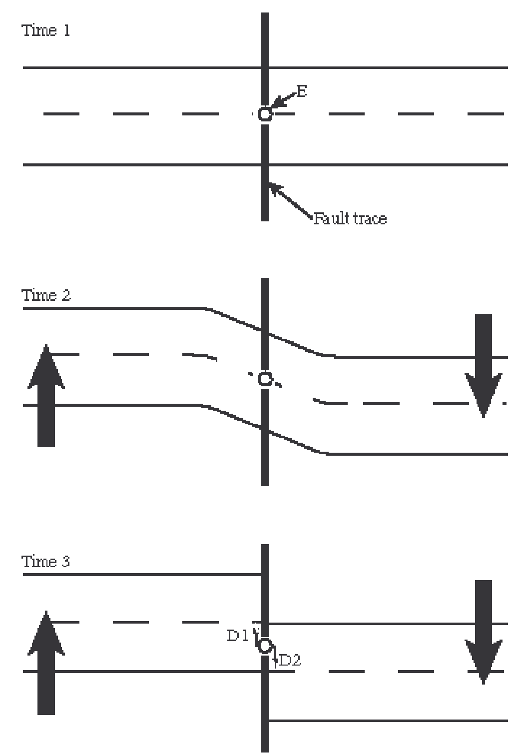
Frictional constitutive law and occurrence of instability

Hiroyuki Noda
Kyoto University

Introduction

Earthquake is explained by "Elastic rebound theory" developed by Reid in 1906 after the great San Francisco Earthquake.

Strain energy stored is released due to **rapid** slip on a fault.



Why and how does such rapid slip occur instead of slow creeping of a fault?

Coupling of friction and the surrounding.

Figure from Wikipedia

Outline

- Physical and experimental basis for frictional constitutive response
- Rate- and state-dependent frictional law
- Conditions for stable versus unstable frictional slip
- Modeling of sequences of earthquakes

Physical and experimental basis for frictional constitutive response

Coulomb’s law or Amonton’s law [1699] on friction

1) Frictional force is independent of apparent area of contact.

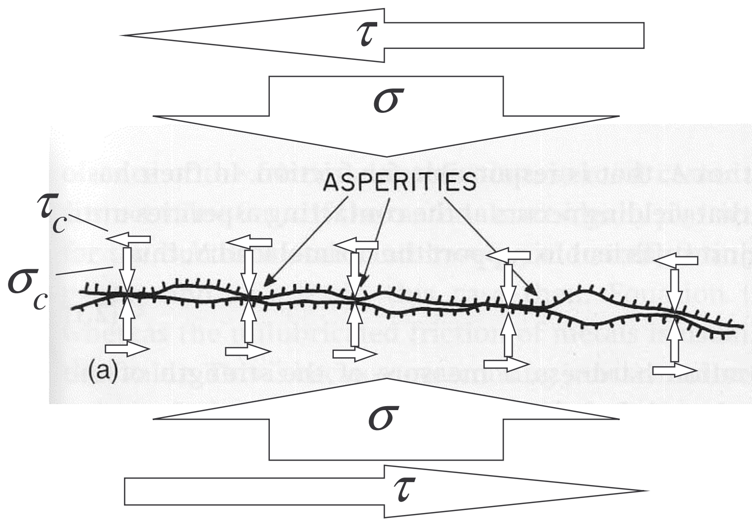
2) Frictional force is proportional to normal load.

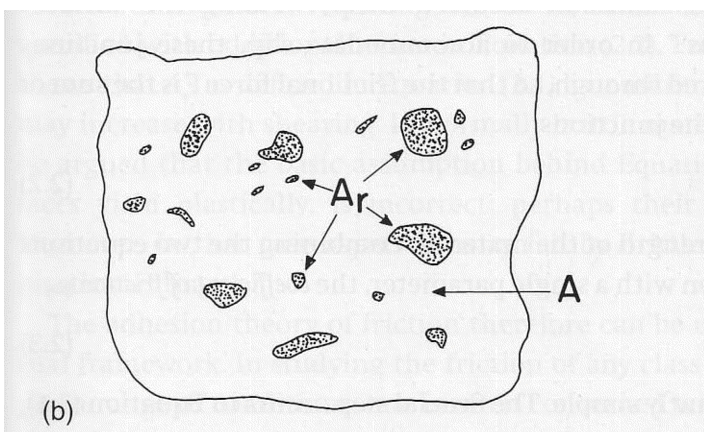
3) Frictional force is independent of slip rate.

1) and 2) were first discovered by ***Leonardo da Vinci*** by about 200 years prior to them.

Existence of static and kinetic friction was also recognized.

“Asperities” on the frictional surface





Figures from Scholz, 1990 with modification

Macroscopically,

$$f = \frac{\tau}{\sigma}\left(\text{or} = \frac{\tau}{\bar{\sigma}}\right) \qquad (\bar{\sigma} = \sigma - p)$$

f : Frictional coefficient
τ : Macroscopic shear stress
σ : Macroscopic normal stress (compressional)
$\bar{\sigma}$: Effective normal stress (compressional)
p : Pore pressure

Microscopically, true contact is achieved only at “asperities”.

A: Fault area
A_r : Area of true contacts
σ_c : Microscopic normal stress (compressional)
τ_c : Contact shear stress

Why and when is f nearly constant?

Adhesion theory [Bowden and Tabor, 1950, 1964]

Shear strength of asperities: τ_c $\quad \Rightarrow \quad \tau = \frac{A_r}{A}\tau_c$

Plastic asperities with hardness: σ_c $\quad \Rightarrow \quad \sigma = \frac{A_r}{A}\sigma_c$ $\quad \Rightarrow f = \frac{\tau}{\sigma} = \frac{\tau_c}{\sigma_c}$

Elastic contacts of spheres [Hertz, 1895]

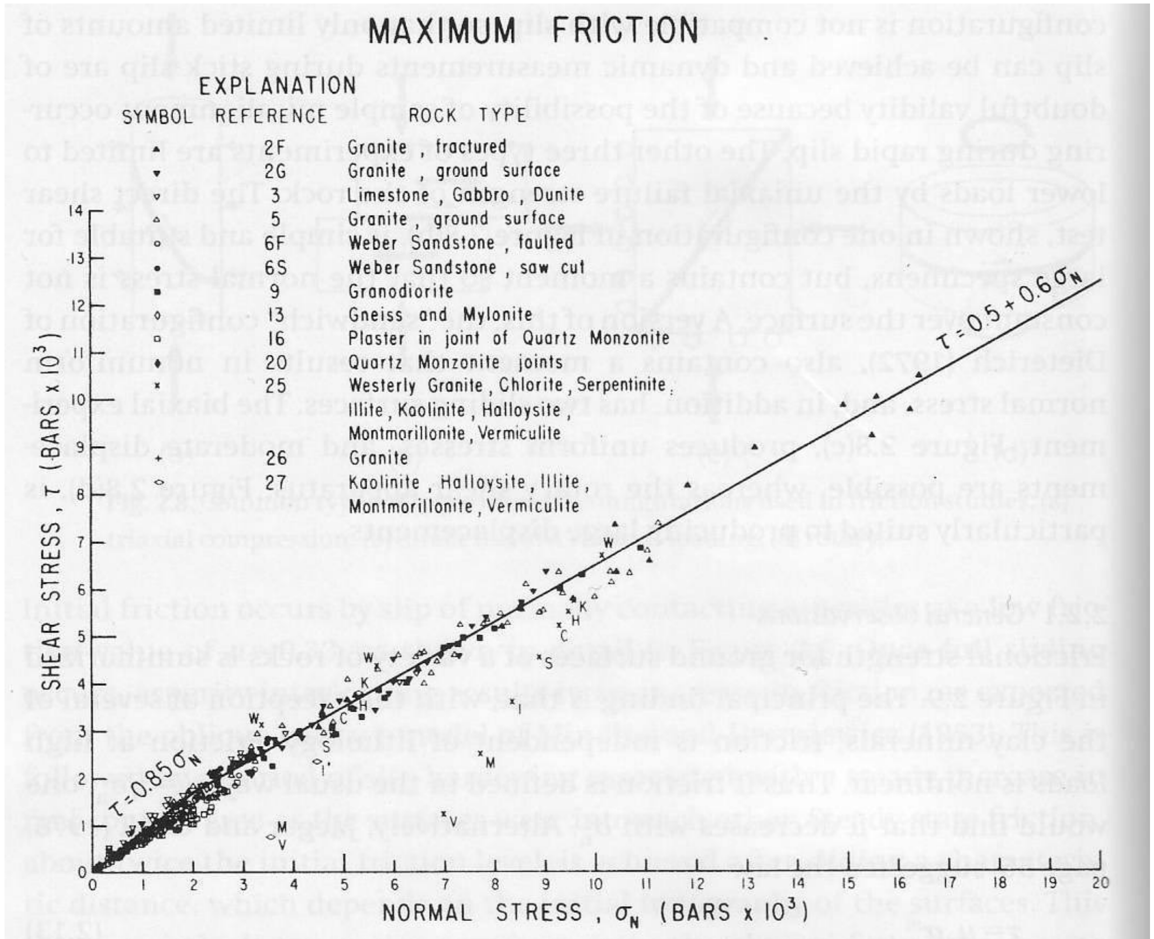
$A_r = K(A\sigma)^{2/3}$ where K combines the elastic constants and a geometrical factor

$\Rightarrow \quad f = \frac{\tau}{\sigma} = K\tau_c(A\sigma)^{-1/3}$ true for diamond pin on diamond surface [Tabor 1964]

with statistics of contacts

[Archard, 1957; Greenwood and Williamson, 1966; Brown and Sholtz, 1985]

Rock friction, Byerlee's law [1978]



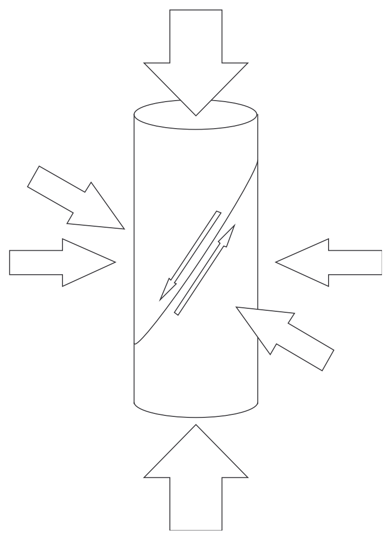
$$\tau = \begin{cases} 0.85\sigma & \sigma < 200\text{MPa} \\ 50\text{MPa} + 0.6\sigma & \text{at higher} \end{cases}$$

for a wide variety of rocks

except clay minerals

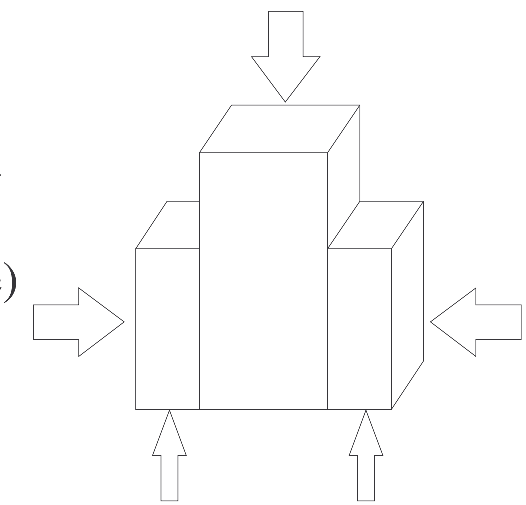
Laboratory experiments

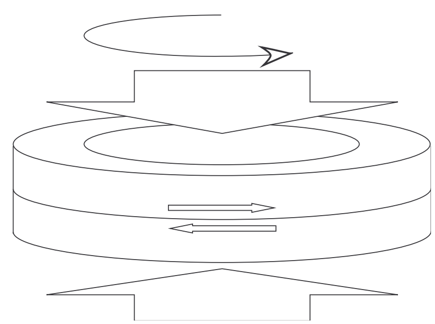
to study rock friction.



