SISSA ISAS

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Spring 2008 - Entrance Examination: Condensed Matter

Solve at least one of the following problems. Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. Do not write your name on the problem sheet, but use extra envelope.

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1. Electron gas at fixed magnetization

Consider a free electron gas at zero temperature in a cubic box of side L with vanishing boundary conditions. Assume that not only the total density $\rho = N/L^3$, N being the total number of electrons, is fixed, but also the density for each spin projection, $\rho_\sigma = N_\sigma/L^3$ with $\sigma =\uparrow, \downarrow$ such that $\rho_\uparrow + \rho_\downarrow = \rho$, namely $N = N_\uparrow + N_\downarrow$. The Hamiltonian is

$$\mathcal{H}_0 = -\sum_{i=1}^{N} \frac{\hbar^2}{2m} \nabla_i^2. \tag{1}$$

(1) Find the ground state wavefunction $|M\rangle$ and energy $E_0(M)$ for fixed average magnetization $M = \rho_\uparrow - \rho_\downarrow$.

(2) At fixed M, calculate the pressure $P(M)$ exerted on each face of the box by the electron gas.

Consider now an interacting gas with a δ-function repulsion. The interaction Hamiltonian is

$$\mathcal{H} = \mathcal{H}_0 + \frac{U}{2} \sum_{i \neq j} \delta\left(x_i - x_j\right), \tag{2}$$

where x_i and x_j are the spatial coordinates of the i and j electrons, respectively.

(3) Calculate at first order in pertubation theory, i.e. at first order in U, the ground state energy of $\mathcal{H}$ at fixed magnetization M.

(4) Determine the values of the electron density for which the ground state energy at first order in U is minimum for $M \neq 0$.

2. Domino pieces on a rectangular checkerboard

Consider a $L \times 2$ rectangular checkerboard, and L domino pieces, each occupying two neighboring squares of the checkerboard.

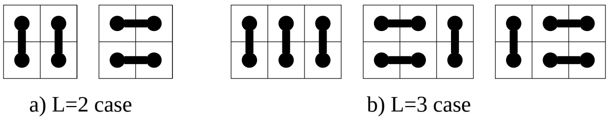

Figure 1: *The possible configurations of domino pieces on* 2×2 *and* 3×2 *checkerboards.*

1) How many inequivalent configurations of domino pieces, N_L, can you form? (**Hint:** Try to find out a recursive relation between N_L, N_{L-1} and N_{L-2}.)

2) Calculate the configuration entropy per site $(S/L) = k_B(\log N_L)/L$ as $L \to \infty$. (**Hint:** Solve the recursive relation via the *Ansatz* $N_L = a^L$ and find out the value of a.)

Imagine now that the system is subject to an external field such that each horizontal domino piece pays an energy $\epsilon_0 > 0$, while vertical dominos do not cost any energy. The system is also held at temperature T.

3) Write down a recursion relation between the partition function at different sizes, specifically Z_L (system of size L), Z_{L-1} and Z_{L-2}.

4) Calculate the free-energy per site

$$f = -k_B T \lim_{L \to \infty} \frac{1}{L} \log Z_L \ .$$

5) **(Optional)** Calculate the average internal energy per site $u(T)$, and entropy per site $s(T)$ at temperature T.

6) **(Optional)** nalyze the large-T and small-T behaviour of $u(T)$ and $s(T)$, and plot the two quantities.

7) **(Optional)** What happens to $u(T)$ and $s(T)$ when the field is turned off, i.e., $\epsilon_0 = 0$? Compare the number of ground states when $\epsilon_0 > 0$ with that for $\epsilon_0 = 0$.