thus the elements $x \in \mathcal{U}_\pm$ generates a group.

DEFINITION 3.1. $\text{pin}(p, q)$ is the group generated by $\mathcal{U}_\pm \subset \mathbb{C}l(p, q)$.

DEFINITION 3.2. $\text{Spin}(p, q)$ is the subgroup of $\text{pin}(p, q)$ of all *even* elements, *i.e.* $\text{Spin}(p, q) = \text{pin}(p, q) \cap \mathbb{C}l^0(p, q)$.

$\text{Spin}^0(p, q)$, the connected component of unity, is generated by the products $v_1 v_2 \cdots v_{2k} u_1 u_2 \cdots u_{2l}$ where $v_i \in \mathcal{H}_+$ and $u_j \in \mathcal{H}_-$, [**230**].

The tensor algebra $\mathcal{T} \equiv \oplus_k \mathcal{V}^{\otimes k}$ has a canonical involutive antiautomorphism[8], namely

$$\overline{x_1 \otimes \cdots \otimes x_p} = x_p \otimes \cdots \otimes x_1.$$

This map leaves invariants the elements of the form $x \otimes x - Q(x)$ and hence the ideal $\mathcal{I}$ in eqn.(1.4). Thus it descends to an involutive antiautomorphism of $\mathbb{C}l(p, q)$. One has

$$\bar{\Gamma}_I = (-1)^{\frac{m(m-1)}{2}} \Gamma_I.$$

where, as always, $m = |I|$.

PROPOSITION 3.1. *Let* $u \in \text{pin}(p, q)$*. Then* $u^{-1} = (\pm 1)^{\deg u} \bar{u}$*. In particular* $u^{-1} = \bar{u}$ *in the subgroup of even elements.*

---

[7] Historically the *Spin* groups were discovered first and took their name from particle physics. Later the *pin* groups were discovered. Since their relations with the *Spin* groups is analogous to that of $U(n)$ (resp. $O(n)$) with $SU(n)$ (resp. $SO(n)$), it looked nice to drop the S in front and call them just *pin*.

[8] A linear map $\mathbb{A} \to \mathbb{A}$, $a \mapsto \bar{a}$ is an *involutive antiautomorphism* if $\overline{(\bar{a})} = a$ and $\overline{ab} = \bar{b}\bar{a}$ for $\forall a, b \in \mathbb{A}$.

Let $u \in \mathcal{U}_\pm$ and $x \in \mathcal{V} \subset \mathbb{C}l(p,q)$. The map $\varphi(u)\colon x \mapsto ux\bar{u}$ sends[9] $\mathcal{V}$ into $\mathcal{V}$. Then $ux\bar{u} \in \mathcal{V}$ for all $u \in \text{pin}(p,q)$. Moreover the map $\varphi(u)$ is a Q–isometry of $\mathcal{V}$: $(ux\bar{u})^2 = (\pm)^{\deg u} ux^2\bar{u} = (\pm)^{\deg u} Q(x)u\bar{u} = Q(x)$. Therefore we have a homomorphism

$$\varphi(u)\colon \text{pin}(p,q) \longrightarrow O(p,q). \tag{3.1}$$

The group $\text{pin}(p,q)$ has a corresponding covariant action on the spinor space $\mathcal{S}(p,q)$, *i.e.* $\psi \mapsto u\,\psi$. Eqn.(3.1) implies that this is a covariant action of the Lorentz group $O(p,q)$. The map

$$\varphi^{\text{ev}}(u)\colon \text{Spin}(p,q) \to SO(p,q) \tag{3.2}$$

is onto in the connected component of the identity; its kernel is $\{\pm 1\}$. One has[10] $\text{Spin}(p,q) \simeq \text{Spin}(q,p)$.

3.2. Bilinear Invariants. In Euclidean signature the matrices representing $\text{Spin}(0,q)$ belong to $O_{\mathbb{K}}(2^M)$ where

$$O_{\mathbb{K}}(N) =_{\text{def}} \begin{cases} SO(N) & \text{for } \mathbb{K}=\mathbb{R} \\ U(N) & \text{for } \mathbb{K}=\mathbb{C} \\ Sp(N) & \text{for } \mathbb{K}=\mathbb{H} \end{cases}$$

The algebra $\mathbb{K}$ and exponent M for the different q's mod 8 can be read in the second column of eqn.(2.11). Indeed we have seen in §.2.1 (discussion around eqn.(2.12)) that the matrices Γ_k are unitary, $\Gamma_k\Gamma_k^\dagger = 1$. There we saw that $\Gamma_k^\dagger = -\Gamma_k$ (in signature $(0,q)$) . Let x_k be the coordinates of a point on the sphere $\sum_k x_k^2 = 1$ and $w = x_k\Gamma_k$ the associated generator of $\text{pin}(0,q)$. One has $w^\dagger = -\sum_k x_k\Gamma_k = -w$ and $w^2 = -1$ so $ww^\dagger = 1$. Thus $w \in U(2^M)$; since[11] (proposition 2.2) $w \in \mathbb{K}(2^{M'})$, $w \in O_{\mathbb{K}} \equiv U(2^M) \bigcap \mathbb{K}(2^{M'})$.

This can be generalized to arbitrary (p,q). The map $\Gamma_k \mapsto (-1)^{p-1}\varepsilon_k\Gamma_k$ is an inner automorphism of the Clifford algebra. So there is a matrix A such that $(-1)^{p-1}\,\varepsilon_k\Gamma_k = A\Gamma_k A^{-1} =$. One can choose $A = \Gamma_P$, the product of all Γ_k's with $\varepsilon_k = 1$. Then $\Gamma_k^\dagger = \varepsilon_k\Gamma_k = (-1)^{p-1}A\Gamma_k A^{-1}$; a general element $y \in \mathbb{C}l(p,q)$

$$y^\dagger = (-1)^{(p-1)\deg y}\, A\,\overline{y}\,A^{-1}. \tag{3.3}$$

Let $w \in$ Spin(p,q) (an *even* element of $\mathbb{C}l(p,q)$ with $\overline{w}w = 1$). Then

$$w^\dagger A\, w = C\overline{w}w = A. \tag{3.4}$$

PROPOSITION 3.2. $\text{Spin}(p,q) \in O_{\mathbb{K}}(2^M, A)$, *the subgroup of* $GL(2^M, \mathbb{K})$ *leaving invariant the bilinear form* $\psi^\sharp C\psi$. *(*$\sharp$ *means* transpose *for* $\mathbb{K}=\mathbb{R}$, Hermitean adjoint *for* $\mathbb{K}=\mathbb{C}$, *and* $\psi \mapsto \epsilon\psi^T\epsilon^T$ *for* $\mathbb{K}=\mathbb{H}$*).*

REMARK. The bilinear $\psi_1^\sharp A\Gamma_{k_1,\cdots k_r}\psi_2$ transforms under $\text{Spin}(p,q)$ as an antisymmetric r–tensor of $SO(p,q)$.

---

[9] $ux\bar{u} = u_i x_j u_k \Gamma_i\Gamma_j\Gamma_k = u_i x_j u_k \Gamma_i(2\delta_{jk} - \Gamma_k\Gamma_j) = u_i x_j u_k(2\delta_{jk}\Gamma_i - \delta_{ik}\Gamma_j)$.

[10] The (complex) morphism $\mathbb{C}l(p,q) \leftrightarrow \mathbb{C}l(q,p)$ given by $\Gamma_k \leftrightarrow i\Gamma_k$ becomes a real isomorphisms when restricted to the even subalgebras: on the basis elements Γ_I (I a set of $2m$ elements) it reduces to $\Gamma_I \leftrightarrow (-1)^m\Gamma_I$.

[11] $M' = M$ except in the quaternionic case where $M = 2M'$.

From $A^2 = (-1)^{p(p-1)/2}$ we see that $A^\sharp = A$ for $p = 0, 1 \mod 4$ and $A^\sharp = -A$ for $p = 2, 3 \mod 4$. Note[12] that tr$A = 0$ if $pq \neq 0$. Therefore for $p = 0, 1 \mod 4$, A gives an invariant symmetric/Hermitean form of signature $(2^{M-1}, 2^{M-1})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and a non–degenerate shewsymmetric form for $\mathbb{K} = \mathbb{H}$ while for $p = 2, 3 \mod 4$ we get a *non–degenerate* shewsymmetric/skew–Hermitean form or, for $\mathbb{K} = \mathbb{H}$, a symmetric form of signature $(2^{M-1}, 2^{M-1})$. The resulting groups are specific real forms of $O_{\mathbb{K}}(2^M)$. We omit the tables with the appropriate groups for the various cases. They can be found in ref.[**231**]. The zoology of spinorial bilinear forms for general space–time signatures (the *spinorial chessboard)* is can be found in the quoted literature.

3.3. More on the Group $\mathrm{Spin}(p, q)$**.** To better describe the Spin groups we have to understand the *even* subalgebra of $\mathbb{C}l(p, q)$. It is described in the following

PROPOSITION 3.3. *We have the isomorphism*

$$\mathbb{C}l^0(p, q) \simeq \mathbb{C}l(p, q-1). \tag{3.5}$$

So the even subalgebra $\mathbb{C}l^0(p, q)$ is just the Clifford algebra in one less spacetime dimension. The virtue of this fact is that it reduces the dimension of the matrix realization of a Spin group. From the definitions, we know that $\mathrm{Spin}(p, q)$ can be realized by the matrices of $\mathbb{C}l(p, q)$. The proposition says that is can also be realized with the smaller matrices of $\mathbb{C}l(p, q-1)$. For instance, in four Euclidean dimensions we have $\mathbb{C}l(0, 4) \simeq \mathbb{H}(2)$, whereas $\mathbb{C}l(0, 3) \simeq \mathbb{H} \oplus \mathbb{H}$. Then to represent Spin(4) we need just a pair of unit quaternions not a 2×2 quaternionic matrix. Of course this reflects the well known isomorphism $\mathrm{Spin}(4) = SU(2) \otimes SU(2)$. Therefore

PROPOSITION 3.4. *The group* $\mathrm{Spin}(p, q)$ *has a realization in terms of (pairs of) matrices over the algebras* $\mathbb{R}$*,* $\mathbb{C}$ *or* $\mathbb{H}$ *of dimension* $2^{\alpha(p,q)}$ *according the following table:*

q-p mod 8	***single/pair***	***algebra***	$2\,\alpha(p,q)$
0	*pair*	$\mathbb{R}$	$p+q-2$
1		$\mathbb{R}$	$p+q-1$
2		$\mathbb{C}$	$p+q-2$
3		$\mathbb{H}$	$p+q-3$
4	*pair*	$\mathbb{H}$	$p+q-4$
5		$\mathbb{H}$	$p+q-3$
6		$\mathbb{C}$	$p+q-2$
7		$\mathbb{R}$	$p+q-1$

(3.6)

REMARK. For $q-p = 0 \mod 4$ we have a pair of matrices, (A, B). Such a pair correspond to a single matrix of twice the dimension of block diagonal form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. However we prefer to use *two* matrices of *smaller* dimension.

[12]Indeed, $A = (-1)^{p-1}\varepsilon_k \Gamma_k^\dagger A \Gamma_k$, so tr$A = (-1)^{p-1}\varepsilon_k \mathrm{tr}(\Gamma^\dagger C \Gamma_k) = (-1)^{p-1}\varepsilon_ktrA$ for all k's.

