4 Laplace equation

4.1 Ill-posedness of Cauchy problem for Laplace equation

In the study of various classes of solutions to the Cauchy problem for the wave equation we were able to establish

- *existence* of the solution in a suitable class of functions;
- *uniqueness* of the solution;
- *continuous dependence* of the solution on the initial data with respect to a suitable topology.

One may ask whether these properties remain valid for other evolutionary PDEs?

Changing the sign in the wave equation one arrives at the *Laplace equation* of elliptic type, which in $\mathbb{R}^2$ reads

$$u_{tt} + a^2 u_{xx} = 0. \tag{4.1.1}$$

Does the change of the type of equation affect seriously the properties of solutions?

To be more specific we will deal with the periodic Cauchy problem

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \tag{4.1.2}$$

with two 2π-periodic smooth initial functions $\phi(x)$, $\psi(x)$. For simplicity let us choose $a = 1$. We will see that the solution to this Cauchy problem *does not* depend continuously on the initial data. For that let us consider the sequence of solutions $u_k(x,t)$, $k \in \mathbb{N}$ to the Cauchy problem with following initial data:

$$u(x,0) = 0, \quad u_t(x,0) = \frac{\sin k\,x}{k}. \tag{4.1.3}$$

The 2π-periodic solution (if any) can be expanded in Fourier series

$$u_k(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} [a_n(t)\cos nx + b_n(t)\sin nx]$$

with some coefficients $a_n(t)$, $b_n(t)$. Substituting the series into equation

$$u_{tt} + u_{xx} = 0$$

we obtain an infinite system of ODEs

$$\begin{aligned} \ddot{a}_n &= n^2 a_n, \quad \forall n \in \mathbb{N} \\ \ddot{b}_n &= n^2 b_n, \quad \forall n \in \mathbb{N} \setminus 0. \end{aligned}$$

The initial data for this infinite system of ODEs follow from the Cauchy problem (4.1.2):

$$\begin{aligned} &a_n(0) = 0, \quad \dot{a}_n = 0, \\ &b_n(0) = 0, \quad \dot{b}_n(0) = \begin{cases} 1/k, & n = k \\ 0, & n \neq k. \end{cases} \end{aligned}$$

The solution has the form

$$\begin{aligned} &a_n(t) = 0 \quad \forall n, \quad b_n(t) = 0 \quad \forall n \neq k \\ &b_k(t) = \frac{1}{k^2}\sinh kt. \end{aligned}$$

So the solution to the Cauchy problem (4.1.2) reads

$$u_k(x,t) = \frac{1}{k^2}\sin kx\,\sinh kt. \tag{4.1.4}$$

Using this explicit solution we can prove the following

Theorem 4.1.1 (J.Hadamard (1922)). *For any positive ϵ, M, t_0 there exists an integer K such that for any $k \geq K$ the initial data* (4.1.3) *satisfy*

$$\sup_{x\in[0,2\pi]} \left(|u_k(x,0)| + |\partial_t u_k(x,0)|\right) < \epsilon \tag{4.1.5}$$

but the solution $u_k(x,t)$ at the moment $t = t_0 > 0$ satisfies

$$\sup_{x\in[0,2\pi]} \left(|u_k(x,t_0)| + |\partial_t u_k(x,t_0)|\right) \geq M. \tag{4.1.6}$$

Proof: Since l.h.s. of (4.1.5) is smaller that $\frac{1}{k}$, choosing an integer K_1 satisfying

$$K_1 > \frac{1}{\epsilon}$$

we will have the inequality (4.1.5) for any $k \geq K_1$. In order to obtain a lower estimate of the form (4.1.6) let us first observe that

$$\sup_{x\in[0,2\pi]} \left(|u_k(x,t_0)| + |\partial_t u_k(x,t_0)|\right) = \frac{1}{k^2}\sinh kt_0 + \frac{1}{k}\cosh kt_0 > \frac{e^{kt_0}}{k^2}$$

where we have used an obvious inequality

$$\frac{1}{k} > \frac{1}{k^2} \quad \text{for} \quad k > 1.$$

The last function of k is monotone increasing for $k > \frac{2}{t_0}$ and grows to ∞, hence for any $t_0 > 0$ and $M > 0$ there exists K_2 such that for any $k > K_2$

$$\frac{e^{k\,t_0}}{k^2} > M.$$

Choosing

$$K = \max(K_1, K_2)$$

we complete the proof. □

The statement of the Theorem is usually referred to as *ill-posedness* of the Cauchy problem (4.1.1), (4.1.2).

A natural question arises: what kind of initial or boundary conditions can be chosen in order to uniquely specify solutions to Laplace equation without violating the continuous dependence of the solutions on the boundary/initial conditions?

4.2 Dirichlet and Neumann problems for Laplace equation on the plane

We consider the (elliptic) *Laplace operator* in the d-dimensional Euclidean space

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2} \tag{4.2.1}$$

and the Laplace equation in an open domain Ω

$$\Delta u(x) = 0, \quad x = (x_1, \dots x_d) \in \Omega \subset \mathbb{R}^d. \tag{4.2.2}$$

The (real) solutions to the Laplace equation are called *harmonic functions* in Ω.

We will formulate the two main boundary value problems (b.v.p.'s), assuming that the boundary $\partial\Omega$ of the domain Ω is a smooth hypersurface. Moreover we assume that the domain Ω does not go to infinity, i.e. Ω belongs to some ball in $\mathbb{R}^d$.

Problem 1 (*Dirichlet problem*). Given a function $f(x)$ defined at the points of the boundary find a function $u = u(x)$ satisfying the Laplace equation on the internal part of the domain Ω and the boundary condition

$$u(x)|_{x\in\partial\Omega} = f(x) \tag{4.2.3}$$

on the boundary of the domain.

Problem 2 (*Neumann problem*). Given a function $g(x)$ defined at the points of the boundary find a function $u = u(x)$ satisfying the Laplace equation on the internal part of the domain Ω and the boundary condition

$$\left(\frac{\partial u(x)}{\partial n}\right)_{x\in\partial\Omega} = g(x) \tag{4.2.4}$$

on the boundary of the domain. Here $n = n(x)$ is the unit external normal vector at every point $x \in \partial\Omega$ of the boundary and the normal derivative is defined as

$$\frac{\partial u(x)}{\partial n} := n \cdot \nabla u = n_1 \frac{\partial u(x)}{\partial x_1} + \cdots + n_d \frac{\partial u(x)}{\partial x_d}. \tag{4.2.5}$$

Recall that the components n_i of n are just equal to the $\cos\alpha_i$, where α_i are the angles between n and the coordinate axes e_i.

Example 1. For $d = 1$ the Laplace operator is just the second derivative

$$\Delta = \frac{d^2}{dx^2}.$$

The Dirichlet b.v.p. in the domain $\Omega = (a, b)$, $\partial\Omega = \{a, b\}$,

$$u''(x) = 0, \quad u(a) = f_a, \quad u(b) = f_b$$

has an obvious unique solution

$$u(x) = \frac{f_b - f_a}{b - a}(x - a) + f_a.$$

The Neumann b.v.p. in the same domain, since $n = n_1\partial_1$ with $n_1(a) = -1$ and $n_1(b) = 1$, is

$$u''(x) = 0, \quad -u'(a) = g_a, \quad u'(b) = g_b,$$

and so its solution is

$$u(x) = g_b x + c,$$

where c is an arbitrary constant, but only provided that

$$g_a + g_b = 0 \tag{4.2.6}$$

since the derivatives of such $u(x)$ are equal at a and at b

$$u'(a) = g_b = u'(b).$$

Lemma 4.2.1. *In two dimensions the Laplace operator*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{4.2.7}$$

in the polar coordinates

$$\left.\begin{aligned} x = & \; r\,\cos\phi \\ y = & \; r\,\sin\phi \end{aligned}\right\} \tag{4.2.8}$$

takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}. \tag{4.2.9}$$

Proof: We have (for brevity we denote $c = \cos\phi, s = \sin\phi$)

$$\partial_r = x_r\partial_x + y_r\partial_y = c\partial_x + s\partial_y \tag{4.2.10}$$

$$\partial_\phi = -rs\partial_x + rc\partial_y.$$

Hence

$$\partial_r^2 = c^2\partial_x^2 + s^2\partial_y^2 + 2sc\partial_x\partial_y$$

and

$$\begin{aligned}\partial_\phi^2 &= \partial_\phi(-rs\partial_x + rc\partial_y) = \partial_\phi(-rs)\partial_x + \partial_\phi(rc)\partial_y - rs\partial_\phi\partial_x + rc\partial_\phi\partial_y \\ &= -rc\partial_x - rs\partial_y + r^2s^2\partial_x^2 + r^2c^2\partial_y^2 - 2r^2sc\partial_x\partial_y.\end{aligned}$$

Then a simple computation gives the result. □

In the particular case of a disc of radius ρ

$$\Omega = B(0,\rho) = \{(x,y)\,|\,x^2+y^2<\rho^2\} = \{(r,\phi)\,|\,0\le r<\rho\} \tag{4.2.11}$$

the Dirichlet b.v.p. is formulated as follows: find a solution $u = u(x,y)$ to the Laplace equation in Ω satisfying the boundary condition

$$u(r,\phi)|_{r=\rho} = f(\phi). \tag{4.2.12}$$

Here we represent the boundary condition defined on the boundary of the disc as a function depending only on the polar angle ϕ. Similarly, the Neumann problem consists of finding a solution to the Laplace equation satisfying

$$\frac{\partial u(r,\phi)}{\partial r}|_{r=\rho} = g(\phi) \tag{4.2.13}$$

for a given function $g(\phi)$. Indeed, the partial derivative ∂_r (4.2.10), is the normal derivative (note that $\sin\phi = \cos(\frac{\pi}{2}-\phi)$ and ϕ and $\frac{\pi}{2}-\phi$ are angles with the axes x_1 and x_2, respectively).

Let us return to the general d-dimensional case. The following identity will be useful in the study of harmonic functions.

Theorem (Green's formula). *For arbitrary smooth functions u, v on a bounded closed domain $\bar\Omega$ with a piecewise smooth boundary $\partial\Omega$*

$$\boxed{\int_\Omega \nabla u\cdot\nabla v\,dV + \int_\Omega u\,\Delta v\,dV = \int_{\partial\Omega} u\,\frac{\partial v}{\partial n}\,dS.} \tag{4.2.14}$$

Here

$$\nabla u\cdot\nabla v = \sum_{i=1}^d \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$$

is the inner product of the gradients of the functions,

$$dV = dx_1 \dots dx_d$$

is the Euclidean volume element, n the external normal and dS is the area element on the hypersurface $\partial\Omega$.

For proof see e.g. [Evans, Appendix C].

Example 1. For $d = 1$ and $\Omega = (a, b)$ the Green's formula reads

$$\int_a^b u_x\, v_x\, dx + \int_a^b u\, v_{xx}\, dx = u\, v_x|_a^b$$

since the oriented boundary of the interval consists of two points $\partial[a, b] = b - a$. This is an easy consequence of integration by parts.