Triaxial test with saw-cut
- Noise from jacket
- Limited displacement
- Easy to control pore pressure
- High normal stress is possible

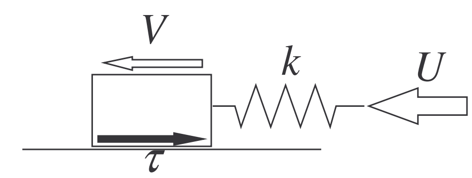
Biaxial double shear test
- Limited displacement
  (Larger than triaxial one)
- Limited Strength of rock
- No jacket



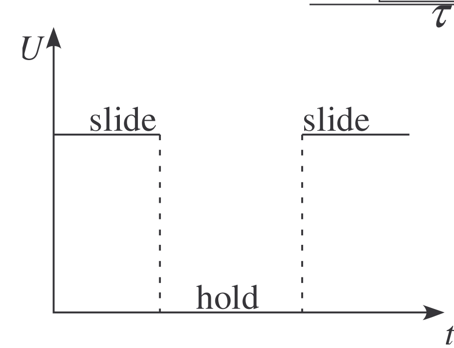


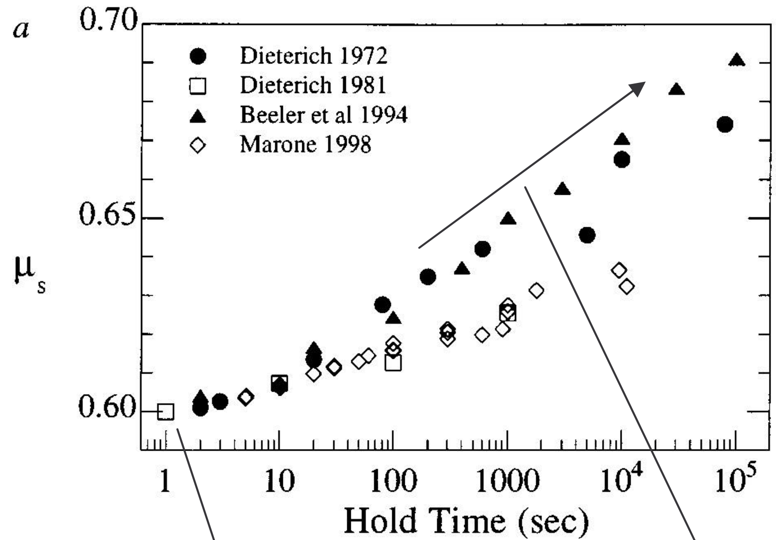
Rotary shear test
- Unlimitted displacement
  (Possibility of high velocity test)
- Technically, challenging
  (misalignment of axis)

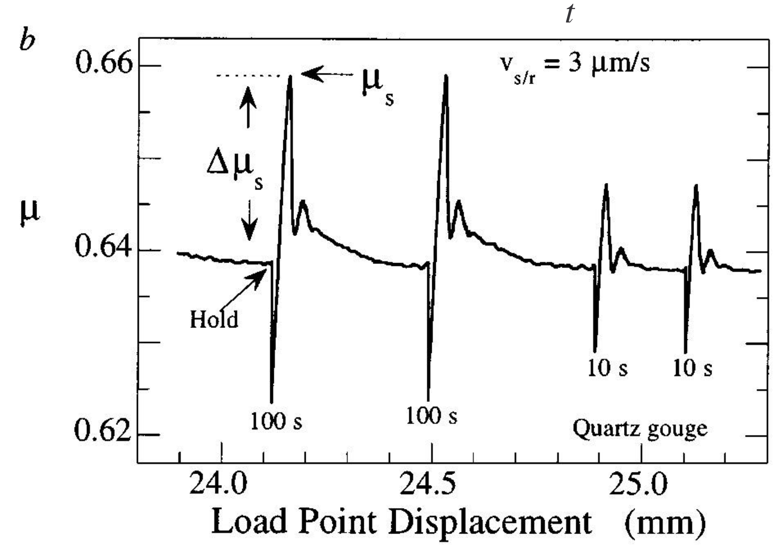
Aging of a fault



Slide-hold-slide test
[Dietrich, 1972, 1981]







Figures from Marone, 1998

"Cut-off time"

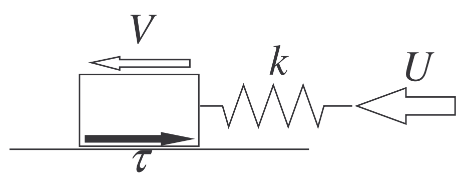
Logarithmic strengthening in time

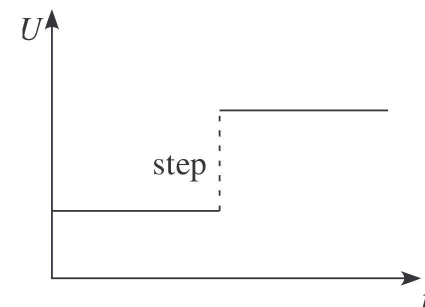
Slower than linear loading.

Asperities become stronger with time.

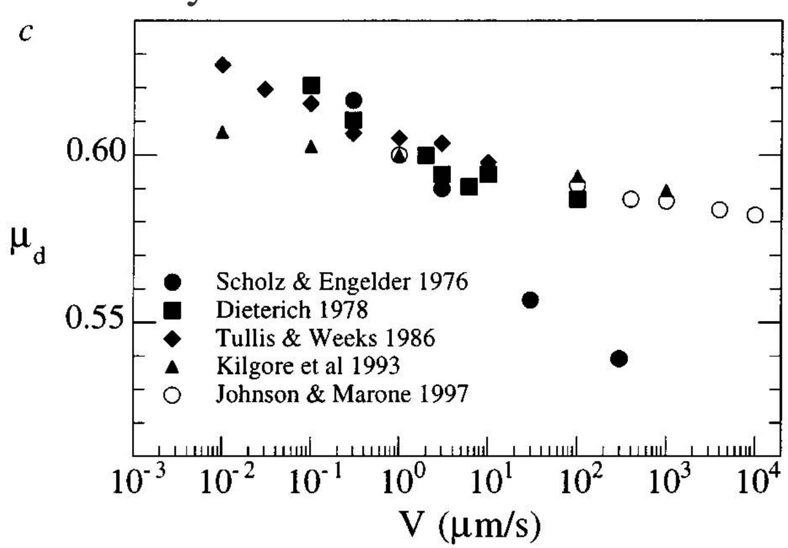
Steady state frictional coefficient

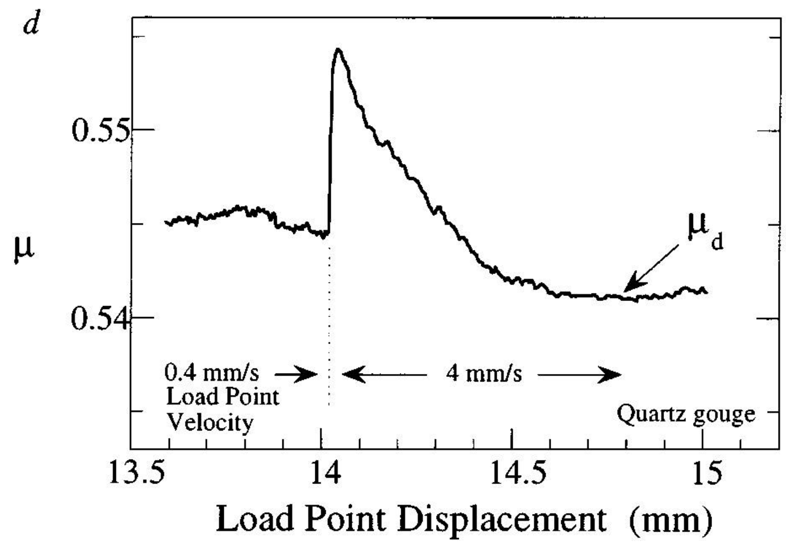
Velocity stepping test [Dietrich, 1978, 1979]





Steady state frictional coefficient





Logarithmic dependency on *V*

Transient behavior

Peak (direct effect) and Decay (evolution effect)

Figures from Marone, 1998

Rate- and state-dependent frictional law

[Dietrich,1979; Ruina, 1983]

