Session 11 - Individual problem solving session: spin waves.

1. **3D-Ferromagnet in an external field**

   Consider a 3D ferromagnet in an external field, assuming nearest neighbor interaction only. The system is described by the following Hamiltonian:

   $$\mathcal{H} = -2\frac{J}{\hbar^2}\sum_{i,\delta}\mathbf{S}_i \cdot \mathbf{S}_{i+\delta} - \frac{g\mu_B B_0}{\hbar}\sum_i S_{iz}, \tag{1}$$

   where δ labels the vectors identifying the nearest-neighbors of site i and the external magnetic field defines the quantization axis.

   a) Write the Hamiltonian in terms of the spin raising and lowering operators, $S_j^{\pm} = S_{jx} \pm iS_{jy}$.

   b) Re-label the site-spin states in terms of the number, n_j, of spin-deviations from the ferromagnetic Ground State: increasing n_j by 1 is the same as decreasing s_{jz} by 1.

   c) Define creation and annihilation operators for states n_j satisfying the harmonic oscillator commutation relations:

   $$[a_i, a_j^+] = \delta_{i,j}, \quad [a_i, a_j] = [a_i^+, a_j^+] = 0. \tag{2}$$

   Given the action of such operators on the state $|n_j\rangle$:

   $$a_j\,|n_j\rangle = \sqrt{n_j}|n_j - 1\rangle, \quad a_j^+|n_j\rangle = \sqrt{n_j+1}|n_j+1\rangle, \quad a_j^+a_j|n_j\rangle = n_j|n_j\rangle, \tag{3}$$

   show that the following, Holstein-Primakoff, transformation holds:

   $$S_j^+ = \hbar\sqrt{2s}\left(1 - \frac{a_j^+a_j}{2s}\right)^{1/2} a_j, \quad S_j^- = \hbar\sqrt{2s}\ a_j^+\left(1 - \frac{a_j^+a_j}{2s}\right)^{1/2}, \tag{4}$$

   and re-write the hamiltonian in terms of the a_j and a_j^+ operators.

   d) Assuming the temperature is small enough that the average number of excitation is negligible ($\langle n_j\rangle \approx 0, \quad \forall j$) expand the Hamiltonian and keep only terms of the lowest non trivial order in creation and annihilation operators.

   Solve the resulting problem by Fourier transforming the operators and extract the magnon dispersion relation in the limit of $k \to 0$ (assume the nearest-neighbor distance is a, and the number of nearest neighbors is Z.)

2. **1D anisotropic Antiferromagnet**

   Consider a one-dimensional chain of N spins in presence of anisotropy, defined by the following Hamiltonian:

   $$\mathcal{H} = -2\frac{J}{\hbar^2}\sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} - D\sum_i S_{iz}^2, \tag{5}$$

   where $J < 0$ represents the antiferromagnetic coupling and $D > 0$ is the anisotropy along the z direction.

   In order to analyze the excitations of this system, apply the linear spin wave theory by following these steps:

a) Apply the Holstein-Primakoff transformation for an antiferromagnet, defined by the following relationships:

$$S_{iz} = (-1)^i (S - n_i), \tag{6}$$

$$S_i^{\pm} = \frac{1 \mp (-1)^i}{2}\, d_i^\dagger \sqrt{2S - n_i} + \frac{1 \pm (-1)^i}{2} \sqrt{2S - n_i}\; d_i, \tag{7}$$

where the d_i's are bosonic operators (satisfying the usual commutation relations) and the n_i's are the corresponding number operators $d_i^\dagger d_i$.

b) Assuming low occupation numbers, take the limit $\langle n \rangle / S \ll 1$ and keep only terms up to order S. The Hamiltonian should now only have two-particle operators.

c) Fourier transform the d_i's into d_k's.

d) Diagonalize this Hamiltonian by means of a Bogoliubov transformation in terms of the new bosonic operators b_k:

$$d_k = \cosh u_k b_k + \sinh u_k b_{-k}^\dagger, \tag{8}$$

choosing the parameters $u_k = u_{-k}$ so that the Hamiltonian is recast in diagonal form.

e) Describe the spectrum of the system, for different values of the anisotropy D.