SISSA - Universita' di Trieste

Corso di Laurea Magistrale in Matematica
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Istituzioni di Fisica Matematica A
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Linear partial differential equations of mathematical physics

Program:

1. Linear partial differential operators:
Definitions and main examples. - Principal symbol of a linear differential operator. - Change of independent variables. - Canonical form of linear differential operators of order 1 and of order 2, with constant coeffcients. - Characteristics. Elliptic and hyperbolic operators. - Reduction to a canonical form of second order linear differential operators in a two-dimensional space. Parabolic operators. - General solution of a second order hyperbolic equation with constant coefficients in the two-dimensional space.

2. Wave equation:
Vibrating string. - Cauchy problem. D'Alembert formula. - Some consequences of the D'Alembert formula. - Semi-infinite vibrating string. - Periodic problem for wave equation. - Introduction to Fourier series. - Finite vibrating string. Standing waves. - Energy of vibrating string. - Solutions in dimension 2 and 3. - Solutions of the inhomogeneous problem.

3. Laplace equation:
Ill-posedness of Cauchy problem for Laplace equation. - Dirichlet and Neumann problems for Laplace equation on the plane. - Properties of harmonic functions: mean value theorem, the maximum principle. - Harmonic functions on the plane and complex analysis.

4. Heat equation:
Derivation of heat equation. - Main boundary value problems for heat equation. - Fourier transform. - Solution of the Cauchy problem for the heat equation on the line. - Mixed boundary value problems for the heat equation. - More general boundary conditions. - Solution of the inhomogeneous heat equation.

5. Statement of the Cauchy-Kowalewska theorem. Abstract Cauchy problem. One-parameter evolution semigroups.

6. Notes on Schroedinger equation, Maxwell equation and Dirac equation.

The following Lecture Notes consist essentially of somewhat modified and corrected Sections 2-5 of the Lecture Notes by Boris Dubrovin http://people.sissa.it/~dubrovin/bd_courses.html, with the addition of the subsections 3.8 and 3.9 (in italian) based on ch. 2.4.1 and 2.4.2 in L.C. Evans, Partial differential equations, Providence, AMS, 1998, section 5 based on ch. 1 of H.O. Fattorini, The Cauchy problem (Enc. Math. Appl. vol 18) Addison-Wesley, 1983 and section 6 for which the suggested reference is W. Thirring, A course in mathematical physics, vol. 3, Springer, 1981.

1 Introduction

Le equazioni alle derivate parziali (PDE's) legano una (o più) funzioni incognite di alcune variabili e le loro derivate parziali. Le PDE's servono tipicamente per formulare e risolvere diversi problemi in fisica, tra cui la propagazione del suono, calore, elasticità, elettrostatica, elettrodinamica (da raggi X, luce, microonde, onde radio, cellulari etc.), aerodinamica, fluidi, meccanica quantistica, biologia (crescita di popolazioni, cellule, etc.), mercati finanziari ('opzioni'), e altro. La stessa equazione, ovvero 'evoluzione', può descrivere fenomeni diversi in diversi campi di applicazione.

2 Linear differential operators

2.1 Definitions and main examples

The Euclidean coordinates on $\mathbb{R}^d$ will be denoted $x_1, \dots, x_d$. We use the notation

$$x \cdot y = x_1 y_1 + \dots + x_d \cdot y_d, \quad x, y \in \mathbb{R}^d$$

for the usual pairing $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. We shall sometimes write $x^2 := x \cdot x$.

Let $\Omega \subset \mathbb{R}^d$ be an open subset. Denote $\mathcal{C}^\infty(\Omega)$ the set of all infinitely differentiable complex valued smooth functions on Ω. We will use short notations for the derivatives

$$\partial_k = \frac{\partial}{\partial x_k}.$$

For $f \in \mathcal{C}^\infty(\Omega)$, f_x or ∇f will denote the gradient of f

$$f_x = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d} \right).$$

For a multiindex

$$\mathbf{p} = (p_1, \dots, p_d)$$

denote

$$\begin{aligned} |\mathbf{p}| &= p_1 + \dots + p_d \\ \mathbf{p}! &= p_1! \dots p_d! \\ \mathbf{x}^{\mathbf{p}} &= x_1^{p_1} \dots x_d^{p_d} \\ \partial^{\mathbf{p}} &= \partial_1^{p_1} \dots \partial_d^{p_d}. \end{aligned}$$

The derivatives define linear operators

$$\partial^{\mathbf{p}} : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega), \quad u \mapsto \partial^{\mathbf{p}} u = \frac{\partial^{|\mathbf{p}|} u}{\partial x_1^{p_1} \dots \partial x_d^{p_d}}.$$

More generally, we will consider *linear differential operators* of the form

$$\begin{aligned} A &= \sum_{|\mathbf{p}| \le m} a_{\mathbf{p}}(x) \partial^{\mathbf{p}}, \quad \text{where} \quad a_{\mathbf{p}} \in \mathcal{C}^\infty(\Omega), \\ A &: \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega), \quad u \mapsto Au. \end{aligned} \tag{2.1.1}$$

We will define the *order* of the linear differential operator by

$$\operatorname{ord} A = \max |\mathbf{p}| \quad \text{such that} \quad a_{\mathbf{p}}(x) \neq 0. \tag{2.1.2}$$

We remark that even more generally one considers operators A whose action on u depends also on u and its derivatives up to the order m and calls them *nonlinear*. If the depndence is only on the derivatives of u up to the order $m-1$, they are called *quasilinear*.

Main examples of A (with constant coefficients) are

1. Laplace operator in $\mathbb{R}^d$

$$\Delta = \partial_1^2 + \dots + \partial_d^2 \tag{2.1.3}$$

2. Heat operator in $\mathbb{R} \times \mathbb{R}^d$

$$\frac{\partial}{\partial t} - \Delta \tag{2.1.4}$$

3. Wave operator in $\mathbb{R} \times \mathbb{R}^d$

$$\frac{\partial^2}{\partial t^2} - \Delta. \tag{2.1.5}$$

4. Schrödinger operator in $\mathbb{R} \times \mathbb{R}^d$

$$i\,\frac{\partial}{\partial t} + \Delta. \tag{2.1.6}$$

5. Dirac operator in $\mathbb{R}^d$

$$-i\sum_j \gamma_j\,\partial_j, \tag{2.1.7}$$

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where $\gamma_j \in \mathrm{Mat}_{2^{[d/2]}}$ satisfy

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}.$$

Here we used coordinates $x = (x_1, \dots, x_d)$ in $\mathbb{R}^d$ and $x = (t, x)$ in $\mathbb{R} \times \mathbb{R}^d$.

2.2 Principal symbol of a linear differential operator

We can associate with A a form which is useful for its study. *Symbol* (called also *total symbol*) of a linear differential operator (2.1.1) is a function $a \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^d)$ given by

$$a(x,\xi) = \sum_{|\mathbf{p}|\le m} i^{|\mathbf{p}|} a_{\mathbf{p}}(x)\xi^{\mathbf{p}}, \quad x \in \Omega \subset \mathbb{R}^d, \quad \xi \in \mathbb{R}^d. \tag{2.2.1}$$

If the order of the operator is equal to m then the *principal symbol* is defined by

$$a_m(x,\xi) = \sum_{|\mathbf{p}|=m} i^{|\mathbf{p}|} a_{\mathbf{p}}(x)\xi^{\mathbf{p}}. \tag{2.2.2}$$

Here $\xi^{\mathbf{p}} = \xi_1^{p_1} \dots \xi_d^{p_d}$ and the symbol (2.2.1), (2.2.2) is a polynomial in d variables $\xi_1, \dots, \xi_d$ with coefficients being smooth functions on Ω.

For the above examples we have the following symbols

1. For the Laplace operator Δ the symbol and principal symbol coincide

$$a = a_2 = -(\xi_1^2 + \dots + \xi_d^2) \equiv -\xi^2.$$

2. For the heat equation

$$a = i\,\tau + \xi^2, \quad a_2 = \xi^2.$$

3. For the wave operator again the total and principal symbol coincide

$$a = a_2 = -\tau^2 + \xi^2.$$

4. For the Schrödinger operator

$$a = -(\tau + \xi^2), \quad a_2 = -\xi^2.$$

5. For the Dirac operator

$$a = a_1 = \sum_j \gamma_j \xi_j.$$

Lemma 2.2.1. *1 Formula for the symbol of a linear differential operator*

$$a(x,\xi) = e^{-i\,x\cdot\xi} A\left(e^{i\,x\cdot\xi}\right). \tag{2.2.3}$$

Proof: Each ∂_j applied to $e^{i\,x\cdot\xi}$ produces $e^{i\,x\cdot\xi}$ (which cancels $e^{-i\,x\cdot\xi}$) times $i\xi_j$. □

We will denote $a(\xi)$ the symbol of a linear differential operator A with *constant* coefficients (it does not depend on x).

Corollary 2.2.2. *For a linear differential operator with constant coefficients the exponential function*

$$u(x) = e^{i\,x\cdot\xi}$$

is a solution to the linear differential equation

$$A\,u = 0$$

iff *the vector ξ satisfies*

$$a(\xi) = 0.$$

Proof: Notice that $e^{i\,x\cdot\xi}$ is invertible and use (2.2.3). □

Lemma 2.2.3. *Let $u(x)$, $S(x)$ be a pair of smooth functions and A a linear differential operator of order m. Then*
i). the expression of the form

$$e^{-i\,\lambda\,S(x)}A\left(u(x)e^{i\,\lambda\,S(x)}\right)$$

is a polynomial in λ of degree m;
ii) the leading coefficient of this polynomial has the following expression

$$\boxed{e^{-i\,\lambda\,S(x)}A\left(u(x)e^{i\,\lambda\,S(x)}\right) = u(x)a_m(x, S_x(x))\lambda^m + O(\lambda^{m-1}).} \tag{2.2.4}$$

Here S_x is the gradient of the function $S(x)$.

Proof: (Hint) The top power m in λ appears only when all the derivatives act on $e^{i\,\lambda\,S(x)}$; thus they do not act on $u(x)$, which can be simply shifted in front of the l.h.s. Next, proceed similarly as in the proof of Lemma (2.2.1), except that each ∂_j produces $\partial_j S$ instead of ξ_j. □

Exercise 2.2.4. *Let A and B be two linear differential operators of orders k and l with the principal symbols $a_k(x,\xi)$ and $b_l(x,\xi)$ respectively. Prove that the superposition $C = A\circ B$ is a linear differential operator of order $\leq k+l$. Prove that the principal symbol of C is equal to*

$$c_{k+l}(x,\xi) = a_k(x,\xi)\,b_l(x,\xi) \tag{2.2.5}$$

in the case $\operatorname{ord} C = \operatorname{ord} A + \operatorname{ord} B$. *In the case of strict inequality* $\operatorname{ord} C < \operatorname{ord} A + \operatorname{ord} B$ *prove that the product* (2.2.5) *of principal symbols is identically equal to zero.*

The formula for computing the full symbol of the product of two linear differential operators is more complicated. We will give here the formula for the particular case of one spatial variable x.

Exercise 2.2.5. *Let $a(x,\xi)$ and $b(x,\xi)$ be the symbols of two linear differential operators A and B with one variable. Prove that the symbol of the superposition $A\circ B$ is equal to*

$$a\star b = \sum_{k\geq 0}\frac{(-i)^k}{k!}\,\partial_\xi^k a\;\partial_x^k b. \tag{2.2.6}$$

2.3 Change of independent variables

Let us now analyze the transformation rules of the principal symbol $a(x, \xi)$ of an operator A under smooth invertible changes of variables

$$y_i = y_i(x), \quad i = 1, \ldots, d. \tag{2.3.1}$$

Recall that the first derivatives transform according to the chain rule

$$\frac{\partial}{\partial x_i} = \sum_{k=1}^{d} \frac{\partial y_k}{\partial x_i} \frac{\partial}{\partial y_k}. \tag{2.3.2}$$

The transformation law of higher order derivatives is more complicated. For example, applying the derivative to $f(y(x))$ and using the Leibniz formula for ∂_i and (2.3.1)

$$\frac{\partial^2}{\partial x_i \partial x_j} = \sum_{k,l=1}^{d} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} \frac{\partial^2}{\partial y_k \partial y_l} + \sum_{k=1}^{d} \frac{\partial^2 y_k}{\partial x_i \partial x_j} \frac{\partial}{\partial y_k}.$$

Similarly for other terms. However it is clear that after the transformation one obtains again a linear differential operator of the same order m. More precisely define the operator

$$\tilde{A} = \sum (-i)^{|\mathbf{p}|}\ \tilde{a}_{\mathbf{p}}(y) \frac{\partial^{|\mathbf{p}|}}{\partial y_1^{p_1} \ldots \partial y_d^{p_d}}$$

by the equation

$$\left(\tilde{A}\,\tilde{f}\right) \circ y = \widetilde{Af} \circ y = Au,$$

where

$$\tilde{f} \circ y = f,$$

i.e. $\forall x \in \Omega$

$$(\tilde{f} \circ y)\,(x) = \tilde{f}(y(x)) = \tilde{f}(y)|_{y=y(x)} := f(x)$$

(depending on the preferred notation).

Warning: the coefficients $\tilde{a}_{\mathbf{p}}$ are usually different from $\widetilde{a_{\mathbf{p}}}$, i.e. the coefficients $a_{\mathbf{p}}$ in coordinates y !

In general the transformation law of the symbol may be complicated, but that of the principal symbol is quite simple, as it follows from the following

Proposition 2.3.1. *Let $a_m(x, \xi)$ be the principal symbol of a linear differential operator A. Denote $\tilde{a}_m(y, \tilde{\xi})$ the principal symbol of the same operator written in the coordinates y, i.e. the principal symbol of the operator $\tilde{A}$. Then*

$$\tilde{a}_m(y(x), \tilde{\xi}) = a_m(x, \xi) \quad \text{provided} \quad \xi_i = \sum_{k=1}^{d} \frac{\partial y_k}{\partial x_i} \tilde{\xi}_k, \quad \text{i.e.} \quad \tilde{\xi}_k = \sum_{i=1}^{d} \frac{\partial x_i}{\partial y_k} \xi_i. \tag{2.3.3}$$

Proof: 1) (Brute force) To simplify the notation denote $m = |p|$ and omit summation over the repeated indices. We compute

$$\frac{\partial^m}{\partial x_1^{p_1} \ldots \partial x_d^{p_d}} = \frac{\partial y_{k_1}}{\partial x_1} \cdots \frac{\partial y_{k_m}}{\partial x_d} \frac{\partial^m}{\partial y_{k_1} \ldots \partial y_{k_m}} + \text{lower order terms.}$$

The product of the first m factors on the right side is

$$\frac{\partial y_{k_1}}{\partial x_1} \cdots \frac{\partial y_{k_{p_1}}}{\partial x_1} \frac{\partial y_{k_{p_1+1}}}{\partial x_2} \cdots \frac{\partial y_{k_{p_1+p_2}}}{\partial x_2} \quad \cdots \quad \frac{\partial y_{k_{p_1+\cdots+p_{d-1}+1}}}{\partial x_d} \cdots \frac{\partial y_{k_m}}{\partial x_d}$$

while

$$\frac{\partial^m}{\partial y_{k_1}\dots\partial y_{k_m}} = (\partial_{y_{k_1}}\dots\partial_{y_{k_{p_1}}})(\partial_{y_{k_{p_1+1}}}\dots\partial_{y_{k_{p_1+p_2}}})\dots(\partial y_{k_{p_1+\dots+p_{d-1}+1}}\dots\partial_{y_{k_m}})$$

and thus the principal symbol can be written as

$$i^m(\frac{\partial y}{\partial x_1}\cdot\tilde{\xi})^{p_1}\dots(\frac{\partial y}{\partial x_d}\cdot\tilde{\xi})^{p_d} = \xi_1^{p_1}\dots\xi_d^{p_d}$$

(The functions $\frac{\partial y}{\partial x_j}$ should be considered as functions of y, i.e. evaluated at $x = x(y)$ via the inverse coordinate trnsformation). Taking into account the coefficients (and their transformation law) the statement follows.