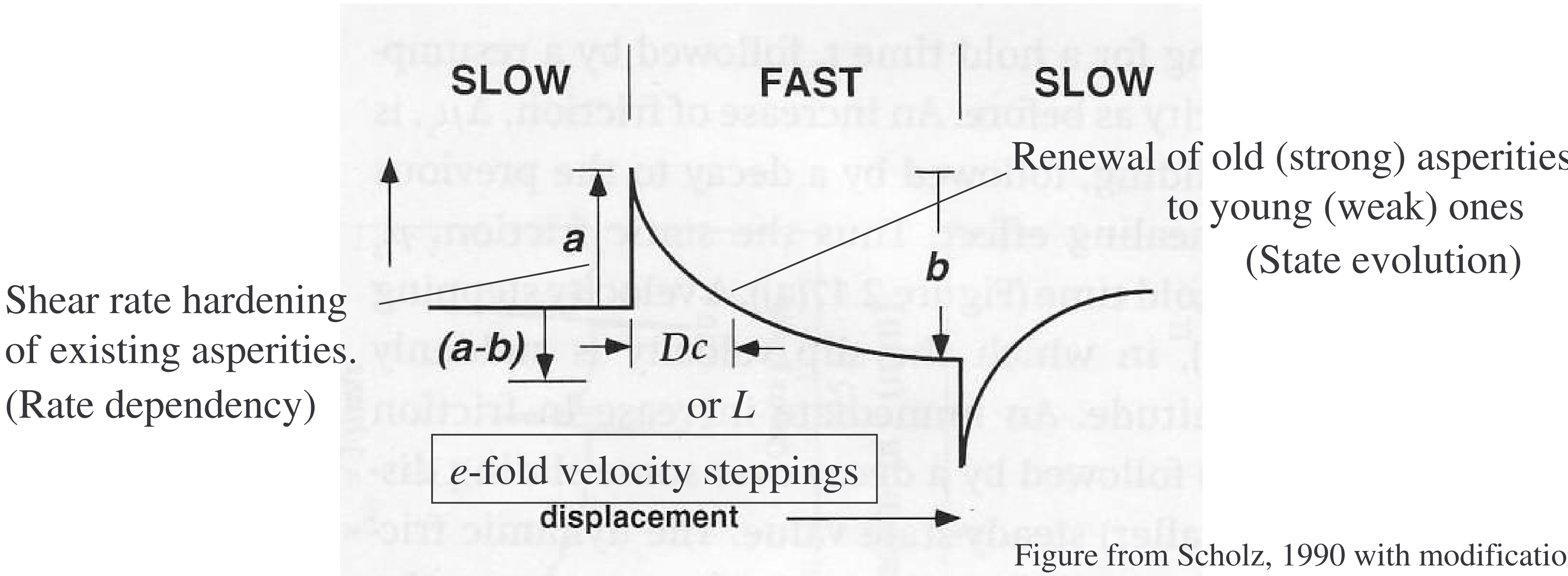


Figure from Scholz, 1990 with modification

a: Nondimensional parameter characterizing direct effect

b: Nondimensional parameter characterizing evolution effect

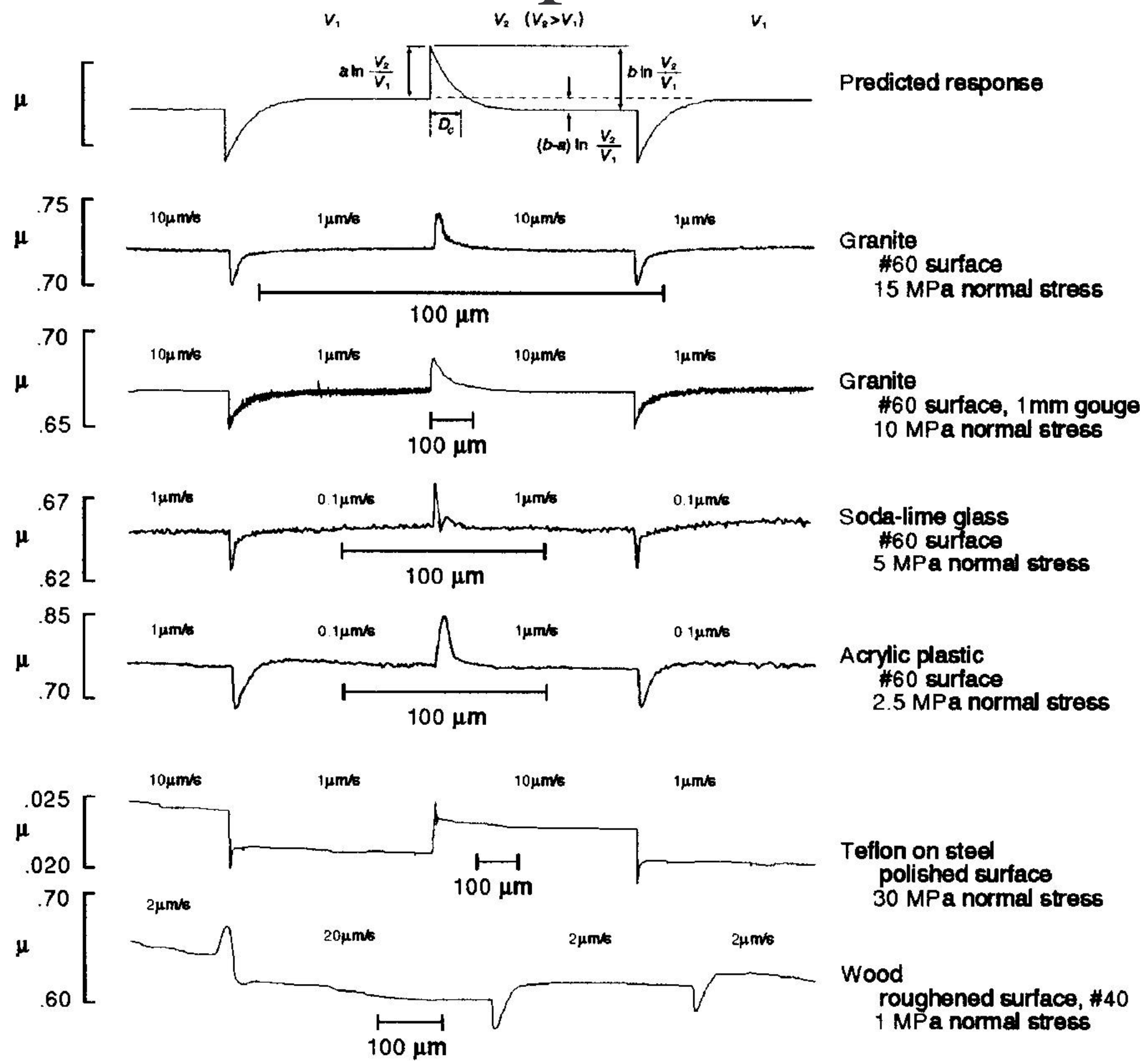
L: Length scale of state evolution

$$\frac{df_{ss}}{d\ln V} = a - b$$

$a-b<0$ Velocity weakening

$a-b>0$ Velocity hardening

Wide applicability of rate- and state-dependent frictional law



Typically, decays occurs for displacement of tens of microns to 1mm.

Figure from Marone, 1998, data from Dieterich and Kilgore 1994

Specific forms of constitutive laws

$$f = f_0 + a\ln\left(\frac{V}{V_0}\right) + b\ln\left(\frac{V_0\theta}{L}\right) \quad \text{with steady state} \quad \theta_{ss} = \frac{L}{V} \qquad f_{ss}(V) = f_0 + (a-b)\ln\left(\frac{V}{V_0}\right)$$

(f_0, V_0): Reference θ: State variable (Time)

State evolution equations

Aging law or Slowness law
(Dietrich-Ruina friction law)

$$\dot{\theta} = 1 - \frac{V\theta}{L} = \frac{V}{L}\left(\frac{L}{V} - \theta\right)$$

Linear aging as V → 0

Dietrich, 1979, 1981

Slip law
(Ruina-Dietrich friction law)

$$\dot{\theta} = -\frac{V\theta}{L}\ln\left(\frac{V\theta}{L}\right)$$

or

$$\left.\dot{f}\right|_V = \frac{V}{L}\left(f_{ss}(V) - f\right)$$

Ruina, 1983

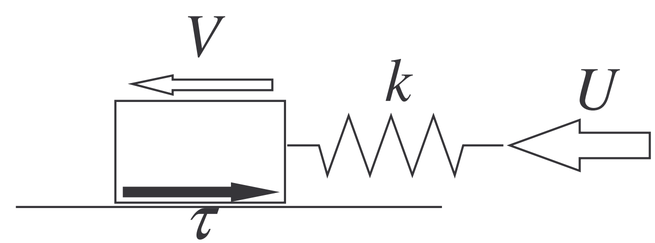
All evolution equations are linearized to: $\dot{\theta} = -\frac{V_0}{L}(\theta - \theta_{ss})$

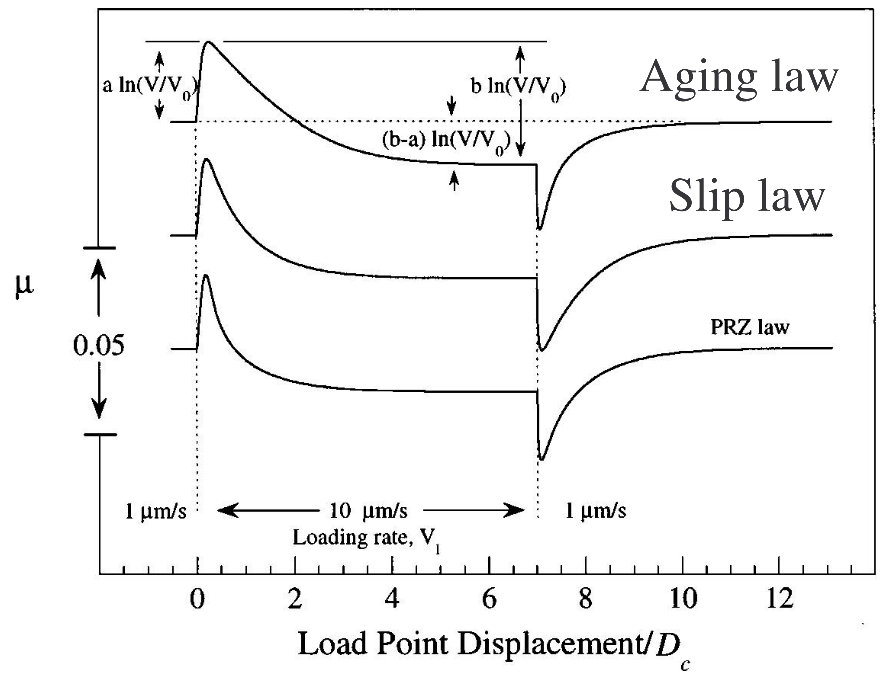
Comparison

Coupled with elastic system with loading (Spring-slider system)

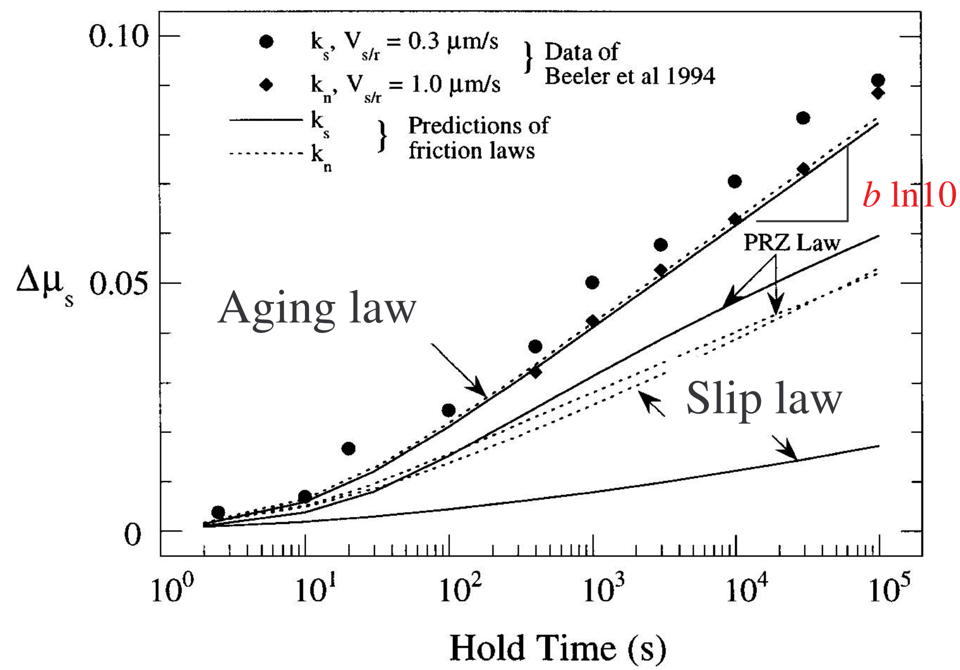
$$\dot{f} = \frac{\dot{\tau}}{\sigma} = \frac{k}{\sigma}(U - V)$$

k: Stiffness of the system, (stress)/(Length)
U: Loading velocity





Symmetry in positive and negative steppings.



Slide-hold-slide test.

Meaning of *b*: Time strengthening in aging law

Physical interpretation of direct effect

Rice et al. 2001

Arrhenius type thermal activation process at the asperity contacts

$$
\begin{aligned}
V &= V_1 \exp\left(-\frac{E_1 - \tau_c \Omega}{k_B T}\right) - V_1 \exp\left(-\frac{E_1 + \tau_c \Omega}{k_B T}\right) \\
&= 2V_1 \exp\left(-\frac{E_1}{k_B T}\right) \sinh\left(\frac{\tau_c \Omega}{k_B T}\right) \\
&\approx V_1 \exp\left(-\frac{E_1 - \tau_c \Omega}{k_B T}\right)
\end{aligned}
$$

Of course, fault can slide back and forth

If applied shear stress is so large that backwards jump is negligible.

