Session 10 - Guided problem solving session: spin waves.

1. **3D-Ferromagnet in an external field**

Let us consider a 3D ferromagnet in an external field, assuming again nearest neighbor interaction only. The system is described by the following Hamiltonian:

$$\mathcal{H} = -2\frac{J}{\hbar^2}\sum_{i,\delta}\mathbf{S}_i\cdot\mathbf{S}_{i+\delta} - \frac{g\mu_B B_0}{\hbar}\sum_i S_{jz}$$

where δ labels the vectors identifying the nearest-neighbors of site i and the external magnetic field defines the quantization axis.

a) Write the Hamiltonian in terms of the spin raising and lowering operators, $S_j^{\pm} = S_{jx} \pm iS_{jy}$, and show that their action on the state $|s_{jz}\rangle$, where the spin quantum number on site j is s_{jz}, is:

$$S_j^{\pm}|s_{jz}\rangle = \hbar[\ s(s+1) - s_{jz}(s_{jz}\pm 1)\ ]^{1/2}|s_{jz}\pm 1\rangle$$

b) Re-label the site-spin states in terms of the number, n_j, of spin-deviations from the ferromagnetic Ground State: increasing n_j by 1 is the same as decreasing s_{jz} by 1. Show that $S_{jz}|n_j\rangle = \hbar(s-n_j)|n_j\rangle$.

c) Define creation and annihilation operators for states n_j satisfying the harmonic oscillator commutation relations:

$$[a_i, a_j^+] = \delta_{i,j}, \quad [a_i, a_j] = [a_i^+, a_j^+] = 0 \tag{1}$$

and show that this implies

$$a_j\,|n_j\rangle = \sqrt{n}_j|n_j-1\rangle, \quad a_j^+|n_j\rangle = \sqrt{n_j+1}|n_j+1\rangle, \quad a_j^+a_j|n_j\rangle = n_j|n_j\rangle. \tag{2}$$

d) Given the definition above, show that the following, Holstein-Primakoff, transformation holds

$$S_j^+ = \hbar\sqrt{2s}\left(1-\frac{a_j^+a_j}{2s}\right)^{1/2} a_j, \quad S_j^- = \hbar\sqrt{2s}\ a_j^+\left(1-\frac{a_j^+a_j}{2s}\right)^{1/2}$$

and re-write the hamiltonian in terms of the a_j and a_j^+ operators.

e) Assuming the temperature is small enough that the average number of excitation is negligible ($\langle n_j\rangle \approx 0, \quad \forall j$) expand the Hamiltonian and keep only terms of the lowest non trivial order in creation and annihilation operators.

Solve the resulting problem and extract the magnon dispersion relation in the limit of $k \to 0$ (assume the nearest-neighbor distance is a, and the number of nearest neighbor is Z.)