2). Or use Lemma 2.2.3. Applying the formula (2.2.4) one easily derives the equality

$$a_m(x, S_x) = \tilde{a}_m(y, \tilde{S}_y)|_{y=y(x)},$$

where S_x and $\tilde{S}_y$ are gradients in coordinates x and y, respectively. Applying the chain rule

$$\frac{\partial S}{\partial x_i} = \sum_{k=1}^{d}\frac{\partial y_k}{\partial x_i}\frac{\partial\tilde{S}}{\partial y_k}$$

we arrive at the transformation rule (2.3.3) for the particular case

$$\xi_i = \frac{\partial S}{\partial x_i}, \quad \tilde{\xi}_k = \frac{\partial\tilde{S}}{\partial y_k}.$$

This proves the proposition since the gradients (at any fixed point) can take arbitrary values.
□

Mini-exercise 2.3.2. *Check some simplest cases, e.g. $\Omega = \mathbb{R}_+$ $(d = 1)$ and $y(x) = x^2$, and others.*

For a linear transformation

$$y_k = \sum_j C_{kj}x_j \tag{2.3.4}$$

we need C to be invertible, that is

$$0 \neq \det\{\frac{\partial y_k}{\partial x_j}\} = \det\{C_{kj}\}$$

in order y to be smooth coordinates. In such a case

$$\xi_j = \sum_k C_{kj}\tilde{\xi}_k \tag{2.3.5}$$

is linear. Vice versa, any linear transformation of ξ is induced by some linear transformation (2.3.4).

2.4 Canonical form of linear differential operators of order ≤ 2 with constant coefficients

Consider a first order linear differential operator

$$A = a_1\frac{\partial}{\partial x_1} + \dots + a_d\frac{\partial}{\partial x_d} \tag{2.4.1}$$

with constant coefficients $a_1, \ldots, a_d$. One can find a linear transformation (2.3.4) of the coordinates and (2.3.5) of $\xi's$ that maps the vector $a = (a_1, \ldots, a_d)$ to the unit coordinate vector $(0, \ldots, 0, 1)$ of the axis y_d (for example take C with $C_{dj} = \frac{a_j}{a \cdot a}$ and, for any fixed $i = 1, \ldots, d-1$, C_{ij} orthogonal to a_j, i.e. $\sum C_{ij} a_j = 0$). After such a transformation the operator A becomes the partial derivative operator (we skip $\tilde{}$ over A)

$$A = \frac{\partial}{\partial y_d}.$$

Lemma 2.4.1. *The general solution of the first order linear differential equation*

$$A\varphi = 0$$

can be written in the form

$$\varphi(y_1, \ldots, y_d) = \varphi_0(y_1, \ldots, y_{d-1}). \tag{2.4.2}$$

Here φ_0 is an arbitrary smooth function of $(d-1)$ variables (functions of class at least $\mathcal{C}^1$ can be also considered).

Corollary 2.4.2. *The general solution to*

$$A\varphi + b\varphi = 0 \tag{2.4.3}$$

with A of the form (2.4.1) *and a constant b reads*

$$\varphi(y_1, \ldots, y_d) = \varphi_0(y_1, \ldots, y_{d-1}) e^{-b\, y_d}$$

for an arbitrary smooth (or at least $\mathcal{C}^1$) function $\varphi_0(y_1, \ldots, y_{d-1})$.

Proof: Clearly, by Leibniz and (2.4.3)

$$A(\varphi e^{b y_d}) = -b\varphi e^{b y_d} + \varphi b e^{b y_d} = 0,$$

hence $\varphi e^{b y_d} =: \varphi_0$ must be independent of y_d, and so $\varphi = \varphi_0 e^{-b y_d}$. □

Consider now a second order linear differential operator of the form

$$A = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c \tag{2.4.4}$$

with constant coefficients. Without loss of generality one can assume the coefficient matrix a_{ij} to be symmetric. Denote

$$Q(\xi) = -a_2(\xi) = \sum_{i,j=1}^{d} a_{ij} \xi_i \xi_j \tag{2.4.5}$$

the quadratic form coinciding with the principal symbol, up to an overall sign. Recall the Sylvester theorem from linear algebra:

Theorem 2.4.3. *There exists a linear (of course invertible) change of variables of the form* (2.3.4) *reducing the quadratic form* (2.4.5) *to the form*

$$Q = \tilde{\xi}_1^2 + \cdots + \tilde{\xi}_p^2 - \tilde{\xi}_{p+1}^2 - \cdots - \tilde{\xi}_{p+q}^2. \tag{2.4.6}$$

The numbers $p \geq 0$, $q \geq 0$, $p + q \leq d$ do not depend on the choice of the reducing transformation.

Corollary 2.4.4. *A second order linear differential operator with constant coefficients can be reduced to the form*

$$A = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_p^2} - \frac{\partial^2}{\partial y_{p+1}^2} - \cdots - \frac{\partial^2}{\partial y_{p+q}^2} + \sum_{k=1}^{d} \tilde{b}_k y_k + c \tag{2.4.7}$$

by a linear transformation of the form (2.3.4)*. The numbers p and q do not depend on the choice of the reducing transformation.*

E se coefficienti non sono costanti ?

2.5 Characteristics. Elliptic and hyperbolic operators.

Let $a_m(x, \xi)$ be the principal symbol of a linear differential operator A.

Definition 2.5.1. *Given a point $x_0 \in \Omega$, the vectors ξ satisfying*

$$a_m(x_0, \xi) = 0 \tag{2.5.1}$$

(2.5.1) *are called* characteristic vectors *of the operator A at the point x_0. The hypersurface in the ξ-space consisting of all characteristic vectors at the point x_0 is called* characteristic cone *at x_0.*

The name "cone" comes from the fact that (2.5.1) is invariant with respect to rescalings

$$\xi \mapsto \lambda \xi, \quad \forall\, \lambda \in \mathbb{R} \tag{2.5.2}$$

since the polynomial $a_m(x_0, \xi)$ is homogeneous of degree m:

$$a_m(x, \lambda\, \xi) = \lambda^m a_m(x, \xi).$$

Definition 2.5.2. *It is said that the operator $A : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega)$ is* elliptic *if*

$$a_m(x, \xi) \neq 0 \quad \text{for any} \quad \xi \neq 0, \quad x \in \Omega. \tag{2.5.3}$$

The characteristic cone (at any x_0) of an elliptic operator is degenerate (consists of just one point $\xi = 0$).

For example the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is elliptic on $\Omega = \mathbb{R}^d$. The *Tricomi operator*

$$A = \frac{\partial^2}{\partial x^2} + x \frac{\partial^2}{\partial y^2} \tag{2.5.4}$$

is elliptic on the right half plane $x > 0$. The Dirac operator is elliptic: it can be seen that $(a_1(\xi))^2 = -\xi^2$.

Next two examples are not elliptic. For the wave operator

$$A = \frac{\partial^2}{\partial t^2} - \Delta, \tag{2.5.5}$$

the characteristic cone at any x_0 is given by the equation

$$\tau^2 - \xi_1^2 - \cdots - \xi_d^2 = 0 \tag{2.5.6}$$

(it coincides with the standard cone in the pseudo-Euclidean space of signature $(1, d)$).

The characteristic cone of the heat operator

$$\frac{\partial}{\partial t} - \Delta \tag{2.5.7}$$

is a line given by

$$\xi_1 = \cdots = \xi_d = 0, \quad \tau \in \mathbb{R}. \tag{2.5.8}$$

Definition 2.5.3. *A hypersurface* Σ *in* $\mathbb{R}^d$ *is called* characteristic surface *or simply* characteristics *for the operator* A *if at every point* x *of the surface the* normal *vector* ξ *is a characteristic vector:*

$$a_m(x, \xi) = 0.$$

Remark: In particular if a hypesurface Σ is given by equation

$$S(x) = 0 \tag{2.5.9}$$

it is a characteristic surface if the smooth function $S(x)$ satisfies the equation

$$a_m(x, S_x(x)) = 0, \quad \forall x \in \Sigma. \tag{2.5.10}$$

Here $S_x = (\partial_1 S, \dots, \partial_d S)$ denotes the gradient of the function S: at any point it is normal to Σ.

As it follows from the Proposition 2.3.1 the characteristics do not depend on the choice of a system of coordinates.

Example. For a first order linear differential operator

$$A = a_1(x)\frac{\partial}{\partial x_1} + \cdots + a_d(x)\frac{\partial}{\partial x_d}, \tag{2.5.11}$$

$$a(x, \xi) = a_1(x, \xi) = a(x) \cdot \xi,$$

where $a(x) := (a_1(x), \dots, a_d(x))$. Hence the function $S(x)$ defining a characteristic hypersurface must satisfy the equation

$$a(x) \cdot S_x(x) = \sum_j a_j(x)\, \partial_j S(x) = 0. \tag{2.5.12}$$

It is therefore a first integral of the following system of ODEs

$$\begin{aligned} \dot{x}_1 &= a_1(x_1, \dots, x_d) \\ &\dots \\ \dot{x}_d &= a_d(x_1, \dots, x_d) \end{aligned} \tag{2.5.13}$$

Namely, the function $S(x)$ is constant along the integral curves of the system (2.5.13). Indeed by (2.5.12)

$$\dot{S}(x) = \sum_j \partial_j S\, \dot{x}_j = \sum_j a_j(x)\, \partial_j S = 0.$$

where 'dot' indicates derivative with respect to parameter of the curve.

It is known from the theory of ordinary differential equations that locally, near a point x^0 such that $\left(a_1(x^0), \dots, a_d(x^0)\right) \neq 0$ there exists a smooth invertible change of coordinates

$$y_k = y_k(x_1, \dots, x_d)$$

such that, in the new coordinates the system (2.5.13) reduces to the form

$$\begin{aligned} \dot y_1 &= 0 \\ &\dots \\ \dot y_{d-1} &= 0 \\ \dot y_d &= 1 \end{aligned} \tag{2.5.14}$$

(the so-called rectification of a vector field). For the particular case of constant coefficients the needed transformation is linear (see above). In these coordinates $\tilde A = \partial_d$ and the general solution to (2.5.12) reads

$$S(y_1, \dots, y_d) = S_0(y_1, \dots, y_{d-1}). \tag{2.5.15}$$

Note now that (2.5.12) is just the equation

$$A\, S(x) = 0$$

and (2.5.15) is its general solution too!

Consider now a linear differential operator A acting on smooth functions on $\Omega \subset \mathbb{R}^{(d+1)}$ with Euclidean coordinates $(t, x_1, \dots, x_d)$. As before, for $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^d$, denote by $a_m(t, x, \tau, \xi)$ the principal symbol of A.

Definition 2.5.4. *A linear differential operator A in $\Omega \subset \mathbb{R}^{d+1}$ is called* hyperbolic *with respect to the time variable t if for any fixed $\xi \neq 0$ and any $(t, x) \in \Omega$ the equation for τ*

$$a_m(t, x, \tau, \xi) = 0 \tag{2.5.16}$$

has m pairwise distinct real roots

$$\tau_1(t, x, \xi), \dots, \tau_m(t, x, \xi).$$

For brevity we will often say that a linear differential operator is hyperbolic if all its characteristics are real and pairwise distinct. For elliptic operators the characteristics are purely imaginary. The wave operator (2.5.5) gives a simple example of a hyperbolic operator. Indeed, the equation

$$\tau^2 = \xi_1^2 + \cdots + \xi_d^2 = \xi^2$$

has two distinct roots

$$\tau = \pm\sqrt{\xi^2}$$

for any $\xi \neq 0$, and so $S(X)$ satisfies

$$\frac{\partial S}{\partial t} = \pm\sqrt{(S_x)^2},$$

which has two families of solutions

$$S(x, t) = t \pm \sqrt{x^2} + C.$$

Now for a general hyperbolic operator, by substituting $(\tau, \xi) = (\partial_t S, \partial_x S)$ to $\tau = \tau_j(t, x, \xi)$ we see that finding the j-th characteristic hypersurface requires knowledge of solutions to the following Hamilton–Jacobi equation for the functions $S = S(x, t)$,

$$\frac{\partial S}{\partial t} = \tau_j\left(t, x, \frac{\partial S}{\partial x}\right). \tag{2.5.17}$$

From the course of analytical mechanics it is known that this problem is reduced to integrating the Hamilton equations

$$\left.\begin{aligned} \dot{x}_i &= \tfrac{\partial H(t,x,p)}{\partial p_i} \\ \dot{p}_i &= -\tfrac{\partial H(t,x,p)}{\partial x_i} \end{aligned}\right\} \tag{2.5.18}$$

with the time-dependent Hamiltonian $H(t,x,p) = \tau_j(t,x,p)$. In the next section we will consider the particular case $d = 1$ and apply it to the problem of canonical forms of the second order linear differential operators in a two-dimensional space.

We close this section by noting that the heat operator (2.5.7) is neither hyperbolic nor elliptic.

2.6 Reduction to a canonical form of second order linear differential operators in a two-dimensional space

Consider a linear differential operator

$$A = a(x,y)\frac{\partial^2}{\partial x^2} + 2b(x,y)\frac{\partial^2}{\partial x \partial y} + c(x,y)\frac{\partial^2}{\partial y^2}, \quad (x,y) \in \Omega \subset \mathbb{R}^2. \tag{2.6.1}$$

The characteristics of these operator are curves which can be parametrized by some parameter. We assume for definiteness that the coefficient $a(x,y) \neq 0$ and that we can choose this paramater to be just x (similar discussion appears if we choose y locally). A tangent vector to the curve is then $(1, \frac{\partial y}{\partial x})$. Then $(-\frac{\partial y}{\partial x}, 1)$ is a normal vector, which in order to be characteristic has to satisfy

$$a(x,y)\left(\frac{dy}{dx}\right)^2 - 2b(x,y)\frac{dy}{dx} + c(x,y) = 0. \tag{2.6.2}$$

This is a quadratic equation for dy/dx. Its discriminant is

$$\delta(x,y) := b^2(x,y) - a(x,y)\,c(x,y). \tag{2.6.3}$$

It is immediate that the operator (2.6.1) is elliptic *iff* $\delta(x,y) < 0$, while it is hyperbolic *iff* $\delta(x,y) > 0$. As said in the previous section, for a hyperbolic operator one has two families of characteristics to be found from the (Hamilton-Jacobi) ODEs

$$\frac{dy}{dx} = \frac{b(x,y) + \sqrt{b^2(x,y) - a(x,y)\,c(x,y)}}{a(x,y)} \tag{2.6.4}$$

$$\frac{dy}{dx} = \frac{b(x,y) - \sqrt{b^2(x,y) - a(x,y)\,c(x,y)}}{a(x,y)}. \tag{2.6.5}$$

Let

$$\phi(x,y) = c_1, \quad \psi(x,y) = c_2 \tag{2.6.6}$$

be the equations of the characteristics (thus a first integral for these ODE taking constant values along the integral curves of this differential equation). Here c_1 and c_2 are two integration constants. Such curves pass through any point $(x,y) \in \Omega$. Moreover, by hyperbolicity, they are not tangent at every point. Let us introduce new local coordinates u, v by

$$u = \phi(x,y), \quad v = \psi(x,y). \tag{2.6.7}$$

Lemma 2.6.1. *The change of coordinates*

$$(x, y) \mapsto (u, v)$$

is locally invertible. Moreover the inverse functions

$$x = x(u, v), \quad y = y(u, v)$$

are smooth.