E_1: Activation energy, Ω: Activation volume, k_B: Boltzmann constant, T: Absolute temperature

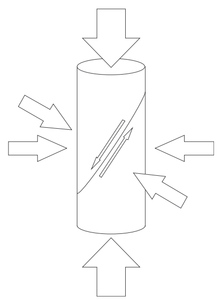
$$V_1 = \text{attempt frequency} \times \text{atom spacing} \times \frac{A_1}{A_r} = c_s \times \frac{A_1}{A_r}$$

A_1: Slipping area

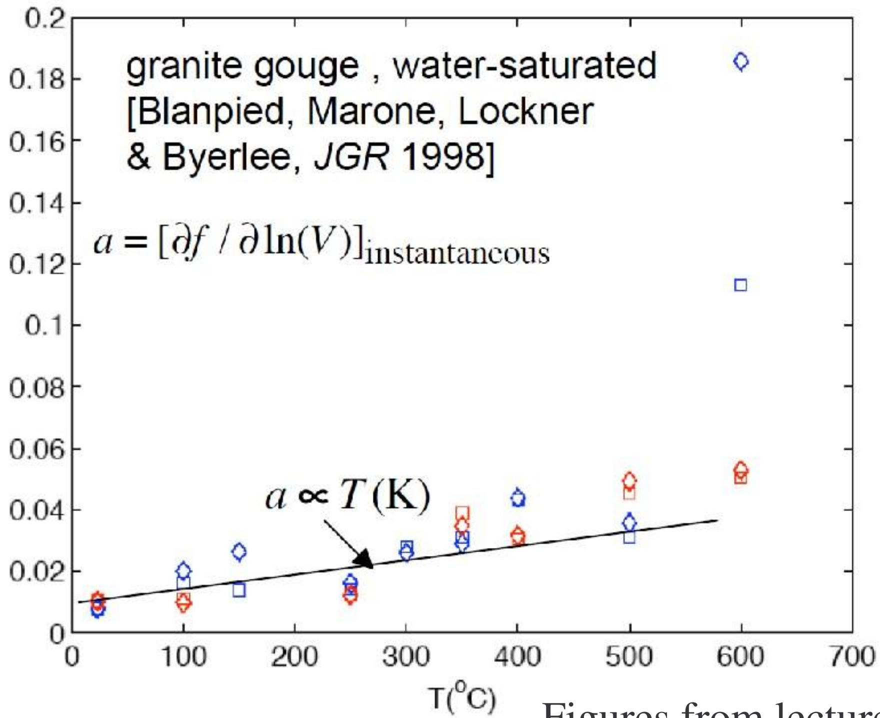
Using $\tau A = \tau_c A_r$, and $\sigma A = \sigma_c A_r$,

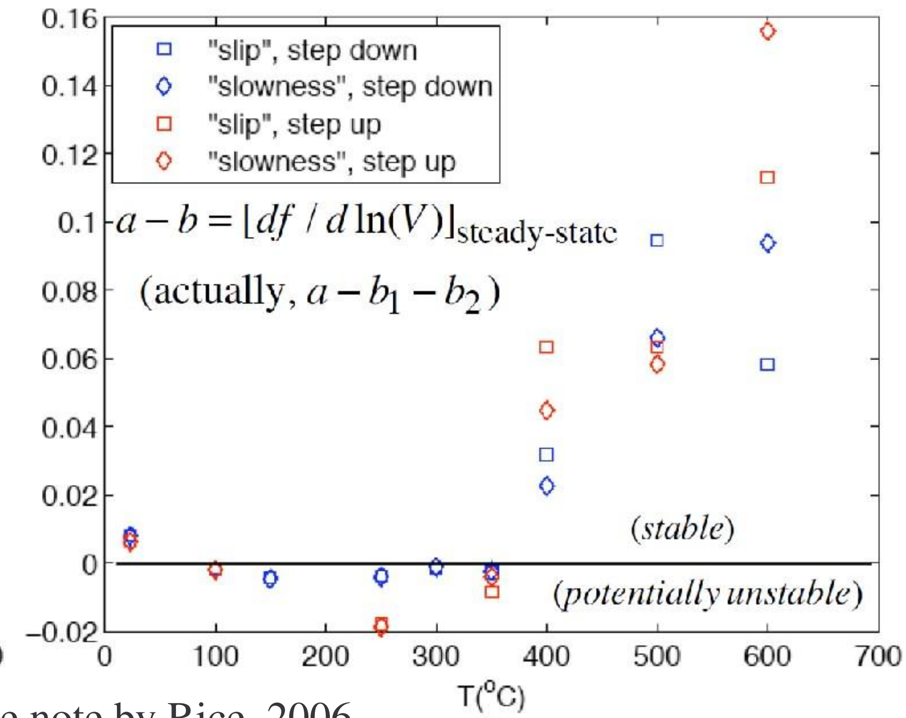
$$\tau = \left[\frac{k_B T}{\sigma_c \Omega} \ln\left(\frac{V}{V_1}\right) + \frac{E_1}{\sigma_c \Omega}\right]\sigma = \left[a \ln\left(\frac{V}{V_1}\right) + \varphi\right]\sigma$$

Where $a = \frac{k_B T}{\sigma_c \Omega}$, $\varphi =$ something else



Experimental data
at elevated temperature





Figures from lecture note by Rice, 2006

Let's move on to "How does instability occur?".

Conditions for stable versus unstable frictional slip

Away from rate- and state-dependent frictional law, what is unstable frictional slip?
(Stick-slip)

In quasistatic spring-slider system with strength τ as a function of slip on the fault.

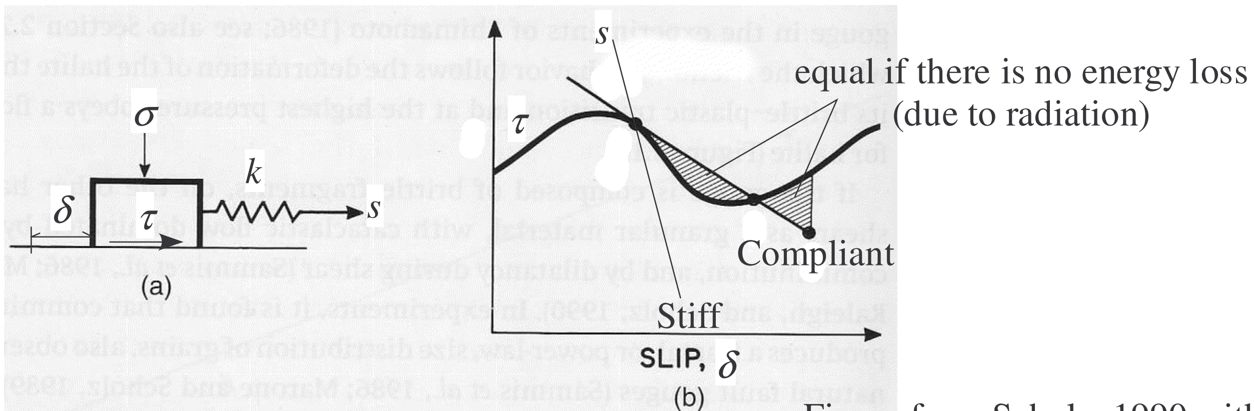
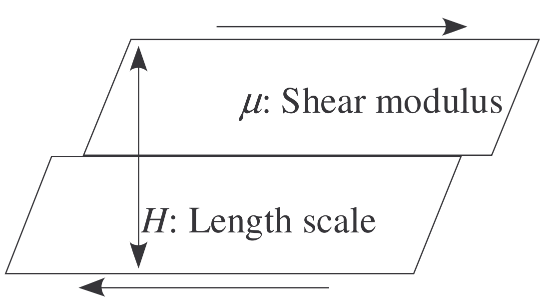


Figure from Scholz, 1990 with modification

If the system (e.g. apparatus) is too compliant, we can't control the slip quasistatically.



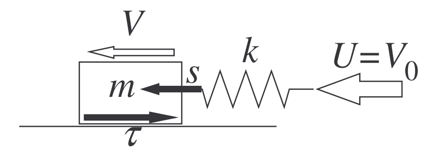
For a fault embedded in elastic body, what is "stiffness"?

$$k = \frac{\mu}{H}$$ Stiffness depends on length scale, and

H ~ (size of a fault, or length scale of a perturbation)

Linearlized analysis around steady state solution

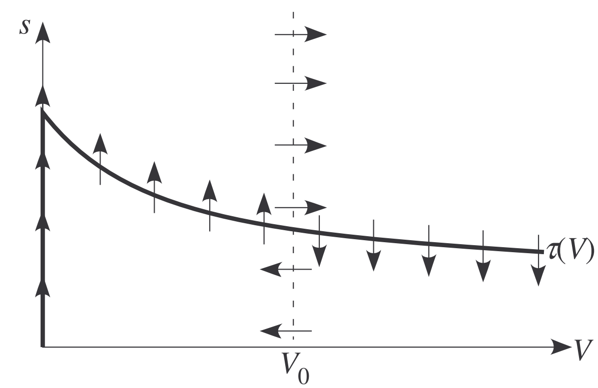
Example: Dynamic spring-slider with pure velocity dependent law
(Simple, but intuitively not very bad)



s: Spring force per area of the fault
m: Mass per area of the fault

$$\dot{s} = k(V_0 - V)$$

$$m\dot{V} = s - \tau(V)$$



Linearize around (V_0, $\tau(V_0) = \tau_0$)

$$\dot{\tilde{s}} = k\tilde{V}$$

$$m\dot{\tilde{V}} = \tilde{s} - \tau'(V_0)\tilde{V}$$

$$\Rightarrow m\ddot{\tilde{V}} + \tau'(V_0)\dot{\tilde{V}} + k\tilde{V} = 0$$

Then, (look for a solution of exp(pt) type)

$\tau'(V_0) < 0 \Leftrightarrow$ unstable

regardless of stiffness, which contradicts frictional experiments.

To discuss frictional instability, pure velocity dependent law is not enough.

Linearization of rate- and state- law with one state variable and constant σ

Most generally, $\tau = F(V,\theta)$ $\quad \dot{\theta} = G(V,\theta)$ $\quad G(V,\theta_{ss}(V)) = 0$

Steady state solution: V_0, $\theta_0 = \theta_{ss}(V_0)$

Time derivative of shear stress:

$$\dot{\tau} = F_V(V,\theta)\dot{V} + F_\theta(V,\theta)\dot{\theta} = F_V(V,\theta)\dot{V} + F_\theta(V,\theta)G(V,\theta)$$

The second term can be linearized to

$$F_\theta(V,\theta)G(V,\theta) \cong -\frac{V_0}{L}\left(\tau - \tau_0 - \tau'_{ss}(V_0)\tilde{V}\right)$$

$$F_V(V,\theta) = \frac{a\sigma}{V} \cong \frac{a\sigma}{V_0}$$

$$\tau'_{ss}(V_0) = \frac{(a-b)\sigma}{V} \cong \frac{(a-b)\sigma}{V_0}$$

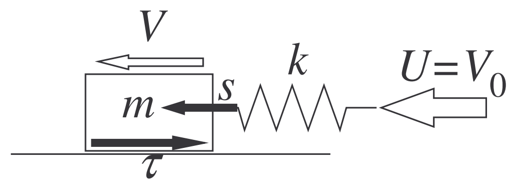
(From logarithmic slip-rate dependency)

Therefore, linearized constitutive law is: $\dot{\tilde{\tau}} = \frac{A}{V_0}\dot{\tilde{V}} - \frac{V_0}{L}\left[\tilde{\tau} - \frac{A-B}{V_0}\tilde{V}\right]$ where $A = a\sigma$, $B = b\sigma$

Regardless of the shape of state evolution equation.

