Special Kähler metrics on ruled surfaces

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ICTP–SISSA Kähler seminar 11/03/2019

Let Σ be a Riemann surface, and let $L \to \Sigma$ be a holomorphic line bundle equipped with a Hermitian fibre metric h. The fibres of $L \to \Sigma$ are copies of $\mathbb{C}$; we can complete L to a fibre bundle with fibres $\mathbb{PC}^1$ by glueing another copy of Σ to the "infinity" of L. More precisely, consider the bundle $\mathcal{O} \oplus L \to \Sigma$, and consider the associated projective bundle $\mathbb{P}(\mathcal{O} \oplus L)$. The total space M of $\mathbb{P}(\mathcal{O} \oplus L) \to \Sigma$ is said to be a *ruled surface*, since the fibres are complex lines.

We propose a study of the cscK equation on the ruled surface $M = \mathbb{P}(\mathcal{O} \oplus L)$ using the *momentum construction* (see in particular [Szé06, chapter 5] and [HS02]), in the special case in which ω represents $c_1(L)$. There are much more powerful techniques to tackle the general case $\mathbb{P}(E) \to \Sigma$ of a rank 2 vector bundle over a complex surface. The general result is

Theorem 1.1. *There is a cscK metric in $\mathbb{P}(E)$ if and only if E is slope–polystable.*

Moreover, you can find explicit results about existence of extremal metrics on Hirzebruch–like ruled surfaces. General references are [ATF06], [ACGTF08], [RT06], [Gau]. A similar result in higher dimension can be found as a corollary in [RT06].

Let z be a holomorphic coordinate on Σ over which L trivializes, and let $\zeta = \zeta(z)$ be a corresponding linear coordinate on the fibres of L. The system of coordinates (z, ζ) on the total space of L is said to be a system of *(holomorphic) bundle–adapted coordinates* on $L \subset M$; the coordinates $(z, \eta) = (z, \zeta^{-1})$ give the rest of the holomorphic atlas for M.

For a fixed Kähler form ω_Σ on Σ, we consider Kähler forms on the total space of the bundle $M \to \Sigma$ that satisfy the following *Calabi ansatz*,

$$\omega_M = \pi^*\omega_\Sigma + \mathrm{i}\,\partial\bar{\partial} f(t) \tag{1.1}$$

where t is the logarithm of the fibrewise norm function, and f is a suitably convex real function. More explicitly, fix a system of bundle–adapted holomorphic coordinates (z, ζ) on M, let $a(z)$ denote the local representative of h on L, and let t be defined as

$$t := \log(a(z)\,\zeta\bar{\zeta}).$$

This is a well–defined function on the total space of L, and if f satisfies some convexity conditions $\mathrm{i}\partial\bar{\partial} f(t)$ is a (globally) well-defined real 2-form on the total space of L, that in some cases can be extended to M. Notice that $\partial_z \partial_{\bar{z}} t = -F(h)_{1\bar{1}}$.

Let's compute a coordinate expression for ω_M; notice that

$$\partial_z \partial_{\bar{z}} f(t) = \partial_z \left(f'(t)\,\partial_{\bar{z}} t\right) = -f'(t)\,F(h) + f''(t)\,\partial_z t\,\partial_{\bar{z}} t$$

so that

$$\omega_M = \pi^*\omega_\Sigma - f'(t)\,\pi^*F(h) + \mathrm{i}\,f''(t)\left(\partial_z t\,\partial_{\bar{z}} t\,\mathrm{d}z \wedge \mathrm{d}\bar{z} + \frac{\partial_z t}{\bar{\zeta}}\mathrm{d}z \wedge \mathrm{d}\bar{\zeta} + \frac{\partial_{\bar{z}} t}{\zeta}\mathrm{d}\zeta \wedge \mathrm{d}\bar{z} + \frac{1}{\zeta\bar{\zeta}}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\right). \tag{1.2}$$

We see that f'' must be strictly positive, and also $\omega_\Sigma - f'(t)\,F(h)$ must be positive. Moreover if we want (1.2) to define a metric on the total space of L, $f''(t)$ must be $O(\zeta\bar{\zeta})$ for $\zeta\bar{\zeta} \to 0$. However, these conditions are easier to interpret if we do a change of variables, taking the *Legendre transform* of f.