Proof: We have to check non-vanishing of the Jacobian

$$\det \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} = \det \begin{pmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{pmatrix} \neq 0. \tag{2.6.8}$$

By definition the first derivatives of the functions ϕ and ψ correspond to two different roots of the same quadratic equation

$$a(x, y)\phi_x^2 + 2b(x, y)\phi_x\phi_y + c(x, y)\phi_y^2 = 0, \quad a(x, y)\psi_x^2 + 2b(x, y)\psi_x\psi_y + c(x, y)\psi_y^2 = 0.$$

The determinant (2.6.8) vanishes *iff* the gradients of ϕ and ψ are proportional:

$$(\phi_x, \phi_y) \sim (\psi_x, \psi_y),$$

that is the characteristics are tangent. This contradicts the requirement to have the roots distinct. □

Let us rewrite the linear differential operator A in the new coordinates:

$$A = \tilde{a}(u, v)\frac{\partial^2}{\partial u^2} + 2\tilde{b}(u, v)\frac{\partial^2}{\partial u \partial v} + \tilde{c}(u, v)\frac{\partial^2}{\partial v^2} + \dots \tag{2.6.9}$$

where the dots stand for the terms with the low order derivatives.

Theorem 2.6.2. *In the new coordinates A reads*

$$A = 2\tilde{b}(u, v)\frac{\partial^2}{\partial u\, \partial v} + \dots$$

Proof: In the new coordinates the characteristic have the form

$$u = c_1, \quad \text{or} \quad v = c_2$$

for arbitrary constants c_1 and c_2. Their tangent vectors can be taken as $(1, 0)$ and $(0, 1)$ and thus the normal vectors $(0, 1)$ and $(1, 0)$ must satisfy the equation for characteristics

$$0 + 0 + \tilde{c}(u, v) = 0 \quad \text{and} \quad \tilde{a}(u, v) + 0 + 0 = 0.$$

This implies $\tilde{a}(u, v) = \tilde{c}(u, v) = 0$. □

For the case of elliptic operator (2.6.1) we have $b^2 - a\,c > 0$ (we will often simplify the notation as $a = a(x, y), \quad b = b(x, y), \quad c = c(x, y)$) and the analogue of the differential equations (2.6.4), (2.6.5) are complex conjugated equations

$$\frac{dy}{dx} = \frac{b \pm i\sqrt{a\,c - b^2}}{a}. \tag{2.6.10}$$

Assuming analyticity of the functions $a(x,y)$, $b(x,y)$, $c(x,y)$ one can prove existence of a complex valued first integral

$$S(x,y) = \phi(x,y) + i\,\psi(x,y) \tag{2.6.11}$$

satisfying

$$\boxed{a\,S_x + \left(b + i\,\sqrt{a\,c - b^2}\right) S_y = 0,} \tag{2.6.12}$$

with

$$S_x^2 + S_y^2 \neq 0. \tag{2.6.13}$$

Let us introduce new system of coordinates by

$$u = \phi(x,y), \quad v = \psi(x,y). \tag{2.6.14}$$

Theorem 2.6.3. *The transformation*

$$(x,y) \mapsto (u,v)$$

is locally smoothly invertible. The operator A in the new coordinates takes the form

$$A = \tilde{a}(u,v)\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + \dots \tag{2.6.15}$$

with some nonzero smooth function $\tilde{a}(u,v)$. Like above the dots stand for the terms with lower order derivatives.

Proof: Substituting $S_x = \phi_x + i\psi_x$ and $S_y = \phi_y + i\psi_y$ to (2.6.12), and looking into the real and imaginary part, we get

$$a\,\phi_x + b\,\phi_y - \sqrt{a\,c - b^2}\psi_y = 0, \quad a\,\psi_x + b\,\psi_y + \sqrt{a\,c - b^2}\phi_y = 0. \tag{2.6.16}$$

Writting this as

$$a\,\phi_x = -b\,\phi_y + \sqrt{a\,c - b^2}\psi_y, \quad a\,\psi_x = -b\,\psi_y - \sqrt{a\,c - b^2}\phi_y, \tag{2.6.17}$$

shows that the determinant of the Jacobian (2.6.8)

$$\phi_x\psi_y - \psi_x\phi_y = a^{-1}\sqrt{a\,c - b^2}(\psi_y^2 + \phi_y^2) \neq 0. \tag{2.6.18}$$

Next, since $S = u + iv$, so $(S_u, S_v) = (1, i)$ which must satisfy

$$\tilde{a} \times 1 + \left(\tilde{b} + i\sqrt{\tilde{a}\,\tilde{c} - \tilde{b}^2}\right) \times i = 0.$$

Hence $\tilde{b} = 0$ and then $\tilde{a} = \sqrt{\tilde{a}\tilde{c}}$. It follows that $\tilde{c} = \tilde{a}$. □

Let us now consider the case of linear differential operators of the form (2.6.1) with identically vanishing discriminant

$$b^2(x,y) - a(x,y)\,c(x,y) \equiv 0. \tag{2.6.19}$$

Operators of this class are called *parabolic*. In this case we have only one characteristic to be found from the equation

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}. \tag{2.6.20}$$

Let $\phi(x,y)$ be a first integral of this equation

$$a\,\phi_x + b\,\phi_y = 0, \quad \phi_x^2 + \phi_y^2 \neq 0. \tag{2.6.21}$$

Choose an arbitrary smooth function $\psi(x,y)$ (always exist!) such that

$$\det\begin{pmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{pmatrix} \neq 0.$$

Thus

$$u = \phi(x,y), \quad v = \psi(x,y),$$

are coordinates in which we have

$$S = u, \quad (S_u, S_v) = (1, 0).$$

Substituting this to the equation for the characteristics we see that $\tilde a(u,v)\times 1 + 0 + 0 = 0$ and so $\tilde a(u,v)$ vanishes. But then the coefficient $\tilde b(u,v)$ must vanish either because of vanishing of the discriminant

$$\tilde b^2 - \tilde a\,\tilde c = 0.$$

Thus we have shown

Theorem 2.6.4. *The canonical form of a parabolic operator is*

$$A = \tilde c(u,v)\frac{\partial^2}{\partial v^2} + \dots \tag{2.6.22}$$

where the dots stand for the terms of lower order.

2.7 General solution of a second order hyperbolic equation with constant coefficients in the two-dimensional space

Consider again a hyperbolic operator

$$A = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\,\partial y} + c\frac{\partial^2}{\partial y^2} \tag{2.7.1}$$

with constant coefficients a, b, c satisfying the hyperbolicity condition

$$b^2 - a\,c > 0.$$

The equations for characteristics (2.6.4), (2.6.5) can be easily integrated

$$y = \lambda_{1,2}x + const,$$

where

$$\lambda_{1,2} = \frac{b \pm \sqrt{b^2 - a\,c}}{a}.$$

This gives two linear first integrals

$$u = y - \lambda_1 x, \quad v = y - \lambda_2 x. \tag{2.7.2}$$

In the new coordinates the hyperbolic equation $A\,\varphi = 0$ reduces to (see Theorem 2.6.2)

$$\frac{\partial^2\varphi}{\partial u\,\partial v} = 0. \tag{2.7.3}$$

The general solution to this equation can be written in the form

$$\varphi = f(y - \lambda_1 x) + g(y - \lambda_2 x) \tag{2.7.4}$$

where f and g are two arbitrary smooth[1] functions of one variable.

For example consider the wave equation

$$\varphi_{tt} = a^2 \varphi_{xx} \tag{2.7.5}$$

where a is a positive constant. The general solution reads

$$\varphi(x, t) = f(x - a\,t) + g(x + a\,t). \tag{2.7.6}$$

Observe that $f(x-a\,t)$ is a right-moving wave propagating with constant speed a. In a similar way $g(x + a\,t)$ is a left-moving wave. Therefore the general solution to the wave equation (2.7.5) is a superposition (= sum) of two such waves.

2.8 Exercises to Section 2

Exercise 2.8.1. *Reduce to the canonical form the following equations*

$$u_{xx} + 2u_{xy} - 2u_{xz} + 2u_{yy} + 6u_{zz} = 0 \tag{2.8.1}$$

$$u_{xy} - u_{xz} + u_x + u_y - u_z = 0. \tag{2.8.2}$$

Exercise 2.8.2. *Reduce to the canonical form the following equations*

$$x^2 u_{xx} + 2x\,y\,u_{xy} - 3y^2 u_{yy} - 2x\,u_x + 4y\,u_y + 16x^4 u = 0 \tag{2.8.3}$$

$$y^2 u_{xx} + 2x\,y\,u_{xy} + 2x^2 u_{yy} + y\,u_y = 0 \tag{2.8.4}$$

$$u_{xx} - 2u_{xy} + u_{yy} + u_x + u_y = 0 \tag{2.8.5}$$

Exercise 2.8.3. *Find general solution to the following equations*

$$x^2 u_{xx} - y^2 u_{yy} - 2y\,u_y = 0 \tag{2.8.6}$$

$$x^2 u_{xx} - 2x\,y\,u_{xy} + y^2 u_{yy} + x\,u_x + y\,u_y = 0. \tag{2.8.7}$$

[1] It suffices to take the functions of the $\mathcal{C}^2$ class.

3 Wave equation

3.1 Vibrating string

We consider small oscillations of an elastic string on the (x, u)-plane. Let the x-axis be the equilibrium state of the string. Denote $u(x, t)$ the displacement of the point x at a time t. It will be assumed to be orthogonal to the x-axis. Thus the shape of the string at the time t is given by the graph of the function $u(x, t)$. The velocity of the string at the point x is equal to $u_t(x, t)$.

We will also assume that the only force to be taken into consideration is the tension directed along the string. In particular the string will be assumed to be totally elastic.

Consider a small interval of the string from x to $x + \delta x$. We will write the equation of motion for this interval. Denote $T = T(x)$ the tension of the string at the point x. The horizontal and vertical components of T at the points x and $x + \delta x$ are equal to

$$T_{\rm hor}(x) = T_1 \cos\alpha, \quad T_{\rm vert}(x) = T_1 \sin\alpha$$
$$T_{\rm hor}(x+\delta x) = T_2 \cos\beta, \quad T_{\rm vert}(x+\delta x) = T_2 \sin\beta$$

where $T_1 = T(x)$, $T_2 = T(x + \delta x)$ (see Fig. 1).

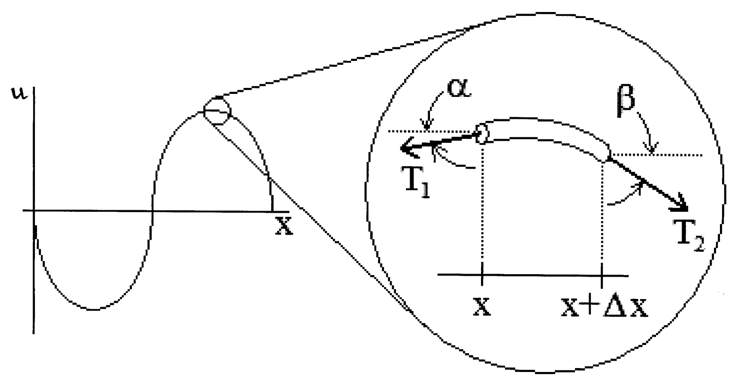

Fig. 1.

The angle α between the string and the x-axis at the point x is given by $\tan\alpha = u_x$, so that

$$\cos\alpha = \frac{1}{\sqrt{1+u_x^2}}, \quad \sin\alpha = \frac{u_x}{\sqrt{1+u_x^2}}.$$

The oscillations are assumed to be *small.* More precisely this means that the term u_x is small. So at the leading approximation we can neglect the square of it to arrive at

$$\cos\alpha \simeq 1, \quad \sin\alpha \simeq u_x(x)$$
$$\cos\beta \simeq 1, \quad \sin\beta \simeq u_x(x+\delta x)$$

So the horizontal and vertical components at the points x and $x + \delta x$ are equal to

$$T_{\rm hor}(x) \simeq T_1, \quad T_{\rm vert}(x) \simeq T_1 u_x(x)$$
$$T_{\rm hor}(x+\delta x) \simeq T_2, \quad T_{\rm vert}(x+\delta x) = T_2 u_x(x+\delta(x),$$

Since the string moves in the u-direction, the horizontal components at the points x and $x + \delta x$ must coincide:

$$T_1 = T(x) = T(x+\delta x) = T_2.$$

Therefore $T(x) \equiv T = \text{const}$.

Let us now consider the vertical components. The resulting force acting on the piece of the string is equal to

$$f = T_2 \sin\beta - T_1 \sin\alpha = T\, u_x(x+\delta x) - T\, u_x(x) \simeq T\, u_{xx}(x)\, \delta x.$$

On the other hand the vertical component of the total momentum of the piece of the string is equal to

$$p \simeq \rho(x) u_t(x,t)\, \delta x$$

where $\rho(x)$ is the linear mass density of the string. The second Newton law

$$p_t = f$$

in the limit $\delta x \to 0$ yields

$$\rho(x) u_{tt} = T\, u_{xx}.$$

In particular in the case of constant mass density one arrives at the equation

$$u_{tt} = a^2 u_{xx} \tag{3.1.1}$$

where the constant a is defined by

$$a^2 = \frac{T}{\rho}. \tag{3.1.2}$$

◇

Remark. Note that (see Corollary 2.2.2): the *plane wave*

$$u(x,t) = A\, e^{i(k\,x + \omega\, t)} \tag{3.1.3}$$

satisfies the wave equation (3.1.1) if and only if the real parameters ω and k satisfy the following *dispersion relation*

$$\omega = \pm a\, k. \tag{3.1.4}$$

The parameter ω is called *frequency* [2] and k is called *wave number* of the plane wave. The arbitrary parameter A is called the *amplitude* of the wave. It is clear that the plane wave is periodic in x with the period (called *wavelength*)

$$L = \frac{2\pi}{|k|} \tag{3.1.5}$$

since the exponential function is periodic with the period $2\pi\, i$. The plane wave is also periodic in t with the period

$$T = \frac{2\pi}{|\omega|}. \tag{3.1.6}$$

Due to linearity of the wave equation and real coefficients the real and imaginary parts of the solution (3.1.3) solve the same equation (3.1.1). Assuming A to be real we thus obtain the real valued solutions

$$\operatorname{Re} u = A\, \cos(k\, x + \omega\, t), \quad \operatorname{Im} u = A\, \sin(k\, x + \omega\, t). \tag{3.1.7}$$

[2] In physics usually $-\omega$ used with $\omega \geq 0$ called frequency.

3.2 D'Alembert formula

Let us start with considering oscillations of an *infinite string*, that is the spatial variable x varies from $-\infty$ to ∞. The Cauchy problem for the equation (3.1.1) is formulated in the following way: find a solution $u(x,t)$ defined for $t \geq 0$ such that at $t = 0$ the initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \tag{3.2.1}$$

hold true.

Remark: In general, for a hyperbolic pde of order m in t, the Cauchy problem requires the $t = 0$ the initial conditions for the the t-derivatives of u up to order $m-1$.