Application of $\dot{\tilde{\tau}} = \frac{A}{V_0}\dot{\tilde{V}} - \frac{V_0}{L}\left[\tilde{\tau} - \frac{A-B}{V_0}\tilde{V}\right]$

Coupling with dynamic spring-slider model



$$\dot{\tilde{s}} = -k\tilde{V}$$

$$m\dot{\tilde{V}} = \tilde{s} - \tilde{\tau}$$

Eliminate $\tilde{s}$ and $\tilde{\tau}$, and look for solutions of $\tilde{V} = \exp(pt)$

$$\Rightarrow \quad \begin{array}{l} k > k_{cr} \Leftrightarrow \text{Perturbation decays.} \\ k < k_{cr} \Leftrightarrow \text{Perturbation grows.} \end{array}$$

where $k_{cr} = \frac{B-A}{L}\left(1 + \frac{mV_0^2}{AL}\right)$

A or *a* and *L* can't be zero

"Critical stiffness"

In quasistatic case, $k_{cr} = \frac{B-A}{L}$

Note $B - A = d\tau_{ss}/d\ln(V)$

At critical stiffness, there is an oscillation mode:

$$\omega = \frac{V_0}{L}\sqrt{\frac{B-A}{AL}}$$

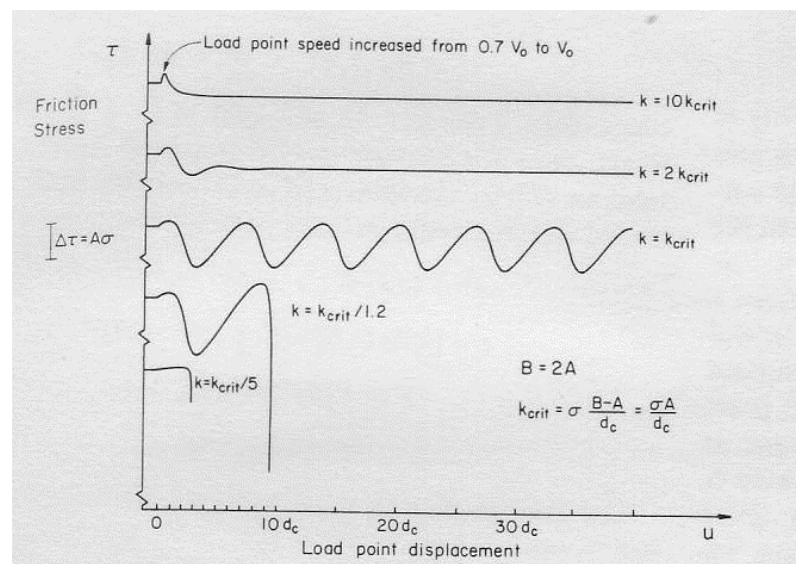
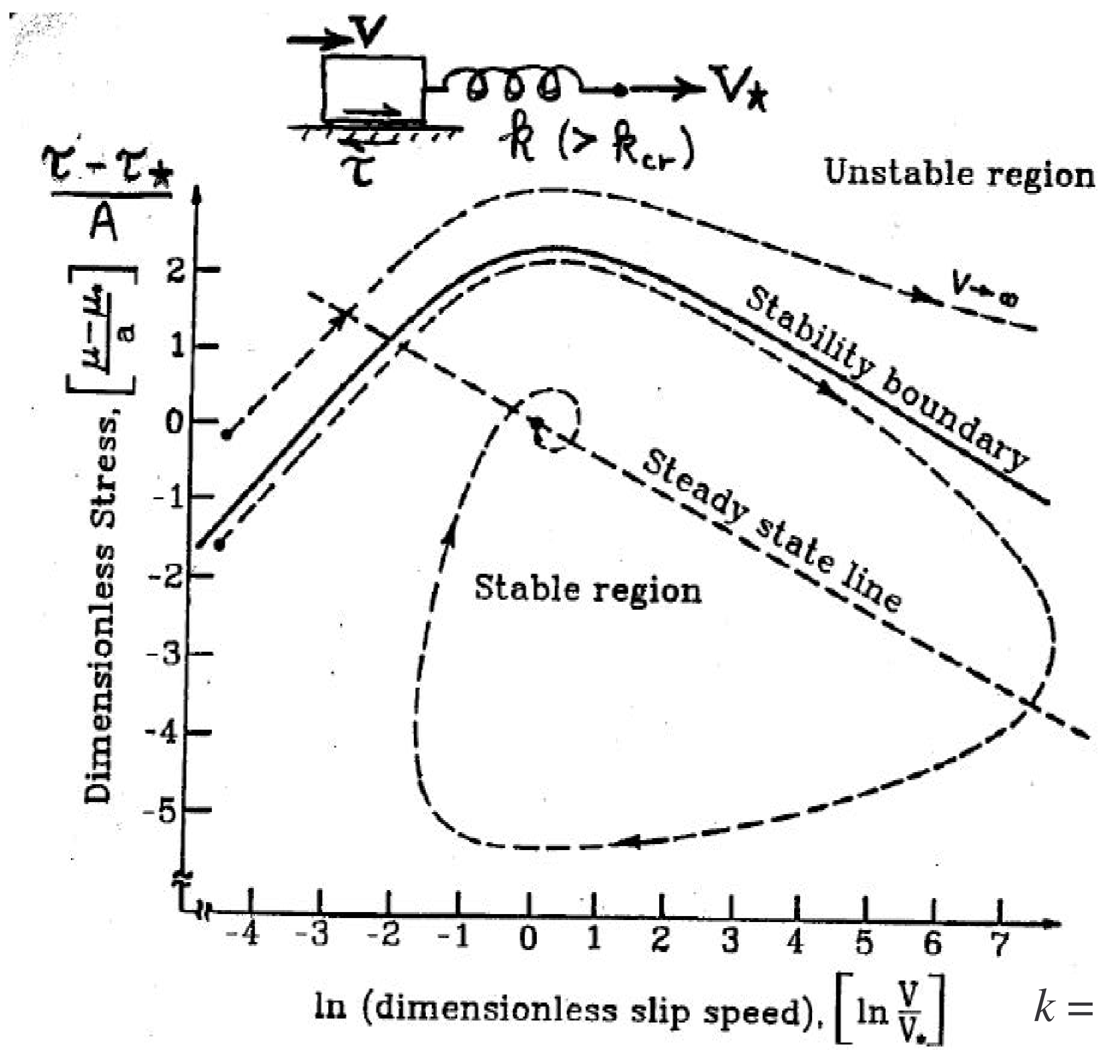


Figure from Ruina 1983

Nonlinear case

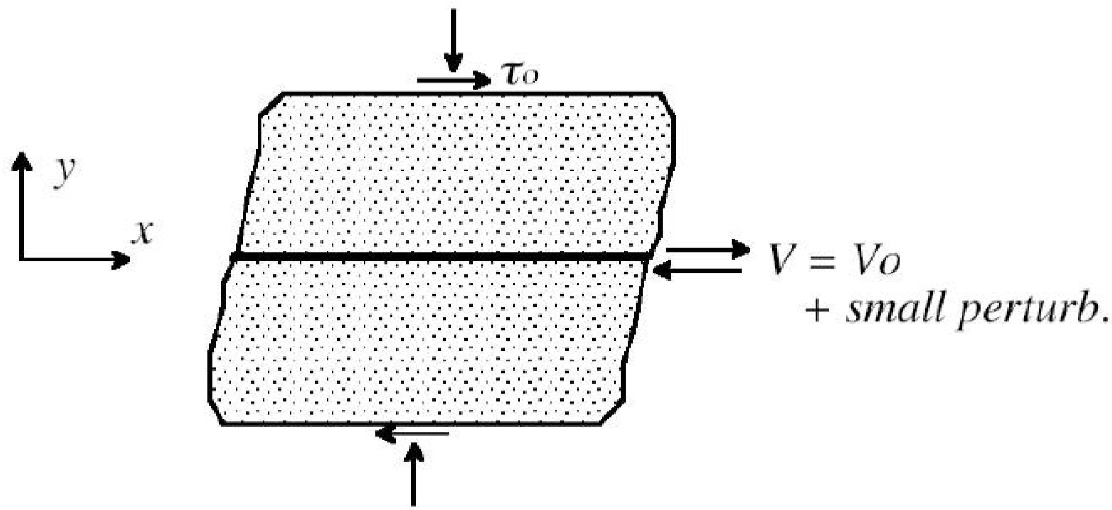


$k = 2k_{cr}$

Figure from Rice and Gu, 1983

Stability of steady state sliding *on a fault*

Rice et al. 2001



Steady state solution: $V = V_0,\ \tau = \tau_0$

with linearlized perturbation on the fault, $\tilde{\tau}, \tilde{V}$

and linearlized frictional constitutive law,

$$\dot{\tilde{\tau}} = \frac{A}{V_0}\dot{\tilde{V}} - \frac{V_0}{L}\left[\tilde{\tau} - \frac{A-B}{V_0}\tilde{V}\right]$$

Long perturbation grows.

I would like to skip the detailed derivation…

Wave equation + linearized Rate- and state- law

Looking for a solution of (steady state sliding) + (Perturbation of $\mathrm{Re}\left[U(y)e^{ikx+pt}\right]$ to displ.)

$$\Rightarrow \underline{k}_{cr} \equiv \frac{2\pi}{\lambda_{cr}} = \frac{2(B-A)}{\mu L}\sqrt{1+q^2} \qquad \text{where} \qquad q = \mu V_0 \Big/ \left[2\sqrt{A(B-A)}\,c_s\right]$$

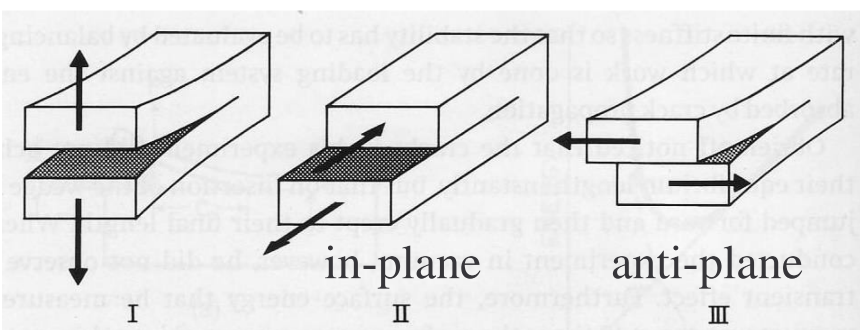
(anti-plane strain)

When $A \to 0$ (no direct effect) or $L \to 0$ (pure velocity weakening),
***ALL perturbation is unstable*.**

In quasistatic limit, $q \to 0$ and

$$\underline{k}_{cr} \equiv \frac{2\pi}{\lambda_{cr}} = \frac{2(B-A)}{\mu L} \qquad \text{(anti-plane strain)}$$

$$\underline{k}_{cr} \equiv \frac{2\pi}{\lambda_{cr}} = \frac{2(1-\nu)(B-A)}{\mu L} \qquad \text{(in-plane strain)}$$



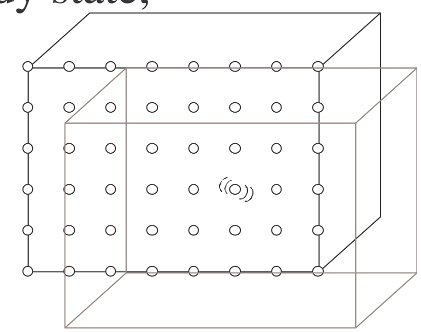
μ: shear modulus, ν: Poisson's ratio

A Fourier mode perturbation with lower $\underline{k}$ (and longer λ) grows.