PROOF OF PROPOSITION 3.3. Recall that the algebra $\mathbb{C}l(p,q)$ has a basis $\{\Gamma_I \,|\, I \subset \{1,\dots,n\}\}$ with multiplication table[13] $\Gamma_I\Gamma_J = (-1)^{\mu(I,J)}\Gamma_{I\Delta J}$. Now consider the linear map $\varpi\colon \mathbb{C}l(p,q) \to \mathbb{C}l^0(p,q+1)$ given by

$$\varpi(\Gamma_I) = \begin{cases} \Gamma_I & \text{if } I \text{ is even} \\ \Gamma_I\Gamma_{n+1} & \text{if } I \text{ is odd.} \end{cases} \tag{3.7}$$

ϖ is an algebra isomorphism provided $\varpi(\Gamma_I)\,\varpi(\Gamma_J) = \varpi(\Gamma_I\Gamma_J)$. This is evident if I, J are both even. If I is even and J odd, $\varpi(\Gamma_I)\,\varpi(\Gamma_J) = \Gamma_I\Gamma_J\Gamma_{n+1} = \varpi(\Gamma_I\Gamma_J)$. The same holds if I is odd and J even, since Γ_{n+1} commutes with Γ_J. Let I, J be odd. Then

$$\varpi(\Gamma_I)\,\varpi(\Gamma_J) = \Gamma_I\Gamma_{n+1}\Gamma_J\Gamma_{n+1} = -\varepsilon_{n+1}\Gamma_I\Gamma_J = -\varepsilon_{n+1}\,\varpi(\Gamma_I\Gamma_J),$$

so ϖ is an isomorphism precisely if the added dimension has negative signature ($\varepsilon_{n+1} \equiv \Gamma_{n+1}^2 = -1$). □

PROPOSITION 3.5. *In the Euclidean case* $\mathrm{Spin}(0,q) \subset O_{\mathbb{K}}(2^{\alpha(0,q)})$. *More generally,* $\mathrm{Spin}(p,q) \subset O_{\mathbb{K}}(2^{\alpha(p,q)}, A)$.

PROOF. If suffices to show that the map ϖ commutes with Hermitean conjugation. It is enough to check on the basis. From $\Gamma_k^\dagger = \varepsilon_k\Gamma_k$ we get $\Gamma_I^\dagger = \phi_I\Gamma_I$, where $\phi_I = (-1)^{m(m-1)/2}\prod_{k\in I}\varepsilon_k$, with $m = |I|$. Then, distinguishing the cases of I even and I odd, and recalling $\varepsilon_{n+1} = -1$

$$\varpi(\Gamma_I^\dagger) = \phi_I\varpi(\Gamma_I) = \phi_I\Gamma_I = \Gamma_I^\dagger = (\varpi(\Gamma_I))^\dagger$$
$$\varpi(\Gamma_I^\dagger) = \phi_I\varpi(\Gamma_I) = \phi_I\Gamma_I\Gamma_{n+1} = (-\varepsilon_{n+1})\Gamma_{n+1}^\dagger\Gamma_I^\dagger = (\varpi(\Gamma_I))^\dagger.$$

□

4. Relation with the Division Algebras

Now we are in a position to discuss the deep relations between space–time dimension and division algebras, a point especially stressed by Duff [**232**]. The connection is already evident from eqn.(3.6) and the presence of $O_{\mathbb{K}}(2^\alpha)$ in proposition 3.5.

The basic idea is that there is a relation between the number of spacetime dimensions and the algebras: adding more dimensions "nothing changes" except that the underlying algebra upgrades. For Euclidean signature, the spin groups in dimension $D = 1, 2, 4$ are: $O_{\mathbb{R}}(1), O_{\mathbb{C}}(1)$, and $O_{\mathbb{H}}(1) \otimes O_{\mathbb{H}}(1)$. Considering the conformal groups for the same Euclidean spaces[14] we get the physical–signature Lorentz groups for $D = 3, 4, 6$:

$$\mathrm{Spin}(1,2) = SL(2,\mathbb{R}), \quad \mathrm{Spin}(1,3) = SL(2,\mathbb{C}), \quad \mathrm{Spin}(1,5) = SL(2,\mathbb{H}) \tag{4.1}$$

again we see the same structure with different algebras. The picture is the following: space–time dimensions appear in series, and in each series we have the same patterns with $\mathbb{R} \leftrightarrow \mathbb{C} \leftrightarrow \mathbb{H}$. This is certainly true for the supersymmetric interactions: the minimal *SUSY* (scalar) models in $D = 3, 4$, and 6 dimensions are based, respectively, on *real*, *complex*, and *quaternionic* differential geometry. Their superalgebras correspondingly have $2, 4, 8$ supercharges.

[13] $\mu(I,J) = \tau(I,J) + \sharp(I \cap J \cap N)$, with N the set of indices k such that $\varepsilon_k = -1$.
[14] See §.5.4 below.

All this is very beautiful and satisfactory except that we would like to continue the series to higher dimensions (8 in Euclidean signature, 10 in Minkowskian) corresponding to $SUSY$ theories with 16 supercharges. On the other hand we know that there exists a fourth division algebra, the *octonions* $\mathbb{O}$. The obvious guess would be a relation like $\mathrm{Spin}(1,9) \simeq SL(2,\mathbb{O})$, but this cannot be true at the face value given that the octonions are not associative. Still it is true that for $q-p=7,8$ spinors are octonionic in "nature". We explain the issue in a language which emphasizes the uniformity of the basic patterns for the diverse algebras.

DEFINITION 4.1. A *normed metric algebra* is a real algebra with unity (not necessarily associative) equipped with a $\mathbb{R}$–linear map $a \mapsto \bar{a}$ such that $a\bar{a} = \bar{a}a = |a|^2 \geq 0$ and $|ab|^2 = |a|^2\,|b|^2$.

THEOREM 4.1 (Hurwitz). $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ *are the only normed algebras.*

More generally we can consider *isometric products* of (Euclidean) vector spaces, *i.e.* linear maps $\alpha\colon \mathcal{V}\otimes\mathcal{W} \to \mathcal{W}$ such that

$$\langle \alpha(x\otimes y)|\alpha(x\otimes y)\rangle_{\mathcal{W}} = \langle x|x\rangle_{\mathcal{V}}\ \langle y|y\rangle_{\mathcal{W}}, \tag{4.2}$$

where $\langle\cdot|\cdot\rangle_{\mathcal{V}}$ and $\langle\cdot|\cdot\rangle_{\mathcal{W}}$ are the (positive–definite) inner products in the two vector spaces. Let e_i, f_a be orthonormal basis for $\mathcal{V}$ and $\mathcal{W}$, respectively. $\alpha(e_i\otimes f_a) = f_b(\Sigma_i)_{ba}$ for some *real* matrices[15] Σ_i. In these basis, eqn.(4.2) reads

$$(\Sigma_i^T\Sigma_j + \Sigma_j^T\Sigma)_{ab} = 2\delta_{ij}\,\delta_{ab} \tag{4.3}$$

which is very similar to the defining property of the Dirac matrices of $\mathbb{C}l(\langle\cdot|\cdot\rangle_{\mathcal{V}})$. It is easy to construct a Clifford algebra out of the matrices Σ. First of all, notice that eqn.(4.3) remains true[16] under $\Sigma_i \leftrightarrow \Sigma_i^T$; set

$$\Gamma_i = \begin{pmatrix} 0 & \Sigma_i^T \\ \Sigma_i & 0 \end{pmatrix}, \tag{4.4}$$

these are *bona fide* Γ–matrices. $\mathcal{W}\oplus\mathcal{W}$ is a Clifford module, usually the direct sum of k copies of the basic one we constructed above. An isometric product of vector spaces defines a Clifford structure on the direct sum of *two copies* of the space. Thus the theory of isometric products of vector space is equivalent to that of Clifford modules.

As a special case of isometric products of vectors space, consider a normed metric algebra $\mathbb{K}$. We take $\mathcal{V} = \mathcal{W} = \mathbb{K}$ and the natural product $\mathbb{K}\otimes\mathbb{K} \to \mathbb{K}$: it is isometric by definition 4.1. Then the above construction produces a (positive signature) Clifford algebra with $\mathcal{V} = \mathbb{K}$ and representation space $\mathbb{K}\oplus\mathbb{K}$ (viewed as real vector spaces). In particular, the octonions $\mathbb{O}$ enter into the game on the same footing of the other three algebras. Comparing dimensions, we see that $\mathbb{R}\oplus\mathbb{R}$, $\mathbb{C}\oplus\mathbb{C}$, and $\mathbb{H}\oplus\mathbb{H}$ contain two copies of the basic spinor space, whereas $\mathbb{O}\oplus\mathbb{O}$ only one. Notice that this octonionic construction of $\mathbb{C}l(8,0)$ gives a rationale of why (for Euclidean signature)

[15]We defined the matrix to act on the right in order to have the correct property under composition: $e_i(e_jf_a) = e_i(f_b(\Sigma_j)_{ba}) = (e_if_b)(\Sigma_j)_{ba} = f_b(\Sigma_i\Sigma_j)_{ba}$, that is the Σ–matrices compose in the right order.

[16] The matrix $(x_i\Sigma_i)/(x_kx_k)^{1/2}$ is orthogonal. So is its transpose. Thus $x_ix_j\Sigma_i^T\Sigma_j = (x_kx_k)\cdot 1$, *i.e.* $\Sigma_i^T\Sigma_j + \Sigma_j^T\Sigma_i = 2\delta_{ij}\cdot 1$.

Majorana–Weyl spinors appear precisely in 8 dimensions: *they are octonions in disguise!*

Let us check this. The basis of a division algebra $\mathbb{K}$ has the form $1, e_1, \dots, e_{\dim \mathbb{K}-1}$, with $e_k^2 = -1$ and $e_k e_h = -e_h e_k$, $h \neq k$. The matrix corresponding to 1 is just $\Sigma_1 = \mathbf{1}$. Inserting this in eqn.(4.3) we get $\Sigma_{e_k}^T = -\Sigma_{e_k}$. Then

$$\Sigma_{e_k}\Sigma_{e_k} + \Sigma_{e_k}\Sigma_{e_k} = -2\delta_{kh} \cdot 1, \tag{4.5}$$

i.e. the Σ–matrices corresponding to the imaginary units e_k of $\mathbb{K}$ generates a $\mathbb{C}l(0, \dim \mathbb{K} - 1)$ Clifford algebra.

The $\mathbb{C}l(\dim \mathbb{K}, 0)$ algebra acting on $\mathbb{K} \oplus \mathbb{K}$ is generated by the real symmetric matrices

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma_{e_k} = \begin{pmatrix} 0 & -\Sigma_{e_k} \\ \Sigma_{e_k} & 0 \end{pmatrix}, \; k = 1, \dots, \dim \mathbb{K} - 1. \tag{4.6}$$

The chirality operator, $\Gamma_{[\dim \mathbb{K}]}$, is equal to

$$\Gamma_{[\dim \mathbb{K}]} = \begin{pmatrix} \Sigma_{[\dim \mathbb{K}-1]} & 0 \\ 0 & \Sigma_{[\dim \mathbb{K}-1]} \end{pmatrix} \tag{4.7}$$

where $\Sigma_{[\dim \mathbb{K}-1]}$ is the chirality operator of $\mathbb{C}l(0, \dim \mathbb{K} - 1)$. Since $\dim \mathbb{K} - 1$ is odd, this element is central. It is symmetric for $\mathbb{H}, \mathbb{O}$. In fact[17]

$$\Sigma_{[\dim \mathbb{H}-1]} = -1, \qquad \Sigma_{[\dim \mathbb{O}-1]} = -1, \tag{4.8}$$

showing that in the representation space $\mathbb{O} \oplus \mathbb{O}$ of $\mathbb{C}l(8, 0)$ the two copies of $\mathbb{O}$ do correspond to the chiral (Majorana–Weyl) subspaces $\mathcal{S}_\pm$.

We shall study $\mathbb{O}$ more explicitly in the last section of the chapter.

5. Spin(8) Triality and Generalizations

Some Spin groups enjoy magical properties. We describe these special groups having in mind a number of applications. First of all, they are fundamental for the consistency of superstring theory (*e.g.* for the equivalence of the NSR and GS formulations). On the mathematical side, the existence of *exceptional holonomy* manifolds — relevant for M–theory — is a consequence of these special structure.