Example 2. For $d = 2$ and a rectangle $\Omega = (a, b) \times (c, d)$ we show the Green's formula

$$\begin{aligned}
&\int_\Omega (u_x v_x + u_y v_y)\, dx\, dy + \int_\Omega u\,(v_{xx} + v_{yy})\, dx\, dy \\
&= \int_c^d \int_a^b (u\, v_x)_x\, dx\, dy + \int_a^b \int_c^d (u\, v_y)_y\, dy\, dx \\
&= \int_c^d (u\, v_x)\,|_{x=a}^{x=b}\, dy + \int_a^b (u\, v_y)\,|_{y=c}^{y=d}\, dx.
\end{aligned}$$

As it should, the right hand side of this formula is a sum (with a sign) of integrals over the four pieces of the boundary $\partial\Omega$.

Remark. These two examples essentially 'prove' (4.2.14) since we can approximate any Ω by a union of rectangulars, and we can analogously proceed in higher dimensions.

Corollary 4.2.2. *The Green's formula*

$$\int_\Omega \Delta v\, dV = \int_{\partial\Omega} \frac{\partial v}{\partial n}\, dS. \tag{4.2.15}$$

we used in Sect 3.8 follows by substitution $u = 1$ in (4.2.14).

Corollary 4.2.3. *For a harmonic function u in a domain Ω with a piecewise smooth boundary $\partial\Omega$ the integral of the normal derivative of u over $\partial\Omega$ vanishes:*

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\, dS = 0 \tag{4.2.16}$$

(use (4.2.15)*). Moreover,*

$$\int_\Omega (\nabla u)^2 = \int_{\partial\Omega} u \partial_n u\, dS = \int_{\partial\Omega} \frac{1}{2} \partial_n u^2\, dS \tag{4.2.17}$$

(substitute $u = v$ in (4.2.14)*).*

Using the last identity we can easily derive the following uniqueness properties of solution to the Dirichlet end Neumann boundary problems.

Theorem 4.2.4. *1) Let u_1, u_2 be two functions harmonic in the domain Ω, smooth in the closed domain $\bar{\Omega}$, and coinciding on the boundary $\partial\Omega$. Then $u_1 \equiv u_2$.*

2) Under the same assumptions about the functions u_1, u_2, if the normal derivatives on the boundary coincide

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n}$$

then the functions differ by a constant.

Proof: Applying to the difference $u = u_2 - u_1$ the identity (4.2.17) one obtains

$$\int_\Omega (\nabla u)^2 dV = 0$$

since the right hand side vanishes. Hence $\nabla u = 0$, and thus the function u is equal to a constant. The value of this constant on the boundary is zero. Therefore $u \equiv 0$. The second statement has a similar proof. □

The following counterexample shows that the uniqueness does not hold true for *infinite* domains. Let Ω be the upper half plane:

$$\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0.\}.$$

The linear function $u(x, y) = y$ is harmonic in Ω and vanishes on the boundary. Clearly u is different from 0, which is obviously harmonic too.

Our goal is to solve the Dirichlet and Neumann boundary value problems. The first result in this direction is the following

Theorem 4.2.5. *For an arbitrary $\mathcal{C}^1$-smooth 2π-periodic function $f(\phi)$ the solution to the Dirichlet b.v.p.* (4.2.12) *in the disc $\Omega = B(0, \rho)$ exists and is unique. Moreover it is given by the following formula*

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho\, r\, \cos(\phi - \psi) + r^2} f(\psi)\, d\psi. \tag{4.2.18}$$

The expression (4.2.18) for the solution to the Dirichlet b.v.p. is called *Poisson formula.*

Proof: We will first use the method of separation of variables in order to construct particular solutions to the Laplace equation. At the second step we will represent solutions to the Dirichlet b.v.p. as a linear combination of the particular solutions.

The method of *separation of variables* starts from looking for solutions to the Laplace equation in the form

$$u = R(r)\Phi(\phi). \tag{4.2.19}$$

Here r, ϕ are the polar coordinates on the plane (see Exercise 4.2.1 above). Using the form (4.2.9) we reduce the Laplace equation $\Delta u = 0$ to

$$R''(r)\Phi(\phi) + \frac{1}{r} R'(r)\Phi(\phi) + \frac{1}{r^2} R(r)\Phi''(\phi) = 0.$$

After division by $\frac{1}{r^2} R(r)\Phi(\phi)$ we can rewrite the last equation in the form

$$\frac{R''(r) + \frac{1}{r} R'(r)}{\frac{1}{r^2} R(r)} = -\frac{\Phi''(\phi)}{\Phi(\phi)}.$$

The left hand side of this equation is independent of ϕ while the right hand side is independent of r. The equality is possible only if both sides are equal to some constant λ. In this way we arrive at two ODEs for the functions $R = R(r)$ and $\Phi = \Phi(\phi)$

$$R'' + \frac{1}{r} R' - \frac{\lambda}{r^2} R = 0 \tag{4.2.20}$$

$$\Phi'' + \lambda\Phi = 0. \tag{4.2.21}$$

We have now to determine the admissible values of the parameter λ. To this end let us begin from the second equation (4.2.21). Its solutions have the form

$$\Phi(\phi) = \begin{cases} A\, e^{\sqrt{-\lambda}\,\phi} + B\, e^{-\sqrt{-\lambda}\,\phi}, & \lambda < 0 \\ A + B\,\phi, & \lambda = 0 \\ A\,\cos\sqrt{\lambda}\,\phi + B\,\sin\sqrt{\lambda}\,\phi, & \lambda > 0 \end{cases} .$$

Since the pairs of polar coordinates (r, ϕ) and $(r, \phi + 2\pi)$ correspond to the same point on the Euclidean plane the solution $\Phi(\phi)$ must be a 2π-periodic function. Hence we must discard the negative values of λ and we have $B = 0$ if $\lambda = 0$. Moreover λ must have the form

$$\lambda = n^2, \quad n = 0,\, 1,\, 2, \dots. \tag{4.2.22}$$

This gives

$$\Phi(\phi) = A\,\cos n\phi + B\,\sin n\phi \tag{4.2.23}$$

($\lambda = n = 0$ corresponds to a constant $\Phi(\phi) = A$).
The first ODE (4.2.20) for $\lambda = n^2$ becomes

$$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R = 0.$$

This is a particular case of Euler equation. One can look for solutions in the form

$$R(r) = r^k.$$

The exponent k has to be determined from the characteristic equation

$$k(k-1) + k - n^2 = 0$$

obtained by the direct substitution of $R = r^k$ into the equation. The roots of the characteristic equation are $k = \pm n$. For $n > 0$ this gives the general solution of the equation (4.2.20) in the form

$$R = a\,r^n + \frac{b}{r^n}$$

with two integration constants a and b. For $n = 0$ the general solution is

$$R = a + b\,\log r.$$

As the solution must be smooth at $r = 0$ one must always choose $b = 0$ for all n. In this way we arrive at the following family of particular solutions to the Laplace equation

$$u_n(r,\phi) = r^n\left(A_n \cos n\phi + B_n \sin n\phi\right), \quad n = 0,\, 1,\, 2, \dots \tag{4.2.24}$$

We want now to represent arbitrary solution to the Dirichlet b.v.p. in the disc of radius ρ as a linear combination of these solutions:

$$\begin{aligned} u(r,\phi) &= \frac{A_0}{2} + \sum_{n\geq 1} r^n\left(A_n \cos n\phi + B_n \sin n\phi\right) \\ u(r,\phi)|_{r=\rho} &= f(\phi). \end{aligned} \tag{4.2.25}$$

The boundary data function $f(\phi)$ must be a 2π-periodic function. Assuming this function to be $\mathcal{C}^1$-smooth let us expand it in Fourier series

$$\begin{aligned} f(\phi) &= \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n \cos n\phi + b_n \sin n\phi\right) \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi\, d\phi, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin n\phi\, d\phi. \end{aligned} \tag{4.2.26}$$

Comparison of (4.2.25) with (4.2.26) yields

$$A_n = \frac{a_n}{\rho^n}, \quad B_n = \frac{b_n}{\rho^n},$$

or, equivalently

$$u(r,\phi) = \frac{a_0}{2} + \sum_{n\geq 1} \left(\frac{r}{\rho}\right)^n (a_n \cos n\phi + b_n \sin n\phi)\,. \tag{4.2.27}$$

Recall that this formula holds true on the disc of radius ρ, i.e., for $r \leq \rho$.

Taking into accout the expression for a_n, b_n the last formula can be rewritten as follows:

$$u(r,\phi) = \frac{1}{\pi}\int_0^{2\pi} \left[\frac{1}{2} + \sum_{n\geq 1}\left(\frac{r}{\rho}\right)^n (\cos n\phi \cos n\psi + \sin n\phi \sin n\psi)\right] f(\psi)\, d\psi$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left[\frac{1}{2} + \sum_{n\geq 1}\left(\frac{r}{\rho}\right)^n \cos n(\phi - \psi)\right] f(\psi)\, d\psi.$$

To compute it we represent the sum in the square bracket as a geometric series converging for $r < \rho$:

$$\frac{1}{2} + \sum_{n\geq 1}\left(\frac{r}{\rho}\right)^n \cos n(\phi-\psi) = \frac{1}{2} + \mathrm{Re}\sum_{n\geq 1}\left(\frac{r}{\rho}\right)^n e^{in(\phi-\psi)}$$

$$= \frac{1}{2} + \mathrm{Re}\,\frac{r\,e^{i(\phi-\psi)}}{\rho - r\,e^{i\,(\phi-\psi)}} = \frac{1}{2} + \frac{1}{2}\left(\frac{r\,e^{i(\phi-\psi)}}{\rho - r\,e^{i\,(\phi-\psi)}} + \frac{r\,e^{-i(\phi-\psi)}}{\rho - r\,e^{-i\,(\phi-\psi)}}\right)$$

$$= \frac{1}{2}\frac{\rho^2 - r^2}{\rho^2 - 2\rho\, r\, \cos(\phi - \psi) + r^2}.$$

□

In a similar way one can treat the Neumann boundary problem. However in this case one has to impose an additional constraint for the boundary value of the normal derivative (cf. (4.2.6) above in dimension 1 and (4.2.16)), namely

$$\int_{\partial\Omega} g\, dS = 0. \tag{4.2.28}$$

We will now prove, for the particular case of a disc domain in the dimension $d = 2$ that this necessary condition of solvability is also sufficient.