Critical cell size, h_* and nucleation of unstable slip

[Rice, 1993, Lapusta et al., 2000]

Suppose a fault and its surrounding are discretized, let it slide in a steady state, and add perturbation to one cell on the fault.



Elastic response is dependent on discretization, and there is a critical cell size decided by critical stiffness:

$$h_* = \gamma\mu / k_{cr}$$

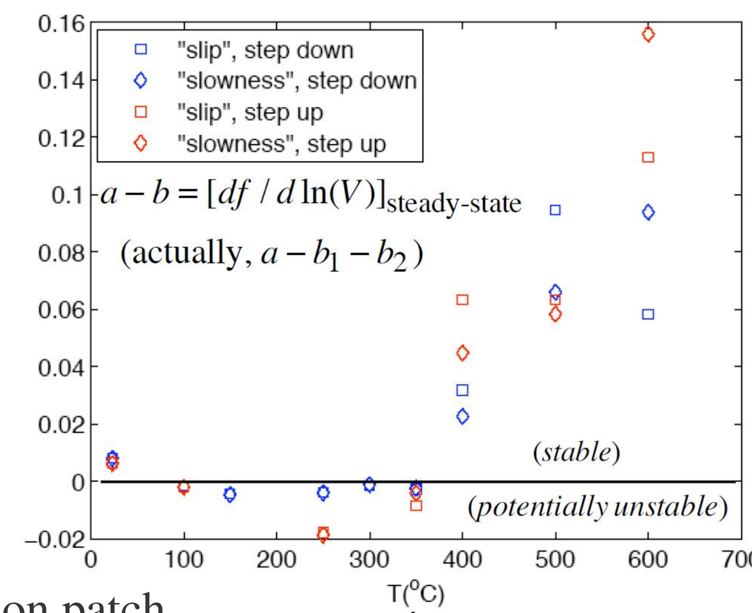
where γ depends on the way of descretization ($\gamma = 2/\pi$ for cellular basis).

Numerically, grid spacing, $h << h_*$.

Physically, nucleation size of unstable slip patch ~ h_*.

$$4h_* = \frac{8\mu L}{\pi(1-\nu)(B-A)} \approx 1\text{km} \times \frac{L}{40\mu\text{m}} \times \frac{1\text{MPa}}{\bar{\sigma}}$$

with $a - b = -0.004$ at low temperature



L ~ tens of microns, $\bar{\sigma}$ = 100 MPa => ~ 10 m of nucleation patch

Earthquakes as frictional instabilities

The deeper, the hotter.

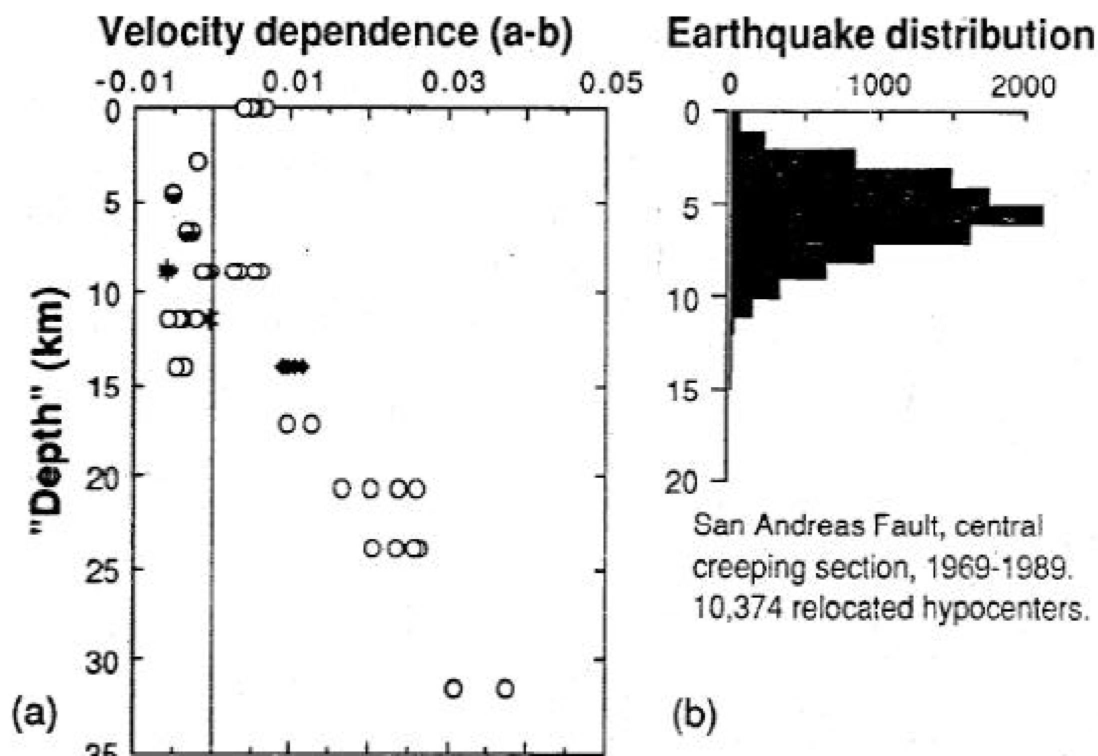
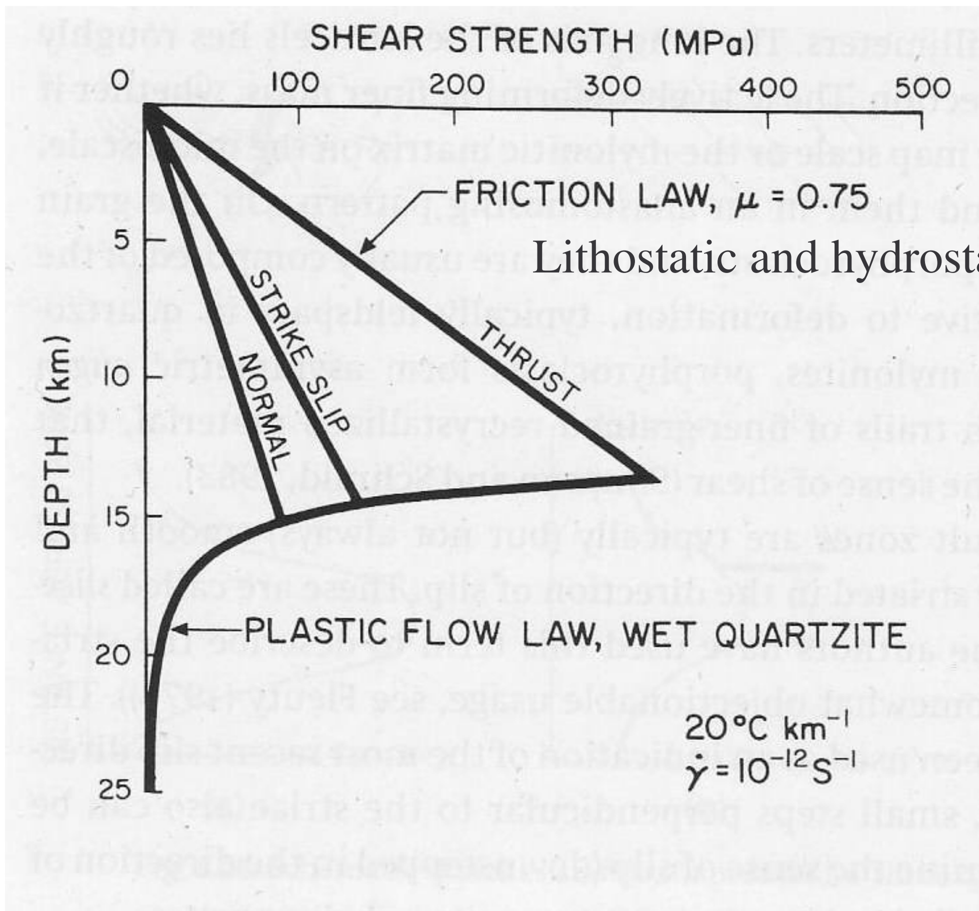


Figure from Blanpied et al. 1991

Earthquake occurs in "negative $a - b$" region.

Brittle-ductile transition

Slip rate range used in experiment is, actually, very high compared to natural slip rate.
Ductile (not frictional i.e. independent of normal stress) deformation dominates at depth.

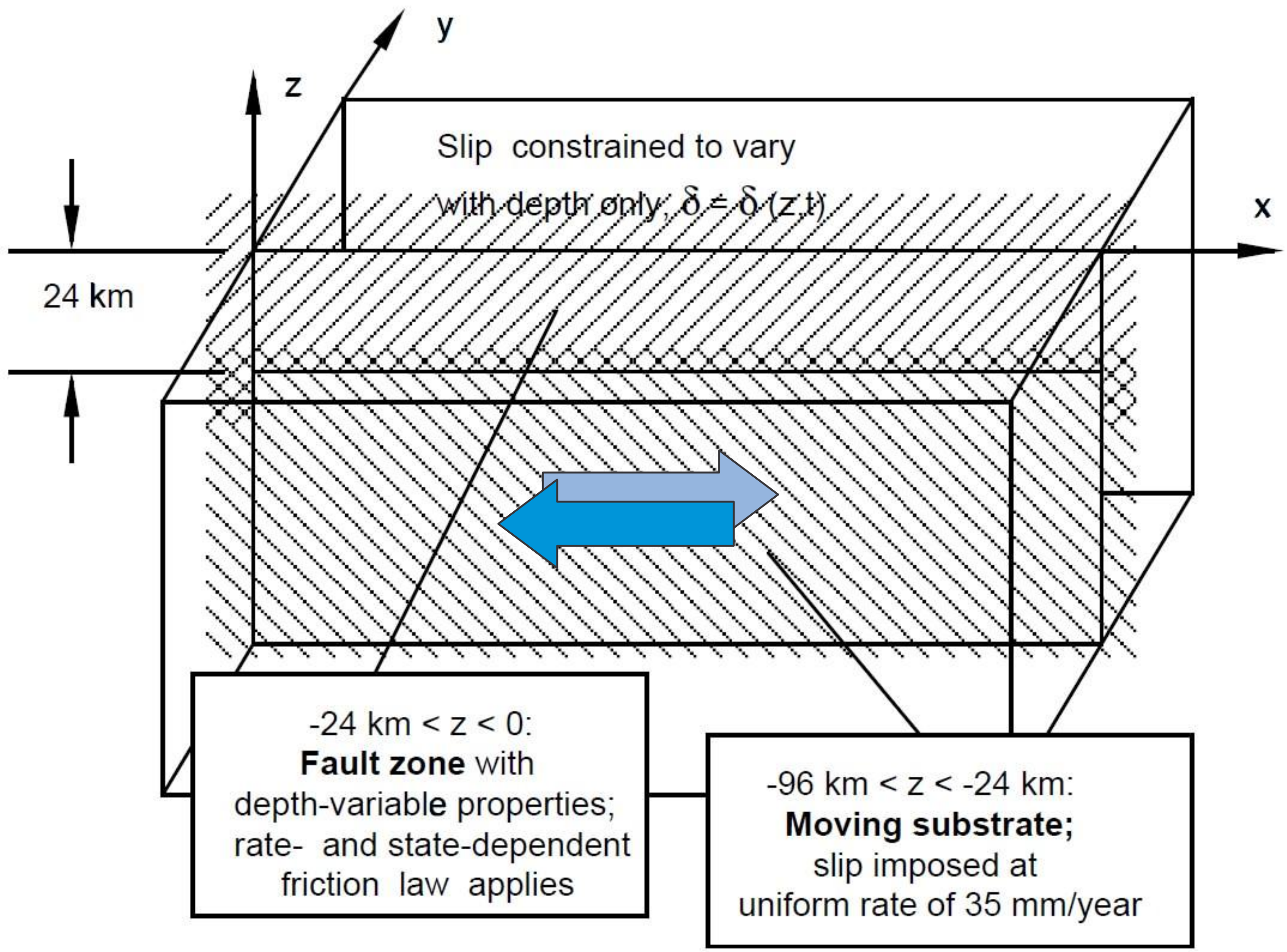


Lithostatic and hydrostatic pressure.