Rather than working with f and t, define τ to be the function $\tau = f'(t)$, and let F be the Legendre transform of f. If we define $\phi := \frac{1}{F''}$, then we have

$$\begin{aligned} \tau &= f'(t) \\ t &= F'(\tau) \\ F(\tau) + f(t) &= t\,\tau \\ f''(t) &= \phi(\tau) \end{aligned}$$

so that the metric $\omega_\phi := \omega_M$ is, with the notation of (1.2)

$$\omega_M = \pi^*\omega_\Sigma - \tau\,\pi^* F(h) + \mathrm{i}\,\phi(\tau)\left(\partial_z t\,\partial_{\bar z} t\,\mathrm{d}z\wedge\mathrm{d}\bar z + \frac{\partial_z t}{\bar\zeta}\mathrm{d}z\wedge\mathrm{d}\bar\zeta + \frac{\partial_{\bar z} t}{\zeta}\mathrm{d}\zeta\wedge\mathrm{d}\bar z + \frac{1}{\zeta\bar\zeta}\mathrm{d}\zeta\wedge\mathrm{d}\bar\zeta\right). \tag{1.3}$$

Definition 1.2. For a real number x, let $\omega_\Sigma(x) := \omega_\Sigma - x\,F(h)$. We say that an interval $I = [a, b]$ is a *compatible momentum interval* if $\omega_\Sigma(x)$ is a Kähler metric for every $x \in I$.

Proposition 1.3 ([HS02], see also [Szé14])**.** *Given a compatible momentum interval $I = [a, b]$, the form ω_ϕ defines a Kähler form on the total space of L if and only if*

1. *ϕ is positive in the interior of I;*
2. *$\phi(a) = 0$, $\phi'(a) = 1$.*

Moreover, it extends to a Kähler metric on the total space of M if and only if

3. *$\phi(b) = 0$, $\phi'(b) = -1$.*

So, let's assume that ϕ satisfies these conditions for a fixed momentum interval. We'll say that ϕ is a *momentum profile* for I. A quick computation shows that the volume form of ω_ϕ is

$$\frac{\omega_\phi\wedge\omega_\phi}{2} = \frac{\phi(\tau)}{\zeta\,\bar\zeta}\omega_\Sigma(\tau)\wedge(\mathrm{i}\,\mathrm{d}\zeta\wedge\mathrm{d}\bar\zeta)$$

where by $\omega_\Sigma(\tau)$ we mean

$$\omega_\Sigma(\tau) := \left(\pi^*\omega_\Sigma(x)\right)\Big|_{x=\tau}.$$

Lemma 1.4. *The volume of (M, ω_ϕ) is $\pi(b-a)\,(2\,\mathrm{Vol}(\Sigma) - (b+a)\mathrm{deg}(L))$.*

Proof. We have to integrate $\frac{1}{2}\omega_\phi^2$ on M; we can split this into a double integral, along the fibres and along the "horizontal sections" of $M \to \Sigma$.

$$\begin{aligned} \int_M \frac{\omega_\phi^2}{2} &= \int_M \frac{\phi(\tau)}{\zeta\,\bar\zeta}\left(\pi^*\omega_\Sigma - \tau\,\pi^*F(h)\right)\wedge(\mathrm{i}\,\mathrm{d}\zeta\wedge\mathrm{d}\bar\zeta) = \\ &= \int_\Sigma\left(\int_{\mathbb{C}}\frac{\phi(\tau)}{\zeta\,\bar\zeta}\mathrm{i}\,\mathrm{d}\zeta\mathrm{d}\bar\zeta\right)\omega_\Sigma - \int_\Sigma\left(\int_{\mathbb{C}}\frac{\tau\phi(\tau)}{\zeta\,\bar\zeta}\mathrm{i}\,\mathrm{d}\zeta\mathrm{d}\bar\zeta\right)F(h) \end{aligned}$$