Theorem 4.2.6. *For an arbitrary $\mathcal{C}^1$-smooth 2π-periodic function $g(\phi)$ satisfying*

$$\int_0^{2\pi} g(\phi)\, d\phi = 0 \tag{4.2.29}$$

the Neumann b.v.p. (4.2.4) *in the unitary disc has a solution unique up to an additive constant. This solution can be represented by the following integral formula*

$$u(r,\phi) = \frac{\rho}{2\pi}\int_0^{2\pi} \log\frac{\rho^2}{\rho^2 - 2\rho\, r\, \cos(\phi-\psi) + r^2}\, g(\psi)\, d\psi. \tag{4.2.30}$$

Proof: Repeating the above arguments one arrives at the following expression for the solution $u = u(r, \phi)$:

$$u = \frac{A_0}{2} + \sum_{n\geq 1} r^n \left(A_n \cos n\phi + B_n \sin n\phi \right)$$
$$\left(\frac{\partial u}{\partial r}\right)_{r=\rho} = g(\phi). \tag{4.2.31}$$

Let us consider the Fourier series of the function $g(\phi)$

$$g(\phi) = \frac{a_0}{2} + \sum_{n\geq 1} \left(a_n \cos n\phi + b_n \sin n\phi \right).$$

Due to the constraint (4.2.29) the constant term vanishes:

$$a_0 = 0.$$

Comparing this series with the boundary condition (4.2.31) we find that up to an arbitrary constant

$$u(r, \phi) = \sum_{n\geq 1} \frac{\rho}{n} \left(\frac{r}{\rho}\right)^n \left(a_n \cos n\phi + b_n \sin n\phi \right),$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \cos n\psi \, g(\psi)\, d\psi, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} \sin n\psi \, g(\psi)\, d\psi.$$

Combining these equations we arrive at the following expression:

$$u(r, \phi) = \frac{\rho}{\pi} \int_0^{2\pi} \sum_{n\geq 1} \frac{1}{n} \left(\frac{r}{\rho}\right)^n \cos n(\phi - \psi) g(\psi)\, d\psi. \tag{4.2.32}$$

It remains to compute the sum of this trigonometric series. The stated result (4.2.30) follows by substituting $\theta = \phi - \psi$, $R = r/\rho$ in the following Lemma. □

Lemma 4.2.7. *Let R and θ be two real numbers, $R < 1$. Then*

$$\sum_{n=1}^{\infty} \frac{1}{n} R^n \cos n\theta = \frac{1}{2} \log \frac{1}{1 - 2R\cos\theta + R^2}. \tag{4.2.33}$$

Proof: The series under consideration can be represented as the real part of a complex series

$$\sum_{n=1}^{\infty} \frac{1}{n} R^n \cos n\theta = \operatorname{Re} \sum_{n=1}^{\infty} \frac{1}{n} R^n e^{in\theta}.$$

The latter can be written as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n} R^n e^{in\theta} = \int_0^R \sum_{n=1}^{\infty} \frac{1}{R} R^n e^{in\theta}\, dR.$$

We can easily compute the sum of the geometric series with the denominator $Re^{i\theta}$. Integrating we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} R^n e^{in\theta} = \int_0^R \frac{e^{i\theta}}{1 - Re^{i\theta}}\, dR = -\log\left(1 - Re^{i\theta}\right).$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n} R^n \cos n\theta = \frac{1}{2}\left[\log \frac{1}{1 - Re^{i\theta}} + \log \frac{1}{1 - Re^{-i\theta}}\right] = \frac{1}{2} \log \frac{1}{1 - 2R\cos\theta + R^2}.$$

□

4.3 Properties of harmonic functions: mean value theorem, the maximum principle

In this section we will establish two fundamental properties of harmonic functions.

Let $\Omega \subset \mathbb{R}^d$ be a domain. Let $x_0 \in \Omega$ be an *internal* point and thus there exists some radius $R > 0$ such that the ball $B(x_0, R)$ with the centre at x_0 entirely belongs to Ω. The radius is chosen small enough to guarantee that the sphere belongs to the domain Ω. We shall work, as in Section 3.8 with the *mean value* of a continuous function $f(x)$ on the sphere $S(x_0, R)$

$$\tilde{f} = \fint_{S(x_0,R)} f(x)\, dS := \frac{1}{d\, \alpha_d\, R^{d-1}} \int_{S(x_0,R)} f(x)\, dS. \tag{4.3.1}$$

In the particular case of a constant function the mean value coincides with the value of the function.

Recal that in dimension $d = 1$ the zero-dimensional "sphere" consists of two points $x_0 \pm R$ and its area is $\alpha_1 = \pi^{1/2}/\Gamma\left(\frac{3}{2}\right) = 2$. So the mean value of a function is just the arithmetic mean

$$\tilde{f} = \frac{f(x_0 + R) + f(x_0 - R)}{2}.$$

In the next case $d = 2$ the sphere is just a circle of radius R with the centre at x_0. The area (i.e., the length) element is $dS = R\, d\phi$ and the mean value is given by ($\alpha_2 = \pi$)

$$\tilde{f} = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi.$$

Theorem 4.3.1. *Let $u = u(x)$ be a function harmonic in Ω. Then the mean value $\tilde{u}$ of u over the sphere $S(x_0, R)$ centered at a point $x_0 \in \Omega$ is equal to the value of the function at this point:*

$$u(x_0) = \fint_{S(x_0,R)} u(x)\, dS. \tag{4.3.2}$$

Proof: We give the proof only in the specific case of dimension $d = 2$. Denote $f(\phi)$ the restriction of the harmonic function u onto the small circle $|x - x_0| = R$. By definition the function $u(x)$ satisfies the Dirichlet b.v.p. inside the circle:

$$\begin{aligned} &\Delta u(x) = 0, \quad |x - x_0| < R \\ &u(x)|_{|x-x_0|=R} = f(\phi). \end{aligned}$$

As we already know from the proof of Theorem 4.2.5 the solution to this b.v.p. can be represented by the Fourier series

$$u(r, \phi) = \frac{a_0}{2} + \sum_{n\geq 1} \left(\frac{r}{R}\right)^n (a_n \cos n\phi + b_n \sin n\phi) \tag{4.3.3}$$

for $r := |x - x_0| < R$ (cf. (4.2.27) above). In this formula

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi$$

is the mean value of the function u on the circle, ie. the r.h.s. of (4.3.3). On the other side the value of the function u at the center of the circle can be evaluated substituting $r = 0$ in the formula (4.3.3):

$$u(x_0) = \frac{a_0}{2}.$$

Comparing the last two equations we arrive at (4.3.2). □

Using the mean value theorem we will now prove another important property of harmonic functions, namely the *maximum principle*. Recall that a function $u(x)$ defined on a domain $\Omega \subset \mathbb{R}^d$ is said to have a *local maximum* at the point x_0 if the inequality

$$u(x) \leq u(x_0) \tag{4.3.4}$$

holds true for any $x \in \Omega$ sufficiently close to x_0. A *local minimum* is defined in a similar way. First the following Main Lemma:

Lemma 4.3.2. *Let the harmonic function $u(x)$ have a local maximum, or local minimum, at an internal point $x_0 \in \Omega$. Then $u(x) \equiv u(x_0)$ on some neighborhood of the point x_0.*

Proof: Let us consider the case of a local maximum. Choosing a sufficiently small sphere $S(x_0, R)$ with the centre at x_0 we can assume the inequality (4.3.4) holds true for all $x \in S(x_0, R)$. Then according to the mean value theorem

$$u(x_0) = \fint_{|x-x_0|=R} u(x)\, dS \leq \fint_{|x-x_0|=R} u(x_0)\, dS = u(x_0).$$

If there exists a point x arbitrarily close to x_0 such that $u(x) < u(x_0)$ then also the inequality is strict (the smaller value can not be compensated by other points). This is a contradiction, which shows that the function $u(x)$ takes constant values on some ball with the centre at x_0. The case of a local minimum can be treated in a similar way. □

Theorem 4.3.3. *(Maximum principle).*
Let a function $u(x)$ be harmonic in a bounded <u>*connected*</u> *domain Ω and continuous in a closed domain $\bar{\Omega}$. Denote*

$$M = \sup_{x \in \partial\Omega} u(x), \quad m = \inf_{x \in \partial\Omega} u(x).$$

Then

1) $m \leq u(x) \leq M$ for all $x \in \Omega$;

2) if $u(x) = M$ or $u(x) = m$ for some internal point $x \in \Omega$ then the function u is constant.

Proof: Denote

$$M' = \sup_{x \in \bar{\Omega}} u(x)$$

the maximum of the function u continuous on the compact $\bar{\Omega}$; it is achieved on some x_0, i.e. $u(x_0) = M'$. We want to prove that $M' \leq M$. Indeed, let us assume the contrary that $M' > M$. Then x_0 can not belong to $\partial\Omega$ but must be <u>internal</u>, $x_0 \in \Omega$. Denote $\Omega' \subset \Omega$ the set of points x of the domain where the function u takes the same value M'. According to the Main Lemma this subset is open. Clearly it is also closed and nonempty. Hence $\Omega' = \Omega$ since the domain is connected. In other words the function u is constant everywhere in Ω. Because of the continuity it assumes the same value M' also at the points of the boundary $\partial\Omega$. But there we have $M' \leq M$, which contradicts our assumption.
Therefore the value of a harmonic function at an internal point of the domain cannot be bigger than the maximal value of this function on the boundary of the domain. Moreover if the harmonic function takes the value M at an internal point then it is constant.
In a similar way we prove that a non-constant harmonic function cannot have a minimum in Ω. □

Corollary 4.3.4. *Given two functions $u_1(x)$, $u_2(x)$ harmonic in a bounded domain Ω and continuous in the closed domain $\bar{\Omega}$. If*

$$|u_1(x) - u_2(x)| \leq \epsilon \quad \text{for} \quad x \in \partial\Omega$$

then

$$|u_1(x) - u_2(x)| \leq \epsilon \quad \text{for any} \quad x \in \Omega \ .$$

Proof: Denote

$$u(x) = u_1(x) - u_2(x).$$

The function u is harmonic in Ω and continuous in $\bar{\Omega}$. By assumption we have $-\epsilon \leq u(x) \leq \epsilon$ for any $x \in \partial\Omega$. So

$$-\epsilon \leq \inf_{x \in \partial\Omega} u(x), \quad \sup_{x \in \partial\Omega} u(x) \leq \epsilon.$$

According to the maximum principle we must also have

$$-\epsilon \leq \inf_{x \in \Omega} u(x), \quad \sup_{x \in \Omega} u(x) \leq \epsilon.$$

□

The Corollary implies that the the solution to the Dirichlet boundary value problem, if exists, *depends continuously* on the boundary data.