For the wave equation the solution is given by the following *D'Alembert formula*:

Theorem 3.2.1. *For arbitrary initial data* $\phi(x) \in \mathcal{C}^2(\mathbb{R})$, $\psi(x) \in \mathcal{C}^1(\mathbb{R})$ *the solution* $u \in \mathcal{C}^2(\mathbb{R})$ *to the Cauchy problem* (3.1.1), (3.2.1) *exists and is unique. Moreover it is given by the formula*

$$u(x,t) = \frac{\phi(x - a\,t) + \phi(x + a\,t)}{2} + \frac{1}{2a}\int_{x-a\,t}^{x+a\,t} \psi(s)\,ds. \tag{3.2.2}$$

Proof: As we have proved in Section 2.7 the general solution to the equation (3.1.1) can be represented in the form

$$u(x,t) = f(x - a\,t) + g(x + a\,t). \tag{3.2.3}$$

Let us impose on the functions f and g the initial conditions (3.2.1). We obtain the following system:

$$\begin{aligned} &f(x) + g(x) = \phi(x) \\ &a\left[g'(x) - f'(x)\right] = \psi(x). \end{aligned} \tag{3.2.4}$$

Integrating the second equation yields

$$g(x) - f(x) = \frac{1}{a}\int_{x_0}^{x} \psi(s)\,ds + C$$

where C is an integration constant. So

$$\begin{aligned} f(x) &= \frac{1}{2}\phi(x) - \frac{1}{2a}\int_{x_0}^{x} \psi(s)\,ds - \frac{1}{2}C \\ g(x) &= \frac{1}{2}\phi(x) + \frac{1}{2a}\int_{x_0}^{x} \psi(s)\,ds + \frac{1}{2}C. \end{aligned}$$

Thus

$$u(x,t) = \frac{1}{2}\phi(x - a\,t) + \frac{1}{2}\phi(x + a\,t) + \frac{1}{2a}\int_{x_0}^{x+a\,t} \psi(s)\,ds - \frac{1}{2a}\int_{x_0}^{x-a\,t} \psi(s)\,ds.$$

This gives the necessary form (3.8.14) of a solution (and uniqueness). It remains to check that, given a pair of functions $\phi(x) \in \mathcal{C}^2$, $\psi(x) \in \mathcal{C}^1$ the D'Alembert formula indeed yields a solution to (3.1.1). In fact the function (3.8.14) is twice differentiable in x and t. It remains to substitute this function into the wave equation and check that the equation is satisfied and the initial data. It is okay ($a \neq 0$) to check that $2au$ is a solution:

$$2au_{tt}(x,t) = a(-a)^2\phi''(x - a\,t) + aa^2\phi''(x + a\,t) + a^2\psi'(x + a\,t) - (-a)^2\psi'(x - a\,t)\,ds,$$

which equals $a^2\times$

$$2au_{xx}(x,t) = a\phi''(x-a\,t) + a\phi''(x+a\,t) + \psi'(x+a\,t) - \psi'(x-a\,t)\,ds.$$

It is also straightforward to verify validity of the initial data (3.2.1). $\square$

Example. For the constant initial data

$$u(x,0) = u_0, \quad u_t(x,0) = v_0$$

the solution has the form

$$u(x,t) = u_0 + v_0 t.$$

This particular solution corresponds to the free motion of the string as a whole with the constant speed v_0.

We show now that the solution to the wave equation is stable with respect to small variations of the initial data. Namely,

Proposition 3.2.2. *For any $\epsilon > 0$ and any $T > 0$ there exists $\delta > 0$ such that if the initial conditions satisfy*

$$\sup_{x\in\mathbb{R}} |\tilde\phi(x) - \phi(x)| < \delta, \quad \sup_{x\in\mathbb{R}} |\tilde\psi(x) - \psi(x)| < \delta, \tag{3.2.5}$$

then the solutions $u(x,t)$ and $\tilde u(x,t)$ of the two Cauchy problems with initial conditions (3.2.1) *and*

$$\tilde u(x,0) = \tilde\phi(x), \quad \tilde u_t(x,0) = \tilde\psi(x) \tag{3.2.6}$$

satisfy

$$\sup_{x\in\mathbb{R},\ t\in[0,T]} |\tilde u(x,t) - u(x,t)| < \epsilon. \tag{3.2.7}$$

Proof:

$$2a \sup_{x\in\mathbb{R},\ t\in[0,T]} |\tilde u(x,t) - u(x,t)| \le$$

$$\sup_{x\in\mathbb{R}} a|\tilde\phi(x-a\,t) - \phi(x-a\,t)| + \sup_{x\in\mathbb{R}} a|\tilde\phi(x+a\,t) - \phi(x+a\,t)| + \sup_{x\in\mathbb{R}, t\in[0,T]} \int_{x-a\,t}^{x+a\,t} |\tilde\psi(s) - \psi(s)|\,ds$$

$$< 2a\delta + \delta \sup_{t\in[0,T]} 2at \;\le\; 2a(1+T)\delta.$$

$\square$

Remark 3.2.3. *The Cauchy problem* (3.1.1), (3.2.1) *which has this property is usually referred to as* well posed. *So the Cauchy problem for the wave equation is well posed. We will discuss later this important property for other equations.*

3.3 Some consequences of the D'Alembert formula

Let (x_0, t_0) be a point of the (x,t)-plane, $t_0 > 0$. As it follows from the D'Alembert formula the value of the solution at the point (x_0, t_0) depends only on the values of $\phi(x)$ at $x = x_0 \pm a\,t_0$ and value of $\psi(x)$ on the interval $[x_0 - a\,t_0, x_0 + a\,t_0]$. The triangle with the vertices (x_0, t_0) and $(x_0 \pm a\,t_0, 0)$ is called *the dependence domain* of the segment $[x_0 - a\,t_0, x_0 + a\,t_0]$. The values of the solution *inside* this triangle are completely determined by the values of the initial data on the segment.

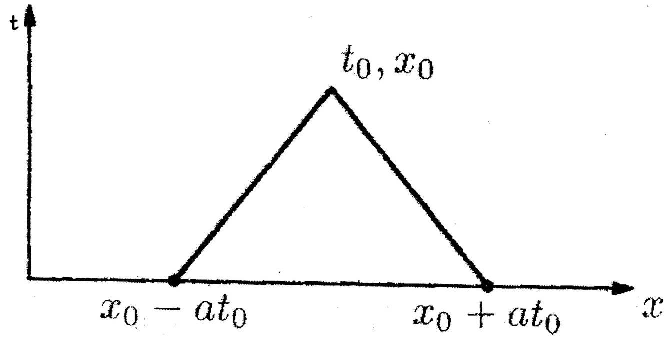

Fig. 2. The dependence domain of the segment $[x_0 - a\,t_0, x_0 + a\,t_0]$.

Another important definition is the *influence domain* for a given segment $[x_1, x_2]$ consider the domain defined by inequalities

$$x + a\,t \geq x_1, \quad x - a\,t \leq x_2, \quad t \geq 0. \tag{3.3.1}$$

Changing the initial data on the segment $[x_1, x_2]$ will not change the solution $u(x,t)$ outside the influence domain.

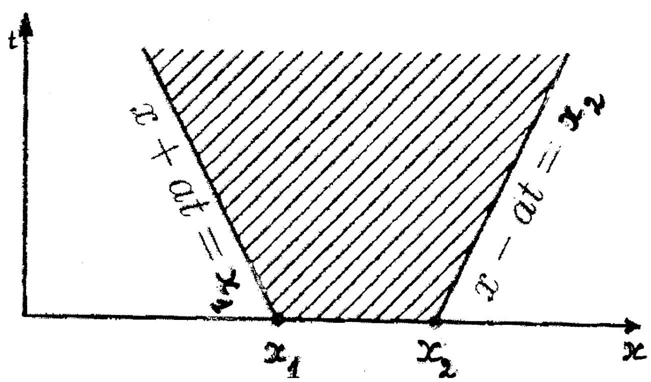

Fig. 3. The influence domain of the segment $[x_1, x_2]$.

Remark. *Il significato fisico è che il 'segnale' non si propaga più velocemente di a, e che vale principio di località e causalità (nessun "evento causa" puo determinare "effetto" fuori dal proprio cono di dipendenza.*

Remark 3.3.1. *It will be convenient to slightly extend the class of initial data admitting piecewise smooth functions $\phi(x)$, $\psi(x)$ (all singularities of the latter must be integrable). Notiamo che* (3.8.14) *può essere scritta*

$$u(x,t) = \frac{1}{2}\left(\phi(x + a\,t) + \phi(x - a\,t)\right) + \frac{1}{2a}\left(F(x + a\,t) - F(x - a\,t)\right), \tag{3.3.2}$$

dove

$$F(x) = \int_{x_0}^{x} \psi(s)\,ds \tag{3.3.3}$$

è la primitiva di ψ. Segue che il valore $u(x,t)$ è determinato da valori di ϕ e di F in punti $x \pm a\,t$. Però le singolarità della primitiva F di ψ sono al massimo quelli (se integrabili !) di ψ. Percio:

If x_j, $j = 1, 2, \ldots$, are the singularities of ϕ and ψ, then the solution $u(x,t)$ given by the D'Alembert formula will satisfy the wave equation outside the lines

$$x = \pm a\,t + x_j, \quad t \geq 0, \quad j = 1, 2, \ldots$$

The above formula says that the singularities of the solution propagate along the characteristics.

Example. Let us draw the profile of the string for the triangular initial data $\phi(x)$ shown on Fig. 4 and $\psi(x) \equiv 0$.

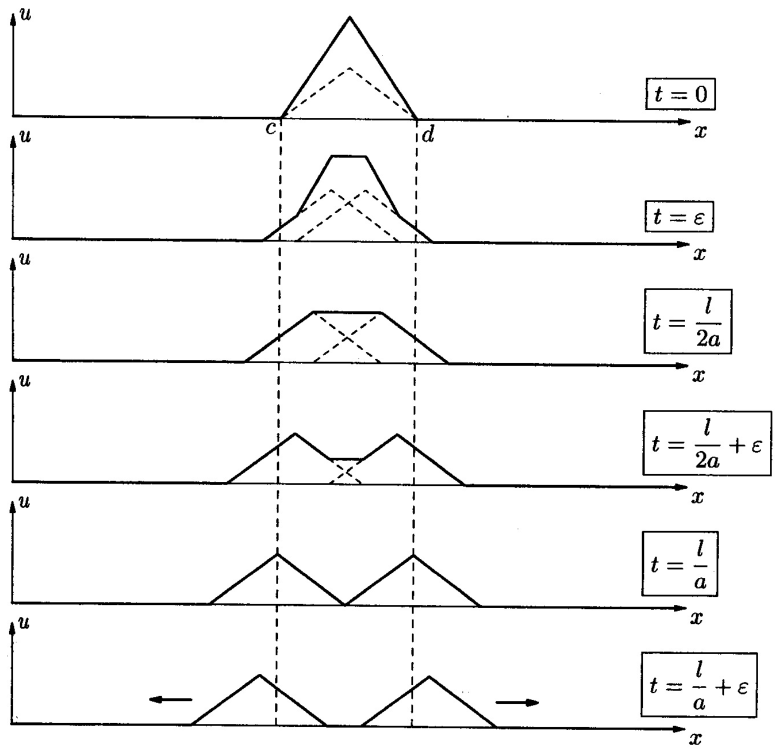

Fig. 4. The solution of the Cauchy problem for wave equation on the real line with a triangular initial profile at few instants of time.

We have the following simple observation.

Lemma 3.3.2. *Let $u(x,t)$ be a solution to the wave equation. Then so are the functions*

$$\pm c' u(\pm c x, \pm c t)$$

for arbitrary choices of all three signs and $c, c' > 0$.

This means that the (linear) wave equation is invariant with respect to the spatial reflection

$$x \mapsto -x,$$

time inversion

$$t \mapsto -t$$

and rescaling

$$(x,t) \mapsto (cx, ct).$$

Mini-exercise 3.3.3. *Verificare l'invarianza dell'equazione delle onde per traslazioni*

$$x \mapsto x + x_0, \quad t \mapsto t + t_0$$

e per trasformazioni di Lorentz (ovvero la relatività speciale)

$$x \mapsto \gamma(x - vt), \quad t \mapsto \gamma(t - \frac{v}{a^2}x),$$

dove $\gamma = (1 - \frac{v}{a^2})^{-\frac{1}{2}}$. *(Onde elettromagnetiche nel vuoto soddisfano l'equazione delle onde con velocitá di propagazione* $a = c \simeq 300000 km/sec$*). Verificare invece che l'equazione delle onde non è invariante per trasformazioni di Galileo*

$$x \mapsto x - vt, \quad t \mapsto t.$$

Ricapitoliamo alcune proprietà salienti dell'equazione delle onde in $\mathbb{R}^2$:
- superposizioni/scomposizioni di soluzioni (in 'modi' semplici)
- esistenza e unicità di soluzioni e problema di Cauchy ben posto
- velocitá finita (nonistantanea) di propagazione
- preservazione di singolarità (non vengono smorzate nel tempo)
- invarianza rispetto inversione di tempo, di spazio, l'omotetia ('riscalamento'), traslazioni e trasformazioni di Lorentz.

3.4 Semi-infinite vibrating string

Let us consider oscillations of a string with a fixed point. Without loss of generality we can assume that the fixed point is at $x = 0$. We arrive at the following Cauchy problem for (3.1.1) on the half-line $x > 0$:

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \quad x > 0. \tag{3.4.1}$$

The solution must also satisfy the boundary condition

$$u(0, t) = 0, \quad t \geq 0. \tag{3.4.2}$$

Of course we require that $\phi(0) = 0$ and $\psi(0) \equiv \phi'(0) = 0$. The problem (3.1.1), (3.4.1), (3.4.2) is often called *mixed problem* since we have both initial conditions and boundary conditions.

The solution to the mixed problem on the half-line can be transformed to the problem on the infinite line by means of the following method.

Lemma 3.4.1. *Let the initial data* $\phi(x)$, $\psi(x)$ *for the Cauchy problem* (3.1.1), (3.2.1) *be odd functions of* x. *Then the solution* $u(x, t)$ *is an odd function for all* t.

Proof: Denote

$$\tilde{u}(x, t) := -u(-x, t).$$

According to Lemma 3.3.2 the function $\tilde{u}(x, t)$ satisfies the same equation. At $t = 0$ we have

$$\tilde{u}(x, 0) = -u(-x, 0) = -\phi(-x) = \phi(x), \quad \tilde{u}_t(x, 0) = -u_t(-x, 0) = -\psi(-x) = \psi(x)$$

since ϕ and ψ are odd functions. Therefore $\tilde{u}(x, t)$ is a solution to the same Cauchy problem (3.1.1), (3.2.1). Due to the uniqueness $\tilde{u}(x, t) = u(x, t)$, i.e. $-u(-x, t) = u(x, t)$ for all x and t. □

We are now ready to present a recipe for solving the mixed problem (3.1.1), (3.4.1), (3.4.2) for the wave equation on the half-line. Let us extend the initial data onto entire real line as odd functions. We arrive at the following Cauchy problem for the wave equation:

$$u(x,0) = \begin{cases} \phi(x), & x > 0 \\ -\phi(-x), & x < 0 \end{cases}, \quad u_t(x,0) = \begin{cases} \psi(x), & x > 0 \\ -\psi(-x), & x < 0 \end{cases} \tag{3.4.3}$$

According to Lemma 3.4.1 the solution $u(x,t)$ to the Cauchy problem (3.1.1), (3.4.3) given by the D'Alembert formula will be an odd function for all t. Therefore

$$u(0,t) = -u(0,t) = 0 \quad \text{for all} \quad t,$$

and thus the restriction of $u(x,t)$ to the nonnegative axis will be the required solution.