Figure from Scholtz, 1990

Modeling of sequences of earthquakes

Lapusta et al. 2000



~10^{-9} m/s

Fault constitutive parameters

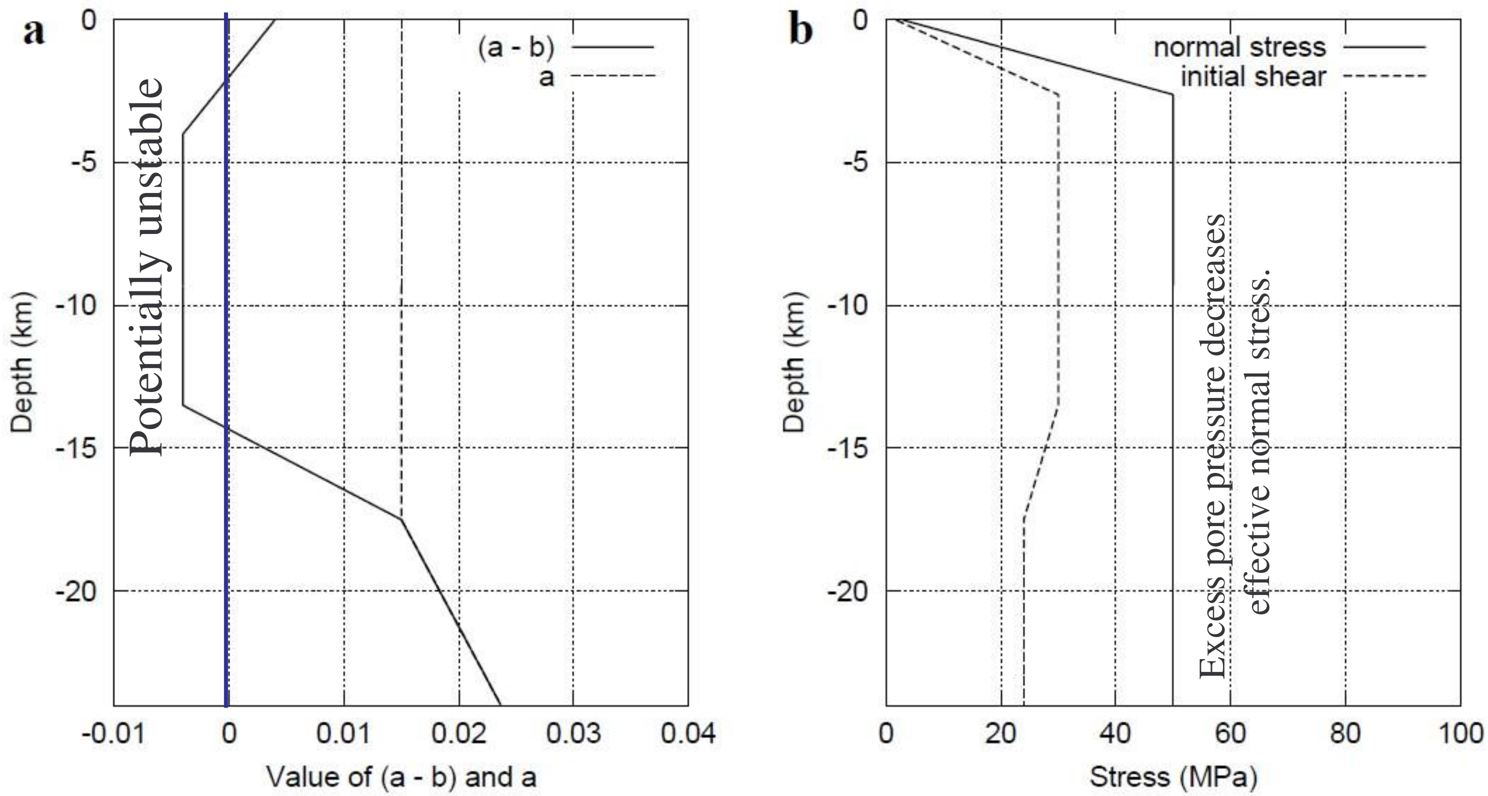
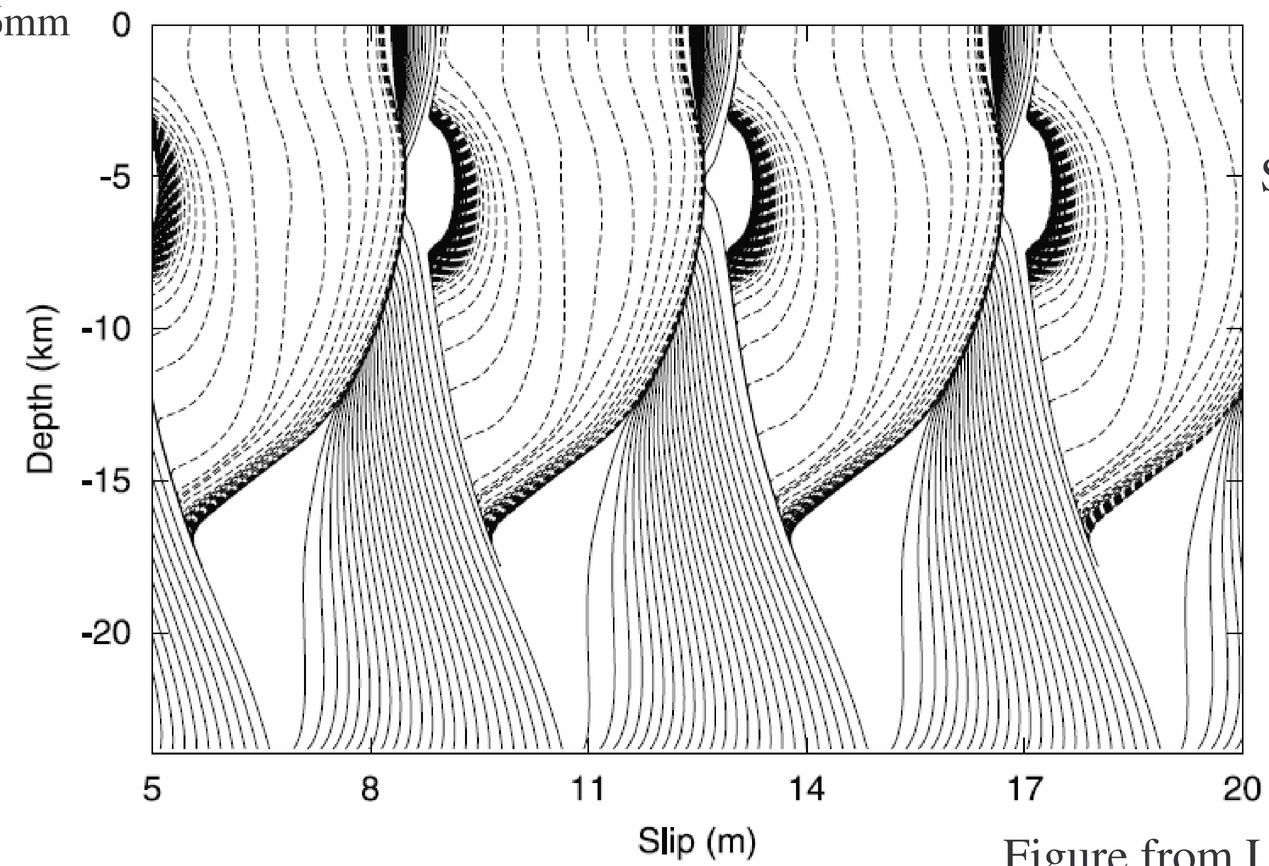


Figure Lapusta et al. 2000

They tested different *L* from 0.14mm to 16mm

Sequences of earthquakes

L: 16mm



Stick-slips
at seismogenic zone.

Stable sliding at depth.

Figure from Lapusta and Rice, 2003

Solid line every 5 years, and dashed line every 1 second.