4.4 Harmonic functions on the plane and complex analysis

Recall that a differentiable complex valued function $f(x, y)$ on a domain in $\mathbb{R}^2$ is called *holomorphic* if its real $u(x, y)$ and imaginary $v(x, y)$ parts satisfy the system of *Cauchy – Riemann equations*

$$\boxed{\left.\begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 0 \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= 0 \end{aligned}\right\}} \tag{4.4.1}$$

or, in the complex form

$$\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y} = 0.$$

Introducing complex combinations of the Euclidean coordinates

$$z = x + iy, \quad \bar{z} = x - iy$$

with inverse

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i} \tag{4.4.2}$$

we have

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right). \end{aligned} \tag{4.4.3}$$

So the Cauchy – Riemann equations can be rewritten in the form

$$\frac{\partial f}{\partial \bar{z}} = 0. \tag{4.4.4}$$

Example. Let $f(x, y)$ be a polynomial

$$f(x, y) = \sum_{k,l} a_{kl} x^k y^l.$$

It is a holomorphic function *iff*, after the substitution (4.4.2) there will be no dependence on $\bar{z}$ (no powers of $\bar{z}$):

$$\sum_{k,l} a_{kl} \left(\frac{z+\bar{z}}{2}\right)^k \left(\frac{z-\bar{z}}{2i}\right)^l = \sum_m c_m z^m.$$

In that case the result will be a polynomial in z. For example a quadratic polynomial

$$f(x,y) = ax^2 + 2bxy + cy^2$$

is holomorphic *iff* $a+c=0$ (from the coefficient of $\bar{z}/2$) and $b = \frac{i}{2}(a-c)$ (from the coefficient of $\bar{z}^2/4$). $\diamond$

Holomorphic functions are usually denoted $f = f(z)$. The partial derivative $\partial/\partial z$ of a holomorphic function is denoted df/dz or $f'(z)$. One also defines *antiholomorphic* functions $f = f(\bar{z})$ as differentiable complex functions satisfying the equation

$$\frac{\partial f}{\partial z} = 0. \tag{4.4.5}$$

Notice that the complex conjugate $\overline{f(z)}$ to a holomorphic function is an antiholomorphic function.

From complex analysis it is known that any function f holomorphic on a neighborhood of a point z_0 is also a *complex analytic* function, i.e., it can be represented as a sum of a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \tag{4.4.6}$$

convergent uniformly and absolutely for sufficiently small $|z-z_0|$. In particular it is continuously differentiable any number of times. Its real and imaginary parts $u(x,y)$ and $v(x,y)$ are infinitely smooth functions of x and y.

Theorem 4.4.1. *The real and imaginary parts of a function holomorphic in a domain Ω are harmonic functions on the same domain.*

Proof: Differentiating the first equation in (4.4.1) in x and the second one in y and adding we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly, differentiating the second equation in x and subtracting the first one differentiated in y gives

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\square$

Corollary 4.4.2. *For any integer $n \geq 1$ the functions*

$$\operatorname{Re} z^n \quad \text{and} \quad \operatorname{Im} z^n \tag{4.4.7}$$

are polynomial harmonic functions (real polynomial solutions to the Laplace equation).

Observe that these polynomials can be represented in the polar coordinates r, ϕ as

$$\operatorname{Re} z^n = r^n \cos n\phi, \quad \operatorname{Im} z^n = r^n \sin n\phi.$$

These are exactly the functions we used to solve the main boundary value problems for the disc.

Polynomial harmonic functions are just called *harmonic polynomials*. We obtain a sequence of harmonic polynomials

$$x,\ y,\ x^2-y^2,\ xy,\ x^3-3xy^2,\ 3x^2y-y^3,\ldots.$$

Remark Note that the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in the coordinates z, $\bar{z}$ becomes

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}. \tag{4.4.8}$$

(use (4.4.3)). Using this representation of the two-dimensional Laplace operator one can describe all complex valued solutions to the Laplace equation.

Theorem 4.4.3. *Any complex valued solution u to the Laplace equation $\Delta u = 0$ on the plane can be represented as a sum of a holomorphic and an antiholomorphic function:*

$$u(x,y) = f(z) + g(\bar{z}). \tag{4.4.9}$$

Proof: Let the $\mathcal{C}^2$-smooth function $u(x,y)$ satisfy the Laplace equation

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$

Denote

$$F = \frac{\partial u}{\partial z}.$$

The Laplace equation implies that this function is holomorphic, $F = F(z)$. From complex analysis it is known that any holomorphic function admits a holomorphic primitive $f(z)$, i.e.

$$F(z) = f'(z).$$

Consider the difference $g := u - f$. It is an antiholomorphic function, $g = g(\bar{z})$. Indeed,

$$\frac{\partial g}{\partial z} = \frac{\partial u}{\partial z} - f' = 0.$$

Thus

$$u = f(z) + g(\bar{z}).$$

$\square$

Now the condition $\bar{u} = u$ that u given by (4.4.9) is real valued, so harmonic, means that

$$\overline{f(z)} - g(\bar{z}) = -\overline{g(\bar{z})} + f(z).$$

Since the right hand side is holomorphic and the left hand side is its complex conjugate (so antiholomorphic), both must be just a constant real function λ. Hence

$$g(\bar{z}) = \overline{f(z)} - \lambda$$

and so

$$u = f(z) + \overline{f(z)} - \lambda\ .$$

Hence

Corollary 4.4.4. *Any harmonic function on the plane can be represented as the real part of a holomorphic function.*

Corollary 4.4.5. *Any harmonic function on the plane is $\mathcal{C}^\infty$ (smooth).*

If a harmonic u is a polynomial, it is a real part of a holomorphic polynomial. (If $f = v + iw$, v must be a polynomial but then by C.R. eqs. also w must be a polynomial). Now a real basis of holomorphic polynomials is z^n and iz^n, and so a real basis of harmonic polynomials is $\mathrm{Re} z^n$ and $\mathrm{Re}\, iz^n$. Since the imaginary part of a holomorphic function $f(z)$ is equal to the real part of the function $-i\, f(z)$ (that is holomorphic as well) we have

Corollary 4.4.6. *Any harmonic polynomial u is a linear combination of the polynomials $\mathrm{Re} z^n$ and $\mathrm{Im}\, z^n$ as given in* (4.4.7).

Another important consequence of the complex representation (4.4.8) of the Laplace operator on the plane is invariance of the Laplace equation under conformal transformation. Recall that a smooth map

$$f : \Omega \to \Omega'$$

is called *conformal* if it preserves the angles between smooth curves (or their tangents). The dilatations

$$(x, y) \mapsto (k\, x, k\, y)$$

with $k \neq 0$, rotations by the angle ϕ

$$(x, y) \mapsto (x \cos\phi - y \sin\phi,\ x \sin\phi + y \cos\phi)$$

and reflections

$$(x, y) \mapsto (x, -y)$$

are examples of linear conformal transformations. These examples and their superpositions exhaust the class of linear conformal maps. The general conformal maps on the plane are described by

Lemma 4.4.7. *Let $f(z)$ be a function holomorphic in the domain Ω with nowhere vanishing derivative:*

$$\frac{df(z)}{dz} \neq 0 \quad \forall z \in \Omega.$$

Then the map

$$z \mapsto f(z)$$

of the domain Ω to $\Omega' = f(\Omega)$ is conformal. Same for antiholomorphic functions. Conversely, if the smooth map $(x, y) \mapsto (u(x, y), v(x, y))$ is conformal then the function $f = u + iv$ is holomorphic or antiholomorphic with nonvanishing derivative.

Proof: Let us consider the differential of the map $(x, y) \mapsto (u(x, y), v(x, y))$ given by the real $u = \mathrm{Re}\, f$ and imaginary $v = \mathrm{Im}\, f$ parts of the holomorphic function f. It is a linear map defined by the Jacobi matrix

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} = \begin{pmatrix} \partial u/\partial x & -\partial v/\partial x \\ \partial v/\partial x & \partial u/\partial x \end{pmatrix}$$

(we have used the Cauchy – Riemann equations). Since

$$0 \neq |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2,$$

we can introduce the numbers $r > 0$ and ϕ by

$$r = |f'(z)|, \quad \cos\phi = \frac{\partial u/\partial x}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}}, \quad \sin\phi = \frac{\partial v/\partial x}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}}.$$

The Jacobi matrix then becomes a combination of the rotation by the angle ϕ and a dilatation with the coefficient r:

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} = r \begin{pmatrix} \cos\phi & -\sin\phi \\ \\ \sin\phi & \cos\phi \end{pmatrix}.$$

This is a linear conformal transformation preserving the angles. A similar computation works for an antiholomorphic map with nonvanishing derivatives $f'(\bar z) \neq 0$.

Conversely, the Jacobi matrix of a conformal transformation must have the form

$$r \begin{pmatrix} \cos\phi & -\sin\phi \\ \\ \sin\phi & \cos\phi \end{pmatrix}$$

or

$$r \begin{pmatrix} \cos\phi & \sin\phi \\ \\ \sin\phi & -\cos\phi \end{pmatrix}.$$

In the first case one obtains the differential of a holomorphic map while the second matrix corresponds to the antiholomorphic map. $\square$

We are ready to prove

Theorem 4.4.8. *Let*

$$f : \Omega \to \Omega'$$

be a conformal map. Then the pull-back of any function harmonic in Ω' is harmonic in Ω.

Proof: According to the Lemma the conformal map is given by a holomorphic or an anti-holomorphic function. Let us consider the holomorphic case,

$$z \mapsto w = f(z).$$

The transformation law of the Laplace operator under such a map is clear from the following formula:

$$\frac{\partial^2}{\partial z \partial \bar z} = |f'(z)|^2 \frac{\partial^2}{\partial w \partial \bar w}. \tag{4.4.10}$$

Thus any function U on Ω' satisfying

$$\frac{\partial^2 U}{\partial w \partial \bar w} = 0$$

will also satisfy

$$\frac{\partial^2 U}{\partial z \partial \bar z} = 0.$$

The case of an antiholomorphic map can be considered in a similar way. $\square$

A conformal map

$$f : \Omega \to \Omega'$$

is called *conformal transformation* if it is one-to-one. In that case the inverse map

$$f^{-1} : \Omega' \to \Omega$$

exists and is also conformal. The following fundamental *Riemann mapping theorem* is the central result of the theory of conformal transformations on the plane.

Theorem 4.4.9. *For any connected and simply connected domain Ω on the plane not coinciding with the plane itself there exists a conformal transformation of Ω to the unit disc.*

The Riemann theorem, together with conformal invariance of the Laplace equation gives a possibility to reduce the main boundary value problems for any connected simply connected domain to similar problems for the unit disc.

4.5 Exercises to Section 4

Exercise 4.5.1. *Find a function $u(x, y)$ satisfying*

$$\Delta u = x^2 - y^2$$

for $r < a$ and the boundary condition $u|_{r=a} = 0$.

Exercise 4.5.2. *Find a harmonic function on the annular domain*

$$a < r < b$$

with the boundary conditions

$$u|_{r=a} = 1, \quad \left(\frac{\partial u}{\partial r}\right)_{r=b} = \cos^2 \phi.$$

Exercise 4.5.3. *Find solution $u(x, y)$ to the Dirichlet b.v.p. in the rectangle*

$$0 \le x \le a, \quad 0 \le y \le b$$

satisfying the boundary conditions

$$\begin{aligned} u(0, y) &= A\, y(b - y), \quad u(a, y) = 0 \\ u(x, 0) &= B \sin \frac{\pi x}{a}, \quad u(x, b) = 0. \end{aligned}$$

Hint: use separation of variables in Euclidean coordinates.