To do these integrals, switch to circle coordinates on $\mathbb{C}$. Notice that, letting $r^2 = \zeta\bar\zeta$ we have $\mathrm{d}\tau = \frac{2\,\phi(\tau)}{r}$; moreover $\mathrm{i}\mathrm{d}\zeta\wedge\mathrm{d}\bar\zeta = 2\,r\,\mathrm{d}r\wedge\mathrm{d}\vartheta$. Then the volume is

$$\int_\Sigma\omega_\Sigma\int_0^{2\pi}\mathrm{d}\vartheta\int_a^b\mathrm{d}\tau - \int_\Sigma F(h)\int_0^{2\pi}\mathrm{d}\vartheta\int_a^b\tau\mathrm{d}\tau = \mathrm{Vol}(\Sigma)\,2\pi\,(b-a) - (2\pi\mathrm{deg}(L))\,2\pi\,\frac{b^2-a^2}{2}.$$

□

2 Constant scalar curvature.

Lemma 2.1. *The scalar curvature of* ω_ϕ *is*

$$S(\omega_\phi) = S(\omega_\Sigma(\tau)) + \frac{2\,\pi^* F(h)}{\omega_\Sigma(\tau)}\phi'(\tau) - \phi''(\tau).$$

Proof. The determinant of ω_ϕ is $g_\Sigma(\tau)_{1\bar{1}}\frac{\phi(\tau)}{\zeta\bar{\zeta}}$, so the Ricci form of ω_ϕ is

$$\rho(\phi) = -\mathrm{i}\partial\bar{\partial}\mathrm{log}\mathrm{det}\omega_\phi = -\mathrm{i}\partial\bar{\partial}\mathrm{log}\left(g_\Sigma(\tau)_{1\bar{1}}\right) - \mathrm{i}\partial\bar{\partial}\mathrm{log}\left(\phi(\tau)\right).$$

To compute the various pieces it will be convenient to fix a coordinate system z on Σ such that, at a fixed point p, $\partial_{\bar{z}}t = 0$. These coordinate system exist around every point, as is easy to check. Notice also that in these coordinates the metric ω_ϕ is diagonal.

Let's compute the various pieces.

$$\partial_z\partial_{\bar{z}}\mathrm{log}\left(g_{\Sigma 1\bar{1}} - \tau\, F(h)_{1\bar{1}}\right) = \partial_z\left(\frac{\partial_{\bar{z}} g_{\Sigma 1\bar{1}} - \tau\,\partial_{\bar{z}}F(h)_{1\bar{1}} - \phi(\tau)\,\partial_{\bar{z}} t\, F(h)_{1\bar{1}}}{g_{\Sigma 1\bar{1}} - \tau\, F(h)_{1\bar{1}}}\right) =$$

$$= \frac{\left(\partial_z\partial_{\bar{z}} g_{\Sigma 1\bar{1}} - \tau\,\partial_z\partial_{\bar{z}}F(h)_{1\bar{1}} + \phi(\tau)\,F(h)^2_{1\bar{1}}\right)\left(g_{\Sigma 1\bar{1}} - \tau\,F(h)_{1\bar{1}}\right) - \left(\partial_{\bar{z}} g_{\Sigma 1\bar{1}} - \tau\,\partial_{\bar{z}}F(h)_{1\bar{1}}\right)\left(\partial_z g_{\Sigma 1\bar{1}} - \tau\,\partial_z F(h)_{1\bar{1}}\right)}{g_\Sigma(\tau)^2_{1\bar{1}}}$$

$$= -\mathrm{Ric}(\omega_\Sigma(x))\Big|_{x=\tau} + \frac{F(h)^2}{g_\Sigma(\tau)}\phi(\tau)$$

$$\partial_z\partial_{\bar{z}}\mathrm{log}\,\phi(\tau) = \partial_z\left(\phi'(\tau)\partial_{\bar{z}} t\right) = -\phi'(\tau)\,F(h)$$

$$\begin{aligned}\partial_\zeta\partial_{\bar{\zeta}}\mathrm{log}\left(g_{\Sigma 1\bar{1}} - \tau\,F(h)_{1\bar{1}}\right) =& \partial_\zeta\left(\frac{-\phi(\tau)\,F(h)_{1\bar{1}}}{\bar{\zeta}\,g_\Sigma(\tau)}\right) = -\frac{F(h)_{1\bar{1}}}{\zeta\bar{\zeta}}\frac{\phi'(\tau)\,\phi(\tau)\,g_\Sigma(\tau) + \phi(\tau)^2 F(h)}{g_\Sigma(\tau)^2}\\ &= -\frac{\phi(\tau)}{\zeta\bar{\zeta}}F(h)_{1\bar{1}}\frac{\phi'(\tau)\,g_\Sigma(\tau) + \phi(\tau)\,F(h)}{g_\Sigma(\tau)^2}\\ \partial_\zeta\partial_{\bar{\zeta}}\mathrm{log}\,\phi(\tau) &= \frac{1}{\bar{\zeta}}\partial_\zeta\left(\phi'(\tau)\right) = \frac{\phi(\tau)}{\zeta\,\bar{\zeta}}\phi''(\tau)\end{aligned}$$