Example. Consider the evolution of a triangular initial profile on the half-line. The graph of the initial function $\phi(x)$ is non-zero on the interval $[l, 3l]$; the initial velocity $\psi(x) = 0$. The evolution is shown on Fig. 5 for few instants of time. Observe the reflected profile (the dotted line) on the negative half-line.

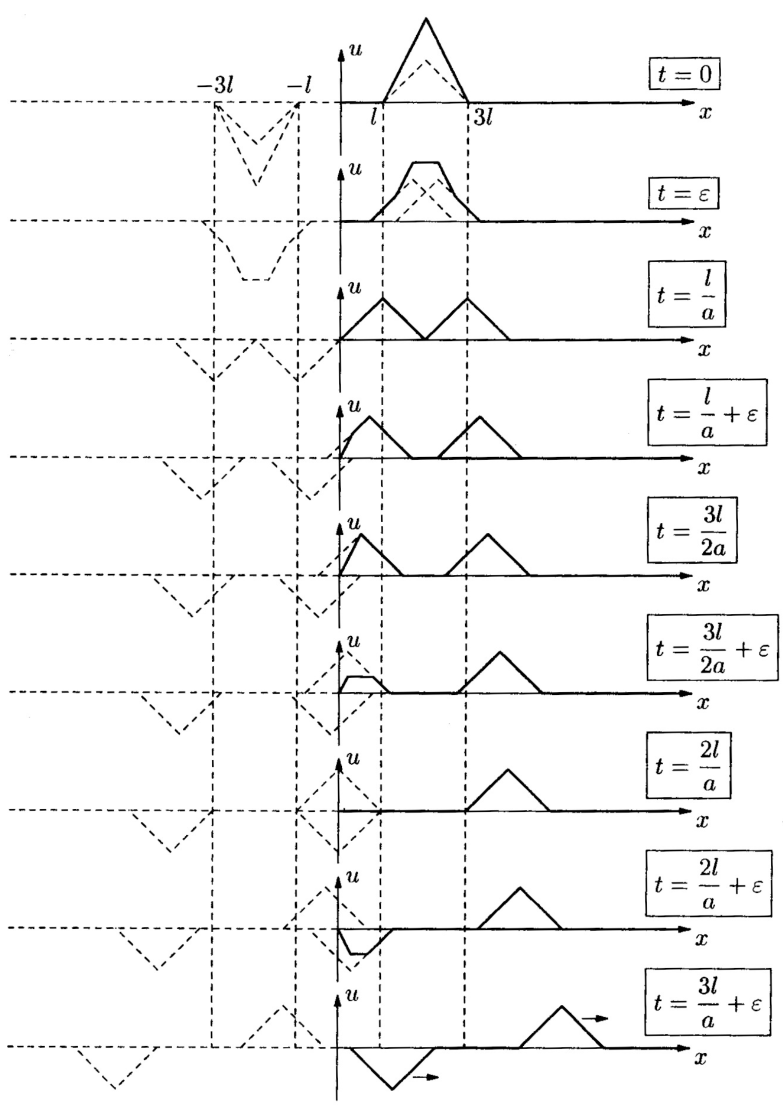

Fig. 5. The solution of the Cauchy problem for wave equation on the half-line with a triangular initial profile.

In a similar way one can treat the mixed problem on the half-line with a free boundary. In this case the vertical component $T\,u_x$ of the tension at the left edge must vanish at all times. Thus the boundary condition (3.4.2) has to be replaced with

$$u_x(0,t) = 0 \quad \text{for all} \quad t \geq 0. \tag{3.4.4}$$

There is a Lemma analogous to 3.4.1: for the initial data $\phi(x)$, $\psi(x)$ for the Cauchy problem (3.1.1), (3.2.1) given by even functions of x the solution $u(x,t)$ is an even function for all t. Therefore one can solve the mixed problem (3.1.1), (3.4.1), (3.4.4) by using *even extension* of the initial data onto the negative half-line. We leave the details of the construction as an exercise for the reader.

3.5 Periodic problem for wave equation. Introduction to Fourier series

Let us look for solutions to the wave equation (3.1.1) periodic in x with a given period $L > 0$. Thus we are looking for a solution $u(x,t)$ of the mixed problem

$$u(x+L,t) = u(x,t) \quad \text{for any} \quad t \geq 0, \tag{3.5.1}$$

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \tag{3.5.2}$$

where the initial data of the Cauchy problem must also be L-periodic functions.

Theorem 3.5.1. *Given L-periodic initial data* $\phi(x) \in \mathcal{C}^2(\mathbb{R})$, $\psi(x) \in \mathcal{C}^1(\mathbb{R})$ *the periodic Cauchy problem* (3.5.1), (3.5.2) *for the wave equation* (3.1.1) *has a unique solution (L-periodic).*

Proof: According to the results of Section 3.2 the solution $u(x,t)$ to the Cauchy problem (3.1.1), (3.5.2) on $-\infty < x < \infty$ exists, is unique and is given by the D'Alembert formula. Denote

$$\tilde{u}(x,t) := u(x+L,t).$$

Since the coefficients of the wave equation do not depend on x the function $\tilde{u}(x,t)$ satisfies the same equation. The initial data for this function have the form

$$\tilde{u}(x,0) = \phi(x+L) = \phi(x), \quad \tilde{u}_t(x,t) = \psi(x+L) = \psi(x)$$

because of periodicity of the functions $\phi(x)$ and $\psi(x)$. So the initial data of the solutions $u(x,t)$ and $\tilde{u}(x,t)$ coincide. From the uniqueness of the solution we conclude that $\tilde{u}(x,t) = u(x,t)$ for all x and t, i.e. the function $u(x,t)$ is periodic in x with the same period L. □

Notare che dalla prove si vede che periodicità di u è garantita (cosa non ovvia a priori) richiedendo solo la periodicità di ϕ e ψ.
We make now few simple observations. Clearly the complex exponential function e^{ikx} is L-periodic *iff* the wave number k has the form

$$k = \frac{2\pi n}{L}, \quad n \in \mathbb{Z}. \tag{3.5.3}$$

In the particular case $L = 2\pi$ the complex exponential

$$e^{\frac{2\pi i n x}{L}}$$

reduces to e^{inx}.

Note that the solution of the periodic Cauchy problem with the Cauchy data

$$u(x,0) = e^{inx}, \quad u_t(x,0) = 0 \tag{3.5.4}$$

is given by the formula

$$u(x,t) = \frac{1}{2}(e^{in(x-at)} + e^{in(x+at)}) = e^{inx}\cos nat. \tag{3.5.5}$$

Instead the solution of the periodic Cauchy problem with the Cauchy data

$$u(x,0) = 0, \quad u_t(x,0) = e^{inx} \tag{3.5.6}$$

is given by the formula

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} e^{ins}\, ds = \begin{cases} \frac{1}{2ina}(e^{in(x+at)} - e^{in(x-at)}), & n \neq 0 \\ \frac{2at}{2a}, & n = 0 \end{cases} \tag{3.5.7}$$

$$= \begin{cases} e^{inx}\frac{\sin nat}{na}, & n \neq 0 \\ t, & n = 0. \end{cases} \tag{3.5.8}$$

Using the theory of *Fourier series* we can represent any solution to the periodic problem to the wave equation as a superposition of the solutions (3.5.5), (3.5.7). Let us first recall some basics of the theory of Fourier series.

Definition Let $f(x)$ be a 2π-periodic continuous complex valued function on $\mathbb{R}$. The *Fourier series* of f is defined by the formula

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}, \tag{3.5.9}$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx. \tag{3.5.10}$$

Sara molto interessante sapere se, in che senso, e a che limite, la serie di Fourier di f converge. Per dire qualcosa ...
We start with recalling some basic material of functional analysis.

Let us introduce *Hermitean inner product* in the space of complex valued 2π-periodic continuous functions:

$$(f,g) = \frac{1}{2\pi}\int_0^{2\pi} \bar f(x)g(x)\, dx. \tag{3.5.11}$$

Here the bar stands for complex conjugation. This inner product satisfies the following properties:

$$(g,f) = \overline{(f,g)} \tag{3.5.12}$$

$$\left.\begin{array}{l}(\lambda f_1 + \mu f_2, g) = \bar\lambda(f_1,g) + \bar\mu(f_2,g) \\ (f, \lambda g_1 + \mu g_2) = \lambda(f,g_1) + \mu(f,g_2)\end{array}\right. \quad \forall \lambda,\, \mu \in \mathbb{C} \tag{3.5.13}$$

$$(f,f) > 0 \text{ for any nonzero continuous function } f(x). \tag{3.5.14}$$

The real nonnegative number (f,f) defines the L_2-norm of f,

$$\|f\| := \sqrt{(f,f)}. \tag{3.5.15}$$

Recall the famous Schwarz inequality
Lemma

$$|(f,g)| \leq \|f\|\,\|g\|.$$

Proof: if $(g,g)=0$, the inequality reads $0\geq 0$, which is true. Assume thus that $(g,g)>0$. Clearly $\forall\,\lambda\in\mathbb{C}$,

$$0\leq(f+\lambda g,f+\lambda g)=(f,f)+\lambda(f,g)+\bar\lambda(g,f)+\lambda\bar\lambda(g,g).$$

Now multiply by (g,g) and substitute $\lambda=-\frac{(g,f)}{(g,g)}$ (so $\bar\lambda=-\frac{(f,g)}{(g,g)}$) to get

$$0\leq\|f\|^2\,\|g\|^2-|(f,g)|^2$$

from which the statement follows. □

Corollary 3.5.2. *The L_2-norm satisfies the triangle inequality:*

$$\|f+g\|\leq\|f\|+\|g\|. \tag{3.5.16}$$

Proof:

$$\|f+g\|^2=(f+g,f+g)\leq\|f\|^2+\|g\|^2+2|(f,g)|\leq\|f\|^2+\|g\|^2+2\|f\|\|g\|=(\|f\|+\|g\|)^2\ .$$

□

Mini-exercise 3.5.3. *Check that the complex exponentials e^{inx} form an orthonormal system with respect to the inner product* (3.5.11)*:*

$$\left(e^{imx},e^{inx}\right)=\delta_{mn}=\begin{cases}1, & m=n\\ 0 & m\neq n\end{cases} \tag{3.5.17}$$

(her and later on we sometimes write explicitly the variable x in the exponent).

Note that the Fourier coefficients (3.5.10) of a continuous 2π-periodic function $f(x)$ can be written as

$$c_n=(e^{inx},f),\quad n\in\mathbb{Z}\ . \tag{3.5.18}$$

This gives a simple interpretation of the Fourier coefficients as the coefficients of decomposition of the function f with respect to the orthonormal system made from exponentials. Observe also that the partial sum of the Fourier series

$$S_N(x)=\sum_{n=-N}^{N}c_ne^{inx} \tag{3.5.19}$$

can be interpreted as the orthogonal projection of the vector f onto the $(2N+1)$-dimensional linear subspace

$$V_N=\text{span}\left(1,e^{\pm ix},e^{\pm 2ix},\dots,e^{\pm iNx}\right) \tag{3.5.20}$$

consisting of all *trigonometric polynomials*

$$P_N(x)=\sum_{n=-N}^{N}p_ne^{inx} \tag{3.5.21}$$

of degree N. Here p_0, $p_{\pm 1}$, $\dots p_{\pm N}$ are arbitrary complex numbers.

Lemma 3.5.4. Bessel inequality *holds true:*

$$\sum_{n=-N}^{N}|c_n|^2\leq\|f\|^2. \tag{3.5.22}$$

Proof: We have

$$0 \leq \|f(x) - \sum_{n=-N}^{N} c_n e^{inx}\|^2 = \left(f(x) - \sum_{n=-N}^{N} c_n e^{inx}, f(x) - \sum_{n=-N}^{N} c_n e^{inx} \right)$$

$$= (f, f) - \sum_{n=-N}^{N} \left[c_n \left(f, e^{inx} \right) + \bar{c}_n \left(e^{inx}, f \right) \right] + \sum_{m,n=-N}^{N} \bar{c}_m c_n \left(e^{imx}, e^{inx} \right).$$

Using (3.5.18) and orthonormality (3.5.17) we recast the right hand side of the last equation in the form

$$(f, f) - \sum_{n=-N}^{N} |c_n|^2.$$

This proves Bessel inequality. □

Note that the proof shows also the equality

$$\boxed{\|f - S_N\|^2 = (f, f) - \sum_{n=-N}^{N} |c_n|^2} \tag{3.5.23}$$

and that geometrically the Bessel inequality says that the square length of the orthogonal projection of a vector onto the linear subspace V_N cannot be longer than the square length of the vector itself.

Corollary 3.5.5. *For any continuous 2π-periodic function $f(x)$ the series of squares of absolute values of Fourier coefficients converges:*

$$\sum_{n \in \mathbb{Z}} |c_n|^2 < \infty , \tag{3.5.24}$$

that is $\{c_n\} \in l^2(\mathbb{Z})$.

The following *extremal property* says that the N-th partial sum of the Fourier series gives the best L_2-approximation of the function $f(x)$ among all trigonometric polynomials of degree N.

Lemma 3.5.6. *For any trigonometric polynomial $P_N(x)$ of degree N the following inequality holds true*

$$\|f - S_N\| \leq \|f - P_N\|. \tag{3.5.25}$$

Here $S_N(x)$ is the N-th partial sum (3.5.19) *of the Fourier series of the function f. The equality in* (3.5.25) *takes place* iff *the trigonometric polynomial P_N coincides with S_N, i.e. iff*

$$p_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx, \quad n = 0, \pm 1, \pm 2, \ldots, \pm N,$$

Proof: From (3.5.18) we derive that for any $Q_N = \sum q_n e^{inx} \in V_N$,

$$(f - S_N, Q_N) = \sum (\bar{c}_n - \bar{c}_n) q_n = 0 .$$

Hence

$$\|f - P_N\|^2 = \|(f - S_N) + (S_N - P_N)\|^2 =$$
$$= (f - S_N, f - S_N) + 0 + 0 + (Q_N, Q_N) \geq \|f - S_N\|^2.$$

Here $Q_N = S_N - P_N$ and so the mixed terms vanish. Clearly the equality takes place *iff* $Q_N = 0$, i.e. $P_N = S_N$. □

We present now a beautiful reinforcement of Bessel inequality:

Lemma 3.5.7. *For any continuous 2π-periodic function the following* Parseval equality *holds true:*

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = \|f\|^2. \tag{3.5.26}$$

The Parseval equality can be considered as an infinite-dimensional analogue of the Pythagoras theorem: of a vector on the coordinate axes is equal to the square length of the vector. The Parseval equality is also referred to as *completeness* of the trigonometric system of functions

$$1, e^{\pm ix}, e^{\pm 2ix}, \dots.$$

For an infinite-dimensional space equipped with a Hermitean (or Euclidean) inner product, an orthonormal system with the property of completeness is the right analogue of the notion of an orthonormal *basis* of the space.

For the proof of the Parseval equality we shall need a very general result about uniform approximation of continuous functions on a compact K in a metric space.
Let $A \subset \mathcal{C}(K)$ be a subset of functions in the space of continuous real- or complex-valued functions on a compact K such that:
1. A is a *subalgebra* in $\mathcal{C}(K)$, i.e. for f, $g \in A$, α, $\beta \in \mathbb{R}$ (or α, $\beta \in \mathbb{C}$) the linear combination and the product belong to A:

$$\alpha\, f + \beta\, g \in A, \quad f \cdot g \in A.$$

2. The functions in A *separate points* in K, i.e., $\forall x$, $y \in K$, $x \neq y$ there exists $f \in A$ such that

$$f(x) \neq f(y).$$

3. The subalgebra is *non-degenerate*, i.e., $\forall x \in K$ there exists $f \in A$ such that $f(x) \neq 0$.

4. The subalgebra A is said to be *self-adjoint* if for any function $f \in A$ the complex conjugate function $\bar{f}$ also belongs to A.