5 Heat equation

5.1 Derivation of heat equation

The heat equation for the temperature $u(x,t)$, as a function of $x \in \mathbb{R}^d$ and $t \in \mathbb{R}_{>0}$, reads

$$\frac{\partial u}{\partial t} = a^2 \Delta u. \tag{5.1.1}$$

Here Δ is the Laplace operator in $\mathbb{R}^d$. We will consider only the case of constant coefficients $a = \text{const}$. For $d = 3$ this equation describes the distribution of temperature in the homogeneous and isotropic media at the moment t.

The derivation of heat equation is based on the following assumptions.

1. The heat δQ necessary for changing from u_1 to u_2 the temperature of a piece of mass m is proportional to the mass and to the difference of temperatures:

$$\delta Q = c_p\, m(u_2 - u_1).$$

The coefficient c_p is called *specific heat capacity.*

2. The *Fourier law* describing the quantity of heat spreading through a surface S during the time interval δt. It says that this quantity δQ is proportional to the area $A(S)$ of the surface, to the time δt and to the derivative of the temperature u along the normal n to the surface:

$$\delta Q = -k\, A(S) \frac{\partial u}{\partial n}\, \delta t.$$

Here the coefficient $k > 0$ is called *thermal conductivity.* The negative sign means that the heat is spreading from hot to cold regions.

In order to derive the heat equation let us consider the heat balance within a domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary $\partial\Omega$. The total change of heat in the domain during the time interval δt is

$$\delta Q = \int_\Omega c_p \rho \left[u(t + \delta t, x) - u(t,x) \right] dV \cdot$$

where ρ is the mass density, such that the mass of the media contained in the volume is equal to

$$m = \int_\Omega \rho\, dV.$$

In the case of a homogeneous media the mass density is constant, so the heat quantity is equal to

$$\delta Q \simeq c_p \rho \int_\Omega \frac{\partial u}{\partial t} \delta t\, dV.$$

This change of heat must be equal, with the negative sign, to the one passing through the boundary $\partial\Omega$

$$\delta Q = \int_{\partial\Omega} k\, \frac{\partial u}{\partial n}\, dS \cdot \delta t.$$

Using Green formula we can rewrite this heat flow in the form

$$k \int_\Omega \Delta u\, dV \cdot \delta t.$$

Dividing by δt we arrive at the equation

$$c_p \rho \int_\Omega \frac{\partial u}{\partial t}\, dV = k \int_\Omega \Delta u\, dV.$$

Since the domain Ω is arbitrary this gives the heat equation with the coefficient (called *thermal diffusivity*)

$$a^2 = \frac{k}{c_p \rho}.$$

The heat equation governs heat diffusion, as well as other diffusive processes, such as particle diffusion or the propagation of action potential in nerve cells. The heat equation is used in probability and describes random walks. For this reason it is also applied in financial mathematics (Black-Scholes or the Ornstein-Uhlenbeck processes). The equation, and various non-linear analogues, has also been used in image analysis. It is also important in Riemannian geometry and surprisingly in topology: it was adapted by R. Hamilton when he defined the Ricci flow that was later used by G. Perelman to solve the topological Poincarè conjecture.

5.2 Main boundary value problems for heat equation

The simplest is the Cauchy problem of finding a function $u(x,t)$ satisfying

$$\begin{aligned} &\frac{\partial u}{\partial t} = a^2 \Delta u \\ &u(x,0) = \phi(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{5.2.1}$$

The physical meaning of this problem is clear: given the initial temperature distribution in the space to determine the temperature at any time $t > 0$ at any point x of the space.

Often we are interested in the temperature distribution only within the bounded domain $\Omega \subset \mathbb{R}^d$. In this case one has to add to the Cauchy data within Ω also the information about the temperature on the boundary $\partial\Omega$ or about the heat flux through the boundary. In this way we arrive at two main mixed problems in a bounded domain:

The *first mixed problem*: find a function $u(x,t)$ satisfying

$$\begin{aligned} &\frac{\partial u}{\partial t} = a^2 \Delta u, \quad t > 0, \quad x \in \Omega \\ &u(0,x) = \phi(x), \quad x \in \Omega \\ &u(x,t) = f(x,t), \quad t > 0, \quad x \in \partial\Omega. \end{aligned} \tag{5.2.2}$$

The *second mixed problem* is obtained from (5.2.2) by replacing the last condition by

$$\left(\frac{\partial u}{\partial n}\right)_{x \in \partial\Omega} = g(x,t), \quad t > 0, \quad x \in \partial\Omega, \tag{5.2.3}$$

where n is the unit external normal to the boundary.

In the particular case of the boundary data independent of time

$$f = f(x) \quad \text{or} \quad g = g(x)$$

one can look for a *stationary solution* u satisfying

$$\frac{\partial u}{\partial t} = 0.$$

In this case the first and the second mixed problem for the heat equation reduce respectively to the Dirichlet and Neumann boundary value problem for the Laplace equation in $\mathbb{R}^d$.

5.3 Fourier transform

Our next goal is to solve the one-dimensional Cauchy problem for heat equation on the line. To this end we will develop a continuous analogue of Fourier series.

Let $f(x)$ be an absolutely integrable complex valued function on the real line, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty. \tag{5.3.1}$$

Definition 5.3.1. *The function $\hat{f}$,*

$$\hat{f}(p) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ipx} dx \tag{5.3.2}$$

of the real variable p is called the Fourier transform *of $f(x)$.*

Due to the condition (5.3.1) the integral converges absolutely and uniformly with respect to $p \in \mathbb{R}$. Thus the function $\hat{f}(p)$ is continuous in p.

Example 1. Let us compute the Fourier transform of the Gaussian function

$$f(x) = e^{-\frac{1}{2}x^2}.$$

We have

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 - ipx} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+ip)^2 - \frac{1}{2}p^2} dx \ .$$

We want to perform a (complex) change of variables

$$z = x + ip.$$

To do this one can consider the integral

$$\oint_C e^{-\frac{1}{2}z^2 - \frac{1}{2}p^2} dz, \quad z = x + iy \tag{5.3.3}$$

over the boundary C of the rectangle on the complex z-plane

$$-R \le x \le R, \quad 0 \le y \le p.$$

This integral is equal to zero since the integrand is holomorphic on the entire complex plane. But it is easy to see that the integrals over the vertical segments $x = \pm R$, $0 \le y \le p$ in (5.3.3) tend to zero when $R \to \infty$ (due to e^{-R^2} and limits on y).

Hence the integrals over the upper and lower horizontal segments

$$\int_{R}^{-R} e^{-\frac{1}{2}(x+ip)^2 - \frac{1}{2}p^2} dx + \int_{-R}^{R} e^{-\frac{1}{2}x^2 - \frac{1}{2}p^2} dx \to 0 \quad \text{as} \quad R \to \infty,$$

so

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+ip)^2 - \frac{1}{2}p^2} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 - \frac{1}{2}p^2} dx \ .$$

Using the Euler integral

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi} \tag{5.3.4}$$

we finally obtain the Fourier transform of the Gaussian function

$$\hat{f}(p) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}p^2}. \tag{5.3.5}$$

Example 2. We compute now the Fourier transform of the function

$$f(x) = \frac{1}{x}$$

(which is in L^2, but not in L^1 at ∞, and at 0 should be considered as a distribution). We calculate first the following auxiliary expression

$$I(a,p) := \int_0^\infty e^{-ax} \frac{\sin px}{x}\, dx, \quad \text{for} \quad a > 0.$$

We have

$$\frac{\partial}{\partial a} I(a,p) = -\int_0^\infty e^{-ax} \sin px\, dx = -Im \int_0^\infty e^{-ax} e^{ipx}\, dx$$

$$= -Im \frac{1}{ip-a} e^{(ip-a)x}|_0^\infty = Im \frac{1}{a-ip}(0-1) = -\frac{p}{a^2+p^2}.$$

Integrating this from 0 to ∞ we get

$$I(\infty,p) - I(0,p) == -\int_0^\infty \frac{p}{a^2+p^2}\, da = -\arctan \frac{a}{p}\ |_0^\infty = -\frac{\pi}{2} \operatorname{sign} p - 0.$$

But $I(\infty,p) = 0$ and hence $I(0,p) = \frac{\pi}{2} \operatorname{sign} p$. This shows that

$$\hat{f}(p) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-ipx}}{x}\, dx = -\frac{i}{\pi} \int_0^\infty \frac{\sin px}{x}\, dx =: -\frac{i}{\pi} I(0,p) = -\frac{i}{2} \operatorname{sign} p\ .$$

Corollary 5.3.2. *Dirichlet integral:*

$$\boxed{\int_0^\infty \frac{\sin x}{x}\, dx = I(0,1) = \frac{\pi}{2}.} \tag{5.3.6}$$

Lemma 5.3.3 (Riemann–Lebesgue)**.** *Let a* <u>*continuous*</u> *function $f(x)$ be absolutely integrable on $\mathbb{R}$. Then*

$$\lim_{\lambda \to \pm\infty} \int_{-\infty}^\infty f(x) e^{i\lambda x} dx = 0. \tag{5.3.7}$$

Proof: Because of convergence of the integral $\int_{-\infty}^\infty f(x)\, dx$ the difference

$$\int_{-\infty}^\infty f(x)\, dx - \int_a^b f(x)\, dx$$

tends to zero when $a \to -\infty$ $b \to \infty$. So it suffices to prove the Lemma for the finite integral. Because of integrability of $f(x)$ there exists, for any given $\epsilon > 0$, a partition of the interval

$$a = x_0 < x_1 < \cdots < x_n = b$$

such that

$$0 < \int_a^b f(x)\, dx - \sum_{j=1}^n m_j\, \delta x_j < \epsilon, \tag{5.3.8}$$

where

$$\delta x_j = x_j - x_{i-1}$$

$$m_j = \inf_{x \in [x_{i-1}, x_j]} f(x).$$

Introduce a step-like function

$$g(x) = m_j \quad \text{for} \quad x \in [x_{i-1}, x_j], \quad i = 1, \dots, n.$$

Then

$$\left| \int_a^b f(x) e^{i\lambda x} dx - \int_a^b g(x) e^{i\lambda x} dx \right| \le \int_a^b |f(x) - g(x)| \left| e^{i\lambda x} \right| dx$$
$$= \int_a^b |f(x) - g(x)| \, dx < \epsilon$$

by (5.3.8). But the integral

$$\int_a^b g(x) e^{i\lambda x} dx = \sum_{j=1}^n \frac{m_j}{i\,\lambda} \left(e^{i\lambda x_j} - e^{i\lambda x_{j-1}} \right)$$

tends to zero when $\lambda \to \pm\infty$ and so also the integral $\int_a^b g(x) e^{i\lambda x} dx$. This shows (5.3.7). $\square$

