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October 2012 - Entrance Examination: Condensed Matter

Solve at least one of the following problems. Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. Do not write your name on the problem sheet, but use extra envelope.

Problem 1: Three spin 1/2 coupled sites

Consider three identical sites $i = 1, 2, 3$, each containing a spin 1/2. The three spins are coupled to each other via an exchange interaction

$$H = \sum_{\langle ij \rangle} J_{ij} \vec{s}_i \cdot \vec{s}_j \ ,$$

(counting each pair (ij) only once).

1. Enumerate and list each of the quantum states of this system, with their total spin and degeneracy.
2. Identify ground state wavefunctions, energy, and degeneracy if all $J_{ij} = -1$
3. Identify ground state wavefunctions, energy and degeneracy if all $J_{ij} = +1$
4. Discuss at least approximately the total ground state spin when $J_{12} = -1$, and $J_{23} = J_{13} = +1$; or alternatively, when $J_{12} = +1$, and $J_{23} = J_{13} = -1$.
5. Adding an external magnetic field B, $H \to H - B \sum_i s_i^z$, say which of these cases is going to retain the same ground state when B becomes very large, which will not, and why.

Hint: Definitions and identities that might be useful are $s^+ = s^x + is^y$, $s^- = s^x - is^y$, $s^z = s^+ s^- - 1/2$, $s^+ s^- + s^- s^+ = 1$.

Problem 2: Bound state in an attractive one-dimensional potential

Consider an attractive potential well in one dimension, with the following properties:

$$V(x) < 0 \qquad \int_{-\infty}^{+\infty} V(x)dx \ \text{ finite} \qquad \int_{-\infty}^{+\infty} x^2 V(x)dx \ \text{ finite} \tag{1}$$

1. By using a variational wave function of the form:

$$\Psi(x) = \left(\frac{\beta}{\pi}\right)^{1/4} e^{-\beta x^2/2} \tag{2}$$

   prove that the potential $V(x)$ has at least one bound state (i.e., a solution exists with $\beta > 0$ and $E_{\mathrm{var}} < 0$).

2. Assuming that the potential is weak, namely $\int_{-\infty}^{+\infty} V(x)dx$ and $\int_{-\infty}^{+\infty} x^2 V(x)dx$ are "small", find the best upper bound state for the energy, within this class of trial functions.

   Hint: Use an approximation for the integrals:
   $\int_{-\infty}^{+\infty} e^{-\beta x^2} V(x)dx = A - \beta B$
   $\int_{-\infty}^{+\infty} e^{-\beta x^2} x^2 V(x)dx = B$

3. Discuss the necessary requirements on the potential $V(x)$ in order to fulfill the conditions in Eq. (1). Can $V(x)$ assume very large values? What is the behavior for $|x| \to \infty$?

Problem 3: Rashba splitting in a magnetic field

In an electron gas confined in two dimensions, the electrons move in a square box of edge L (L is very large) with periodic boundary conditions. The electrons have spin that couples with an external magnetic field of magnitude B and direction perpendicular to the square. Moreover they interact through spin-orbit coupling with an external potential. Neglecting electron-electron interaction, their behavior can be described by the one-particle Hamiltonian:

$$H = \frac{p_x^2 + p_y^2}{2m} + \mu_B B \sigma_z + b\left(p_x \sigma_y - p_y \sigma_x\right) \tag{3}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0, & -i \\ i, & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix},$$

$\mathbf{p} = (p_x, p_y) = -i\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is the electron momentum, m is the electron effective mass, b a parameter that describes the strength of the spin-orbit coupling and μ_B is the Bohr magneton. In this exercise, we use units such that $\hbar = 1$, $m = 1$, $\mu_B = 1$, and we assume that $b = 1$ and B is positive.

1. Find the band structure of this system, searching the eigenfunctions of the Hamiltonian in the form of two dimensional plane waves multiplied by two-component spinors: $\psi_{\mathbf{k}}(x, y) = Ae^{ik_x x + ik_y y}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Plot the eigenvalues $\epsilon(\mathbf{k})$ along the line $\mathbf{k} = (k, 0)$ and discuss the shape of these energy bands as a function of B. Start by plotting $\epsilon(\mathbf{k})$ for the cases $B = 0$, $B = 1/2$, $B = 1$, $B \gg 1$ and then consider the general case.

2. Deduce from previous plots and from the symmetry properties of the system the form of the eigenvalues along the lines $\frac{1}{\sqrt{1+\alpha^2}}(k, \alpha k)$, where α is an arbitrary constant.

3. At the point $\mathbf{k} = (k, 0)$ plot the eigenvalues as a function of B. Discuss the different possibilities at different k.

4. Compute the average values of the σ_z operator in the eigenstates of H:$\langle\psi_{\mathbf{k}}|\sigma_z|\psi_{\mathbf{k}}\rangle$. Plot this average value along the line $\mathbf{k} = (k, 0)$, considering the same values of B used at the point 1. Comment your results.