TRIALITY is usually described in terms of the Dynking diagram of the Lie algebra D_4. We will follow that approach in chapter 13. Here we follow a different line (borrowing from [**227**]). Our starting point is the previous section: we have three copies of the division algebra which we label as $\mathbb{K}_\mathcal{V}, \mathbb{K}_+, \mathbb{K}_-$ and identify, respectively, with the vector space $\mathcal{V}$ and the two chiral spaces $\mathcal{S}_\pm$. We can attach labels $(\mathcal{V}, +, -)$ in different ways to the three $\mathbb{K}$'s appearing in the product $\mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$. Permutation of labels leads to relations between three inequivalent representations of Spin$(\dim \mathbb{K})$. The interesting case is $\mathbb{K} = \mathbb{O}$: roughly speaking the three 8–dimensional representations of Spin(8) appear symmetrically.

[17]The chiral element is the matrix representation of the element $e_1 e_2 \cdots e_{\dim \mathbb{K}-1}$ on the basis f_a. For $\mathbb{O}$ this requires some care since it is not associative.

5.1. The Structure of spin(dim $\mathbb{K}$)**.** We rephrase the results of section 4. Let $\mathbb{K}$ a normed metric algebra, *i.e.* a unital real algebra with an involution $x \mapsto x^*$ (conjugation) such that $xx^* = |x|^2 > 0$ for $x \neq 0$, and $|xy|^2 = |x|^2|y|^2$. The correspondence[18]

$$x \mapsto \hat{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \tag{5.1}$$

defines a Clifford algebra (not necessarily equivalent to the universal one). Indeed $\hat{x}^2 = |x|^2$. From eqn.(5.1) we see that the spin group associated to this Clifford algebra is generated by the elements $\hat{1}\hat{u}$ with $u \in \mathbb{K}$, $|u| = 1$. We set $\tilde{u} = \hat{1}\,\widehat{u^*}$. Explicitly[19]

$$\tilde{u} = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}. \tag{5.2}$$

The Clifford involution (denoted by a bar) acts on $\tilde{u}$ as the $\mathbb{K}$ conjugation (denoted by a star): $\overline{\tilde{u}} = \widehat{u^*}\,\hat{1} = \hat{1}\,\widehat{u} = \widetilde{u^*}$.

The action of a generator of the spin group, $\tilde{u}$, on an element of the Clifford algebra, $\hat{x}$, is given (§.3.1) by $\hat{x} \mapsto \tilde{u}\hat{x}\,\overline{\tilde{u}} = \tilde{u}\hat{x}\widetilde{u^*}$ *i.e.*

$$\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & uxu \\ u^*x^*u^* & 0 \end{pmatrix}. \tag{5.3}$$

We denote a spinor $\in \mathbb{K}_+ \oplus \mathbb{K}_-$ as $\begin{pmatrix} \psi \\ \chi^* \end{pmatrix}$. Multiplication by $\tilde{u}$ gives

$$\begin{pmatrix} \psi \\ \chi^* \end{pmatrix} \longmapsto \begin{pmatrix} u\psi \\ u^*\chi^* \end{pmatrix} \equiv \begin{pmatrix} u\psi \\ (\chi u)^* \end{pmatrix}. \tag{5.4}$$

The natural product $\mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$ defines three linear operators on $\mathbb{K}$. They are: *left* multiplication, $L_u\colon x \to ux$; *right* multiplication, $R_u\colon x \to xu$; and the composition $T_u = L_u \circ R_u = R_u \circ L_u\colon x \to uxu$. Eqns.(5.3) and (5.4) imply the following

PROPOSITION 5.1. *The actions of a general element* $a = \tilde{u}_1\tilde{u}_2\cdots\tilde{u}_{2r}$ *of* Spin(dim $\mathbb{K}$) *on* $\mathbb{K}$ *— identified, respectively, with* $\mathbb{K}_+$, $\mathbb{K}_-^*$, *and* $\mathbb{K}_\mathcal{V}$ *— are given by*

$$\psi \longmapsto L_{u_1} \circ L_{u_2} \circ \cdots \circ L_{u_{2r}}\, \psi \tag{5.5}$$

$$\chi \longmapsto R_{u_1} \circ R_{u_2} \circ \cdots \circ R_{u_{2r}}\, \chi \tag{5.6}$$

$$x \longmapsto T_{u_1} \circ T_{u_2} \circ \cdots \circ T_{u_{2r}}\, x. \tag{5.7}$$

Now we are in a position to give a first statement of TRIALITY. Other versions will be given elsewhere through the book. We write[20] $\mathcal{L}_a$ for

[18] Since $\mathbb{K}$ is NOT associative in general, the algebra generated by the $\hat{x}$'s is NOT a matrix algebra. The (associative) matrix algebra is correctly given by $x \mapsto x_i\Gamma_i$ (where Γ_i are the matrices constructed in sect.4). However the formulas below are correct, since for these specific expression all associators vanish. Alternatively one can work with the *bona fide* matrices Γ_i, Σ_i, but it is rather tedious.

[19] The map $\mathbb{C}l(0, \dim\mathbb{K} - 1) \ni u \longmapsto \tilde{u} \in \mathbb{C}l^0(\dim\mathbb{K})$ in eqn.(5.2) is the isomorphism in eqn.(3.5).

[20] The cumbersome notation is needed because the product in $\mathbb{K}$ is not associative: $L_{u_1u_2\cdots u_{2r}}$ is ill–defined without parenthesis and it is not equal $L_{u_1} \circ \cdots \circ L_{u_{2r}}$.

$L_{u_1} \circ \cdots \circ L_{u_{2r}}$; $\mathcal{R}_a$, $\mathcal{T}_a$ stand for the other two strings of operators in the proposition.

THEOREM 5.1 (TRIALITY). *Let* $a \in \mathrm{Spin}(\dim \mathbb{K})$ *and* $x, y \in \mathbb{K}$. *Then*

$$\mathcal{T}_a(xy) = \mathcal{L}_a x \cdot \mathcal{R}_a y. \tag{5.8}$$

Here $\cdot$ stands for multiplication in $\mathbb{K}$. It suffices to check eqn.(5.8) for $a = \tilde{u}$ ($u \in \mathbb{K}$ of norm 1). In this case (5.8) reduces to

$$u \cdot (xy) \cdot u = (ux) \cdot (yu) \tag{5.9}$$

If $\mathbb{K}$ were associative (5.9) would be trivially true. But the octonions are NOT associative. The associator $(x, y, z) = (xy)z - x(yz) \neq 0$ in general. However a normed metric algebra is necessarily *alternative, i.e.* the associator (x, y, z) is totally antisymmetric in its arguments[21]. In particular the associator vanishes if two arguments are equal. More generally all expressions containing only two elements $y, z \in \mathbb{K}$ satisfy the identities valid in an associative algebra. In fact the subalgebra generated by y and z is contained in quaternionic subalgebra $\subset \mathbb{K}$. All expressions in y, z, y^* and z^* belong to this associative subalgebra. For instance $y \cdot z^3 = (yz \cdot z)z$; replacing $z \to x + u$ and keeping only terms of 1^{st}–order in x, we get the identity

$$y \cdot uxu = (yu \cdot x)u. \tag{5.10}$$

PROOF (of eqn.(5.9))

$$\begin{aligned} &ux \cdot yu - u \cdot xy \cdot u = [ux \cdot yu - (ux \cdot y)u] + [ux \cdot y - u \cdot xy]u = \\ &= [-y \cdot uxu + (y \cdot ux)u] + [yu \cdot x - y \cdot ux]u \quad \text{(antisymmetry of associators)} \\ &= -y \cdot uxu + (yu \cdot x)u = 0 \quad \text{(identity in eqn.(5.10))}. \end{aligned}$$

5.2. Generalization to Spin$(\dim \mathbb{K} + 1)$**.** The result above can be extended to one more dimensions [**227**]. The way we add one more dimension is a generalization of the observation that if γ_μ are the four dimensional Dirac matrices then $\{\gamma_\mu, \gamma_5\}$ is a representation of the five dimensional Clifford algebra. We identify the $(\dim \mathbb{K} + 1)$–dimensional space with $\mathbb{K}^\oplus \equiv \mathbb{R} \oplus \mathbb{K}$. Then[22] [23]

$$y \oplus x \mapsto \widehat{y \oplus x} = y\Gamma_{[\dim \mathbb{K}]} + \widehat{x^*} = \begin{pmatrix} -y & x^* \\ x & y \end{pmatrix}, \tag{5.11}$$

i.e. traceless "matrices" Hermitean in the $\mathbb{K}$ sense. They satisfy $(\widehat{y \oplus x})^2 = y^2 + |x|^2 \equiv |y \oplus x|^2$. The elements of $\mathbb{K}^\oplus$ act on $\mathbb{K} \oplus \mathbb{K}^*$ as

$$\widehat{y \oplus x} \begin{pmatrix} \psi \\ \chi^* \end{pmatrix} = \begin{pmatrix} -y\psi + x^*\chi^* \\ (y\chi + \psi^* x^*)^* \end{pmatrix}, \tag{5.12}$$

where the products are dictated by the algebra structure of $\mathbb{K}$. The *pin* and *Spin* groups are generated, respectively, by elements of the form $\hat{u}_1$ and $\hat{u}_1\hat{u}_2$, where $u_i = v_i \oplus w_i$, with $v_i^2 + |w_i|^2 = 1$. The action of a *pin* generator

$$\widehat{y \oplus x} \longmapsto \hat{u}\, \widehat{y \oplus x}\, \hat{u} \tag{5.13}$$

[21] See sect.8.

[22] The replacement $x \leftrightarrow x^*$ is a choice of conventions in order to agree with [**227**].

[23] For an interesting interpretation of this formula in string theory, see eqns.(4.13)-(4.17) in [**229**].

reads in components

$$y \longmapsto -y(1-2v^2) + 2\langle x, w\rangle \tag{5.14}$$

$$x \longmapsto -v^2 x + w x^* w + 2vyw, \tag{5.15}$$

where all products are in $\mathbb{K}$ (the inner product is defined by the norm of $\mathbb{K}$: $\langle x|y\rangle \equiv (\bar{x}y + \bar{y}x)/2$). Spin$(\dim\mathbb{K}+1)$ enjoys a property analogue to TRIALITY [**227**].

5.3. Extension to Spin$(\dim\mathbb{K}+1,1)$**.** To go up another dimension, we need Minkowskian signature. This sequence contains, as we already mentioned, the most interesting examples: signature $(2,1)$, (3,1) (the phenomenological word), (5,1) (exotic chiral theories) and (9,1) (the critical dimension of the superstring); these also are the maximal dimensions in which a physical theory with 2, 4, 8, and 16 supercharges may exist.