We will now establish, under certain additional assumptions, validity of the *inversion formula* for the Fourier transform:

$$\int_{-\infty}^{\infty} \hat{f}(p) e^{ipx} dp = f(x). \tag{5.3.9}$$

Theorem 5.3.4. *Let the absolutely integrable function $f(x)$ be differentiable at any point $x \in \mathbb{R}$. Then*

$$\lim_{R \to \infty} \int_{-R}^{R} \hat{f}(p) e^{ipx} dp = f(x). \tag{5.3.10}$$

Proof: Denote $I_R(x)$ the integral in the left hand side of (5.3.10). Using continuity and uniform convergence of the Fourier integral (5.3.2) we can apply Fubini theorem to this integral and thus rewrite it as follows:

$$I_R(x) = \int_{-R}^{R} \hat{f}(p) e^{ipx} dp = \int_{-R}^{R} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) e^{-ipy} dy \right) e^{ipx} dp$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \left(\int_{-R}^{R} e^{ip(x-y)} dp \right) dy = \frac{1}{\pi} \int_{-\infty}^{\infty} f(y) \frac{\sin R(x-y)}{x-y} dy$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+s) \frac{\sin Rs}{s} ds = \frac{1}{\pi} \int_0^{\infty} [f(x+s) + f(x-s)] \frac{\sin Rs}{s} ds.$$

Using the Dirichlet integral (5.3.6) we can rewrite the difference of the l.h.s. and r.h.s. in the form

$$I_R(x) - f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{f(x+s) - 2f(x) + f(x-s)}{s} \sin Rs \, ds.$$

Because of the differentiablity requirement on f we have

$$\lim_{s \to 0} \frac{f(x+s) - 2f(x) + f(x-s)}{s} = \lim_{s \to 0} \frac{f(x+s) - f_+(x)}{s} + \lim_{s \to 0} \frac{f(x-s) - f_-(x)}{s}$$
$$= f'_+(x) - {f_-}'(x) = 0,$$

and so the part

$$F(s; x) = \begin{cases} \frac{f(x+s) - 2f(x) + f(x-s)}{s}, & s \neq 0 \\ 0, & s = 0 \end{cases}$$

of the integrand is a continuous function in s depending on the parameter x.

Now represent the last integral in the form

$$\int_0^\infty \frac{f(x+s)-2f(x)+f(x-s)}{s}\sin R\,s\,ds$$
$$=\int_0^1 F(s;x)\sin R\,s\,ds+\int_1^\infty \frac{f(x+s)+f(x-s)}{s}\sin R\,s\,ds-2f(x)\int_1^\infty \frac{\sin R\,s}{s}\,ds.$$

Now in the limit $R\to\infty$ the first integral in the r.h.s. vanishes according to the proof of Riemann–Lebesgue lemma. The same is true for the second integral (there is no pole for $s\in[1,\infty]$). Finally the last integral by a change of integration variable $x=R\,s$ reduces to

$$\int_1^\infty \frac{\sin R\,s}{s}\,ds=\int_R^\infty \frac{\sin x}{x}\,dx\to 0 \quad \text{for} \quad R\to\infty$$

(since the Dirichlet integral converges). □

Proposition 5.3.5. *Let $f(x)$ be an absolutely integrable piecewise continuous function of $x\in\mathbb{R}$ differentiable on every interval of continuity. Let us also assume that at every discontinuity point x_0 the left and right limits $f_-(x_0)$ and $f_+(x_0)$ exists and, moreover, the left and right derivatives*

$$\lim_{s\to 0-}\frac{f(x_0+s)-f_-(x_0)}{s} \quad \text{and} \quad \lim_{s\to 0+}\frac{f(x_0+s)-f_+(x_0)}{s}$$

exist as well. Then there is following modification of the inversion formula for the Fourier transform

$$\lim_{R\to\infty}\int_{-R}^{R}\hat f(p)e^{ipx}dp=\begin{cases} f(x), & x \text{ is a continuity point}\\ \frac{f_-(x)+f_+(x)}{2}, & x \text{ is a discontinuity point}\end{cases} \tag{5.3.11}$$

Proof: Vedere le modifiche in rosso e sostituire $f=\frac{f_+(x)+f_-(x)}{2}$ and $0=f'_+(x)-f'_-(x)$ nella prova precedente. □

The main property of Fourier transform used for solving linear PDEs is given by the following formula:

Lemma 5.3.6. *Let $f(x)$ be an absolutely integrable continuously differentiable function with absolutely integrable derivative $f'(x)$. Then*

$$\int_{-\infty}^{\infty} f'(x)e^{-ipx}dx=ip\,\hat f(p). \tag{5.3.12}$$

Proof: From the integrability of $f'(x)$ it follows that there exist limits

$$f(\pm\infty):=\lim_{x\to\pm\infty}f(x)=f(0)+\lim_{x\to\pm\infty}\int_0^x f'(y)\,dy.$$

Because of absolute integrability of f the limiting values $f(\pm\infty)$ must be equal to zero. Integrating by parts

$$\int_{-\infty}^{\infty} f'(x)e^{-ipx}dx=\left(e^{-ipx}f(x)\right)_{-\infty}^{\infty}+ip\int_{-\infty}^{\infty}f(x)e^{-ipx}dx=ip\hat f(p),$$

but the first term on the r.h.s. vanishes and we arrive at the needed formula. □

Denote $\mathcal{F}$ the linear map of the space of functions in x variable to the space of functions in p variable given by the Fourier transform:

$$\mathcal{F}(f) = \hat{f}\ . \tag{5.3.13}$$

The property formulated in the above Lemma says that the operator of x-derivative transforms to i times the operator of multiplication by the independent variable

$$\widehat{\frac{d}{dx}f}\,(p) = ip\,\hat{f}(p). \tag{5.3.14}$$

This property of the Fourier transform will be used in the next section for solving the Cauchy problem for heat equation.

A similar calculation gives the formula

$$\widehat{xf}(p) = i\frac{d}{dp}\hat{f}(p) \tag{5.3.15}$$

valid for functions $f = f(x)$ absolutely integrable together with $xf(x)$. Moreover, the product of two functions transforms to the convolution product of their transforms

$$\widehat{fg}(p) = 2\pi\left(\hat{f} * \hat{g}\right)(p) := 2\pi\int \hat{f}(q)\hat{g}(p-q)\,dq\ . \tag{5.3.16}$$

We leave the proof of this formulae as an exercise for the reader. Finally we remark that the Fourier transform in $\mathbb{R}^d$ is defined as the operation of Fourier transform in each coordinate $\mathbb{R}$.

5.4 Solution to the Cauchy problem for heat equation on the line

Let us consider the one-dimensional Cauchy problem for the heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \quad t > 0 \\ &u(x,0) = \phi(x), \quad x \in \mathbb{R}. \end{aligned} \tag{5.4.1}$$

Theorem 5.4.1. *Let the initial data $\phi(x)$ be absolutely integrable function on $\mathbb{R}$. Then the Cauchy problem* (5.4.1) *has a unique solution $u(x,t)$ absolutely integrable in $x \in \mathbb{R}$ for all $t > 0$ represented by the formula*

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y;t)\,\phi(y)\,dy. \tag{5.4.2}$$

where

$$\boxed{G(x;t) = \frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2 t}}.} \tag{5.4.3}$$

The integral representation (5.4.2) of solutions to the Cauchy problem is called *Poisson integral* and its integral kernel is called *heat kernel*There are similar expressions in $\mathbb{R}^d$, one has to change the prefactor to $(4\pi t)^{-\frac{d}{2}}a^{-1}$ and x to $|x|$.

Proof: Denote

$$\hat{u}(p,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-ipx}dx$$

the Fourier-image of the unknown solution. According to Lemma 5.3.6 the function $\hat{u}(p,t)$ satisfies equation

$$\frac{\partial \hat{u}(p,t)}{\partial t} = -a^2p^2\hat{u}(p,t).$$

This equation can be easily solved

$$\hat{u}(p,t) = \hat{u}(p,0)e^{-a^2p^2t}.$$

Due to the initial condition we obtain

$$\hat{u}(p,0) = \hat{\phi}(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(y)e^{-ipy}dy.$$

Thus

$$\hat{u}(p,t) = \hat{\phi}(p)e^{-a^2p^2t}. \tag{5.4.4}$$

It remains to apply the inverse Fourier transform to this formula:

$$u(x,t) = \int_{-\infty}^{\infty} e^{ixp}\hat{\phi}(p)e^{-a^2p^2t}dp = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{ixp-a^2p^2t}\int_{-\infty}^{\infty} e^{-ipy}\phi(y)\,dy\right)dp$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(y)\left(\int_{-\infty}^{\infty} e^{ip(x-y)-a^2p^2t}dp\right)dy$$

The integral in p is nothing but the (inverse) Fourier transform of the Gaussian function. A calculation similar to Example 1 gives the value for this integral

$$\int_{-\infty}^{\infty} e^{ip(x-y)-a^2p^2t}dp = \frac{\sqrt{\pi}}{a\sqrt{t}}e^{-\frac{(x-y)^2}{4a^2t}}$$

(change the variables $p = q/\sqrt{2t}a$). This completes the proof of the Theorem. $\square$

Remark 5.4.2. *The formula* (5.4.2) *can work also for not necessarily absolutely integrable functions. For example for the constant initial data $\phi(x) \equiv \phi_0$ we obtain $u(x,t) \equiv \phi_0$ due to the following integral*

$$\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4a^2t}}dy \equiv 1. \tag{5.4.5}$$

We will now use this integral and the Poisson integral (5.4.2) in order to prove an analogue of the maximum principle for solutions to the heat equation.

Theorem 5.4.3. *Let $m = \inf_{x\in\mathbb{R}}\phi(x)$ and $M = \sup_{x\in\mathbb{R}}\phi(x)$. The solution to the Cauchy problem represented by the Poisson integral* (5.4.2) *for all $t > 0$ satisfies*

$$m \le u(x,t) \le M. \tag{5.4.6}$$

Moreover, if some of the inequalities becomes equality for some $t > 0$ and $x \in \mathbb{R}$ then $u(x,t) \equiv$ const.

Proof: Multiply $m \le \phi(y) \le M$ by a (positive) $G(x-y;t)$, integrate over y and use (5.4.5). The equality $u(x,t) = M$ for some x,t means that $\int G(x-y;t)(M-\phi(y))dy = 0$, which due to the same positivity can have place only if $\phi(x) \equiv M$ but then also $u(x,t) \equiv M$. Similar reasoning applies if $u(x,t) = m$. $\square$

Corollary 5.4.4. *The solution to the Cauchy problem* (5.4.1) *for the heat equation depends continuously on the initial data.*

Proof: Let $u_1(x,t)$, $u_2(x,t)$ be two solutions to the heat equation with the initial data $\phi_1(x)$ and $\phi_2(x)$ respectively. If the initial data differ by ϵ, i.e.

$$|\phi_1(x) - \phi_2(x)| \leq \epsilon \quad \forall x \in \mathbb{R}$$

then from the maximum principle applied to the solution $u(x,t) = u_1(x,t) - u_2(x,t)$ it follows that

$$|u_1(x,t) - u_2(x,t)| \leq \epsilon.$$

$\square$

5.5 Mixed boundary value problems for the heat equation

Let us begin with the 2π-periodic problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \\ &u(x + 2\pi, t) = u(x,t), \quad t > 0, \\ &u(x,0) = \phi(x), \end{aligned} \tag{5.5.1}$$

where $\phi(x)$ is a smooth 2π-periodic function.

