April 2011 - Entrance Examination: Condensed Matter

Solve at least one of the following problems. Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. Do not write your name on the problem sheet, but use extra envelope.

Problem 1: Electrons in a 2D Box

Consider electrons moving in two dimensions inside a square box with hard walls and side length a.

1. Solve Schrödinger's equation and determine the energy, the eigenfunctions, and the symmetry of the three lowest energy levels.
2. Fill the lowest levels with three electrons first, then with four electrons, always assumed to be non-interacting but obeying Pauli's principle. Give the resulting overall total energy, and degeneracy (each state with its total spin) in both cases.
3. Imagine now that the box could deform from square to a general rectangular form of sides (a_x, a_y), at an energy cost $(1/2)k[(a_x - a)^2 + (a_y - a)^2]$, where $k > 0$ is a box stiffness parameter. Treating a_x and a_y as classical variables, minimize total energy to determine their optimal value as a function of the stiffness k for three and for four electrons, specifying again in each case the resulting symmetry, degeneracy, and spin.
4. Introduce now electron-electron Coulomb repulsion as a small perturbation. Describe what will happen in the four electron case due to that repulsion, as the inverse box stiffness k^{-1} is gradually increased from zero.
5. Returning finally to the square box of side a, and ignoring again electron repulsion, increase now the electron number to a macroscopically large $2N$. Calculate the total energy as a function of N and a, and from the latter derive the electron chemical potential $\mu(N, a)$ and also the pressure $P(N, a)$ exerted by the electrons on the box walls.

Problem 2: Estimating the number of bound states

The following theorem concerns the energy eigenvalues E_n (with $E_1 < E_2 < E_3 < \ldots$) of the Schrödinger equation in one dimensional systems: if the potential $V_1(x)$ gives the eigenvalues $E_n^{(1)}$ and the potential $V_2(x)$ gives the eigenvalues $E_n^{(2)}$, and $V_1(x) \leq V_2(x)$ for all x, then $E_n^{(1)} \leq E_n^{(2)}$.

- Prove the theorem.
  Hint: Consider a potential $V(\lambda, x)$, where $V(0, x) = V_1(x)$ and $V(1, x) = V_2(x)$ and $\frac{dV}{d\lambda} \geq 0$, and calculate $\frac{dE_n}{d\lambda}$.

- Now consider the potential

$$V(x) = \begin{cases} \frac{k}{2}x^2 & |x| < a \\ \frac{ka^2}{2} & |x| \geq a \end{cases} \tag{1}$$

  We want to determine rigorous upper bounds for the number of bound states N of this potential.

  1. Estimate an upper bound by using an harmonic potential and the previous theorem.
  2. Do the same by using a square box potential.
     Hint: you have to estimate the maximum number of bound state that this latter potential can hold.
  3. What is the most useful bound?

Problem 3: Magnetic susceptibility of a spin system

Let us consider a system with spin angular momentum $\mathbf{S}$ and consider the subspace $\{|+\rangle, |-\rangle\}$ of the two eigenstates of $\mathbf{S}^2$ and S_z with eigenvalues $3/4\hbar^2$ and $\pm\hbar/2$, respectively. Let assume that the Hamiltonian of the system is:

$$H_0 = aS_z + \frac{b}{\hbar}S_z^2 \tag{2}$$

where a and b are two given parameters.

1. Write H_0 as a 2×2 matrix in the basis $\{|+\rangle, |-\rangle\}$ and find the eigenvalues.

2. Apply a static magnetic field $\mathbf{B}$ to the system. Assume that the field is in the direction $\mathbf{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ given by the polar angles θ and ϕ. Write the Hamiltonian of the system assuming that the magnetic moment of the system is $\mathbf{M} = \gamma\mathbf{S}$ where γ is the gyromagnetic ratio, a negative number.

3. Write the term of the Hamiltonian found at point 2 in the basis $\{|+\rangle, |-\rangle\}$.

4. Assuming that the term written at previous point is a small perturbation on H_0 find the perturbed wavefunction for the ground state of the system at first order in the magnetic field B.

5. Calculate, at first order in the magnetic field, the expectation value of the $\mathbf{M}$ operator on the perturbed wavefunction found at previous point.

Problem 4: Elasticity of a rubber string

Suppose that a rubber string can be described by a linear chain of N building blocks, each one of which can be in two different states, a and b. The energy ϵ and length l of a block in states a and b are: $\epsilon_a = \epsilon_0$, $\epsilon_b = \epsilon_0 + \Delta$, $l_a = l_0$ and $l_b = l_0 + \delta$. The total length and energy of the string are thus: $E_0 = n_a \epsilon_a + n_b \epsilon_b$ and $L = n_a l_a + n_b l_b$. Also suppose that the string is subject to an external force, which adds a term $-fL$ to the total energy: $E(f) = E_0 - fL$.

1. Calculate the partition function of the string, as a function of temperature.

2. Calculate the internal energy of the string, as a function of temperature, total number of blocks, and force.

3. Calculate the length of the string as a function of the same variables. Find and comment the value of the length when $\delta = 0$.