So the scalar curvature is

$$\begin{aligned}S(\omega_\phi) =& \frac{1}{g_\Sigma(\tau)}\left(\mathrm{Ric}(\omega_\Sigma(x))\Big|_{x=\tau} - \frac{F(h)^2}{g_\Sigma(\tau)}\phi(\tau) + \phi'(\tau)\,F(h)\right) +\\ &+ \frac{\zeta\,\bar{\zeta}}{\phi(\tau)}\left(\frac{\phi(\tau)}{\zeta\bar{\zeta}}F(h)_{1\bar{1}}\frac{\phi'(\tau)\,g_\Sigma(\tau) + \phi(\tau)\,F(h)}{g_\Sigma(\tau)^2} - \frac{\phi(\tau)}{\zeta\,\bar{\zeta}}\phi''(\tau)\right) =\\ =& S(g_\Sigma(\tau)) + \frac{2\,F(h)}{g_\Sigma(\tau)}\phi'(\tau) - \phi''(\tau).\end{aligned}$$

□

So we see that *if* $F(h) = \lambda\omega$ for some constant λ, the scalar curvature of ω_ϕ is a linear differential expression in $\phi(\tau)$:

$$S(\omega_\phi) = \frac{1}{1-\lambda\,\tau}S(\omega_\Sigma) + \frac{2\,\lambda}{1-\lambda\,\tau}\phi'(\tau) - \phi''(\tau)$$

well, we should actually fix the background metric ω_Σ on Σ. We'll assume that ω_Σ is a cscK metric, $S(\omega_\Sigma) = \hat{s}$; we may also assume that the volume of Σ with this metric is 2π, so that $\hat{s} = \chi(\Sigma)$ (by Gauss–Bonnet) and $\lambda = \deg(L)$. Hence we find

$$S(\omega_\phi) = \frac{\hat{s}}{1-\lambda\,\tau} + \frac{2\,\lambda}{1-\lambda\,\tau}\phi'(\tau) - \phi''(\tau) = \frac{\chi(\Sigma)}{1-\lambda\,\tau} - \frac{1}{1-\lambda\,\tau}\left[(1-\lambda\,\tau)\phi(\tau)\right]'' \tag{2.1}$$

and

$$\mathrm{Vol}(M) = \pi(b-a)\left(2-\lambda(b+a)\right)\mathrm{Vol}(\Sigma)$$

Lemma 2.2. *The average of $S(\omega_\phi)$ is*

$$\widehat{S(\omega_\phi)} = \frac{2\,\chi(\Sigma)}{2-\lambda(b+a)} + \frac{2}{b-a}.$$

Proof. We just have to integrate equation (2.1). We'll use the same change of coordinates used to compute the volume of (M, ω_ϕ).

$$\begin{aligned}\int_M S(\omega_\phi)\frac{\omega_\phi^2}{2} &= \int_\Sigma \omega_\Sigma \int_{\mathbb{C}} \left(\frac{\chi(\Sigma)}{1-\lambda\tau} - \frac{1}{1-\lambda\tau}\left[(1-\lambda\tau)\phi(\tau)\right]''\right)\frac{2\phi}{r}(1-\lambda\tau)\mathrm{d}r\mathrm{d}\vartheta = \\ &= \int_\Sigma \omega_\Sigma \int_0^{2\pi}\mathrm{d}\vartheta \int_a^b \left(\chi(\Sigma) - \left[(1-\lambda\tau)\phi(\tau)\right]''\right)\mathrm{d}\tau = \mathrm{Vol}(\Sigma)\,2\pi\left[\chi(\Sigma)\,\tau - (1-\lambda\tau)\phi'(\tau) + \lambda\,\phi(\tau)\right]_a^b = \\ &=\mathrm{Vol}(\Sigma)\,2\pi\left(\chi(\Sigma)(b-a) + (1-\lambda\,b) + (1-\lambda\,a)\right) = \mathrm{Vol}(\Sigma)\,2\pi\left(\chi(\Sigma)(b-a) + 2 - \lambda(b+a)\right).\end{aligned}$$