Theorem 3.5.8 (Stone – Weierstrass)**.** *Given an algebra of functions $A \subset \mathcal{C}(K)$ that separates points, is non-degenerate and is self-adjoint then A is an everywhere dense subset in $\mathcal{C}(K)$.*

Recall that 'everywhere dense' means that for any $F \in \mathcal{C}(K)$ and arbitrary $\epsilon > 0$ there exists $f \in A$ such that

$$\sup_{x\in K} |F(x) - f(x)| < \epsilon,$$

or that any $F \in \mathcal{C}(K)$ can be uniformly (in x) approximated by elements in A. In the particular case of algebra of polynomials one obtains the classical Weierstrass theorem about polynomial approximations of continuous functions on a finite interval. For the needs of the theory of Fourier series one applies the Stone–Weierstrass theorem to the subalgebra of trigonometric polynomials in the space of continuous 2π-periodic functions. Such functions can be thought as functions on $[0, 2\pi]/$"$0 \sim 2\pi$" or on $S^1 = \{z \in \mathbb{C}|\, |z| = 1\}$, and $z = e^{ix}$ separates points of S^1.

Proof: of Lemma 3.5.7 [Parseval]
According to Stone – Weierstrass theorem any continuous 2π-periodic function can be uniformly

approximated by Fourier polynomials

$$P_N(x) = \sum_{n=-N}^{N} p_n e^{inx}. \tag{3.5.27}$$

That means that for a given function $f(x)$ and any $\epsilon > 0$ there exists a trigonometric polynomial $P_N(x)$ of some degree N such that

$$\sup_{x\in[0,2\pi]} |f(x) - P_N(x)| < \epsilon.$$

Then

$$\|f - P_N\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x) - P_N(x)|^2 dx < \epsilon^2.$$

Therefore, due to the extremal property (see Lemma 3.5.6), we obtain

$$\|f - S_N\|^2 < \epsilon^2.$$

But using (3.5.23) we get

$$\|f\|^2 - \sum_{n=-N}^{N} |c_n|^2 < \epsilon^2$$

and so we arrive at the proof of Lemma. □

Corollary. *$S_N \to f$ in $\| \|_{\mathcal{L}^2(I)}$ iff $f \in \mathcal{L}^2(I)$.*

Corollary 3.5.9. *Two continuous 2π-periodic functions $f(x)$, $g(x)$ with all equal Fourier coefficients identically coincide.*

Proof: Indeed, the difference $h(x) = f(x) - g(x)$ is continuous function with zero Fourier coefficients. The Parseval equality implies $\|h\|^2 = 0$. So $h(x) \equiv 0$ a.e., and by continuity everywhere □

We can state now a fundamental result of the theory of Fourier series.

Theorem 3.5.10. *For any 2π-periodic continuously differentiable complex valued function $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{C})$ the Fourier series is uniformly convergent to the function $f(x)$.*

In particular we conclude that any $\mathcal{C}^1$-smooth 2π-periodic function $f(x)$ can be represented as a sum of uniformly convergent Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx} dx. \tag{3.5.28}$$

Proof: Denote c'_n the Fourier coefficients of the derivative $f'(x)$. Integrating by parts we derive the following formula:

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\, dx = -\frac{1}{2\pi in} f(x)e^{-inx}\,|_0^{2\pi} + \frac{1}{2\pi in}\int_0^{2\pi} f'(x)e^{-inx}\, dx = -\frac{i}{n}c'_n.$$

This implies convergence of the series $\sum_{n\in\mathbb{Z}} |c_n|$. Indeed,

$$|c_n| = \frac{|c'_n|}{n} \le \frac{1}{2}\left(|c'_n|^2 + \frac{1}{n^2}\right)$$

(usare $0 \le \frac{1}{2}\left(|c'_n| - \frac{1}{n}\right)^2$). The series $\sum |c'_n|^2$ converges according to the Corollary 3.5.5; convergence of the series $\sum \frac{1}{n^2}$ is well known. Using Weierstrass theorem we conclude that the Fourier series converges absolutely

$$\sum_{n\in\mathbb{Z}} \left|c_n e^{inx}\right| = \sum_{n\in\mathbb{Z}} |c_n| < \infty$$

and by inspection we see that also uniformly. This last property assures that the sum of this series, which we denote $g(x)$ is a continuos function. The Fourier coefficients of g coincide with those of f: $\left(e^{inx}, g\right) = c_n$. Hence, by Corollary 3.5.9, $f(x) \equiv g(x)$. □

For the specific case of real valued function the Fourier coefficients satisfy the following property.

Lemma 3.5.11. *A 2π-periodic function $f \in \mathcal{C}^1$ is real valued* iff *its Fourier coefficients satisfy*

$$\bar{c}_n = c_{-n} \quad \text{for all} \quad n \in \mathbb{Z}. \tag{3.5.29}$$

Proof: Reality of the function can be written in the form

$$\bar{f}(x) = f(x).$$

Since

$$\overline{e^{inx}} = e^{-inx}$$

we have

$$\bar{c}_n = \frac{1}{2\pi}\int_0^{2\pi} \bar{f}(x) e^{inx} dx = c_{-n}.$$

Inverting this reasoning and using Theorem 3.5.10, we get the inverse implication. □

Note that the coefficient

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$$

is always real if $f(x)$ is a real valued function.

Let us establish the correspondence of the complex form (3.5.28) of the Fourier series of a real valued function with the real form.

Lemma 3.5.12. *Let $f(x)$ be a real valued 2π-periodic* <u>*smooth*</u> *function. Denote c_n its Fourier coefficients* (3.5.10). *Introduce coefficients*

$$a_n = c_n + c_{-n} \quad = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx, \quad n = 0, 1, 2, \dots \tag{3.5.30}$$

$$b_n = i(c_n - c_{-n}) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx, \quad n = 1, 2, \dots \tag{3.5.31}$$

Then the function $f(x)$ is represented as a sum of uniformly convergent Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n\ge 1}\left(a_n \cos nx + b_n \sin nx\right). \tag{3.5.32}$$

Proof:

$$c_0 + \sum_{n>0}(c_n + c_{-n})\cos(nx) + i(c_n - c_{-n})\sin(nx) = c_0 + \sum_{n>0}\left(c_n e^{inx} + c_{-n} e^{-inx}\right) = f(x)$$

by Theorem 3.5.10. □

Lemma 3.5.13. *For any real valued continuous 2π-periodic function $f(x)$ there is the following version*[3] *of Bessel inequality* (3.5.22)*:*

$$\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2) \le \frac{1}{\pi}\int_0^{2\pi} f^2(x)\,dx \tag{3.5.33}$$

and Parseval equality (3.5.26)

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_0^{2\pi} f^2(x)\,dx. \tag{3.5.34}$$

Proof: We show (3.5.26):

$$\frac{a_0^2}{2} + \sum_{n>0}^{\infty}(a_n^2 + b_n^2) = 2c_0^2 + \sum_{n>0}^{\infty}(c_n + c_{-n})^2 - (c_n - c_{-n})^2 = 2c_0^2 + \sum_{n>0}^{\infty} 4c_n c_{-n} =$$

$$2\left(c_0^2 + \sum_{n>0} c_n c_{-n} + \sum_{n<0} c_n c_{-n}\right) = 2\frac{1}{2\pi}\|f\| = \frac{1}{\pi}\int_0^{2\pi} f^2(x)\,dx\ ,$$

where in the penultimate eqaution we used (3.5.29) (since f is real) and the Parseval identity. Clearly the inequality (3.5.22) follows. □

For non-smooth functions the problem of convergence of Fourier series is more delicate. Let us consider an example giving some idea about the convergence of Fourier series for piecewise smooth functions. Consider the function

$$\operatorname{sign} x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}. \tag{3.5.35}$$

This function will be considered on the interval $[-\pi, \pi]$ and then continued 2π-periodically onto entire real line. The Fourier coefficients of this function can be easily computed:

$$a_n = 0, \quad b_n = \frac{2}{\pi}\frac{(1 - (-1)^n)}{n}.$$

So the Fourier series of this functions reads

$$\frac{4}{\pi}\sum_{k\ge 1}\frac{\sin(2k-1)x}{2k-1}. \tag{3.5.36}$$

One can prove that this series converges to the sign function at every point of the interval $(-\pi, \pi)$. Moreover this convergence is uniform on every closed subinterval non containing 0 or $\pm\pi$. However the character of convergence near the discontinuity points $x = 0$ and $x = \pm\pi$ is more complicated as one can see from the following graph of a partial sum of the series (3.5.36).

[3] Notice a change in the normalization of the L_2 norm.

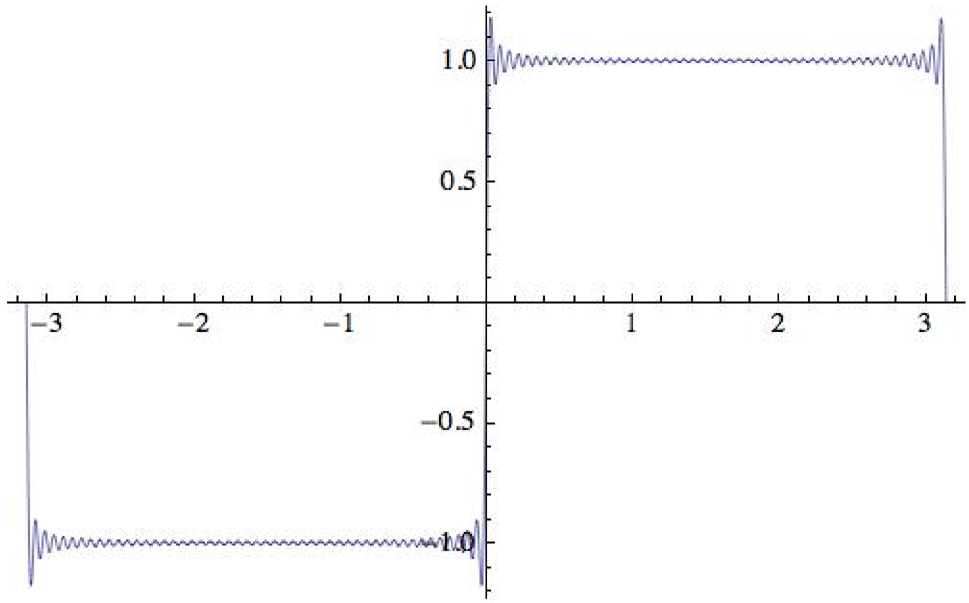

Fig. 6. Graph of the partial sum $S_n(x) = \frac{4}{\pi}\sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}$ for $n = 50$.

In general for piecewise smooth functions $f(x)$ with some number of discontinuity points one can prove that the Fourier series converges to the mean value $\frac{1}{2}\left(f(x_0+0)+f(x_0-0)\right)$ at every first kind discontinuity point x_0. The non vanishing oscillatory behavior of partial sums near discontinuity points is known as *Gibbs phenomenon* (see Exercise 3.10.9 below).

For more general classes of functions, the situation is more complicated. A natural functional space turns out to be the $\mathcal{L}^2$ space in which case the assignment (Fourier transform) $f \rightsquigarrow \{c_n\}$ becomes unitary isomorphism (isometry) onto $l^2(\mathbb{Z})$. Carlson: for $f \in \mathcal{L}^2(I)$ (so e.g. $f \in \mathcal{C}$), $S_N(x) \to f(x)$ a.e. But for any x, most of $\mathcal{C}$ does not converge at x. Actually $f \rightsquigarrow \{c_n\}$ works for $f \in \mathcal{L}^1$ ($\mathcal{L}^2(I) \subset \mathcal{L}^1(I)$), but $S_N(x)$ may diverge everywhere [Kolmogorov]. Non si sa condizioni per c_n di$f \in \mathcal{C}$.

Let us return to the wave equation. Using the theory of Fourier series we can represent any periodic solution to the Cauchy problem (3.5.2) as a superposition of solutions of the form (3.5.5), (3.5.7). Namely, let us expand the initial data in Fourier series:

$$\phi(x) = \sum_{n\in\mathbb{Z}} \phi_n e^{inx}, \quad \psi(x) = \sum_{n\in\mathbb{Z}} \psi_n e^{inx}. \tag{3.5.37}$$

Then the solution to the periodic Cauchy problem reads

$$\boxed{u(x,t) = \sum_{n\in\mathbb{Z}} \phi_n e^{inx}\cos ant + \psi_0 t + \frac{1}{a}\sum_{n\in\mathbb{Z}\setminus 0} \psi_n e^{inx}\frac{\sin ant}{n}.} \tag{3.5.38}$$

Remark 3.5.14. *The formula* (3.5.38) *says that the solutions*

$$\begin{gathered} u_n^{(1)}(x,t) = e^{inx}\cos ant \\ u_n^{(2)}(x,t) = \begin{cases} t, & n = 0 \\ e^{inx}\frac{\sin ant}{n}, & n \neq 0 \end{cases} \end{gathered} \tag{3.5.39}$$

for $n \in \mathbb{Z}$ form a basis in the space of 2π-periodic solutions to the wave equation. Observe that all these solutions can be written in the so-called separated form

$$u(x,t) = X(x)T(t) \tag{3.5.40}$$

for some smooth functions $X(x)$ and $T(t)$. A rather general method of separation of variables *for solving boundary value problems for linear PDEs has this observation as a starting point. This method will be explained later on.*

3.6 Finite vibrating string. Standing waves

To discuss a finite string we start with two simple remarks. When working with functions with an arbitrary period, we can just use the scale transformed solutions to the wave equation as in Lemma 3.3.2

$$\tilde{u}(x,t) = u\,(c\,x, c\,t)\,, \quad c \neq 0, \tag{3.6.1}$$

which is periodic in x with the period $\frac{2\pi}{c}$ if $u(x,t)$ was 2π-periodic.

Next, let r_y denote the reflection of the x axis with respect to some fixed point y, that is $r_y : x \mapsto 2y - x$. Then the composition $r_{y+l}r_y$ of two reflections is just the translation $t_{2l} : x \mapsto x + 2l$ of x by $2l$ (to the right if $l > 0$). Therefore, if f is symmetric with respect to both r_y and r_{y+l}, or if f is antisymmetric with respect to both r_y and r_{y+l}, then it is invariant under t_{2l}, i.e. it is $2l$-periodic. Note actually that since $r_y^2 = 1$ we have also $r_y = r_{y+l}t_{2l}$ and $r_{y+l} = t_{2l}r_y$. Therefore any two of the three properties of f

- r_y-symmetry (respectively r_y-antisymmetry)
- r_{y+l}-symmetry (respectively r_{y+l}-antisymmetry)
- $2l$-periodicity

imply the third one.

Let us proceed to considering the oscillations of the string of the length l with initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in [0,l] \tag{3.6.2}$$

We impose additionally the boundary conditions, either

i) fixed endpoints, or

ii) free endpoints.

So we have to solve the following mixed problem for the wave equation (3.1.1): (3.6.2) together with either

$$u(0,t) = 0, \quad u(l,t) = 0 \quad \text{for all} \quad t > 0, \tag{3.6.3}$$

or

$$u_x(0,t) = 0, \quad u_x(l,t) = 0 \quad \text{for all} \quad t > 0. \tag{3.6.4}$$

The idea of solution, again is a suitable extension of the problem onto entire line.