We identify $\mathbb{R}^{(1,\dim\mathbb{K}+1)}$ with $\mathbb{R}\oplus\mathbb{K}^{\oplus}$. An element of $\mathbb{R}\oplus\mathbb{K}^{\oplus}$ is written $(t, y\oplus x)$; its Minkowskian square is $Q[(t,y\oplus x)] = t^2 - |y\oplus x|^2$. The map $\mathbb{R}\oplus\mathbb{K}^{\oplus} \ni z \mapsto \hat{z} \in \mathbb{C}l(1,\dim\mathbb{K}+1)$ is given by

$$\widehat{(t, y\oplus x)} = \begin{pmatrix} 0 & t - \widehat{y\oplus x} \\ t + \widehat{y\oplus x} & 0 \end{pmatrix}. \tag{5.16}$$

Repeating the argument used for TRIALITY, we see that Spin$^0(1,\dim\mathbb{K}+1)$ is generated by elements of the form

$$\widetilde{(s, v\oplus w)} \equiv \begin{pmatrix} s + \widehat{v\oplus w} & 0 \\ 0 & (s - \widehat{v\oplus w}) \end{pmatrix} = \begin{pmatrix} s + \widehat{v\oplus w} & 0 \\ 0 & (s + \widehat{v\oplus w})^{-1} \end{pmatrix} \tag{5.17}$$

where $1 = Q[(s, v\oplus w)] = (s+\widehat{v\oplus w})(s-\widehat{v\oplus w})$, while for $Q[(s,v\oplus w)] = -1$ we must replace $(s+\widehat{v\oplus w})$ in the first matrix with $\sigma_3(s+\widehat{v\oplus w})$, and $(s-\widehat{v\oplus w})$ by $-\sigma_3(s-\widehat{v\oplus w})$. A general element of Spin$^0(1,\dim\mathbb{K}+1)$ is a product $\tilde{u}_1\tilde{u}_2\cdots\tilde{u}_r$ where $u_k\in\mathbb{R}\oplus\mathbb{K}^{\oplus}$ with $Q[u_k] = \pm 1$, in perfect analogy with the Spin$(\dim\mathbb{K})$ case. Formulae analogous to eqn.(5.5)–(5.7) hold, with now $L_{(s,v\oplus w)}$ meaning the action on the *left* of the column vector $(\boldsymbol{\Psi}_L)_\alpha \in \mathbb{K}\oplus\mathbb{K}^*$ by the operator $s+\widehat{v\oplus w}$ [resp. $\sigma_3(s+\widehat{v\oplus w})$]; $R_{(s,v\oplus w)}$ is multiplication on the *right* of the vector $(\boldsymbol{\Psi}_R^T\epsilon)_{\dot\alpha}$ by the operator $[(s+\widehat{v\oplus w})^{-1}]^T$ (resp.$[(s+\widehat{v\oplus w})^{-1}]^T\sigma_3$); $T_{(s,v\oplus w)}$ acts on a vector $(t, y\oplus x)$ as

$$\begin{aligned} &(t-\widehat{y\oplus x}) \longmapsto (s+\widehat{v\oplus w})(t-\widehat{y\oplus x})(s+\widehat{v\oplus w}) \\ \text{resp.}\quad &(t-\widehat{y\oplus x}) \longmapsto \sigma_3(s+\widehat{v\oplus w})(t-\widehat{y\oplus x})(s+\widehat{v\oplus w})\sigma_3. \end{aligned} \tag{5.18}$$

Note that $(t+\widehat{y\oplus x})$ is the general 2×2 matrix Hermitean in the $\mathbb{K}$ sense

$$\begin{pmatrix} t-y & x^* \\ x & t+y \end{pmatrix} \qquad t,\ y\in\mathbb{R},\ x\in\mathbb{K}, \tag{5.19}$$

so eqns.(5.18) have the structure $A\mapsto UAU^\dagger$, $A\in \text{Herm}(\mathbb{K},2)$. This describes Spin$(\dim\mathbb{K}+1,1)$ in terms of the multiplication in $\mathbb{K}$. For future reference, we note that

$$\det\left(L_{(s,v\oplus w)}\right) = \begin{cases} \det\begin{pmatrix} s-v & w \\ w^* & s+v \end{pmatrix} & = \quad Q[(s,v\oplus w)] \\ \det\sigma_3\cdot\det\begin{pmatrix} s-v & w \\ w^* & s+v \end{pmatrix} & = -Q[(s,v\oplus w)] \end{cases} = 1. \tag{5.20}$$

5.4.∗ Geometrical interpretation in $P^1(\mathbb{K})$.

In the preceding sections we have given a description of $\mathrm{Spin}(\dim\mathbb{K}-1)$, $\mathrm{Spin}(\dim\mathbb{K})$, $\mathrm{Spin}(\dim\mathbb{K}+1)$, and $\mathrm{Spin}(\dim\mathbb{K}+1,1)$in terms of the properties of the division algebra $\mathbb{K}$. These constructions can be also understood, more geometrically, in terms of the projective line over $\mathbb{K}$.

Given an *associative* algebra $\mathbb{K}$, one defines $P^1(\mathbb{K})$ as the equivalence classes of pairs $(z_1, z_2) \in \mathbb{K}^2$, not both zero, under the identification $(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$ for all $\lambda \in \mathbb{K}$, $\lambda \neq 0$. Equivalently $P^1(\mathbb{K})$ is the space of lines in $\mathbb{K}^2$. Unfortunately this definition does not work for $\mathbb{O}$. We adopt a different point of view: each line in the affine plane $\mathbb{K}^2$ corresponds to a one–dimensional subspace V; let $P_V\colon \mathbb{K}^2 \to V$ be the orthogonal projection. There is a one–to–one correspondence between points in $P^1(\mathbb{K})$ and projectors in $\mathbb{K}^2$ of rank 1. This motivates the following

Definition 5.1. $P^1(\mathbb{K})$, the projective line over a division algebra $\mathbb{K}$, is the space of the $\mathbb{K}$–Hermitean 2×2 matrices

$$X = \begin{pmatrix} y_1 & x \\ x^* & y_2 \end{pmatrix} \qquad y_i \in \mathbb{R},\ x \in \mathbb{K} \tag{5.21}$$

which are idempotent, $X^2 = X$ and have $\mathrm{Tr}(X) = 1$.

Remark. X is a projector of rank $1 \Leftrightarrow X^2 = X$, $X^* = X$ and $\mathrm{Tr}(X) = 1$. This definition works fine for octonions too. We can parameterize X in terms of $(z_1, z_2) \in \mathbb{K}^2$

$$y_i = \frac{|z_i|^2}{|z_1|^2 + |z_2|^2}, \qquad x = \frac{z_1\bar{z}_2}{|z_1|^2 + |z_2|^2}, \tag{5.22}$$

which shows (for $\mathbb{K}$ associative) the equivalence with the standard approach.

For $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $P^1(\mathbb{K})$ is diffeomorphic, respectively, to S^1, S^2 (the Riemann sphere), and S^4. Not surprisingly we get S^8 for $\mathbb{O}$. Indeed, from eqn.(5.21) and $\mathrm{Tr}(X) = 1$ we see that

$$X = \frac{1}{2}(1 + \hat{Y}) \tag{5.23}$$

where $\hat{Y}$ is a traceless matrix with exactly the structure in eqn.(5.11):then $\hat{Y}$ is the image in $\mathbb{C}l(\dim\mathbb{K}+1)$ of an element of $\mathbb{K}^{\oplus}$. $X^2 = X$ is equivalent to $\hat{Y}^2 = 1$ *i.e.* $Y \in S^{\dim\mathbb{K}} \subset \mathbb{K}^{\oplus}$: $\hat{Y}$ is a generator of $\mathrm{pin}(\dim\mathbb{K}+1)$!!

Several groups act naturally on the projective line. If we think $P^1(\mathbb{K})$ as the sphere $S^{\dim\mathbb{K}}$ we have the isometry group $SO(\dim\mathbb{K}+1)$. There is the stability group of a point on the sphere, $SO(\dim\mathbb{K})$, and the group of its conformal transformations, $SO(\dim\mathbb{K}+1, 1)$. The actions of these geometrically defined groups on $P^1(\mathbb{K})$ correspond to the special constructions in the previous sections. The standard action of the spin group $\mathrm{Spin}(\dim\mathbb{K}+1)$ on $\hat{Y}$ can be interpreted as the transformation on $P^1(\mathbb{K})$ given by $X \mapsto (\hat{u}_1\hat{u}_2)X(\hat{u}_2\hat{u}_1)$ with $\hat{u}_i$ as in §.5.2. Spin(dim $\mathbb{K}$) corresponds to $(\hat{u}_1\hat{u}_2)$ block diagonal.

In the associative case, see §.6, the conformal group is just $SL(2,\mathbb{K})$ acting on the homogenous coordinates (z_1, z_2) or, equivalently, *á la* Mobius $(az+b)(cz+d)^{-1}$ on the affine coordinate $z = z_1 z_2^{-1} \equiv y_2^{-1}x$ (for $\mathbb{H}$ the order matters). Comparing with eqn.(5.22), we see that the conformal action on $X \in P^1(\mathbb{K})$ is

$$X \mapsto \frac{1}{\mathrm{Tr}(UXU^\dagger)} UXU^\dagger \tag{5.24}$$

where U is as in eqn.(5.18). The RHS of (5.24) is a 2×2 Hermitean matrix with zero determinant and trace 1, and hence a projector[24]. It is remarkable that the Lorentz group for the superstring critical dimension, Spin(9, 1), is the automorphims group of the octonion projective line.

6. Explicit Constructions and Examples

The results of the previous section allow us to describe Spin(dim $\mathbb{K}-1$), Spin(dim $\mathbb{K}$), Spin(dim $\mathbb{K}+1$), and Spin(dim $\mathbb{K}+1,1$) in a uniform way in terms of the algebraic structure of $\mathbb{K}$. We consider some relevant examples.

Spin(4) $\mathbb{R}^4$ is identified with $\mathbb{H}_\mathcal{V}$. $\dim_\mathbb{R}(\mathbb{H}_+ \oplus \mathbb{H}^*_-) = 8$, thus $\binom{\psi}{\chi^*}$ has the same content as an (Euclidean) *Dirac* spinor. From prop.3.4 we know that an element of Spin(4) corresponds to a *pair* of unit quaternions, ξ, η, that is to a block diagonal matrix in $\mathbb{H}(2)$. Since $\mathbb{H}$ is associative, the expressions in proposition 5.1 simplify to

$$\psi \longmapsto L_{u_1u_2\cdots u_r}\, \psi = \xi\, \psi \tag{6.1}$$

$$\chi \longmapsto R_{u_ru_{r-1}\cdots u_1}\, \chi = \chi\, \eta \tag{6.2}$$

$$x \longmapsto L_{u_1u_2\cdots u_r} \circ R_{u_ru_{r-1}\cdots u_1}\, x = \xi\, x\, \eta. \tag{6.3}$$

where $\xi = u_1u_2\cdots u_r$ and $\xi = u_ru_{r-1}\cdots u_1$ (in general they are independent quaternions). The map $\psi \mapsto \xi\psi$, with $\xi, \psi \in \mathbb{H}$, $|\xi| = 1$, defines the standard action on $\mathbb{H}$ of the group of unit quaternions, $\xi \in S^3 \subset \mathbb{H}$. Therefore

$$\mathrm{Spin}(4) = S^3 \times S^3,$$

and only one of the two S^3 acts on the spinors of each chirality. The spinors, ψ, χ^*, are quaternions. Then we can write them as 2×2 complex matrices subjected to the *symplectic–Majorana* condition $\psi_{\alpha a} = \epsilon_{\alpha\beta}\epsilon_{ab}\psi^{\beta b}$ (resp. $\chi^*_{\dot\alpha a} = \epsilon_{\dot\alpha\dot\beta}\epsilon_{ab}\chi^{*\,\dot\beta b}$) where the raising of the indices stands for *componentwise* COMPLEX *conjugation.* The Σ_1, Σ_{e_k} matrices acting on them, written in the 2×2 form, are just Pauli matrices

$$\begin{aligned}(\Sigma_1)^{\dot\alpha\beta} &= \delta^{\dot\alpha\beta},\ (\Sigma_{e_k})^{\dot\alpha\beta} = -i(\sigma_k)^{\dot\alpha\beta}\\ (\Sigma_1)^{\alpha\dot\beta} &= \delta^{\alpha\dot\beta},\ (\Sigma_{e_k})^{\alpha\beta} = i(\sigma_k)^{\alpha\dot\beta}.\end{aligned} \tag{6.4}$$

Spin(3) The subgroup Spin(3)⊂ Spin(4) is given by the elements of $\mathbb{C}l(0,3) \simeq \mathbb{C}l^0(0,4)$ which are even in the $3D$ sense. Going trough the formulae of sect.5, we see that they are generated by quaternions of the

[24] All expressions make sense for $\mathbb{O}$ because only two octonions are involved.

form $u_1 u_2$ where $u_i^2 = -1$ (purely imaginary of norm 1). Since $u_i^* = -u_i$, in this case we have

$$\xi = u_1 u_2 \cdots u_{2r} = (u_{2r}^* u_{2r-1}^* \cdots u_1^*)^* = (-1)^{2r}(u_{2r}u_{2r-1}\cdots u_1)^* = \eta^{-1}$$

and we reduce to the action of a *single* S^3. The two spinor representations become equivalent $\psi \mapsto \xi\psi$, $\chi^* \mapsto \eta^*\chi^* \equiv \xi\chi^*$. On the vectors $x \mapsto \xi x \xi^{-1}$, so Spin(3)=$S^3 = SU(2)$. The same results are obtained by viewing Spin(3) as Spin($\dim\mathbb{C}+1$).