So the average is

$$\widehat{S(\omega_\phi)} = \frac{2\left(\chi(\Sigma)(b-a) + 2 - \lambda(b+a)\right)}{(b-a)\left(2-\lambda(b+a)\right)} = \frac{2\,\chi(\Sigma)}{2-\lambda(b+a)} + \frac{2}{b-a}.$$

□

The constant scalar curvature equation for ω_ϕ is

$$\frac{\chi(\Sigma)}{1-\lambda\tau} - \frac{1}{1-\lambda\tau}\left[(1-\lambda\tau)\phi(\tau)\right]'' = \frac{2\,\chi(\Sigma)}{2-\lambda(b+a)} + \frac{2}{b-a}. \tag{2.2}$$

Equation (2.2) can be explicitly integrated:

$$\phi(\tau) = \frac{\chi(\Sigma)\,\tau^2}{2(1-\lambda\tau)} - \left(\frac{2\,\chi(\Sigma)}{2-\lambda(b+a)} + \frac{2}{b-a}\right)\frac{\tau^2}{2(1-\lambda\tau)}\left(1-\frac{\lambda}{3}\tau\right) + \frac{A\,\tau + B}{1-\lambda\tau} \tag{2.3}$$

and now we should impose the boundary conditions to verify if this solution gives a cscK metric on the whole space of M.

Before going on with the proof, we should compute the Kähler class in which we want to solve the cscK equation, since we want to keep it fixed. This will cut down some freedom in the choice of parameters. The space $H^2(M,\mathbb{R})$ is generated by the Poincaré duals of a fibre of $M \to \Sigma$ and of the infinity section Σ_∞. Hence we can write $[\omega_\phi] = k_1 F^* + k_2 \Sigma_\infty^*$ for some coefficients. Since the intersection formulas for F and Σ_∞ are

$$F\cdot F = 0 \quad F\cdot\Sigma_\infty = 1 \quad \Sigma_\infty\cdot\Sigma_\infty = -\deg(L) = -\lambda$$

then we find (by integration of ω_ϕ on F and Σ_∞)

$$[\omega_\phi] = 2\pi\left((1-\lambda\,a)F^* + (b-a)\Sigma^*\right).$$

Hence to fix the Kähler class of ω_ϕ we should fix the product $\lambda\,a$ and the width of the momentum interval $[a, b]$, remembering that $1-\lambda\,b$ and $1-\lambda\,a$ should be positive numbers.

Flat line bundle. First, let's study the case in which L has zero degree, i.e. $\lambda = 0$ in equation (2.3). The equation becomes

$$\phi(\tau) = \frac{\chi(\Sigma)}{2}\,\tau^2 - \left(\chi(\Sigma) + \frac{2}{b-a}\right)\frac{\tau^2}{2} + A\,\tau + B = -\frac{\tau^2}{b-a} + A\,\tau + B$$

and the unique choice of A, B that satisfies the boundary conditions gives

$$\phi(\tau) = -\frac{(\tau-b)(\tau-a)}{b-a}.$$

General case. Assume now that $\lambda \neq 0$. It will be more convenient to write $b = a + m$ for a positive number m (the length of the momentum interval). Solving the linear system of conditions, we find that the boundary conditions are satisfied precisely when

$$\begin{aligned} A =& \frac{-4\lambda^2 m^2 + m^2\chi^2 - 16m\chi + 48}{16\lambda m} \\ B =& \frac{4\lambda^2 m^2 - m^2\chi^2 + 8m\chi - 16}{16\lambda^2 m} \\ a =& \frac{-2\lambda m - m\chi + 4}{4\lambda} \end{aligned}$$

and *cannot* be satisfied if $2\,\lambda + \chi = 0$. Notice in particular that from the last identity we get

$$1 - \lambda\, a > 0 \iff 1 - \frac{-2\lambda m - m\chi + 4}{4} > 0 \iff m(2\,\lambda + \chi) > 0.$$

However, since we are assuming that $[a, b]$ is a compatible momentum interval we should also check that $1 - \lambda\, b > 0$, i.e. that $1 - \lambda\, a - \lambda\, m > 0$. Unfortunately, this last condition is equivalent to

$$-\frac{m}{4}(2\,\lambda - \chi) > 0$$

i.e. $2\,\lambda - \chi < 0$. Since all these quantities are integers, we conclude that *it is never possible to find a cscK metric, unless L is flat.*

To summarize, we have Table 1.

