April 2009 - Entrance Examination: Condensed Matter

Solve one of the following problems. Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. Do not write your name on the problem sheet, but use extra envelope.

Problem 1: 2D rotor in an electric field

A 2-dimensional rotor obeys the Schrœdinger equation

$$-\frac{\hbar^2}{2I}\,\frac{\partial^2\,\psi(\phi)}{\partial\phi^2} = E\,\psi(\phi)$$

where ϕ is the angular variable, and I the moment of inertia.

1. Find the eigenvalues E_n and eigenstates ψ_n, with their degeneracies.

Now suppose the rotor to possess a dipole moment of magnitude d, and add a weak electric field E, described by the perturbing potential

$$V(\phi) = -E\,d\,\cos\phi.$$

2. Find the matrix elements of V betweeen the unperturbed states ψ_n.
3. Use perturbation theory to calculate the perturbed energies $E_n + \delta E_n$ and perturbed wavefunctions $\psi_n + \delta\psi_n$ to their lowest respective relevant order.
4. Describe the different probabilities to find the dipole parallel, orthogonal, or antiparallel to the electric field
   a) starting in the unperturbed ground state ψ_0;
   b) starting in the first excited state ψ_1.

Problem 2: Energy levels of the He atom

The He atom has two electrons and, in atomic units, its Hamiltonian is:

$$H = \frac{\mathbf{p}_1^2}{2} + \frac{\mathbf{p}_2^2}{2} - \frac{Z}{|\mathbf{r}_1|} - \frac{Z}{|\mathbf{r}_2|} + \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \tag{1}$$

where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the momenta of the two electrons, $\mathbf{r}_1$ and $\mathbf{r}_2$ their positions and Z the nuclear charge.

1. Neglecting the electron-electron interaction, but not the electron spin, write the wave-functions of the $1s^2$, $1s2s$, $1s2p$ states of the He atom and find, in this approximation, their energies and degeneracy.

   Hint: The $1s$, $2s$ and $2p$ energies of a hydrogenic atom with nuclear charge Z are $E_{1,0} = -\frac{Z^2}{2}$, $E_{2,0} = E_{2,1} = -\frac{Z^2}{8}$ where the indexes on the energies are the principal quantum number n and the orbital angular momentum l. The wave-functions can be factorized as $\psi_{n,l,m}(\mathbf{r}) = \psi_{n,l}(r)Y_{l,m}(\Omega_{\mathbf{r}})$, where $Y_{l,m}(\Omega_{\mathbf{r}})$ are the spherical harmonics and $(r, \Omega_{\mathbf{r}})$ are the components of $\mathbf{r}$ in spherical coordinates. Here m indicates the projection of the orbital angular momentum along a quantization axis.

2. Calculate the expectation value of the electron-electron interaction between states with the same energy found at point [1.].

   Hint: $Y_{0,0} = 1/\sqrt{4\pi}$ and the following relationship could be useful:

   $$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{l,m}^*(\Omega_{\mathbf{r}_1}) Y_{l,m}(\Omega_{\mathbf{r}_2}) \tag{2}$$

   where $r_<$ and $r_>$ are the smallest and the largest between $|\mathbf{r}_1|$ and $|\mathbf{r}_2|$ respectively. You can call $U_{n,l}$ and $J_{n,l}$ the integrals $\int_0^\infty dr_2\ r_2^2 \psi_{n,l}^2(r_2) \int_0^\infty dr_1\ r_1^2 \psi_{1,0}^2(r_1) \frac{1}{r_>}$ and and $\frac{1}{2l+1} \int_0^\infty dr_2\ r_2^2 \psi_{1,0}(r_2)\psi_{n,l}(r_2) \int_0^\infty dr_1\ r_1^2 \psi_{1,0}(r_1)\psi_{n,l}(r_1) \frac{(r_<)^l}{(r_>)^{l+1}}$, respectively. The evaluation of these radial integrals is not required.

3. Using first order perturbation theory, find the energy shifts and the splitting of the levels found at point [1.] due to the electron-electron interaction.

4. Among the levels found at point [3.] identify those that could be split by the spin-orbit interaction.

5. Optional: Evaluate $U_{1,0}$.

   Hint: The radial component of the wave-function is: $\psi_{1,0}(r) = 2Z^{3/2}e^{-Zr}$.

Problem 3: Vibrational broadening of an electronic excitation

An electronic system can have two states, $|a\rangle$ and $|b\rangle$, (with energies $E_a < E_b$). The system is coupled with a vibrational degree of freedom, x, described by an harmonic oscillator with normal frequency ω_0, whose equilibrium position changes by an amount λ when the system is excited from the state $|a\rangle$ to state $|b\rangle$.

The total Hamiltonian can be written as:

$$H = \left[E_a + \left(\frac{p^2}{2m} + \frac{m\omega_0^2}{2}x^2\right)\right]|a\rangle\langle a| + \left[E_b + \left(\frac{p^2}{2m} + \frac{m\omega_0^2}{2}(x-\lambda)^2\right)\right]|b\rangle\langle b|$$

Suppose that the system is initially in the electronic ground state $|a\rangle$ and in the vibrational n-th excited state, ϕ_n. The system is then *suddenly* excited to the electronic excited state $|b\rangle$. The vibrational wavefunction does not change during the short time involved in the electronic excitation.

1) Write the energy of the system in the initial state.

2) Calculate the average energy of the system in the final state immediately after the electronic transition and show that the average transition energy is equal to the vertical one.

3) Show that the squared width of the emission peak (computed as the the second moment of the transition energy around its average value) is proportional to the mean square fluctuation of the particle around its equilibrium position in the electronic ground state. Using the virial theorem express the width of the emission peak in terms of the vibrational eigenvalue in the initial state.

4) (Optional) From the previous results obtain the temperature behavior of the emission peak width and indicate explicitly its high and low temperature limits.

Problem 4: Low-temperature specific heat and elementary excitations

At low temperature, the specific heat of EuO is proportional to $T^{\frac{3}{2}}$.

1) Can EuO be a conventional metal with a finite electron density of states at the chemical potential?

2) What is the low-energy form of a suitable density of states of excitations that can give rise to such a low temperature specific heat?

3) Suppose that the density of states found at point 2. is due to some form of boson quasi-particle. What form of the energy vs. momentum dispersion can give rise to such a density of states?

4) Can you guess the nature of the elementary excitations corresponding to these quasi-particles? What can be said about the nature of the ground state?