Spin(3,1) *(Two–component formalism).* Spin(3,1) can be obtained by "analytic continuation", $\Sigma_1 \leftrightarrow i\Sigma_1$, of eqns.(6.4) for the Euclidean case, or as Spin($\dim\mathbb{C}+1, 1$). Since $\mathbb{C}$ is associative, the operator $L_{(s,v\oplus w)}$ in eqn.(5.20) is just a matrix in $\mathbb{C}(2)$ with determinant 1. The general Spin(3,1) element is a product of such operators, and hence a complex matrix ξ with $\det\xi = 1$. Therefore $\mathrm{Spin}(3,1) = SL(2,\mathbb{C})$. The spinors of definite chirality can be written as two–component *complex* vectors ψ_α, $\bar\chi^{\dot\alpha}$. The invariant tensors $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$, $\epsilon^{\dot\alpha\dot\beta}$ $\epsilon_{\dot\alpha\dot\beta}$ can be used to rise/lower indices. From the remarks preceding eqn.(5.18) we see that the complex conjugate of a right–handed Weyl spinor $(\bar\chi^{\dot\alpha})^*$ transforms as a left–handed one. In $4D$ we have Majorana (real) fermions; their chiral components are each other conjugate.

Spin(5,1) We know (prop.3.4) that Spin(5,1) is given by *two* $\mathbb{H}(2)$ matrices (block diagonal form). The explicit construction of §.5.3 confirms the general theory. The actions of Spin($\dim\mathbb{H}+1, 1$) is described around eqn.(5.18). As in the Spin(4) case, the formulae simplify for $\mathbb{K} = \mathbb{H}$, since quaternions are associative. Let $\Xi \in \mathbb{H}(2)$ be the matrix

$$\Xi \equiv (s_1 + \widehat{v_1 \oplus w_1})(s_2 + \widehat{v_2 \oplus w_2}) \cdots (s_r + \widehat{v_r \oplus w_r}), \quad s_i^2 - v_i^2 - |w_i|^2 = 1 \quad (6.5)$$

By eqn.(5.20)

$$\Xi \in SL(2,\mathbb{H}),$$

The action of Spin(5,1) is given by

$$\mathbf{\Psi}_L \longmapsto \Xi\,\mathbf{\Psi}_L \quad (6.6)$$

$$(\mathbf{\Psi}_R^T\epsilon) \longmapsto (\mathbf{\Psi}_R^T\epsilon)\,\Xi^{-1} \quad (6.7)$$

$$\begin{pmatrix} y_- & x \\ x^* & y_+ \end{pmatrix} \longmapsto \Xi \begin{pmatrix} y_- & x \\ x^* & y_+ \end{pmatrix} \Xi, \quad (6.8)$$

confirming our previous claim that Spin(5,1)$\simeq SL(2,\mathbb{H})$.

Spin(5) Prop.3.4 predicts Spin(5)$\in \mathbb{H}(2)$. The construction in §.5.2 realizes Spin(5) as the maximally compact subgroup of Spin(5,1) (as described in §.5.3). The general element of Spin(5) is the product of an *even* number of matrices of the form in eqn.(5.17). One gets

$$\begin{pmatrix} \Xi & 0 \\ 0 & \Xi \end{pmatrix}$$

and the two spinorial representations coincide. Eqns.(6.5)–(6.8) still hold, but now $\Xi \in \mathbb{H}(2)$ is restricted to have the form

$$\Xi = \sigma_3\widehat{\mathbf{v}_1}\,\sigma_3\widehat{\mathbf{v}_2}\cdots\sigma_3\widehat{\mathbf{v}_{2r}}, \quad \text{where } \mathbf{v}_i = \lambda_i \oplus x_i, \text{ with } \lambda_i^2 + |x_i|^2 = 1\ , \quad (6.9)$$

which implies $\Xi^\sharp\,\Xi = 1$ ($\sharp$ denotes Hermitean conjugation in the $\mathbb{H}$ sense). Therefore Spin(5)$\simeq$ $Sp(2)$ = the maximal compact subgroup of $SL(2,\mathbb{H})$. Notice the parallel with the relation of Spin(3,1) with Spin(2)=$SU(2)$ the maximal compact subgroup of $SL(2,\mathbb{C})$.

7. Spin(7)

Spin(7) and its subgroup G_2 are the two *exceptional holonomy groups*; they are also relevant for M–theory.

As in the case of Spin(3), to get Spin(dim $\mathbb{O}-1$) from Spin(dim $\mathbb{O}$) one restricts the orthogonal operators $\mathcal{L}_a$, $\mathcal{R}_a$ $\mathcal{T}_a$ to a's which are the product of an *even* number of $\hat{u}$'s where now $u \in \mathbb{O}$, $u^2 = -1$. Referring back to eqn.(5.1), we see that

$$\hat{u}\hat{v} = \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix}\begin{pmatrix} 0 & -v \\ v & 0 \end{pmatrix} = -\mathbf{1}\cdot L_u\, L_v$$

and, again, the two chiral representations become equivalent. We denote by $\mathbb{O}_{\mathcal{S}}$ the octonions identified with this unique spinorial representation. By the same argument as in the Spin(3) case, $(\mathcal{R}_a\xi)^* = \mathcal{L}_a\xi^*$ for all $\xi \in \mathbb{O}_{\mathcal{S}}$. The vector representation $\mathbb{O}_{\mathcal{V}}$ decomposes as $\mathbf{1}\oplus\mathbf{7}$: the $\mathbf{7}$–vector belongs to the subspace of *purely imaginary* octonions, $x^2 = -|x|^2$, while the real octonions are invariant under Spin(7). In fact

$$T_u \circ T_v(1) = u\cdot v1v\cdot u = -u^2 = 1.$$

Given two spinors $\psi, \chi \in \mathbb{O}_{\mathcal{S}}$, we can construct the antisymmetric bilinear $2\,\psi\wedge\chi = \psi\chi^* - \chi\psi^*$. Since it is purely imaginary, it should be a vector. Indeed, from $\mathcal{R}_a\chi^* = (\mathcal{L}_a\chi)^*$ and Spin(8) TRIALITY we get

$$\mathcal{L}_a\psi\wedge\mathcal{L}_a\chi = \mathcal{T}_a(\psi\wedge\chi), \tag{7.1}$$

which is the correct transformation for a vector. Given four spinors, $\psi, \chi, \lambda, \zeta$ we can construct two vectors, $\psi\wedge\chi$ and $\lambda\wedge\zeta$. Consider their inner product $-\Re(\psi\wedge\chi\cdot\lambda\wedge\zeta)$, it is obviosly a Spin(7) invariant. We know that Spin(8), and hence its subgroup Spin(7), acts on $\mathbb{O}_{\mathcal{S}}$ by *orthogonal* matrices. Thus the inner products $\langle\psi|\chi\rangle \equiv (\psi\chi^* + \chi\psi^*)/2$ are also invariants. Therefore the 4–tensor

$$\Omega(\psi,\chi,\lambda,\zeta) =_{\rm def} -\Re(\psi\wedge\chi\cdot\lambda\wedge\zeta) - \langle\psi|\lambda\rangle\langle\chi|\zeta\rangle + \langle\psi|\zeta\rangle\langle\chi|\lambda\rangle \tag{7.2}$$

is a Spin(7) invariant. Let f_α be an orthonormal basis of $\mathbb{O}_{\mathcal{S}}$ ($f_0 = 1, f_\beta = e_\beta$ for $\beta = 1,\dots,7$). We write

$$\Omega_{\alpha\beta\gamma\delta} = \Omega(f_\alpha, f_\beta, f_\gamma, f_\delta). \tag{7.3}$$

We claim that this invariant 4–tensor is in fact a antiself– dual 4–form. While one can check this directly (it takes a few minutes), we argue on general grounds. For Spin(8) the matrix elements of the positive chirality rotation operators, $\mathcal{L}_a$, are 8×8 orthogonal matrices. TRIALITY requires all orthogonal matrices to arise as $\mathcal{L}_a$ for some $a \in$ Spin(8). Stated in a different way: the only (independent) invariant tensors for the action of Spin(8) on tensor products of several copies of $\mathbb{O}_+$ are $\delta_{\alpha\beta}$ and $\epsilon_{\alpha_1\alpha_2\cdots\alpha_8}$. When we restrict a to a *subgroup* $\mathcal{G} \subset$ Spin(8), other invariant tensor appear. A given subgroup $\mathcal{G}$ can be identified with the set of orthogonal 8×8 matrices leaving invariant — besides the Kronecher and the Levi–Civita tensors — some additional tensor characteristic of the specific subgroup at hand. Then

THEOREM 7.1. *Spin(7) is the subgroup of SO(8) consisting of orthogonal* 8×8 *matrices leaving invariant the antiself–dual 4–form* Ω *in eqn.*(7.3).

One has $e_i e_j = -\delta_{ij} + C_{ijk} e_k$ for some structure constants C_{ijk} The tensor $\Omega_{\alpha\beta\gamma\delta}$ can be written in terms of the C_{ijk}'s. For instance $\Omega_{0ijk} = C_{jki}$. In the next section we show that $\Omega_{\alpha\beta\gamma\delta}$ is totally antisymmetric and anti–selfdual.

8. The Cayley Numbers $\mathbb{O}$ and the Group G_2

We have discussed the properties of spin groups in terms of division algebras with particular insistence on the octonions, but we have not explained their structure nor proved their existence. We close the gap here. We do this because we will need the group of automorphism of $\mathbb{O}$ which is a fancy Lie group, G_2.

8.1. Algebra Doubling. There is a procedure, known as *algebra doubling*, which from $\mathbb{R}$ produces $\mathbb{C}$ and from $\mathbb{C}$ gives $\mathbb{H}$. Let $\mathbb{A}$ an algebra over the reals with an involutive antiautomorphism $a \mapsto \bar{a}$. We equip the vector space $\mathbb{A}^2$ with the product

$$(a, b)\,(c, d) = (au - \bar{v}b, \bar{b}u + va) \tag{8.1}$$

making it an algebra. The map $a \mapsto (a, 0)$ injects $\mathbb{A} \to \mathbb{A}^2$ and allows to identify the original algebra $\mathbb{A}$ with a subalgebra of $\mathbb{A}^2$. Then the element $(a, 0)$ is written simply a. Let $e = (0, 1)$. One writes $(a, b) = a + b\,e$ and the product is defined by the following rules

$$a(be) = (ba)e, \quad (ae)b = (a\bar{b})e, \quad (ae)(be) = -\bar{b}a. \tag{8.2}$$

REMARK. These rules have dramatic consequences: if in $\mathbb{A}$ the involution $a \mapsto \bar{a}$ is not the identity, $\mathbb{A}^2$ is not commutative. Thus the conjugation being non trivial in $\mathbb{C}$, $\mathbb{C}^2 \equiv \mathbb{H}$ is not commutative. Moreover if $\mathbb{A}$ is not commutative, then $\mathbb{A}^2$ will be *not associative*. Therefore the octonions $\mathbb{O} \equiv \mathbb{H}^2$ are neither commutative nor associative.

The conjugation in $\mathbb{A}^2$ is given by

$$\overline{a + be} = \bar{a} - be. \tag{8.3}$$

One has $\overline{ab} = \bar{b}\bar{a}$.