Lemma 3.6.1. *Let the initial data $\phi(x)$, $\psi(x)$ of the Cauchy problem* (3.2.1) *for the wave equation on $\mathbb{R}$ be symmetric (resp. antisymmetric) with respect to r_0 and r_l, and hence $2l$-periodic functions. Then the solution $u(x,t)$ will also be a function symmetric (resp. antisymmetric) with respect to r_0 and r_l, (hence $2l$-periodic) for all t, satisfying the boundary conditions* (3.6.4)*.*

Proof: Straightforward adaptation of the material in Sect. 3.4. □

The above Lemma gives an algorithm for solving the mixed problem (3.6.2, 3.6.3) or (3.6.2, 3.6.4) for the wave equation. Namely, we extend the initial data $\phi(x)$, $\psi(x)$ from the interval $[0, l]$ onto the real axis as symmetric (resp. antisymmetric) with respect to r_0 and r_l (hence $2l$-periodic) functions. After this we apply D'Alembert formula to the extended initial data. The resulting solution will satisfy the initial conditions (3.6.3) on the interval $[0, l]$ as well as the boundary conditions (3.6.4) at the end points of the interval.

We will apply now the technique of Fourier series to the mixed problems (3.6.2), (3.6.3)-(3.6.4).

Lemma 3.6.2. *Let a 2π-periodic functions $f(x)$ be represented as the sum of its Fourier series*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The function $f(x)$ is even/odd iff *the Fourier coefficients satisfy*

$$c_{-n} = \pm c_n$$

respectively.

Proof: For an even/odd function one must have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \pm f(-x) e^{-inx} dx = \frac{1}{2\pi}\int_{\pi}^{-\pi} \pm f(x) e^{inx} d(-x) = \pm c_{-n},$$

where we used that $f(x) = \pm f(-x)$ and in the second integral we changed the integration variable $x \mapsto -x$. □

Corollary 3.6.3. *Any even/odd smooth 2π-periodic function can be expanded in Fourier series in cosines/sines:*

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos nx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx, \quad \text{if } f \text{ is even} \tag{3.6.5}$$

$$f(x) = \sum_{n\geq 1} b_n \sin nx, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx, \quad \text{if } f \text{ is odd.} \tag{3.6.6}$$

Proof: Let us consider the case of an odd function. In this case we have $c_{-n} = -c_n$, and, in particular, $c_0 = 0$, so we rewrite the Fourier series in the following form

$$\begin{aligned} f(x) &= \sum_{n\geq 1} c_n e^{inx} + \sum_{n\leq -1} c_n e^{inx} \\ &= \sum_{n\geq 1} c_n \left(e^{inx} - e^{-inx}\right) = 2i \sum_{n\geq 1} c_n \sin nx. \end{aligned}$$

Denote

$$b_n = 2i c_n, \quad n \geq 1$$

and compute it

$$b_n = \frac{2i}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{i}{\pi}\int_0^{\pi} f(x) e^{-inx} dx + \frac{i}{\pi}\int_{-\pi}^{0} f(x) e^{-inx} dx.$$

In the second integral we change the integration variable $x \mapsto -x$ and use $f(-x) = -f(x)$ to arrive at

$$b_n = \frac{i}{\pi}\int_0^\pi f(x)e^{-inx}dx + \frac{i}{\pi}\int_\pi^0 f(x)e^{inx}dx$$

$$= \frac{i}{\pi}\int_0^\pi f(x)\left[e^{-inx} - e^{inx}\right] dx = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\, dx.$$

Similarly one shows the case of an even function. □

Let us return to the solution to the wave equation on the interval $[0, l]$ with fixed endpoints boundary condition. Using the rescaling $x \mapsto \frac{\pi}{l}x, t \mapsto \frac{\pi}{l}t$, the last corollary and (the imaginary parts of) (3.5.5), (3.5.7), we arrive at the following

Theorem 3.6.4. *Let $\phi(x) \in \mathcal{C}^2([0, l])$, $\psi(x) \in \mathcal{C}^1([0, l])$ be two arbitrary smooth functions. Then the solution to the mixed problem* (3.6.2), (3.6.3) *for the wave equation is written in the form*

$$\boxed{u(x,t) = \sum_{n\geq 1} \sin\frac{\pi n x}{l}\left(b_n \cos\frac{\pi a n t}{l} + \dot b_n \sin\frac{\pi a n t}{l}\right)} \tag{3.6.7}$$

$$b_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{\pi n x}{l}\, dx, \quad \dot b_n = \frac{2}{\pi a n}\int_0^l \psi(x)\sin\frac{\pi n x}{l}\, dx.$$

Particular solutions ($b_m = \delta_{mn}, \dot b_m = 0$ or $b_m = 0, \dot b_m = \delta_{mn}$) to the wave equation giving a basis in the space of all solutions satisfying the boundary conditions (3.6.3) have the form

$$u_n^{(1)}(x,t) = \sin\frac{\pi n x}{l}\cos\frac{\pi a n t}{l}, \quad u_n^{(2)}(x,t) = \sin\frac{\pi n x}{l}\sin\frac{\pi a n t}{l}, \quad n = 1, 2, \dots \tag{3.6.8}$$

and $b_n u_n^{(1)}$, $\dot b_n u_n^{(2)}$ are called *standing waves*. Observe that these solutions have the separated form (3.5.40). The shape of these waves essentially does not change in time, only the size does change. In particular the location of the *nodes*

$$x_k = k\,\frac{l}{n}, \quad k = 0, 1, \dots, n \tag{3.6.9}$$

of the n-th solution $u_n^{(1)}(x,t)$ or $u_n^{(2)}(x,t)$ does not depend on time. The n-th standing waves (3.6.8) has $(n + 1)$ nodes on the string. The solution takes zero values at the nodes at all times.

Similarly, the solution to the wave equation on the interval $[0, l]$ with free endpoints boundary condition, using the rescaling $x \mapsto \frac{\pi}{l}x$, the last corollary and (the real parts of) (3.5.5), (3.5.7), we arrive at the following

Theorem 3.6.5. *Let $\phi(x) \in \mathcal{C}^2([0, l])$, $\psi(x) \in \mathcal{C}^1([0, l])$ be two arbitrary smooth functions. Then the solution to the mixed problem* (3.6.2), (3.6.4) *for the wave equation is written in the form*

$$\boxed{u(x,t) = \dot a_0 t + \sum_{n\geq 1} \cos\frac{\pi n x}{l}\left(a_n \cos\frac{\pi a n t}{l} + \dot a_n \sin\frac{\pi a n t}{l}\right)} \tag{3.6.10}$$

$$a_n = \frac{2}{l}\int_0^l \phi(x)\cos\frac{\pi n x}{l}\, dx, \quad \dot a_0 = \frac{1}{l}\int_0^l \psi(x)\, dx, \quad \dot a_n = \frac{2}{\pi a n}\int_0^l \psi(x)\cos\frac{\pi n x}{l}\, dx, \text{ for } n \geq 1\ .$$

Particular solutions to the wave equation giving a basis in the space of all solutions satisfying the boundary conditions (3.6.3) have the form

$$v^{(0)}(x,t) = t, \quad v_n^{(1)}(x,t) = \cos\frac{\pi n x}{l}\cos\frac{\pi a n t}{l}, \quad v_n^{(2)}(x,t) = \cos\frac{\pi n x}{l}\sin\frac{\pi a n t}{l}, \quad n \geq 1 \tag{3.6.11}$$

and $a_n v_n^{(1)}$, $\dot{a}_n v_n^{(2)}$ are also called *standing waves*. Observe that these solutions have the separated form (3.5.40) too and again the shape of these waves again essentially does not change in time, only the size does change. In particular the location of the *nodes*

$$x_k = (2k+1)\frac{l}{2n}, \quad k = 0, 1, \ldots, n-1 \tag{3.6.12}$$

of the n-th solution $v_n^{(1)}(x,t)$ or $v_n^{(2)}(x,t)$ does not depend on time. The n-th standing waves (3.6.11) has n nodes on the string. The solution takes zero values at the nodes at all times.

3.7 Energy of vibrating string

Consider the energy functional of the vibrating string with fixed points $x = 0$ and $x = l$. It is clear that the kinetic energy of the string at the moment t is equal to

$$K = \frac{1}{2}\int_0^l \rho\, u_t^2(x,t)\, dx. \tag{3.7.1}$$

Let us now compute the potential energy U of the string. By definition U is equal to the work done by the elastic force moving the string from the equilibrium $u \equiv 0$ to the actual position given by the graph $u(x)$. The motion can be described by the one-parameter family of curves $v(x;s), \quad v \in \mathcal{C}^\infty([0,l]\times[0,1])$, where the parameter s changes from $s = 0$ (the equilibrium) to $s = 1$ (the actual position of the string). (For instance we can take $v(x;s) = s\, u(x)$).
As we already know the vertical component of the force acting on the interval of the string $v(x;s)$ between x and $x + \delta x$ is equal to

$$F = T\,\left(v_x(x+\delta x; s) - v_x(x;s)\right) \simeq T\, v_{xx}(x,s)\,\delta x.$$

The work U_s to move this interval from the position $v(x;s)$ to $v(x;s+\delta s)$ is therefore equal to

$$U_s = -F\cdot[v(x;s+\delta s) - v(x;s)] \simeq -T\, v_{xx}(x,s)\delta x\, v_s(x,s)\delta s$$

(the negative sign since the direction of the force is opposite to the direction of the displacement). The total work of the elastic forces for moving the string of length l from the equilibrium $s = 0$ to the given configuration at $s = 1$ is obtained by integration:

$$U = -\int_0^1 ds \int_0^l T\, v_{xx}(x,s) v_s(x,s)\, dx\ .$$

Integrating by parts we get

$$U = \int_0^1 ds \int_0^l T\, v_x(x,s) v_{xs}(x,s)\, dx = \frac{1}{2}\int_0^1 ds \int_0^l T\,(v_x^2)_s(x,s)\, dx = \frac{1}{2}\int_0^l T\, v_x^2(x,s)\, dx|_{s=0}^{s=1}$$

and using the boundary conditions,

$$v(x,1) = u(x),\ v(x,0) = 0$$

and their x-derivatives, we finally arrive at the following expression for the potential energy:

$$U = \frac{1}{2}\int_0^l T\, u_x^2(x)\, dx. \tag{3.7.2}$$

The sum of (3.7.1) and (3.7.2) gives the formula for the total energy $E = E(t)$ of the vibrating string at the moment t

$$E = K + U = \frac{1}{2}\int_0^l \left(\rho\, u_t^2(x,t) + T\, u_x^2(x,t)\right)\, dx. \tag{3.7.3}$$

Remark 3.7.1. *Note that U and E does not depend on the path $s \mapsto v(x,s)$.*

We will now prove that the total energy E of vibrating string with fixed end points does not depend on time.

Lemma 3.7.2. *Let the function $u(x,t)$ satisfy the wave equation $u_{tt} = a^2 u_{xx}$. Then the following identity holds true*

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\, u_t^2(x,t) + \frac{1}{2} T\, u_x^2(x,t)\right) = \frac{\partial}{\partial x}\left(T\, u_x u_t\right). \tag{3.7.4}$$

Proof: A straightforward differentiation using the wave equation yields

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\, u_t^2(x,t) + \frac{1}{2} T\, u_x^2(x,t)\right) = \rho\, a^2 u_t u_{xx} + T\, u_x u_{xt}.$$

Recalling that

$$a^2 = \frac{T}{\rho}$$

we rewrite the last equation in the form

$$= T\left(u_t u_{xx} + u_{tx} u_x\right) = T\,\left(u_t u_x\right)_x.$$

□

Corollary 3.7.3. *Denote $E_{[a,b]}(t)$ the energy of a segment of vibrating string*

$$E_{[a,b]}(t) = \int_a^b \left(\frac{1}{2}\rho\, u_t^2(x,t) + \frac{1}{2} T\, u_x^2(x,t)\right) dx. \tag{3.7.5}$$

The following formula describes the dependence of this energy on time:

$$\boxed{\frac{d}{dt} E_{[a,b]}(t) = T\, u_t u_x|_{x=b} - T\, u_t u_x|_{x=a}.} \tag{3.7.6}$$

Remark 3.7.4. *In physics literature the quantity*

$$\frac{1}{2}\rho\, u_t^2(x,t) + \frac{1}{2} T\, u_x^2(x,t) \tag{3.7.7}$$

is called energy density*. It is equal to the energy of a small piece of the string from x to $x + dx$ at the moment t. The total energy of a piece of a string is obtained by integration of this density in x. Another important notion is the* flux density

$$T\, u_t u_x. \tag{3.7.8}$$

The formula (3.7.6) *says that the change of the energy of a given piece of the string for the time dt is given by the total flux through the boundary of the piece.*

Finally we arrive at the *conservation law* of the total energy of a vibrating string with fixed end points.

Theorem 3.7.5. *The total energy* (3.7.3) *of the vibrating string with fixed end points does not depend on t:*

$$\frac{d}{dt}E = 0.$$

Proof: The formula (3.7.6) for the particular case $a = 0$, $b = l$ gives

$$\frac{d}{dt}E = T\,\left(u_t(l,t)u_x(l,t) - u_t(0,t)u_x(0,t)\right) = 0$$

since

$$u_t(0,t) = \partial_t u(0,t) = 0, \quad u_t(l,t) = \partial_t u(l,t) = 0$$

due to the boundary conditions $u(0,t) = u(l,t) = 0$, $\forall t$. □

The conservation law of total energy means that the vibrating string is a *conservative system.*

Theorem 3.7.6. *The formula for the total energy remains the same for a vibrating string of finite length with free boundary conditions $u_x(0,t) = u_x(l,t) = 0$ and the previous proof of the conservation law is valid as well.*

Proposition 3.7.7. *The energy of the vibrating string represented as sum* (3.6.7) *of standing waves $b_n u_n^{(1)}$, $\dot b_n u_n^{(2)}$ (see* (3.6.8)*) is equal to the sum of energies of these standing waves. The energy of the vibrating string represented as sum* (3.6.10) *is equal to the energy of the uniformly moving solution $\dot a_0\, t$ plus the sum of energies of the standing waves $a_n v_n^{(1)}$, $\dot a_n v_n^{(2)}$ (see* (3.6.11)*).*

Proof: Since E is conserved it suffices to compute E at $t = 0$,

$$E(0) = \frac{1}{2}\int_0^l \left(\rho\, u_t^2(x,0) + T\, u_x^2(x,0)\right)\, dx$$

$$= \frac{T}{2}\frac{\pi^2}{l^2}\sum_{m,n>0}\int_0^l mn\left(\dot b_m \dot b_n \sin\frac{\pi m x}{l}\sin\frac{\pi n x}{l} + b_m b_n \cos\frac{\pi m x}{l}\cos\frac{\pi n x}{l}\right)\, dx$$

$$= \frac{\pi T}{4}\frac{\pi}{l}\sum_{n>0} n^2\left(b_n^2 + \dot b_n^2\right)$$

due to the othogonality of cosines and of sines. The result is however just the sum of the energies of the individual contributions of standing waves. Similarly one shows the second statement. □

The conservation of total energy can be used for proving uniqeness of solution for the wave equation. Indeed, if $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ are two solutions vanishing at $x = 0$ and $x = l$ with the same initial data. The difference

$$u(x,t) = u^{(2)}(x,t) - u^{(1)}(x,t)$$

solves wave equation, satisfies the same boundary conditions and has zero initial data $u(x,0) = \phi(x) = 0$, $u_t(x,0) = \psi(x) = 0$. The conservation of energy for this solution gives

$$E(t) = \int_0^l \left(\frac{1}{2}\rho\, u_t^2(x,t) + \frac{1}{2}T\, u_x^2(x,t)\right)\, dx = E(0) = \int_0^l \left(\frac{1}{2}\rho\, \psi^2(x) + \frac{1}{2}T\, \phi_x^2(x)\right)\, dx = 0.$$

Hence $u_x(x,t) = u_t(x,t) = 0$ for all x, t. Using the boundary conditions one concludes that $u(x,t) \equiv 0$.

