SISSA ISAS

Spring 2003 – Entrance Examination: Condensed Matter

Solve at least one of the following problems. Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. Do not write your name on the problem sheet, but use extra envelope.

Problem 1: One-dimensional binding

A particle moves in a one-dimensional δ-function attractive potential

$$H_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - v\delta(x) \ . \tag{1}$$

1) Consider its bound state ψ_1, and find its energy E_1 as a function of v. [Hints: assume $\psi \sim e^{-|x|/\lambda}$; recall that $2\delta(x) = d^2|x|/dx^2$]

A second δ-function attraction is now added some large distance a apart

$$H_T = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - v\delta(x) - \alpha v\delta(x-a) \ , \tag{2}$$

where $\alpha > 0$ is a parameter measuring its strength. Obviously, neglecting the first delta, $-v\delta(x)$, you would have a problem similar to the previous one, with $H_2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha v\delta(x-a)$, admitting a bound state ψ_2 with energy E_2. Consider the overall ground state energy E_T of H_T as a function of α, and define a "binding energy" Δ as the difference between the lowest among the ground states E_1 and E_2 of the two separate δ-functions, and the true ground state E_T, $\Delta = min(E_1, E_2) - E_T$.

2) Sketch the behaviour of E_T as α varies from 0 to ∞, and indicate where Δ will be the largest. [Hint: do not attempt a calculation of E_T. Use only intuition and symmetry arguments.]

3) Use perturbation theory to determine the approximate value of Δ for the value of α where you believe Δ is maximum. [Hint: use (ψ_1, E_1) and (ψ_2, E_2) as unperturbed states.]

Problem 2. A ring with magnetic flux

Consider a one-dimensional ring of length L with coordinate x ($0 \le x \le L$), and a magnetic field confined in a solenoid through its center, as shown in figure. The vector potential A can be considered constant on the ring and the magnetic flux inside the ring is $\Phi = LA$. For a single particle, setting $\hbar = e = c = 1$, the Hamiltonian is
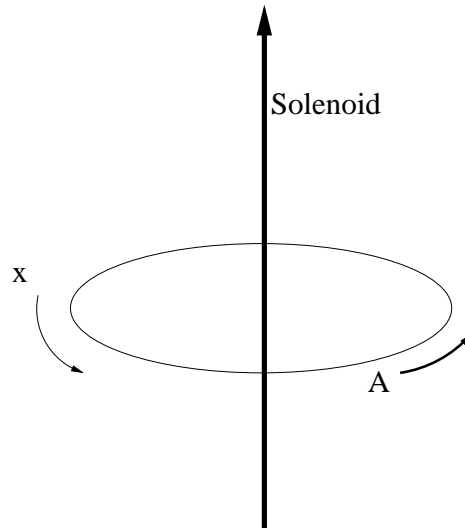
simply:

$$H = \frac{1}{2m}(-i\partial_x + A)^2$$

and one should solve $H\psi(x) = E\psi(x)$ with the requirement that $\psi(x + L) = \psi(x)$.

1. Show that the field A can be eliminated by considering the wavefunction:

$$\psi(x) = e^{-iAx}\Psi(x)$$

2. What boundary conditions must be satisfied by the wavefunction $\Psi(x)$? Determine accordingly the spectrum of H for a generic flux Φ.

3. Which values of the flux Φ are consistent with a ground state with no current flowing? [Recall that the current is $j(x) = -(1/2m)\psi^*(x)(-i\partial_x + A)\psi(x) + \text{c.c.}$]

4. Consider now *two non-interacting electrons* with opposite spins subject to the same H. For which values of the flux Φ do you have a ground state with no current flowing?

Problem 3. A diatomic molecule

The adiabatic interatomic potential for the Na_2 molecule, as a function of the distance R between the two atoms, can be represented in the form:

$$V(R) = A \left[e^{-2a(R-R_0)} - 2e^{-a(R-R_0)} \right] \tag{3}$$

with $A = 0.75$ eV, $R_0 = 3.07$ Å, and $a = 0.84$ Å^{-1}.

1. Determine the $T = 0$ equilibrium distance of the molecule, and its binding energy, neglecting quantum effects.

2. Determine the vibrational energy $\hbar\omega$ of the molecule (for small oscillations) and its moment of inertia I.
   [The mass of Na is ≈ 23 times mass of the proton m_p, which is in turn $m_p \approx 1.66 \times 10^{-27}$kg. It is useful to know that $\hbar^2/m_p \approx 4.1$ meV Å^2.]

3. For a gas of such molecules at $T = 300K^o$, evaluate the ratio between the number of molecules in the first excited state N_1 and that in the ground state N_0, for both rotational and vibrational excitations. [Recall that $300K^o \approx 25.85$ meV.]

Problem 4. Atomic polarizability

Consider a hydrogen atom immersed in electromagnetic radiation. The Hamiltonian of the electron in the 'electric dipole' approximation is:

$$H = \frac{\mathbf{p}^2}{2m} + V(r) + \frac{eEp_z}{m\omega} sin\omega t \tag{4}$$

where $\mathbf{p}$ and $\mathbf{r}$ are the electron momentum and position ($r = |\mathbf{r}|$), m and e are the electron mass and charge respectively, E is the radiation electric field intensity (the field is parallel to the z axis), and ω is the angular frequency of the electromagnetic radiation.

1. First take $V(r) = \frac{1}{2}m\omega_0^2 r^2$, where ω_0 is some positive frequency. Solve the classical equations of motion and find $z(t)$.

2. The polarizability $P(\omega)$ of the hydrogen atom is defined as the ratio between the piece of the time-dependent induced dipole moment, $ez(t)$, at frequency ω, and the applied electric field. Find the classical expression of the polarizability. Take into account only the steady state part of the solution.

3. Now solve the quantum mechanical problem with $V(r) = -e^2/r$. Denote with $|\phi_i\rangle$ the eigenstates of the hydrogen atom Hamiltonian $H_0 = \frac{\mathbf{p}^2}{2m} + V(r)$ and with E_i its eigenvalues. Use first order perturbation theory and find the wavefunction $|\psi_0(t)\rangle$ solution of the Schrödinger equation with Hamiltonian (1) coinciding, at $t = 0$, with the ground state of the hydrogen atom. Consider only the steady state part of the solution.

4. Find the mean value of z in the state $|\psi_0(t)\rangle$ at linear order in the electric field and compute the quantum mechanical expression of the polarizability.

5. (Optional) Compare the classical and quantum expressions of the polarizability and discuss the result.

Hint: The following identity might be useful: $\langle\phi_i|p_z|\phi_j\rangle = im\frac{(E_i - E_j)}{\hbar}\langle\phi_i|z|\phi_j\rangle$.