Assume that $\mathbb{A}$ is metric, i.e. $a\bar{a} = |a|^2 \geq 0$ with $= 0$ only if $a = 0$. Then $\mathbb{A}^2$ is also metric

$$(a + be)\,\overline{(a + be)} = (a + be)(\bar{a} - be) = a\bar{a} + b\bar{b}.$$

8.2. Structure of $\mathbb{O} \equiv \mathbb{H}^2$. The basis of $\mathbb{O}$ is given by 1 and the seven imaginary units

$$i,\ j,\ k,\ e,\ f = ie,\ g = je,\ h = ke, \tag{8.4}$$

having square equal -1. (When considering all imaginary units together we call them e_k, the index going from 1 to 7). Their product is completely determined by this definition and the rules (8.2). Since $\mathbb{O}$ is metric, distinct imaginary units *anti*commute

$$e_k e_h + e_h e_k = -2\delta_{kh} \tag{8.5}$$

and the e_k's are odd under conjugation, $\bar{e}_k = -e_k$.

LEMMA 8.1. *Let $x \in \mathbb{O}$ be purely imaginary, $\bar{x} = -x$. Then $(e_h x)x = -e_h\,|x|^2 = x(xe_h)$.*

PROOF. Let $e_\alpha = i, j, k$ and $x = a + be$ with $\bar{a} = -a$. Using the rules (8.2) we get

$$e_\alpha x \cdot x = (e_\alpha a + be_\alpha \cdot e)(a + be) = -e_\alpha|a|^2 + be_\alpha a + be_\alpha\bar{a} - \bar{b}be_a = -e_\alpha|x|^2$$
$$ex \cdot x = -(ae + \bar{b})(a + be) = -|a|^2 e - \bar{b}a + \bar{b}a - b\bar{b}e = -|x|^2 e$$
$$(e_\alpha e \cdot x)x = -(e_\alpha a \cdot e + \bar{b}e_\alpha)(a + be) = -e_\alpha|a|^2 + \bar{b}e_\alpha a - \bar{b}e_\alpha a - \bar{b}be_\alpha e = -|x|^2 e_\alpha e$$

Thus $e_k x \cdot x = -|x|^2 e_k$. Taking the conjugate, we get $x \cdot xe_k = -|x|^2 e_k$. □

LEMMA 8.2. *The associator $(a, b, c) \equiv (ab)c - a(bc)$ is: (1) totally antisymmetric in its arguments; (2) purely imaginary.*

PROOF. (1) By linearity it is enough to show that it vanishes when two arguments are equal. It suffices to check for imaginary arguments. Let $\bar{x} = -x$. By the previous lemma

$$(x, x, e_k) = -e_k|x|^2 - x(xe_k) = 0, \qquad (e_k, x, x) = (e_k x)x + e_k|x|^2 = 0.$$

(2) $[(e_k e_h)e_j - e_k(e_h e_j)]^* = (e_j e_h)e_k - e_j(e_h e_k) = -[(e_k e_h)e_j - e_k(e_h e_j)]$ by antisymmetry. □

COROLLARY 8.1. *All expressions containing only two octonions, x, y, together with their conjugates $\bar{x}, \bar{y}$, satisfy the identities valid in an associative algebra isomorphic to $\mathbb{H}$.*

This result was stated (without proof) in §.5.1. There we deduced from this two important identities:

COROLLARY 8.2 (Moufang identities). *$z \cdot xy \cdot z = zx \cdot yz$ and $x \cdot zyz = (xz \cdot y)z$ for all $x, y, z \in \mathbb{O}$.*

A metric algebra is *normed* if $|xy| = |x| \cdot |y|$.

THEOREM 8.1. *$\mathbb{O}$ is normed.*

PROOF. By the corollary 8.1, the computation of $|xy|^2 \equiv (xy) \cdot (\bar{y}\bar{x})$ can be done in the quaternionic subalgebra containing x, y. But in $\mathbb{H}$ $(xy)(\bar{y}\bar{x}) = (x\bar{x})(y\bar{y})$. □

REMARK. The (already mentioned) theorem of Hurwitz states that the only normed algebra are $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$. In particular the doubling of $\mathbb{O}$ is not normed. In fact given a normed *metric* algebra $\mathbb{A}$ the construction of §.5.1 gives a representation of $\mathbb{C}l(\dim \mathbb{A})$ in terms of matrices of dimension (at most) $2 \dim \mathbb{A}$. This implies $2^{\dim \mathbb{A}} \leq 4(\dim \mathbb{A})^2$ which can hold only for $\dim \mathbb{A} \leq 8$.

We have $e_k e_h = -\delta_{kh} + C_{khj} e_j$ for some (real) structure constant $C_{khj} = (\Sigma_{e_k})_{e_j e_h}$ in the notation of sect.4. By eqn.(8.5), $C_{khj} = -C_{hkj}$. The associator

$$\begin{aligned}(e_k e_h)e_j - e_k(e_h e_j) = (C_{hjk} - C_{khj}) + \\ + (C_{khm}C_{mjl} + C_{hjm}C_{mkl} - \delta_{kh}\delta_{jl} + \delta_{hj}\delta_{kl})\, e_l,\end{aligned} \tag{8.6}$$

is purely imaginary (lemma 8.2). Then

PROPOSITION 8.1. *C_{khj} is totally antisymmetric in its indices.*

PROOF. The real part of the associator (8.6) vanish. So C_{khj} is invariant under cyclic permutation of its indices. Since it is antisymmetric in the first two indices, it is totally antisymmetric. □

We consider the tensor

$$\Omega_{khjl} = C_{khm}C_{mjl} - \delta_{jk}\delta_{hl} + \delta_{kl}\delta_{hj}, \tag{8.7}$$

it is the same tensor $\Omega_{\alpha\beta\gamma\delta}$ of sect.7 restricted to $k, h, j, l = 1, \dots, 7$. Indeed let x, y, w, z be purely imaginary octonions; the definition (7.2) can be rewritten in the form

$$\begin{aligned}x_k\, y_h\, w_j\, z_l\, \Omega_{khjl} = \Omega(x, y, w, z) = \\ - \Re\big[z \cdot (xy \cdot w)\big] + \langle x|y\rangle\, \langle w|z\rangle - \langle w|x\rangle\, \langle y|z\rangle + \langle x|z\rangle\, \langle y|w\rangle.\end{aligned} \tag{8.8}$$

PROPOSITION 8.2. *(1) Ω_{hkjl} is totally antisymmetric in its indices. (2) The associator is given by $(e_k, e_h, e_j) = 2\,\Omega_{khjl}\, e_l$.*

PROOF. (1) The tensor is antisymmetric in the two pairs of indices and symmetric with respect to the interchange of pairs. To show that it is totally antisymmetric it suffices to show antisymmetry in the two central indices, or, equivalently, that $\Omega(x, y, y, z) = 0$. But

$$\Omega(x, y, y, z) = -\Re\big(z \cdot (xy \cdot y)\big) + |y|^2\, \langle x|z\rangle,$$

while $\Re[z \cdot (xy \cdot y)] = \Re[z \cdot (x \cdot y^2)] = -|y|^2 \Re(zx) = |y|^2\, \langle x|z\rangle$ since x, y, z are purely imaginary. Then $\Omega(x, y, y, z) = 0$. (2) Comparing eqn.(8.6) with definition (8.7), we get $(e_k, e_h, e_j) = (\Omega_{khjl} + \Omega_{hjkl})e_l = 2\,\Omega_{khjl}\, e_l$. □

PROPOSITION 8.3. *One has*[25]

$$\Omega_{i_1 i_2 i_3 i_4} = -\epsilon_{i_1 i_2 i_3 i_4 i_5 i_6 i_7} C_{i_5 i_6 i_7}. \tag{8.9}$$

PROOF. We recall a property of the chiral operator $\Gamma_{[n]} \in \mathbb{C}l(0, n)$: $\Gamma_{i_1 \cdots i_k} = (-1)^{\tau(k)+k} \epsilon_{i_1 \dots i_n} \Gamma_{i_{k+1} \cdots i_n}$. We apply this formula to the Σ–matrices of $\mathbb{C}l(0, 7)$ constructed in sect.4. Then

$$\Sigma_{i_1 i_2 i_3 i_4} = \epsilon_{i_1 i_2 \cdots i_6 i_7} \Sigma_{i_5 i_6 i_7} \Sigma_{[7]} = -\epsilon_{i_1 i_2 \cdots i_6 i_7} \Sigma_{i_5 i_6 i_7}. \tag{8.10}$$

In $\mathbb{O}$ we fix the basis $f_0 = 1, f_k = e_k$, and take the 00 component of both sides of eqn.(8.10). From the definitions

$$(\Sigma_i)_{0j} = -\delta_{ij}, \quad (\Sigma_i)_{j0} = \delta_{ij}, \quad (\Sigma_i)_{jk} = C_{ikj}.$$

[25]The sign depends on a choice of orientation. Some authors have a different sign.

we get

$$\begin{aligned}&(\Sigma_{i_1i_2})_{0i_3} - (\Sigma_{i_1i_2})_{i_30} = C_{i_1i_2i_3}\\&(\Sigma_{i_5i_6i_7})_{00} = -(\Sigma_{i_6})_{i_5i_7} = -C_{i_5i_6i_7}\\&(\Sigma_{i_1i_2i_3i_4})_{00} = -(\Sigma_{i_2i_3})_{i_1i_4} = -C_{i_2ki_1}C_{i_3i_4k} - \text{traces} = -\Omega_{i_1i_2i_3i_4}.\end{aligned} \tag{8.11}$$

□

8.3. The Group G_2.

We are interested in the automorphism group of $\mathbb{O}$ called G_2. Since $\mathbb{O}$ is a metric algebra, any automorphism maintains norms and leaves $\mathbb{R} \subset \mathbb{O}$ invariant. Hence $G_2 \subset O(7)$. We denote by S^6 the sphere of imaginary octonions of norm 1, *i.e.* the elements of $\mathbb{O}$ with $x^2 = -|x|^2$.

The group of automorphisms of $\mathbb{O}$ is easily described:

PROPOSITION 8.4. *Let $\xi,\ \eta,\ \zeta \in S^6$ such that $\Re(\xi\eta) = 0$ and $\Re(\xi\zeta) = \Re(\eta\zeta) = \Re(\xi\eta\cdot\zeta) = 0$. Then there exist a (unique) automorphism Φ of $\mathbb{O}$ such that*

$$\xi = \Phi(i), \qquad \eta = \Phi(j), \qquad \zeta = \Phi(e).$$

PROOF. Since $(\xi)^2 = -1$, $\mathbb{R}[\xi] \simeq \mathbb{C}$. The elements of the form $a + b\xi + c\eta + d\xi\eta$ (a, b, c, d real) form a subalgebra $\widetilde{\mathbb{H}}$ isomorphic to $\mathbb{H}$. By hypothesis, ζ is orthogonal to $\widetilde{\mathbb{H}}$. It remains to show that $\forall A, B \in \widetilde{\mathbb{H}}$

$$A\cdot B\zeta = BA\cdot\zeta, \qquad A\zeta\cdot B = A\bar{B}\cdot\zeta, \qquad A\zeta\cdot B\zeta = -\bar{B}A, \tag{8.12}$$

these relations reproduce the rules (8.2) which define the product in $\mathbb{O}$ in terms of that of $\mathbb{H}$. Note that $A\zeta$ is purely imaginary[26] so $-\zeta\bar{A} = \overline{A\zeta} = -A\zeta$, and the first two identities (8.12) are related by conjugation. In showing the identities we can assume B purely imaginary, since for B real they are obvious.