	L positive	L flat	L negative
$g(\Sigma) = 0$	no	yes	no
$g(\Sigma) = 1$	no	yes	no
$g(\Sigma) > 1$	no	yes	no

Table 1: The possible cases.

3 Extremal metrics.

For extremal metrics, Table 1 has a much more positive outlook. Recall that a metric is extremal if and only if the gradient of $S(\omega)$ is holomorphic. So, what is the gradient of $S(\omega_\phi)$?

A quick computation using (2.1) shows that

$$\mathrm{grad}^{1,0} S(\omega_\phi) = \partial_\tau S(\omega_\phi)\, \zeta \partial_\zeta$$

so the gradient is holomorphic if and only if $S(\omega_\phi)'' = 0$. In other words, there must be two constants A, B such that

$$S(\omega_\phi) = A\,\tau + B$$

substituting the expression for $S(\omega_\phi)$ that we found in equation (2.1) and integrating, we find that ϕ must be of the form

$$\phi(\tau) = \frac{\tau^2}{12(1 - \lambda\,\tau)}\,(6\,\chi - A\,\tau(1 - \lambda\,\tau) - 2\,B(3 - \lambda\,\tau)) + \frac{D + C\,\tau}{1 - \lambda\,\tau}$$

for constants A, B, C, D. Their value can be found by imposing the boundary conditions.

It will be notationally more convenient to assume $\lambda < 0$ and $[a, b] = [0, m]$ for some positive m. This, together with the boundary conditions, give us

$$\phi(\tau) = \frac{\tau(m - \tau)}{2m(\lambda m(\lambda m - 6) + 6)(1 - \lambda\tau)} \left(\lambda \left(\lambda m^2(\tau\chi + 2) - m(\lambda\tau(\tau(2\lambda + \chi) - 8) + 12) + 4\tau(\lambda\tau - 3)\right) + 12\right).$$

The only thing we should check to see that ϕ defines an extremal metric is that its sign should be positive in $]0,m[$. The fractional term is always positive, so we just have to study the sign of

$$k(m,t,\lambda,g) := \lambda\left(\lambda m^2(\tau\chi+2) - m(\lambda\tau(\tau(2\lambda+\chi)-8)+12) + 4\tau(\lambda\tau-3)\right) + 12$$

Lemma 3.1. *For $g(\Sigma)=0,1$, this quantity is positive in $[0,m]$.*

Proof. For the sphere, the quantity becomes

$$2\lambda^2m^2 + 2\lambda^2mt(m-t) - 12\lambda m - 2\lambda^3mt^2 + 8\lambda^2mt + 4\lambda^2t^2 - 12\lambda t + 12$$

and it is a sum of positive terms. For the torus, it is

$$2\lambda^2m^2 - 12\lambda m - 2\lambda^3mt^2 + 8\lambda^2mt + 4\lambda^2t^2 - 12\lambda t + 12$$

which is again a sum of positive terms. □

For genus greater than 1, write $k(m,t,\lambda,g)$ as a polynomial in m. We get

$$\left(4\lambda^2t^2 - 12\lambda t + 12\right) + m\left(2g\lambda^2t^2 - 12\lambda - 2\lambda^3t^2 - 2\lambda^2t^2 + 8\lambda^2t\right) + m^2\lambda^2\left(-2gt+2+2t\right)$$

notice that the only possibly nonpositive terms appear in the m^2 coefficient. So, we find at least that *for m small enough, we can always find an extremal metric.* The whole picture is somewhat more complicated, but for now we leave this to future discussion.

So, let's update Table 1 to the situation for extremal metrics.

	L positive	L flat	L negative
$g(\Sigma)=0$	yes	yes	yes
$g(\Sigma)=1$	yes	yes	yes
$g(\Sigma)>1$	for small enough m	yes	for small enough m

Table 2: Existence of extremal metrics.

