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CRACKS, SHEAR BANDS AND LAMELLAR INCLUSIONS IN HOMOGENEOUSLY PRESTRESSED MATERIALS

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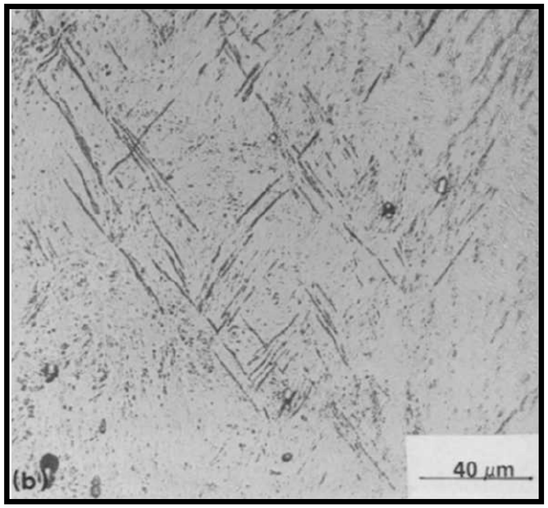
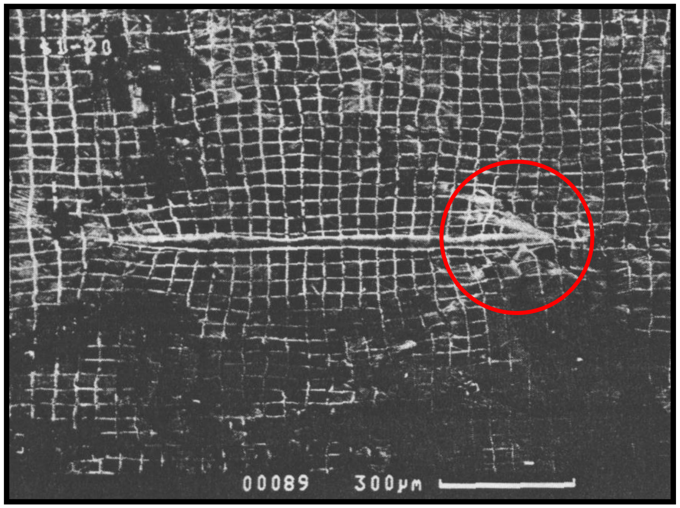
LOCALIZATION OF DEFORMATION

Localized deformation patterns are experimentally observed to prelude failure in many ductile and quasi-brittle materials.



Maraging steel
(Anand & Spitzig, 1980)

Presence of a second phase may promote failure due to stress concentrations at the inclusion boundaries.



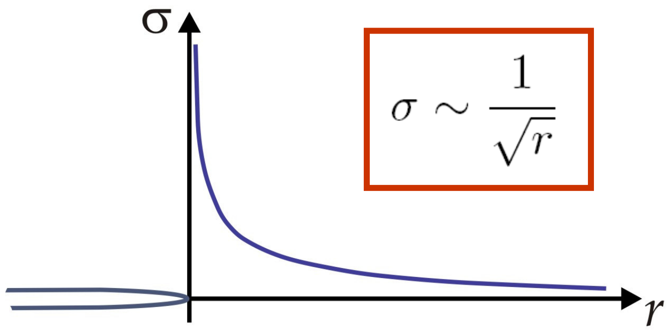
Cu-W laminates
(Öztürk et al., 1991)

In these cases numerical approaches can hardly have the appropriate resolution to detail the highly inhomogeneous mechanical fields and to disclose the failure mechanisms.

MAIN RESULTS FROM INFINITESIMAL ELASTICITY

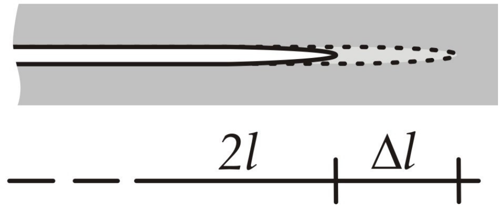
CRACK

- Square root singularity in the stress fields at the crack tips



$$\sigma \sim \frac{1}{\sqrt{r}}$$

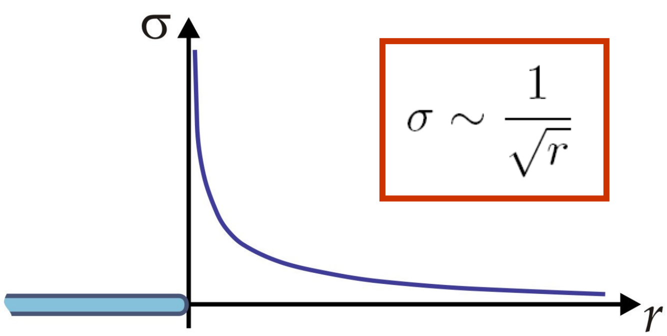
- Favorited crack growth (predicted decrease of total potential energy of the system for crack advance)