The antisymmetry of the associators implies

$$A\cdot B\zeta + A\cdot\zeta B = AB\cdot\zeta + A\zeta\cdot B$$

Replace $B \leftrightarrow \bar{B}$ and take the difference

$$A\cdot\big(B\zeta - \zeta\bar{B}\big) + A\cdot\big(\zeta B - \bar{B}\zeta\big) = (AB - A\bar{B})\cdot\zeta + A\zeta\cdot(B - \bar{B}).$$

The LHS vanish since $\zeta\bar{B} = B\zeta$ for all B in the quaternionic subalgebra. Then, taking B purely imaginary, we have $AB\cdot\zeta = -A\zeta\cdot B = A\zeta\cdot\bar{B}$ which is the second identity in eqn.(8.12). The last identity is easily reduced to TRIALITY: $A\zeta\cdot B\zeta = \zeta\bar{A}\cdot B\zeta = \zeta\cdot(\bar{A}B)\cdot\zeta = -\overline{\bar{A}B} = -\bar{B}A$. □

The above result gives a clear geometrical picture of G_2. We have a surjective map $G_2 \to S^6$ given by $\Phi \mapsto \Phi(i)$. Then S^6 is diffeomorphic to the coset G_2/K where K is the group of automorphisms of $\mathbb{O}$ leaving i invariant. Viewing $\mathbb{O}$ as the iterated algebra doubling $(\mathbb{C}^2)^2$, we see that the space spanned by the six imaginary units orthogonal to i is a complex space with basis j, e and $je = g$. The automorphisms leaving i fixed are precisely the $\mathbb{C}$–linear transformations of this three–dimensional space. Since

[26] $A\zeta$ has the form $(a_o + \sum_\lambda a_\lambda e_\lambda)(\sum_\Lambda \zeta_\Lambda e_\Lambda)$ where the indices λ and Λ takes values in *disjoint* sets.

these transformations should preserve norms and the relation $j \cdot e \cdot g = -1$, $K = SU(3)$ and

$$G_2/SU(3) \simeq S^6. \tag{8.13}$$

For all automorphisms, $\Phi \in SU(3)$, the element $\eta \in \Phi(j)$ is orthogonal to i and hence belongs to S^5 (the equator of S^6). Thus we have a surjective map from $K = SU(3)$ to the sphere S^5 given by $\Phi \mapsto \Phi(j)$. Then

$$K/L \simeq S^5 \tag{8.14}$$

where L is the subgroup of $SU(3)$ leaving invariant j. Now the element $\zeta = \Phi(e)$ is orthogonal to i, j, k and hence belongs to a sphere $S^3 \subset S^6$. By prop.8.4, Φ is a diffeomorphism from L to S^3. Therefore G_2 is the total space of a principal $SU(3)$ bundle with base S^6: the fiber itself is the total space of an $SU(2)$ principal bundle with basis S^5. We have established

PROPOSITION 8.5. *The group G_2 has dimension 14. It is connected and simply connected.*

The discussion in the previous section shows that Spin(7) acts transitively on the S^7 sphere of unit octonions. The isotropy group of 1 is clearly G_2. Then

$$S^7 = \mathrm{Spin}(7)/G_2. \tag{8.15}$$

8.4. The Lie Algebra $\mathfrak{g}_2$. For the applications we are expecially interested in the Lie algebra $\mathfrak{g}_2$ of G_2. We know that $\mathfrak{g}_2 \subset \mathfrak{so}(7)$, *i.e.* that the elements of $\mathfrak{g}_2$ can be represented by antisymmetric 7×7 matrices, but we are interested to know which antisymmetric matrices belong to $\mathfrak{g}_2$.

The elements of $\mathfrak{g}_2$ are precisely the antisymmetric matrices preserving the 3–form C_{khj} (or, equivalently, its dual 4–form Ω). One can also state this as the invariance of the $\Im\mathbb{O} \times \Im\mathbb{O} \to \Im\mathbb{O}$ exterior product given by $\Im(xy^*)$, (the restriction to imaginary octonions of the spinorial wedge product introduced in the context of Spin(7), sect.7). Thus an antisymmetric matrix a_{kh} belong to $\mathfrak{g}_2$ if and only if

$$a_{km}\, C_{mhl} + a_{hm}\, C_{kml} + a_{lm}\, C_{khm} = 0. \tag{8.16}$$

In our conventions the non–vanishing components of C_{khl} are

$$C_{123} = C_{176} = C_{145} = C_{246} = C_{257} = C_{365} = C_{347} = 1, \tag{8.17}$$

and thus we get the following seven constraints on the a_{kh}:

$$\begin{aligned} &a_{16} + a_{52} + a_{43} = 0 && a_{17} + a_{24} + a_{53} = 0 \\ &a_{41} + a_{27} + a_{63} = 0 && a_{51} + a_{62} + a_{73} = 0 \\ &a_{12} + a_{47} + a_{65} = 0 && a_{32} + a_{54} + a_{67} = 0 \\ &a_{31} + a_{57} + a_{46} = 0. \end{aligned}$$

From TRIALITY we have two inequivalent embeddings $\alpha, \beta\colon \mathrm{Spin}(7) \to SO(8)$, through the vector and spinor representations. $G_2 = \alpha(\mathrm{Spin}(7)) \cap \beta(\mathrm{Spin}(7)) \subset SO(8)$. In particular $\dim G_2 = 14$.

APPENDIX D

Poor man's version of the Berger theorem

1. A *poor man's* Berger theorem

For the applications to supergravity discussed in the the present course one does not need the full–fledged Berger theorem. A weaker result is sufficient for (almost) all purposes. The weaker result is easy to prove.

PROPOSITION 1.1. *Let $\mathcal{M}$ be an irreducible Riemannian manifold. Assume one of the following vector bundle isomorphisms:*

(1) $(\mathbb{C}\otimes T\mathcal{M})\big|_{(1,0)} \simeq \mathcal{V}_1\otimes\mathcal{V}_2$ *with* $2<\operatorname{rank}\mathcal{V}_1\leq\operatorname{rank}\mathcal{V}_2$;
(2) $(\mathbb{C}\otimes T\mathcal{M})\big|_{(1,0)} \simeq \wedge^k\mathcal{V}$, *with* $2\leq k\leq\operatorname{rank}\mathcal{V}/2$;

then $\mathcal{M}$ is SYMMETRIC.

This result is directly applicable to supergravity. One has just to compare the tangent bundle isomorphisms we listed in chapt. 2 with the proposition.

EXAMPLE. Let $D=4$ $\mathcal{N}=6,8$. One has $(\mathbb{C}\otimes T\mathcal{M})\big|_{(1,0)}\simeq\wedge^2\Psi^\vee$ and, respectively, $\wedge^4\Psi^\vee$. Then $\mathcal{M}$ is symmetric.

As a consequence one has

PROPOSITION 1.2. *Let $\mathcal{M}$ be a simply–connected,* IRREDUCIBLE, NON–SYMMETRIC *Riemannian manifold with* $\dim\mathcal{M}=n$. *Then the Lie group Hol($\mathcal{M}$) is one of the following:*

(1) *a simple subgroup $H\subseteq SO(n)$ and $T\mathcal{M}$ is an irreducible real representation of degree n;*
(2) *a simple subgroup $H\subseteq SU(n/2)$ and $T_{\mathbb{C}}\mathcal{M}\big|_{(1,0)}$ is an irreducible non–real representation of degree $n/2$;*
(3) *$U(1)\times H$, where H is a simple subgroup of $SU(n/2)$ and $T_{\mathbb{C}}\mathcal{M}\big|_{(1,0)}$ is an irreducible complex representation of degree $n/2$;*
(4) *$Sp(2)\times H$, where H is a simple subgroup of $Sp(n/4)$ and $T_{\mathbb{C}}\mathcal{M}\big|_{(1,0)}=V_2\otimes V_H$, where V_2 is the defining 2–dimensional representation of $Sp(2)$ and V_H a quaternionic representation of degree $n/2$ (irreducible at least over $\mathbb{R}$).*

1.1. Proofs.

1.1.1. *Proposition 1.1.* **(1)** We consider the case in which $(\mathbb{C}\otimes T\mathcal{M})\big|_{(1,0)}$ is a *complex* irreducible representation of Hol($\mathcal{M}$). The cases in which this irreducible representation is real or symplectic are essentially identical and the obvious modifications to the argument are left to the reader.

We replace the indices i and $\bar{\imath}$ of the complexified tangent bundle, $(\mathbb{C}\otimes T\mathcal{M})_{(1,0)}\oplus(\mathbb{C}\otimes T\mathcal{M})_{(0,1)}$ by double indices

$$i \to a\,\alpha, \qquad\qquad \bar{\imath} \to \bar{a}\,\bar{\alpha}$$

corresponding to the tensor product $(\mathbb{C}\otimes T\mathcal{M})\big|_{(1,0)} \simeq \mathcal{V}_1\otimes\mathcal{V}_2$. Let $\mathfrak{hol}$ be the Lie algebra of the holonomy group $\mathrm{Hol}(\mathcal{M})$. Since

$$\mathfrak{hol} \subset \mathrm{End}(\mathcal{V}_1)\oplus\mathrm{End}(\mathcal{V}_2), \tag{1.1}$$

an element $L \in \mathfrak{hol} \subset \mathrm{End}(T_{\mathbb{C}}\mathcal{M}\big|_{(1,0)})$ takes the following form in the double–index notation

$$L_{a\,\alpha\;\bar{b}\,\bar{\beta}} = \delta_{a\bar{b}}\,A_{\alpha\bar{\beta}} + \delta_{\alpha\bar{\beta}}\,B_{a\bar{b}}. \tag{1.2}$$

The Riemann tensor in an element of $\odot^2\mathfrak{hol}$. Hence, up to permutations of the indices, the only non–vanishing components of the Riemann tensor have the form

$$R_{a\alpha\,b^*\beta^*\;c\gamma\,d^*\delta^*}.$$

From eqn.(1.2) we see that this component may be $\neq 0$ only if $a = b$ or $\alpha = \beta$. Assume $a = b$ (otherwise make $\mathcal{V}_1 \leftrightarrow \mathcal{V}_2$). We remain with two kinds of possibly non–vanishing components: *(i)* $R_{a\alpha\,a^*\beta^*\;b\gamma\,b^*\delta^*}$, which by eqn.(1.2) is independent of a, b, and *(ii)* $R_{a\alpha\,a^*\beta^*\;b\gamma\,c^*\gamma^*}$, which is independent of a,γ. For components of the first kind, therefore, we are free to assume $a \neq b$, provided $\mathrm{rank}\,\mathcal{V}_1 > 1$. Now

$$R_{a\alpha\,a^*\beta^*\;b\gamma\,b^*\delta^*} = -R_{a^*\beta^*\;b\gamma\,a\alpha\,b^*\delta^*} - R_{b\gamma\,a\alpha\,a^*\beta^*\;b^*\delta^*}$$

which is non vanishing only if $\beta = \gamma$ and $\alpha = \delta$. Thus, up to index permutations, the only non–vanishing components of the first kind are

$$R_{a\alpha\,a^*\beta^*\;b\beta\,b^*\alpha^*} = -R_{a^*\beta^*\;b\beta\,a\alpha\,b^*\alpha^*} = R_{b\beta\,a^*\beta^*\;a\alpha\,b^*\alpha^*}, \tag{1.3}$$

which are manifestly independent of a, b, α, β. Then, provided $\mathrm{rank}\,\mathcal{V}_1 > 1$ and $\mathrm{rank}\,\mathcal{V}_2 > 1$, we are free to take $a \neq b$ and $\alpha \neq \beta$.

The terms of the second kind, $R_{a\alpha\,a^*\beta^*\;b\gamma\,c^*\gamma^*}$, are

$$R_{a\alpha\,a^*\beta^*\;b\gamma\,c^*\gamma^*} = -R_{a^*\beta^*\;b\gamma\,a\alpha\;\,c^*\gamma^*} - R_{b\gamma\,a\alpha\,a^*\beta^*\;c^*\gamma^*}$$

where again, if $\mathrm{rank}\,\mathcal{V}_1 > 1$, $\mathrm{rank}\,\mathcal{V}_2 > 1$, in the LHS we are free to take $a \neq b$ and $\gamma \neq \alpha$. This expression is non–zero only if $\gamma = \beta$ and $c = a$. Since this component does not depend on the value of a, if $\mathrm{rank}\,\mathcal{V}_1 > 2$ we can take $a \neq b, c$ and then it should vanish.