4 Futaki invariant computations

We can also make some explicit computations for the Futaki invariant of ω_ϕ, to check again that there are no cscK metrics on our Hirzebruch–like ruled surfaces.

Recall the definition (actually, one of the many) of the Futaki invariant: for a holomorphic vector field X having a *holomorphy potential* $f \in \mathcal{C}^\infty$, i.e. $X = \mathrm{igrad}^{1,0}(f)$, the Futaki character of X is

$$\mathcal{F}(X) = \int f\left(S(\omega) - \widehat{S(\omega)}\right)\frac{\omega^n}{n!}.$$

A theorem of Calabi, generalizing a previous result of Futaki in the contest of KE metrics, is that $\mathcal{F}(X)$ *does not* depend upon the choice of ω in its Kähler class. Therefore, if there is a cscK metric in $[\omega]$ the character $\mathcal{F}$ must vanish identically.

For this fact to make sense one should also note that the space of holomorphic vector fields with a holomorphy potential does not depend on the Kähler form, but just on the complex structure; indeed, it can be characterized as the space of holomorphic vector fields that have a zero somewhere.

On our ruled surfaces we have a natural holomorphic vector field to consider, the generator of the $\mathbb{C}^*$–action on the fibres of $\mathbb{P}(L\oplus\mathcal{O})\to\Sigma$. It turns out that in local bundle–adapted coordinates

$X = \zeta\partial_\zeta$, and the holomorphy potential of X with respect to ω_ϕ is the function τ itself. So, to compute its Futaki character we have to compute the integral

$$\begin{aligned}\mathcal{F}(X) =& \int_M \tau \left(\frac{\chi}{1-\lambda\tau} - \frac{1}{1-\lambda\tau}[(1-\lambda\tau)\phi(\tau)]'' \right)(1-\lambda\tau)\frac{\phi(\tau)}{\zeta\bar{\zeta}}\pi^*\omega_\Sigma \wedge (\mathrm{i}d\zeta \wedge d\bar{\zeta}) = \\ =& 2\pi\,\mathrm{Vol}(\Sigma)\int_a^b \left(\chi\tau - \tau[(1-\lambda\tau)\phi(\tau)]'' - \widehat{S(\omega_\phi)}(\tau-\lambda\tau^2)\right)\mathrm{d}\tau = \\ =& 2\pi\,\mathrm{Vol}(\Sigma)\left[\frac{\chi}{2}\tau^2 - \tau[(1-\lambda\tau)\phi]' + (1-\lambda\tau)\phi - \widehat{S(\omega_\phi)}\left(\frac{1}{2}\lambda^2 - \frac{\lambda}{3}\tau^3\right)\right]_a^b.\end{aligned}$$

After some simplifications we can find

$$\mathcal{F}(X) = \frac{1}{6}\lambda(b-a)^2\left(\frac{\chi(b-a)}{2-\lambda(a+b)} - 2\right).$$

And we see immediately that $\mathcal{F}(X) = 0$ if L is a flat bundle, i.e. $\lambda = 0$. Otherwise, since $[a,b]$ is a compatible momentum interval $\frac{b-a}{2-\lambda(a+b)}$ is always positive. Hence $\mathcal{F}(X)$ is different from 0 if $g(\Sigma) = 1$ or $g(\Sigma) > 1$, since in these two cases the term in parentheses is strictly negative, confirming the fact that there are no cscK metrics in these cases.

The other possibility is that $g(\Sigma) = 0$, i.e. Σ is the sphere. In this case the Futaki invariant vanishes if and only if

$$\frac{b-a}{2-\lambda(a+b)} = 1 \iff \lambda = -1 + \frac{2+2a}{a+b} \iff (\lambda+1)(a+b) = 2+2a.$$

Assume that $\lambda > 0$. Then the compatibility conditions are $a, b < \lambda^{-1}$, so we have

$$(\lambda+1)(a+b) - 2a = (\lambda-1)a + (\lambda+1)b < \frac{\lambda-1}{\lambda} + \frac{\lambda+1}{\lambda} < 2$$

and so $\mathcal{F}(X) \neq 0$. So also in this case we can show that there are no cscK metrics in the class $[\omega_\phi]$.

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