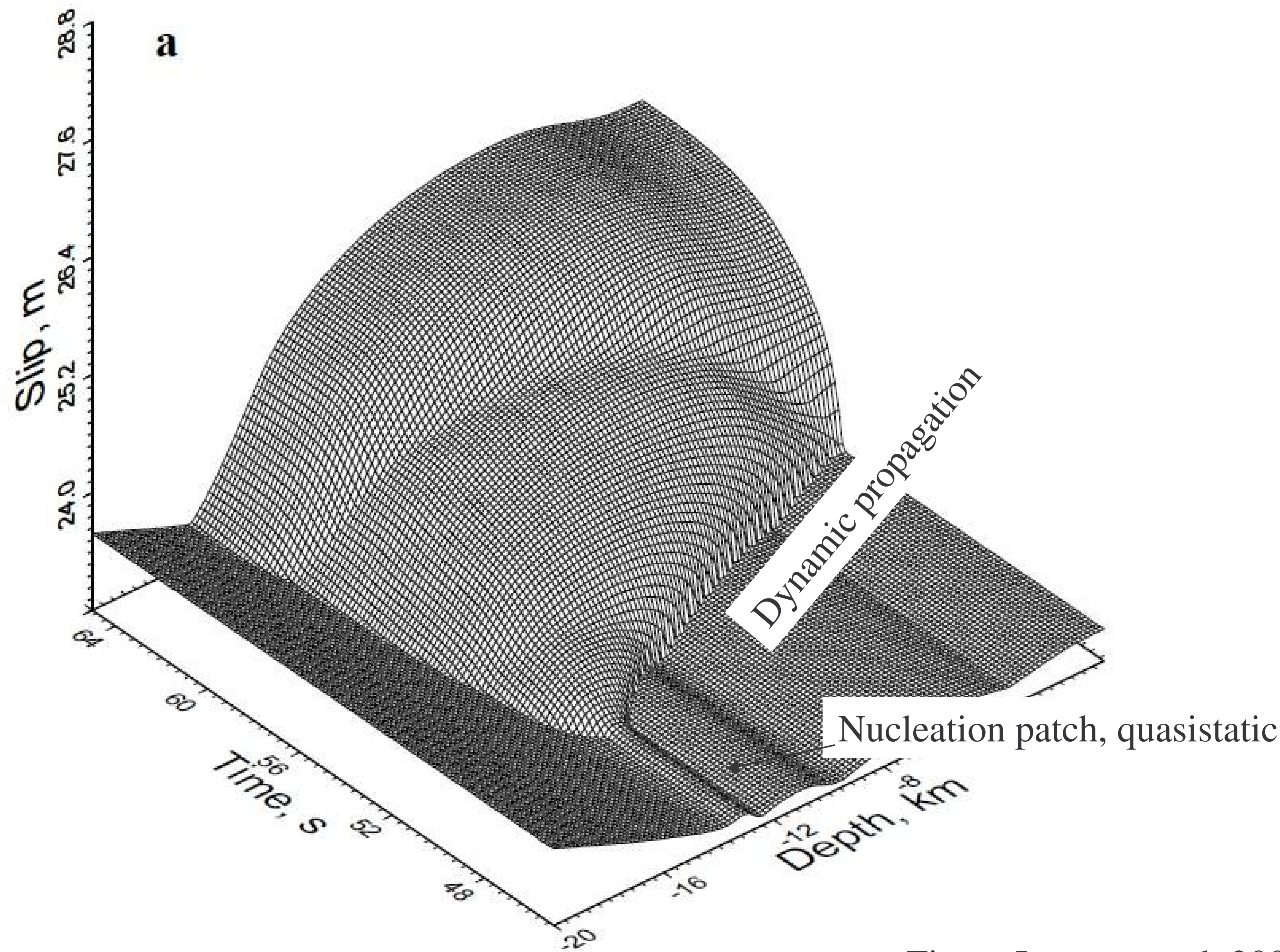
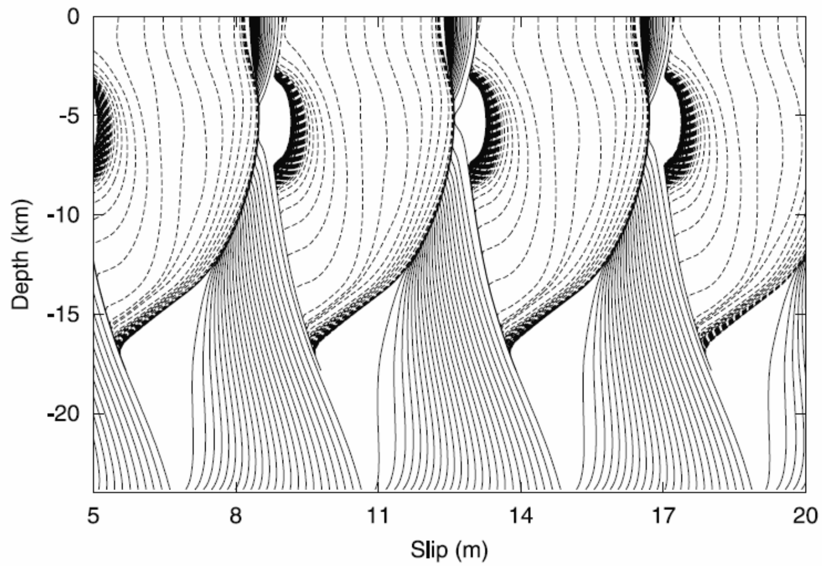
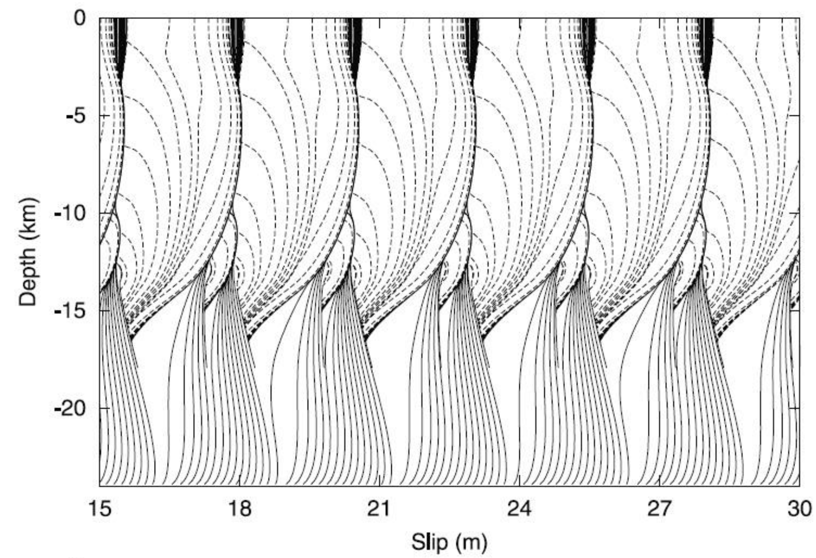


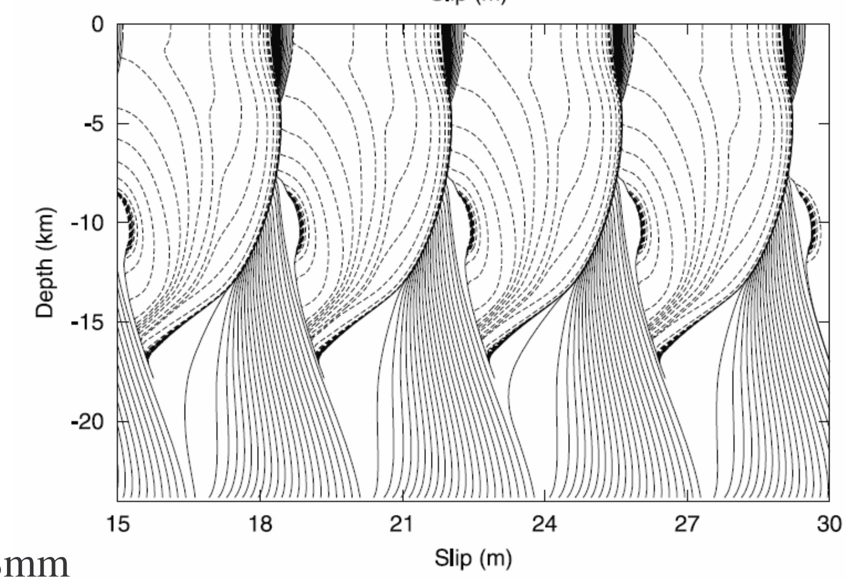
Figure Lapusta et al. 2000

L: 16mm

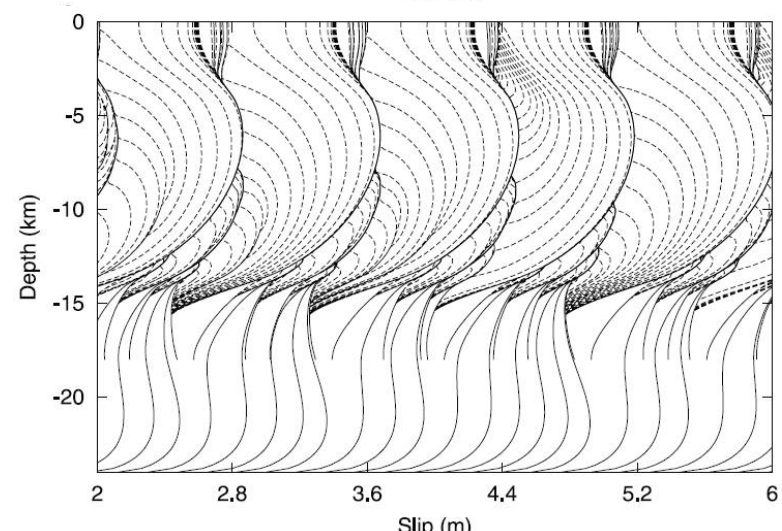


2mm



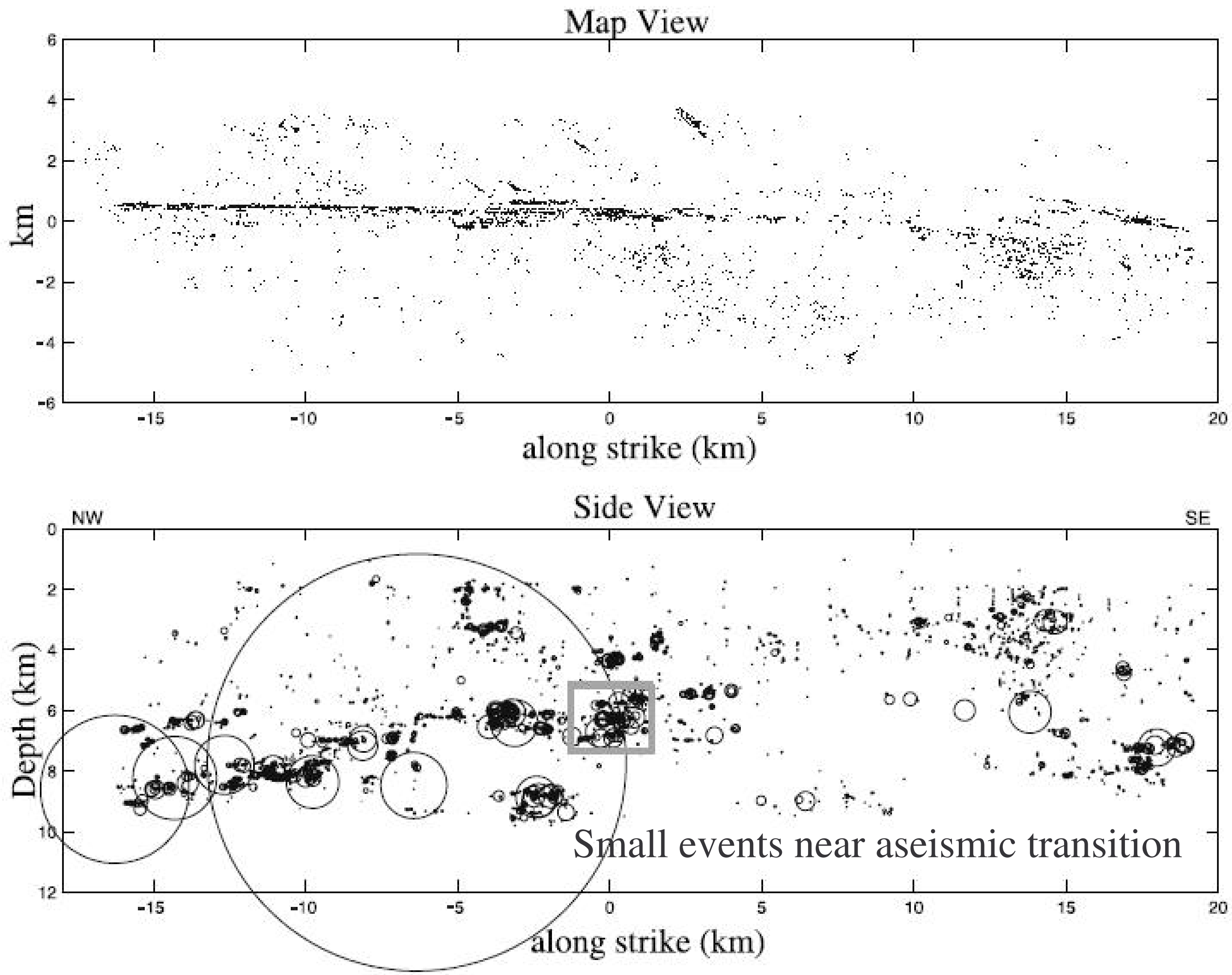


8mm



0.14mm (computational limit, not fully dynamic)

By changing *L*, not simply the size and interval of events,
but the pattern of events occurrence changes.
Small events near aseismic transition.



Calaveras fault, Schaff et al. [2002]

Summary

- Absolute value of frictional coefficient of rock is around 0.6-0.85, and independent of rock type (except clay minerals).
- Frictional constitutive behavior is well modeled by rate- and state-dependent law.
- Rate- and state-dependent feature (direct effect and evolution effect) is essentially important to discuss stability of frictional sliding.
- There is critical stiffness and thus critical nucleation size of patch of unstable sliding.
- Sequence of earthquakes can be modeled by rate- and state- law and temperature (and thus depth) dependent constitutive parameter.

Thank you.