Next compute the covariant derivative of the non–vanishing components of type *(i)*: $\nabla_{c\gamma}R_{a\alpha\,a^*\beta^*\;b\beta\,b^*\alpha^*}$. If $\mathrm{rank}\,\mathcal{V}_1 > 2$ we are free to choose $a \neq c$ and $b \neq a, c$. By the Bianchi identity this expression is equal to

$$-\nabla_{a\alpha}R_{a^*\beta^*\;c\gamma\,b\beta\,b^*\alpha^*} - \nabla_{a^*\beta^*}R_{c\gamma\,a\alpha\,b\beta\,b^*\alpha^*} = -\nabla_{a\alpha}R_{a^*\beta^*\;c\gamma\,b\beta\,b^*\alpha^*}$$

which is manifestly zero unless $\gamma = \beta$; in this case the RHS becomes $\nabla_{a\alpha}R_{b\beta\,b^*\alpha^*\;c\beta\,a^*\beta^*}$, *i.e.* a component of the the kind *(ii)*, which also vanishes if $\mathrm{rank}\,\mathcal{V}_1 > 2$. Therefore if $2 < \mathrm{rank}\,\mathcal{V}_1 \leq \mathrm{rank}\,\mathcal{V}_2$ the Riemann tensor is parallel, and hence $\mathcal{M}$ symmetric.

(2) A tangent vector is written $v^{a_1 a_2\cdots a_k}$, totally anti–symmetric in its indices. For short we write $v^{\mathbf{A}}$ where $\mathbf{A}$ is the order set $\{a_1, a_2, \ldots, a_k\}$. We may forget about the index order at the price of an overall sign ambiguity. From the embedding $\wedge^k\mathcal{V} \to \otimes^k\mathcal{V}$, we see that a matrix element of the $\mathfrak{Hol}$ Lie algebra, $L_{\mathbf{A}}{}^{\mathbf{B}}$, vanishes unless $\sharp(\mathbf{A}\cap\mathbf{B}) \geq k-1$. Then a non vanishing element has the form $\pm L_{(\{a\}\cup\mathcal{A})\,(\{b\}\cup\mathcal{A})}$ for some index set $\mathcal{A}$ of cardinality $k-1$. Up to sign, this matrix element is independent of the index set $\mathcal{A}$, provided it is an allowed one, that is $\mathcal{A}\cap\{b\} = \mathcal{A}\cap\{a\} = \emptyset$. Let $n = \mathrm{rank}\,\mathcal{V}$. The number of allowed (unordered) index sets $\mathcal{A}$ is equal to $\binom{n-1}{k-1}$ for $a = b$ and $\binom{n-2}{k-1}$ otherwise.

A non–vanishing component of the Riemann tensor has the form

$$R_{(\{a\}\cup\mathcal{A})\ (\{b\}\cup\mathcal{A})^*\ (\{c\}\cup\mathcal{B})\ (\{d\}\cup\mathcal{B})^*},$$

and it is independent, up to sign, of $\mathcal{A},\mathcal{B}$ as long as they are allowed. Now

$$\begin{aligned}&R_{(\{a\}\cup\mathcal{A})\ (\{b\}\cup\mathcal{A})^*\ (\{c\}\cup\mathcal{B})\ (\{d\}\cup\mathcal{B})^*}=\\&=-R_{(\{b\}\cup\mathcal{A})^*\ (\{c\}\cup\mathcal{B})\ (\{a\}\cup\mathcal{A})\ (\{d\}\cup\mathcal{B})^*}\end{aligned}\tag{1.4}$$

which is non–zero only if $(\{b\}\cup\mathcal{A})\cap(\{c\}\cup\mathcal{B})$ and $(\{a\}\cup\mathcal{A})\cap(\{d\}\cup\mathcal{B})$ have, at least, cardinality $k-1$. If $\binom{n-2}{k-1}>1$ we can choose $\mathcal{A}\neq\mathcal{B}$; then our component (1.4) is $\neq 0$ only if $b=c$, $a=d$, and the cardinality of $\mathcal{A}\cap\mathcal{B}$ is at least $k-2$. Hence we can write $\mathcal{A}=\{i\}\cup A$ and $\mathcal{B}=\{j\}\cup A$. The non–vanishing components are then

$$\begin{aligned}&R_{(\{a\}\cup\{i\}\cup A)\ (\{b\}\cup\{i\}\cup A)^*\ (\{b\}\cup\{j\}\cup A)\ (\{a\}\cup\{j\}\cup A)^*}=\\&=\pm R_{(\{b\}\cup\{j\}\cup A)\ (\{b\}\cup\{i\}\cup A)^*\ (\{a\}\cup\{i\}\cup A)\ (\{a\}\cup\{j\}\cup A)^*}\end{aligned}$$

This component is independent (up to sign) of a,b,i,j and A in the allowed index range. Following the argument of part **(1)**, we get

$$\begin{aligned}&\nabla_{(\{c\}\cup\{h\}\cup C)}\,R_{(\{a\}\cup\{i\}\cup\mathcal{A})\ (\{b\}\cup\{i\}\cup A)^*\ (\{b\}\cup\{j\}\cup A)\ (\{a\}\cup\{j\}\cup A)^*}=\\&=\nabla_{(\{a\}\cup\{i\}\cup A)}\,R_{(\{c\}\cup\{h\}\cup C)\ (\{b\}\cup\{i\}\cup A)^*\ (\{b\}\cup\{j\}\cup A)\ (\{a\}\cup\{j\}\cup A)^*}.\end{aligned}$$

which is non zero only if the set $(\{c\}\cup\{h\}\cup C)\equiv(\{a\}\cup\{j\}\cup A)\equiv\{a\}\cup\mathcal{B}$. If $\binom{n-2}{k-1}>2$ we can certainly choose a $\mathcal{B}\neq\mathcal{A}$ such that this equality does not hold; thus $\binom{n-2}{k-1}>2$ implies that $\mathcal{M}$ is symmetric. This inequality always holds for $k>1$ and $n\geq 2k$.

REMARK. From the logic of the proof, it appears that $\mathcal{M}$ should be symmetric whenever $T_{\mathbb{C}}\mathcal{M}$ is isomorphic to a subrepresentation of $\otimes^k\mathcal{V}$ (with $k\geq 2$) *unless the argument fails for shortage of free index–values.* This more general statement, supplemented with a precise definition of what 'index–shortage' means, is equivalent to the full Berger theorem. It is very easy to follow the above argument to show, say, that the only genuine spin representation which can be the holonomy group of a Riemannian manifold is $Spin(7)$ in eight dimensions. Although it is hard to make the existing evidence into a complete proof, it is easy to convice oneself that, for the exceptional Lie groups G_2, F_4, E_6, E_7, E_8, whose fundamental representations can be written as subrepresentations of $\otimes^2\mathcal{V}$ only G_2 passes the index–shortage criterion.

1.1.2. *Proposition 1.2.* $\mathrm{Hol}(\mathcal{M})$ is a compact Lie group, and hence it must have the form

$$\mathrm{Hol}(\mathcal{M})=U(1)^k\times G_1\times G_2\times\cdots G_l,\tag{1.5}$$

with G_r non–Abelian *simple.* By de Rham's theorem, $T_{\mathbb{C}}(\mathcal{M})\big|_{(1,0)}$ is an *irreducible* representation of $\mathrm{Hol}(\mathcal{M})$, and hence it should be of the form

$$T_{\mathbb{C}\,\phi}\mathcal{M}\big|_{(1,0)}\simeq V_0\otimes V_1\otimes V_2\otimes\cdots\otimes V_l\tag{1.6}$$

where V_0 is an irreducible representation of the Abelian group $U(1)^k$ (with $\dim V_0=1$), while V_r is an irreducible representation of G_r. Since V_0 is one–dimensional, only one linear combination of the $\mathfrak{u}(1)$ generators may act non–trivially. Thus, $k=0,1$. On the other hand, for $r\geq 1$, either $\dim V_r\geq 2$ or V_r is the trivial representation. So we may assume $\dim V_r\geq 2$ for all $r\geq 1$.

$T_{\mathbb{C}}\mathcal{M}$ is either a *real* (irreducible) representation or a *complex* representation of the holonomy group. The third possibility allowed by the Frobenius–Schur (FS) criterion[1], namely a *quaternionic* representation, is ruled out since the holonomy group leaves invariant the metric, which is a *symmetric* pairing.

Consider first the *real* case. Then $k = 0$ and the V_r's are either real irreducible or quaternionic irreducible; moreover, by the FS criterion, the number of quaternionic V_r's should be *even.* The real non–trivial representations have dimension at least 3, since $SO(2) \simeq U(1)$ is Abelian and it is ruled out. By proposition 1.1**(1)**, if $\mathcal{M}$ is not–symmetric, there is *at most one* real irreducible representation, V_r, in the tensor product (1.6). On the other hand, if $T\mathcal{M}$ is isomorphic to the tensor product of a real irreducible representation V_r with the product of $2k \geq 2$ quaternionic ones, $V' \simeq \otimes_i^{2k} Q_i$, $\mathcal{M}$ should be symmetric, since $\dim V' \geq 2^{2k} \geq 4$ and $\dim V_r \geq 3$. Thus, in the real case, either $\mathrm{Hol}(\mathcal{M}) = G$ is a simple Lie group and $T\mathcal{M}$ an irreducible[2] real G–representation, or $T\mathcal{M}$ is a direct product of an even number of irreducible quaternionic representations. Now, by proposition 1.1**(1)**, at most one representation factor may have dimension ≥ 2. The product of two dimension 2 representations is a dimension 4 representation (for the product group), and hence we can have at most one pair of such representations. Taking into account that the total number of factors is even, we remain with only one possibility, namely $T\mathcal{M} = V_2 \otimes V_H$, where V_2 is a quaternionic representation of dimension 2 and V_H is a quaternionic irreducible representation of dimension $n/2$. Since $Sp(2)$ is the only Lie group having an irreducible quaternionic representation of degree 2, we conclude that

$$Sp(2) \times H \tag{1.7}$$

where H is a <u>simple</u> Lie group with an irreducible quaternionic representation V_H of dimension $n/2$.

Next consider the case in which $T\mathcal{M}$ is a complex (irreducible) representation. Arguing as above, proposition 1.1**(1)** leaves us four possibility for $\mathrm{Hol}(\mathcal{M})$:

(1) $\mathrm{Hol}(\mathcal{M}) = H$;
(2) $\mathrm{Hol}(\mathcal{M}) = U(1) \times H$;
(3) $\mathrm{Hol}(\mathcal{M}) = SU(2) \times H$;
(4) $\mathrm{Hol}(\mathcal{M}) = SU(2) \times U(1) \times H$;

where H a <u>simple</u> subgroup of $SU(n/2)$ acting *irreducibly* on $\mathbb{C}^{n/2}$ (the fundamental of $SU(n/2)$).

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[1] The *Frobenius–Schur criterion* ([**233**] **theorem III.5.1** and **theorem VII.9.8**; [**120**] **theorem 45.1**): An *irreducible* complex representation R of a compact Lie group G is REAL, COMPLEX, OR QUATERNIONIC according if

$$\int_G dg\, \mathrm{Tr}_R(g^2) = \begin{cases} 1 & R \text{ real} \\ 0 & R \text{ complex} \\ -1 & R \text{ quaternionic.} \end{cases}.$$

This also implies that the three cases are *mutually exclusive.*

[2] Irreducible in the complex sense!

In cases (3) and (4), one writes $T_{\mathbb{C}}\mathcal{M} \simeq V_2 \otimes W$, where V_2 is the fundamental representation of $SU(2)$ which is quaternionic and W is some representation (reducible in general) of H or $U(1) \times H$. The canonical real structure on the complex tangent bundle, combined with the quaternionic structure of V_2, gives a quaternionic structure on W. Since this quaternionic structure is preserved by the holonomy group, H or, respectively, $U(1) \times H$ should be contained in $Sp(n/2)$. In particular, case (4) is ruled out. □

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