Theorem 5.5.1. *There exists a unique solution $\mathcal{C}^2$ in x and $\mathcal{C}^1$ in t to the problem* (5.5.1). *It can be represented in the form*

$$u(x,t) = \frac{1}{2\pi} \int_0^{2\pi} \Theta(x - y; t) \phi(y)\, dy, \quad t > 0 \tag{5.5.2}$$

where

$$\boxed{\Theta(x;t) = \sum_{n \in \mathbb{Z}} e^{-a^2 n^2 t + inx}.} \tag{5.5.3}$$

Proof: Let us expand the unknown periodic function $u(x,t)$ in the Fourier series:

$$u(x,t) = \sum_{n \in \mathbb{Z}} \hat{u}_n(t) e^{inx},$$

where

$$\hat{u}_n(t) = \frac{1}{2\pi} \int_0^{2\pi} u(x,t) e^{-inx} dx.$$

The substitution to the heat equation yields

$$\frac{\partial \hat{u}_n(t)}{\partial t} = -a^2 n^2 \hat{u}_n(t),$$

so

$$\hat{u}_n(t) = \hat{u}_n(0) e^{-a^2 n^2 t}, \quad n \in \mathbb{Z}.$$

At $t = 0$ one must meet the initial conditions, so

$$\hat{u}_n(0) = \hat{\phi}_n = \frac{1}{2\pi} \int_0^{2\pi} \phi(y) e^{-iny} dy.$$

For the function $u(x,t)$ we obtain

$$u(x,t) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_0^{2\pi} e^{-a^2 n^2 t + in(x-y)} \phi(y) dy.$$

In order to show (5.5.2) it suffices to exchange the sum with the integral, which is possible since the series (5.5.3) converges absolutely and uniformly for all $x \in \mathbb{R}$ and all $t > \epsilon$ for any $\epsilon > 0$. This easily follows from the uniform convergence of the integral (which superates the series $\Theta - 1$)

$$\int_0^\infty e^{-a^2 y^2 t} dy = \frac{1}{a\sqrt{t}} \int_0^\infty e^{-y^2} dy < \infty \quad \text{for} \quad t > \epsilon > 0.$$

So far we have uniqueness: if the solution exists, it must have the form (5.5.2). In a similar way one can prove that the series (5.5.3) can be differentiated any number of times (the series of derivatives converges uniformly). Moreover (5.5.2) is $\mathcal{C}^1$ in t and $\mathcal{C}^2$ in x (we can derive under the integration sign) and so it is indeed a solution. The theorem is proved. □

We shall dedicate now some time to see the relations with other mathematical structures. The functioned defined by the series (5.5.3) is called *theta-function.* It is expressed via the *Jacobi theta-function*

$$\theta_3(\phi \,|\, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n \phi} \tag{5.5.4}$$

by a change of variables

$$\Theta(x;t) = \theta(\phi \,|\, \tau), \quad \phi = \frac{1}{2\pi} x, \quad \tau = i\frac{a^2 t}{\pi}. \tag{5.5.5}$$

The convergence of the series (5.5.4) for Jacobi theta function takes place for all complex values of τ provided

$$\operatorname{Im} \tau > 0 \quad (t > 0). \tag{5.5.6}$$

The function $\Theta(x;t)$ is periodic in x with the period 2π while the Jacobi theta-function is periodic in ϕ with the period 1. It satisfies many remarkable properties. Let us see some of them.

Lemma 5.5.2. *$\Theta(x;t)$ is a real valued function on $\mathbb{R} \times \mathbb{R}_+$ and*

$$\int_0^{2\pi} \Theta(x;t)\, dx = 2\pi. \tag{5.5.7}$$

Proof: Change of the summation variable $n \to -n$ in (5.5.3) shows that

$$\overline{\Theta(x;t)} = \Theta(x;t).$$

Exchanging the integral and sum,

$$\sum \int e^{-n^2 t + inx}\, dx = \sum 2\pi \delta_{n,0} = 2\pi \ .$$

□

Lemma 5.5.3. *The series*

$$\boxed{\sum_{n \in \mathbb{Z}} e^{-a^2 n^2 t + inz}} \tag{5.5.8}$$

converges for any complex number $z = x + iy$ uniformly on the strips $|\operatorname{Im} z| = y \le M$ for any positive M. So the theta-function (5.5.3) can be analytically continued to a function $\Theta(z;t)$ holomorphic on the entire complex z-plane.

Proof: Adapt the proof of theorem 5.5.1

Lemma 5.5.4. *The function $\Theta(z;t)$ is even and 2π-periodic in z, and satisfies the identity*

$$\Theta\left(z+2ia^2t;t\right)=e^{a^2t-iz}\Theta(z;t). \tag{5.5.9}$$

The complex number $2ia^2t$ is called *quasi-period* of the theta-function.

Proof: Since $z\mapsto -z$ together with $n\mapsto -n$ leaves invariant (5.5.3), Θ is even in z. The periodicity under $z\mapsto z+2\pi$ is evident. Moreover,

$$\sum_{n\in\mathbb{Z}}e^{-a^2n^2t+in(z+2ia^2t)}=\sum_{n\in\mathbb{Z}}e^{-a^2t(n^2+2n+1-1)}e^{i(n+1-1)z}=e^{a^2t-iz}\sum_{n\in\mathbb{Z}}e^{-a^2(n+1)^2t+i(n+1)z}$$

shows the quasi-periodicity. □

Lemma 5.5.5. *The theta-function has zeroes at the points*

$$z=z_{k\,l}=\pi(2k+1)+ia^2t\,(2l+1),\quad k,\,l\in\mathbb{Z}. \tag{5.5.10}$$

Proof: Due to the 2π-periodicity and $2ia^2t$ quasi-periodicity it suffices to check $k=0=l$. Set $\omega=\pi+ia^2t$. We have

$$\Theta(-z+\omega)=\Theta(-z-\omega+2\pi+2ia^2t)=\Theta(-z-\omega)e^{a^2t+i(z+\omega)}=\Theta(z+\omega)(-e^{iz}),$$

where in the last equality we used the parity of Θ and $e^{i\pi}=-1$. Setting $z=0$ we get the statement

$$\Theta(\omega)=-\Theta(\omega)=0.$$

□

Lemma 5.5.6. *The theta-function has no other zeroes on the complex plane. Moreover*

$$\Theta(x;t)>0\quad\text{for}\quad x\in\mathbb{R}. \tag{5.5.11}$$

Proof: If $z=z_i$, $i=1,\dots,n$ are zeroes of a function f holomorphic in (bounded) Ω, then $F(z)=f(z)(z-z_1)\dots(z-z_n)$, where f is holomorphic and has no zeroes in Ω, and

$$\frac{1}{2\pi i}\oint_{\partial\Omega}\frac{dF(z)}{F(z)}=\frac{1}{2\pi i}\oint_{\partial\Omega}\sum_i\frac{dz}{z-z_i}=n.$$

Now, this integral for $F(z)=\Theta(z;t)$ (parametric in t) and Ω given as the rectangle $\{0\le x\le 2\pi,\ 0\le y\le 2a^2t\}$ on the complex z-plane, $z=x+iy$, gives the result 1. This comes as follows. From the 2π-periodicity of Θ the contributions of the oriented vertical segments of $\partial\Omega$ cancel. From the quasi-periodicity of $\Theta(z;t)$, the contributions of the oriented horizontal segments cancel, except (due to the Leibniz rule) the contribution of the upper segment

$$\frac{1}{2\pi i}\int_{2\pi}^{0}\frac{d\,e^{-iz}}{e^{-iz}}=(-)(-)\frac{i}{2\pi i}\int_0^{2\pi}dx=1.$$

Since $\Theta(x;t)$ is real, has no zeroes on $\mathbb{R}$, and $\Theta(0;t)>0$, it must be positive. □

Another proof of positivity of the theta-function follows from the following *Poisson summation formula* that is of course of interest on its own.

Lemma 5.5.7. *Let $f(x)$ be a continuously differentiable absolutely integrable function satisfying the inequalities*

$$|f(x)|<C(1+|x|)^{-1-\epsilon},\quad |\hat{f}(p)|<C(1+|p|)^{-1-\epsilon}$$

for some positive C and ϵ. Here $\hat{f}(p)$ is the Fourier transform of $f(x)$. Then

$$\sum_{n\in\mathbb{Z}}f(2\pi n)=\sum_{m\in\mathbb{Z}}\hat{f}(m). \tag{5.5.12}$$

Proof: We will actually prove a somewhat more general formula

$$\sum_{n\in\mathbb{Z}} f(x+2\pi n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)e^{imx}, \quad x\in\mathbb{R}. \tag{5.5.13}$$

Of course since f is not necessarily periodic, the right hand side need not to be its Fourier series and we have to work out various properties anew. The series on the l.h.s. and the series on the r.h.s. (as well as the series of derivatives of their terms) converge uniformly due to the decay conditions on f and $\hat{f}$. Thus on both sides we have differentiable functions which are moreover 2π-periodic in x. Therefore, it suffices to check that the Fourier coefficients c_m of the function on the left side coincide with the Fourier coefficients of the function on the right side, which are clearly $\hat{f}(m)$. Indeed, the m-th Fourier coefficient of the left hand side is equal to

$$c_m = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n\in\mathbb{Z}} f(x+2\pi n)\right)e^{-imx}dx.$$

Due to absolute and uniform (in x) convergence of the series

$$\sum_{n\in\mathbb{Z}} f(x+2\pi n)$$

one interchange the order of summation and integration to arrive at

$$c_m = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^{2\pi} f(x+2\pi n)e^{-imx}dx.$$

By shifting in the n-th summand the integration variable

$$y = x+2\pi n,$$

(and accordingly shifting the integration limits) one rewrites the sum as follows:

$$c_m = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{2\pi n}^{2\pi(n+1)} f(y)e^{-imy-2\pi imn}dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{-imy}dy = \hat{f}(m),$$

since $e^{-2\pi imn}=1$. $\square$

Using the Poisson summation formula we can prove the following remarkable identity for the theta-function.

Proposition 5.5.8. *The theta-function* (5.5.1) *satisfies the following identity*

$$\Theta(x;t) = \frac{1}{a}\sqrt{\frac{\pi}{t}}\sum_{n\in\mathbb{Z}} e^{-\frac{(x+2\pi n)^2}{4a^2t}}. \tag{5.5.14}$$

Proof: It can be obtained by applying the Poisson summation formula (in the stronger version (5.5.13)) to the function

$$f(x) = \frac{1}{a}\sqrt{\frac{\pi}{t}}\,e^{-\frac{x^2}{4a^2t}}$$

which gives the r.h.s. of (5.5.14). Its Fourier transform

$$\hat{f}(p) = e^{-a^2p^2t},$$

we computed in Theorem 5.4.1, and this yields the defining formula for (5.5.1). The required assumptions on f and $\hat{f}$ are clearly satisfied. $\square$

Now, looking at (5.5.14), we can affirm that $\Theta(x;t)$ is positive.