3.8 Equazione delle onde in $\mathbb{R}^n$

L'equazione delle onde in $\mathbb{R}^2$ che descrive l'evoluzione temporale della corda vibrante unidimensionale, in $\mathbb{R}^3$ descrive le vibrazioni della membrana e in $\mathbb{R}^4$ del corpo elastico. Più in generale consideriamo il problema di Cauchy per $u \in \mathcal{C}^m(\mathbb{R}^n \times [0,\infty))$:

$$\begin{aligned} u_{tt} - \Delta u = 0 \qquad & \text{su} \qquad \mathbb{R}^n \times (0,\infty) \\ u = \phi,\, u_t = \psi \qquad & \text{su} \qquad \mathbb{R}^n \times \{t=0\}. \end{aligned} \tag{3.8.9}$$

Studieremo prima la media di u sulla sfera $S(x,r)$ in $\mathbb{R}^n$ con centro nel punto x e raggio $r > 0$. Definiamo una funzione di r, t con parametro x

$$U(x;r,t) := \fint_{S(x,r)} u(y;t) dS(y),$$

dove $\fint$ è l'integrale 'normalizzato', ovvero

$$\fint_{S(x,r)} f dS := \frac{1}{n\alpha_n r^{n-1}} \int_{S(x,r)} f dS,$$

con

$$\alpha_n := \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

dato dal volume di una palla unitaria in $\mathbb{R}^n$ (uguale anche a $\frac{1}{n}$ per l'area di una sfera unitaria).

Definiamo anche

$$\Phi(x;r) := \fint_{S(x,r)} \phi(y) dS(y),$$

$$\Psi(x;r) := \fint_{S(x,r)} \psi(y) dS(y).$$

Lemma 3.8.8. *(Eulero-Poisson-Darboux)*
Per ogni x (fisso), $U \in \mathcal{C}^m([0,\infty) \times (0,\infty))$ e inoltre

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \quad \text{su} \quad \mathbb{R}_+ \times (0,\infty), \tag{3.8.10}$$

$$U = \Phi, \quad U_t = \Psi \quad \text{su} \quad \mathbb{R}_+ \times \{t=0\}. \tag{3.8.11}$$

Proof: Con il cambio di variabile di integrazione $y = x + rz$ e grazie alla normalizzazione abbiamo

$$U_r(x;r,t) = \partial_r \fint_{S(x,r)} u(x+rz;t) dS(z) = \fint_{S(x,r)} \nabla u(x+rz;t) \cdot z \, dS(z)$$

che si può riscrivere con formula di ([Evans, App. C]) come

$$\frac{r}{n} \fint_{B(x,r)} \Delta u(y;t) d(y), \tag{3.8.12}$$

dove $\fint_{B(x,r)} = \frac{1}{\alpha_n r^n} \int_{B(x,r)}$ è l'integrale "normalizzato" sulla palla con centro x e raggio r.

Questa espressione converge a 0 quando $r \to 0$, il che dimostra che U_r è continua su $\mathbb{R}_+ \times (0,\infty)$ e si estende per contunuità anche sul bordo $r = 0$. Analogamente si dimostrano le stesse proprietà per U_{rr}, U_{rt}, etc., perciò $u \in \mathcal{C}^m([0,\infty) \times [0,\infty)$.

Inoltre, da (3.8.12) segue che

$$U_r(x;r,t) = \frac{r}{n} \fint_{B(x,r)} u_{tt}(y;t)d(y)$$

e perciò

$$r^{n-1}U_r(x;r,t) = \frac{1}{n\alpha_n} \int_{B(x,r)} u_{tt}(y;t)d(y). \tag{3.8.13}$$

Derivando (3.8.13) rispetto r, da un lato otteniamo

$$(r^{n-1}U_r(x;r,t))_r = (n-1)r^{n-2}U_r(x;r,t)) + r^{n-1}U_{rr}(x;r,t))$$

e dall'altro

$$\frac{1}{n\alpha_n}\partial_r \int_{B(x,r)} u_{tt}(y;t)d(y) = \frac{1}{n\alpha_n} \int_{S(x,r)} u_{tt}(y;t)dS(y) = r^{n-1} \fint_{S(x,r)} u_{tt}(y;t)d(y) = r^{n-1}U_{tt}$$

che dimostra (3.8.10). Inoltre (3.8.11) segue dalla definizione. □

Trasformeremo ora (3.8.10), (3.8.11) nel problema che già sapiamo a risolvere. Concentriamoci sul caso $n = 3$. Sia U soluzione di (3.8.10). Poniamo

$$\tilde{U} = rU, \quad \tilde{\Phi} = r\Phi, \quad \tilde{\Psi} = r\Psi$$

e calcoliamo

$$\tilde{U}_{tt} = rU_{tt} = rU_{rr} + 2U_r = (rU_r + U)_r = (\tilde{U}_r)_r$$

che è niente altro che l'equazione delle onde in dimensione 2 (!). Inoltre, ci sono i dati iniziali

$$\tilde{U}(r,0) = \tilde{\Phi}(r), \quad \tilde{U}_t(r,0) = \tilde{\Psi}(r) \quad \text{su} \quad \mathbb{R}^n \times \{t=0\}$$

e per definizione di $\tilde{U}$ abbiamo anche la condizione al contorno

$$\tilde{U}(0,t) = 0, \forall t,$$

ovvero un problema misto con bordo fissato. Ma questo problema sappiamo che si risolve con la formula di D'Alembert estesa per antisimmetria su tutti i valori di r reali. In particolare, per $r < t$ abbiamo

$$\tilde{U}(x;r,t) = \frac{1}{2}\left(\tilde{\Phi}(r+t) - \tilde{\Phi}(t-r)\right) + \frac{1}{2}\int_{t-r}^{r+t} \tilde{\Psi}(s)\, ds. \tag{3.8.14}$$

Ci interessa la regione $r < t$ perchè vorremmo calcolare

$$u(x,t) = lim_{r\to 0_+} U(x;r,t) = lim_{r\to 0_+} \frac{1}{r}\tilde{U}(x;r,t) = \tilde{\Phi}'(t) + \tilde{\Psi}(t)$$

$$= \partial_t \left(t \fint_{S(x,t)} \phi(y)\, dS(y) \right) + t \fint_{S(x,t)} \psi(y)\, dS(y).$$

Usando la regola di Leibniz e con il cambio della variabile di integrazione $y = x + rz$, l'ultima espressione diventa

$$\fint_{S(x,t)} (t\psi(y) + \phi(y))\; dS(y) + t \fint_{S(0,1)} \nabla\phi(x+tz)\cdot z\, dS(z).$$

Tornando alle variabili y con cambio inverso $z = \frac{y-x}{t}$ otteniamo la formula di Kirchhoff

$$\boxed{u(x,t) = \fint_{S(x,t)} (t\psi(y) + \phi(y) + \nabla\phi(y)\cdot(y-x))\; dS(y), \quad x \in \mathbb{R}^3,\, t > 0,}$$

per la soluzione del problema di Cauchy (3.8.9).

Questo procedimento non funziona per $n = 2$. Però usando le soluzioni per $n = 3$, che sono costanti lungo x_3, dopo alcuni passagi [Evans, p. 73-74] si arriva alla formula di Poisson:

$$u(x,t) = \frac{1}{2} \fint_{B(x,t)} \left(t\phi(y) + t^2\phi(y) + t\nabla\psi(y)\cdot(y-x) \right) \left(t^2 - \|y-x\|^2 \right)^{-\frac{1}{2}} \, dS(y),$$

Si possono scrivere anche soluzioni per $n \geq 4$, bisogna però assumere regolarità di ϕ e ψ più alta (vedi [Evans]).

3.9 Equazione delle onde nonomogenea

Sia $u(x,t;s)$ una famiglia a un parametro $s > 0$ di soluzioni, ovvero $\forall s$

$$\begin{aligned} &u_{tt}(\cdot,\cdot,s) - \Delta u(\cdot,\cdot,s) = 0 && \text{su} \quad \mathbb{R}^n \times (s,\infty) \\ &u(\cdot,s;s) = 0,\, u_t = \psi(\cdot,s;s) = f(\cdot;s), && \text{su} \quad \mathbb{R}^n, \end{aligned} \tag{3.9.15}$$

dove $f \in \mathcal{C}^{[\frac{n}{2}]+1}(\mathbb{R}^n \times (0,\infty))$.

Theorem 3.9.9 ("principio di Duhamel"). *Poniamo*

$$u(x,t) = \int_0^t u(x,t;s)ds, \quad x \in \mathbb{R}^n,\, t \geq 0.$$

Tale u soddisfa
i). $u \in \mathcal{C}^2(\mathbb{R}^n \times (0,\infty))$
ii). $u_{tt} - \Delta u = f$ su $\mathbb{R}^n \times (0,\infty)$
iii). $lim_{x,t\to x_0,0}\, u(x,t) = 0$, $lim_{x,t\to x_0,0}\, u_t(x,t) = 0$, $\forall x_0 \in \mathbb{R}^n$.

Proof:
i). segue dalla proprietà delle soluzioni per $n \geq 2$
ii). calcoliamo

$$u_t(x,t) = u(x,t;t) + \int_0^t u_t(x,t;s)ds = \int_0^t u_t(x,t;s)ds,$$

e di conseguenza

$$u_{tt}(x,t) = u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)ds = f(x,t) + \int_0^t \Delta u\,(x,t;s)ds = f(x,t) + \Delta u\,(x,t)$$

iii). chiaramente $u(x,0) = 0$ e $u_t(x,t) = 0$, $\forall x \in \mathbb{R}^n$. □

Esempio.
Assumiamo $n = 1$. Usiamo la formula di D'Alembert sostituendo t con $t - s$

$$u_t(x,t;s) = \frac{1}{2} \int_{x-t-s}^{x+t-s} f(y,s)dy, \quad \text{per} \quad t > s,$$

perciò

$$u(x,t) = \frac{1}{2} \int_0^t \int_{x-t-s}^{x+t-s} f(y,s)\,dy\,ds = \frac{1}{2} \int_0^t ds \int_{x-s}^{x+s} f(y,t-s)\,dy.$$

Esempio.
Assumiamo $n = 3$. Usiamo la formula di Kirchhoff con $\phi = 0$, e sostituendo $\phi(y)$ con $f(y,s)$ e t con $t - s$

$$u_t(x,t;s) = (t-s) \fint_{S(x,t-s)} f(y,s)\,dS(y)$$

perciò

$$u(x,t) = \int_0^t (t-s) \left(\fint_{S(x,t-s)} f(y,s)\, dS(y) \right) ds =$$

$$\frac{1}{4\pi} \int_0^t \int_{S(x,t-s)} (t-s)^{-1} f(y,s)\, dS(y)\, ds =$$

$$\frac{1}{4\pi} \int_0^t \int_{S(x,r)} r^{-1} f(y,t-r)\, dS(y)\, dr =$$

$$\frac{1}{4\pi} \int_{B(x,t)} \frac{f(y,t-|x-y|)}{|x-y|}\, dy.$$

Vediamo che f gioca il ruolo di 'potenziale' e che l'ultima espressione contiene il 'potenziale ritardato' per il tempo $t = |x-y|$.

Osservazione.
Per risolvere

$$u_{tt} - \Delta u = f$$
$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) \tag{3.9.16}$$

basta sommare la soluzione dell'equazzione omogenea con dati iniziali ϕ, ψ (p. es. la formula di D'Alembert o di Kirchhoff) e la soluzione nonomogenea (Duhamel) con dati iniziali nulli.

3.10 Exercises to Section 3

Exercise 3.10.1. *For few instants of time $t \geq 0$ make a graph of the solution $u(x,t)$ to the wave equation with the initial data*

$$u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 1, & x \in [x_0, x_1] \\ 0 & \text{otherwise} \end{cases}, \quad -\infty < x < \infty.$$

Exercise 3.10.2. *Let the initial data $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$ of the Cauchy problem for the wave equation on $-\infty < x < \infty$ have the following form: the graph of $\phi(x)$ consists of two isosceles triangles with the non-overlapping bases $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ (i.e., $\beta_1 < \alpha_2$) of the heights h_1 and h_2 respectively, and $\psi(x) \equiv 0$. Denote $u(x,t)$ the solution to the problem. Find*

$$\max_{x \in \mathbb{R},\, t > 0} u(x,t).$$

Compare this number with

$$\max_{x \in \mathbb{R},\, t \geq 0} u(x,t).$$

Exercise 3.10.3. *For few instants of time $t \geq 0$ make a graph of the solution $u(x,t)$ to the wave equation on the half line $x \geq 0$ with the free boundary condition*

$$u_x(0,t) = 0$$

and with the initial data

$$u(x,0) = \phi(x), \quad u_t(x,0) = 0, \quad x > 0$$

where the graph of the function $\phi(x)$ is an isosceles triangle of height 1 and the base $[l, 3l]$.

Exercise 3.10.4. *For few instants of time $t \geq 0$ make a graph of the solution $u(x,t)$ to the wave equation on the half line $x \geq 0$ with the fixed point boundary condition*

$$u(0,t) = 0$$

and with the initial data

$$u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 1, & x \in [l, 3l] \\ 0, & \text{otherwise} \end{cases}, \quad x > 0.$$

Exercise 3.10.5. *Prove that*

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2} \quad \text{for} \quad 0 < x < 2\pi.$$

Compute the sum of the Fourier series for all other values of $x \in \mathbb{R}$.

Exercise 3.10.6. *Compute the sums of the following Fourier series:*

$$\sum_{n=1}^{\infty} \frac{\sin 2nx}{2n}, \quad 0 < x < \pi;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx, \quad |x| < \pi.$$

Exercise 3.10.7. *Prove that*

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx, \quad |x| < \pi.$$

Exercise 3.10.8. *Compute the sums of the following Fourier series:*

$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

Exercise 3.10.9. *Denote*

$$S_n(x) = \frac{4}{\pi} \sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}$$

the n-th partial sum of the Fourier series (3.5.36). *Prove that*

1) for any $x \in (-\pi, \pi)$

$$\lim_{n \to \infty} S_n(x) = \operatorname{sign} x.$$

Hint: derive the following expression for the derivative

$$S_n'(x) = \frac{2}{\pi} \frac{\sin 2nx}{\sin x}.$$

2) Verify that the n-th partial sum has a maximum at

$$x_n = \frac{\pi}{2n}.$$

3) Prove that

$$S_n(x_n) = \frac{2}{\pi} \sum_{k=1}^{n} \frac{\pi}{n} \cdot \frac{\sin \frac{(2k-1)\pi}{2n}}{\frac{(2k-1)\pi}{2n}} \to \frac{2}{\pi} \int_0^{\pi} \frac{\sin x}{x}\, dx \simeq 1.17898$$

for $n \to \infty$*.*

Thus for the trigonometric series (3.5.36)

$$\limsup_{n\to\infty} S_n(x) > 1 \quad \text{for} \quad x > 0.$$

In a similar way one can prove that

$$\liminf_{n\to\infty} S_n(x) < -1 \quad \text{for} \quad x < 0.$$