Remark 5.5.9. *The formula* (5.5.14) *is the clue to derivation of the transformation law for the Jacobi theta-function under* modular transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad a,\, b,\, c,\, d \in \mathbb{Z}, \quad ad - bc = 1.$$

Let us now consider the first mixed problem for heat equation on the interval $[0, l]$ with zero boundary conditions:

$$\begin{aligned} &\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le l, \quad t > 0 \\ &u(0,t) = u(l,t) = 0 \\ &u(x,0) = \phi(x), \quad 0 \le x \le l. \end{aligned} \tag{5.5.15}$$

Analogously to Section 3.6 let us extend the initial data $\phi(x)$ to the real line as an odd $2l$-periodic function. We leave as an exercise for the reader to check that the solution to this periodic Cauchy problem will remain an odd periodic function for all times and, hence, it will vanish at the points $x = 0$ and $x = l$. In this way one arrives at the following

Theorem 5.5.10. *The mixed b.v.p.* (5.5.15) *has a unique solution for an arbitrary smooth function $\phi(x)$. It can be represented by the following integral*

$$u(x,t) = \frac{1}{l} \int_0^l \widetilde{\Theta}(x, y; t) \phi(y)\, dy \tag{5.5.16}$$

where

$$\widetilde{\Theta}(x, y; t) = 2 \sum_{n=1}^{\infty} e^{-a^2 n^2 t} \sin \frac{\pi n x}{l} \sin \frac{\pi n y}{l}. \tag{5.5.17}$$

5.6 More general boundary conditions for the heat equation. Solution to the inhomogeneous heat equation

In the previous section the simplest b.v.p. for the heat equation has been considered. We will now address the more general problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad 0 < x < l \\ &u(0,t) = f_0(t), \quad u(l,t) = f_1(t), \quad t > 0 \\ &u(x,0) = \phi(x), \quad 0 < x < l. \end{aligned} \tag{5.6.1}$$

The following simple procedure reduces the above problem for $u(x,t)$ to the b.v.p. for $v(x,t)$ with zero boundary condition for the *inhomogeneous* heat equation in $d = 1$. Namely, let $u = v + w$, where

$$w(x,t) := \left[f_0(t) + \frac{x}{l}(f_1(t) - f_0(t)) \right]. \tag{5.6.2}$$

(Note for the next case that w is a (unique) harmonic in $d = 1$). Note also that $w(x,t)$ takes the needed values $f_0(t)$ and $f_1(t)$ at the endpoints of the interval. Now, $v = u - w$ indeed satisfies

$$\begin{aligned} &\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + F(x,t), \quad t > 0, \quad 0 < x < l \\ &v(0,t) = v(l,t) = 0, \quad t > 0 \\ &v(x,0) = \Phi(x), \quad 0 < x < l\ , \end{aligned} \tag{5.6.3}$$

where the functions $F(x,t)$, $\Phi(x)$ are given by

$$F(x,t) = -\frac{\partial w(x,t)}{\partial t} = -\left[\frac{df_0(t)}{dt} + \frac{x}{l}\left(\frac{df_1(t)}{dt} - \frac{df_0(t)}{dt}\right)\right]$$
$$\Phi(x) = \phi(x) - w(x,0) = \phi(x) - \left[f_0(0) + \frac{x}{l}(f_1(0) - f_0(0))\right]. \tag{5.6.4}$$

In the more general case of d-dimensional heat equation with non-vanishing boundary conditions

$$\begin{aligned} &\frac{\partial u}{\partial t} = a^2 \Delta u, \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^d \\ &u(x,t)|_{x\in\partial\Omega} = f(x,t), \quad t > 0 \\ &u(x,0) = \phi(x), \quad x \in \Omega \end{aligned} \tag{5.6.5}$$

the procedure is similar to the above one. Namely, denote $w(x,t)$ the solution to the Dirichlet boundary value problem for the Laplace equation in x depending on t as on the parameter:

$$\begin{aligned} &\Delta w = 0, \quad x \in \Omega \subset \mathbb{R}^d \\ &w(x,t)|_{x\in\partial\Omega} = f(x,t). \end{aligned} \tag{5.6.6}$$

We already know that the solution to the Dirichlet boundary value problem is unique and depends continuously on the boundary conditions. Therefore the solution $w(x,t)$ is a continuous function on $\Omega \times \mathbb{R}_{>0}$. One can also prove that this functions is smooth, if the boundary data $f(x,t)$ are so. Then the substitution

$$u(x,t) = v(x,t) + w(x,t) \tag{5.6.7}$$

reduces the mixed b.v.p. (5.6.5) to the one with zero boundary conditions

$$v(x,t)|_{x\in\partial\Omega} = 0, \quad t > 0$$

with the modified initial data

$$v(x,0) = \phi(x) - w(x,0), \quad x \in \Omega$$

but the heat equation becomes inhomogeneous one:

$$\frac{\partial v}{\partial t} = a^2 \Delta v + F(x,t), \quad F(x,t) = -\frac{\partial w(x,t)}{\partial t}, \quad x \in \Omega.$$

We will now explain a simple method for solving the inhomogeneous heat equation. For the sake of simplicity let us consider in details the case of one spatial variable. Moreover we will concentrate on the infinite line case. So the problem under consideration is in finding a function $u(x,t)$ on $\mathbb{R} \times \mathbb{R}_{>0}$ satisfying

$$\begin{aligned} &\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in \mathbb{R}, \quad t > 0 \\ &u(x,0) = \phi(x). \end{aligned} \tag{5.6.8}$$

Theorem 5.6.1. *The solution to the inhomogeneous problem* (5.6.8) *has the form*

$$u(x,t) = \int_0^t ds \int_{-\infty}^{\infty} G(x-y; t-s)\, f(y,s) dy + \int_{-\infty}^{\infty} G(x-y;t)\, \phi(y)\, dy \tag{5.6.9}$$

where the function $G(x;t)$ was defined in (5.4.3).

Proof: The first term

$$u_1(x,t) = \int_0^t ds \int_{-\infty}^{\infty} G(x-y; t-s)\, f(y,s) dy$$

clearly vanishes at $t = 0$. Let us prove that it satisfies the inhomogeneous heat equation

$$\frac{\partial u_1}{\partial t} = a^2 \frac{\partial^2 u_1}{\partial x^2} + f(x,t).$$

Denote

$$v(x,t;s) = \int_{-\infty}^{\infty} G(x-y; t-s)\, f(y,s) dy.$$

Like in the Theorem 5.4.1 we derive that this is a solution to the homogeneous heat equation in x, t depending on the parameter s. This solution is defined for $t \geq s$; for $t = s$ it satisfies the initial condition

v_t in 2.9 but there we had ∂_t^2

$$v(x,s;s) = f(x,s).$$

Now applying $\frac{\partial}{\partial t}$ to the first term

$$u_1(x,t) = \int_0^t v(x,t;s)\, ds$$

one obtains

$$\begin{aligned}\frac{\partial}{\partial t} u_1(x,t) &= v(x,t;t) + \int_0^t \frac{\partial}{\partial t} v(x,t;s)\, ds = f(x,t) + \int_0^t a^2 \frac{\partial^2}{\partial x^2} v(x,t;s)\, ds \\ &= f(x,t) + a^2 \frac{\partial^2}{\partial x^2} u_1(x,t)\ .\end{aligned}$$

Concerning the the second term

$$u_2(x,t) = \int_{-\infty}^{\infty} G(x-y;t)\, \phi(y)\, dy$$

in (5.6.9), we already know from Theorem 5.4.1 that it solves the homogeneous heat equation and satisfies initial condition

$$u_2(x,0) = \phi(x).$$

This concludes the proof. □

Ricapitoliamo alcune proprietà salienti dell'equazione del calore in $\mathbb{R}^2$:
- superposizioni/scomposizioni di soluzioni (in 'modi' semplici)
- esistenza e unicità di soluzioni e problema di Cauchy ben posto
- propagazione istantanea
- smorzatura di singolarità nel tempo (per il principio del massimo)
- nel caso periodico legame con funzioni theta.

5.7 Exercises to Section 5

Exercise 5.7.1. *Let the function $f(x)$ belong to the class $\mathcal{C}^k(\mathbb{R})$ and, moreover, all the functions $f(x)$, $f'(x)$, ..., $f^{(k)}(x)$ be absolutely integrable on $\mathbb{R}$. Prove that then*

$$\hat{f}(p) = \mathcal{O}\left(\frac{1}{p^k}\right) \quad \text{for} \quad |p| \to \infty. \tag{5.7.1}$$

Exercise 5.7.2. *Let $\hat{f}(p)$ be the Fourier transform of the function $f(x)$. Prove that $e^{iap}\hat{f}(p)$ is the Fourier transform of the shifted function $f(x+a)$.*

Exercise 5.7.3. *Find Fourier transforms of the following functions.*

$$f(x) = \Pi_A(x) = \begin{cases} \frac{1}{2A}, & |x| < A \\ 0, & \text{otherwise} \end{cases} \tag{5.7.2}$$

$$f(x) = \Pi_A(x)\cos\omega x \tag{5.7.3}$$

$$f(x) = \begin{cases} \frac{1}{A}\left(1 - \frac{|x|}{A}\right), & |x| < A \\ 0, & \text{otherwise} \end{cases} \tag{5.7.4}$$

$$f(x) = \cos ax^2 \quad \text{and} \quad f(x) = \sin ax^2 \quad (a > 0) \tag{5.7.5}$$

$$f(x) = |x|^{-\frac{1}{2}} \quad \text{and} \quad f(x) = |x|^{-\frac{1}{2}}e^{-ax} \quad (a > 0). \tag{5.7.6}$$

Exercise 5.7.4. *Find the function $f(x)$ if its Fourier transform is given by*

$$\hat{f}(p) = e^{-k|p|}, \quad k > 0. \tag{5.7.7}$$

Exercise 5.7.5. *Let $u = u(x,y)$ be a solution to the Laplace equation on the half-plane $y \geq 0$ satisfying the conditions*

$$\begin{aligned} &\Delta u(x,y) = 0, \quad y > 0 \\ &u(x,0) = \phi(x) \\ &u(x,y) \to 0 \quad \text{as} \quad y \to +\infty \quad \text{for every} \quad x \in \mathbb{R} \end{aligned} \tag{5.7.8}$$

1) Prove that the Fourier transform of u in the variable x

$$\hat{u}(p,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,y)e^{-ipx}dx$$

has the form

$$\hat{u}(p,y) = \hat{\phi}(p)e^{-y\,|p|}.$$

Here $\hat{\phi}(p)$ is the Fourier transform of the boundary function $\phi(x)$.

2) Derive the following formula for the solution to the b.v.p. (5.7.8)

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-s)^2 + y^2}\phi(s)\,ds. \tag{5.7.9}$$