$$G > 0$$

STIFFENER

- Square root singularity in the stress fields at the stiffener tips

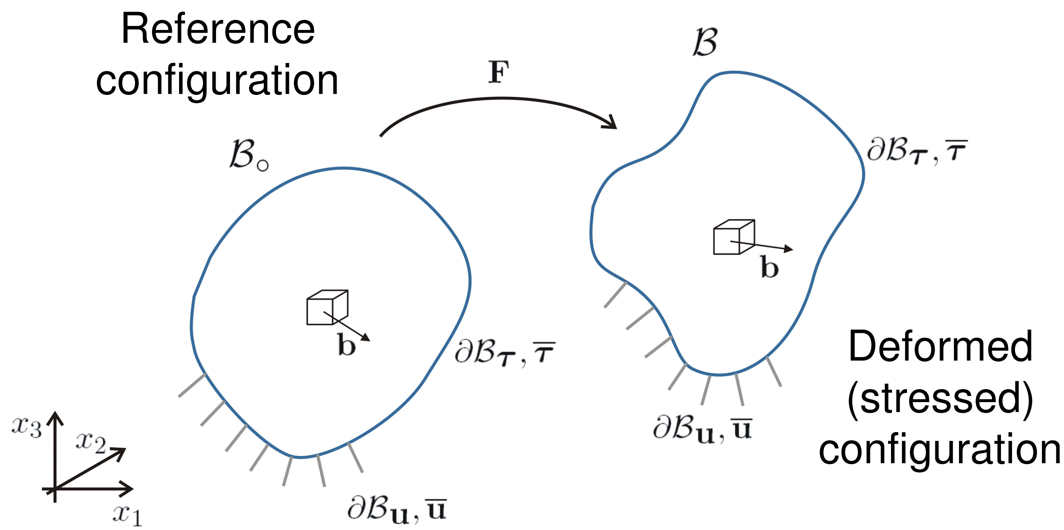


$$\sigma \sim \frac{1}{\sqrt{r}}$$

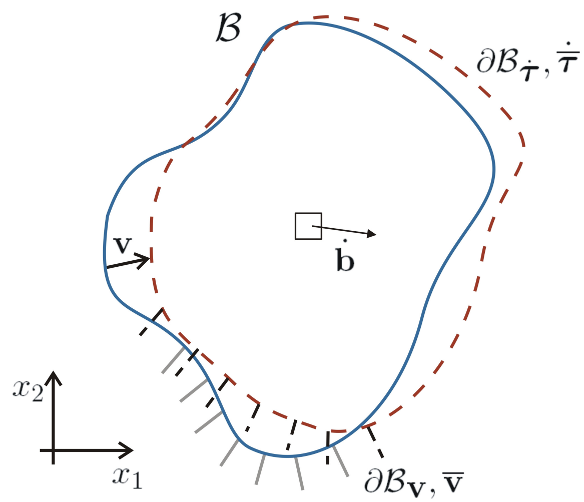
INCOMPRESSIBLE INCREMENTAL NONLINEAR ELASTICITY

INCOMPRESSIBLE INCREMENTAL NONLINEAR ELASTICITY [1]

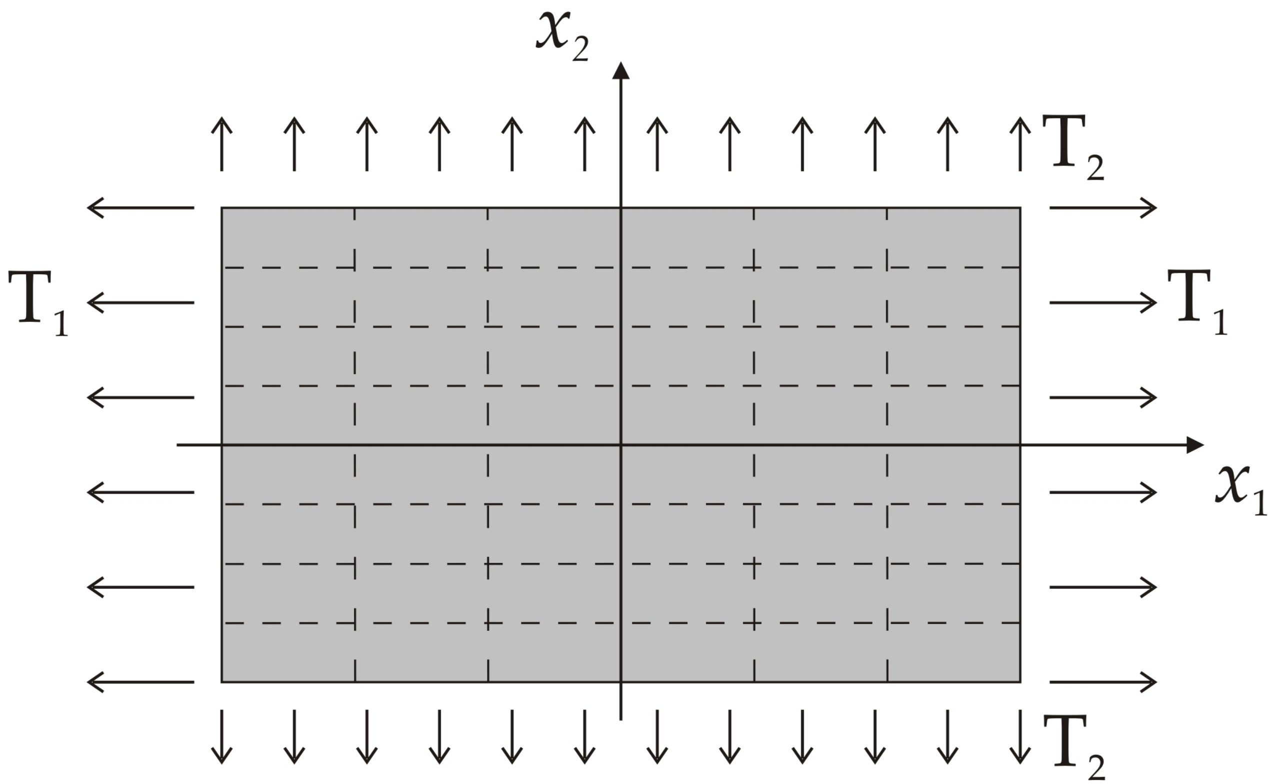
Incompressible material subject to large deformation



Incremental problem:
A generic (plane) perturbation field superimposed upon the deformed configuration



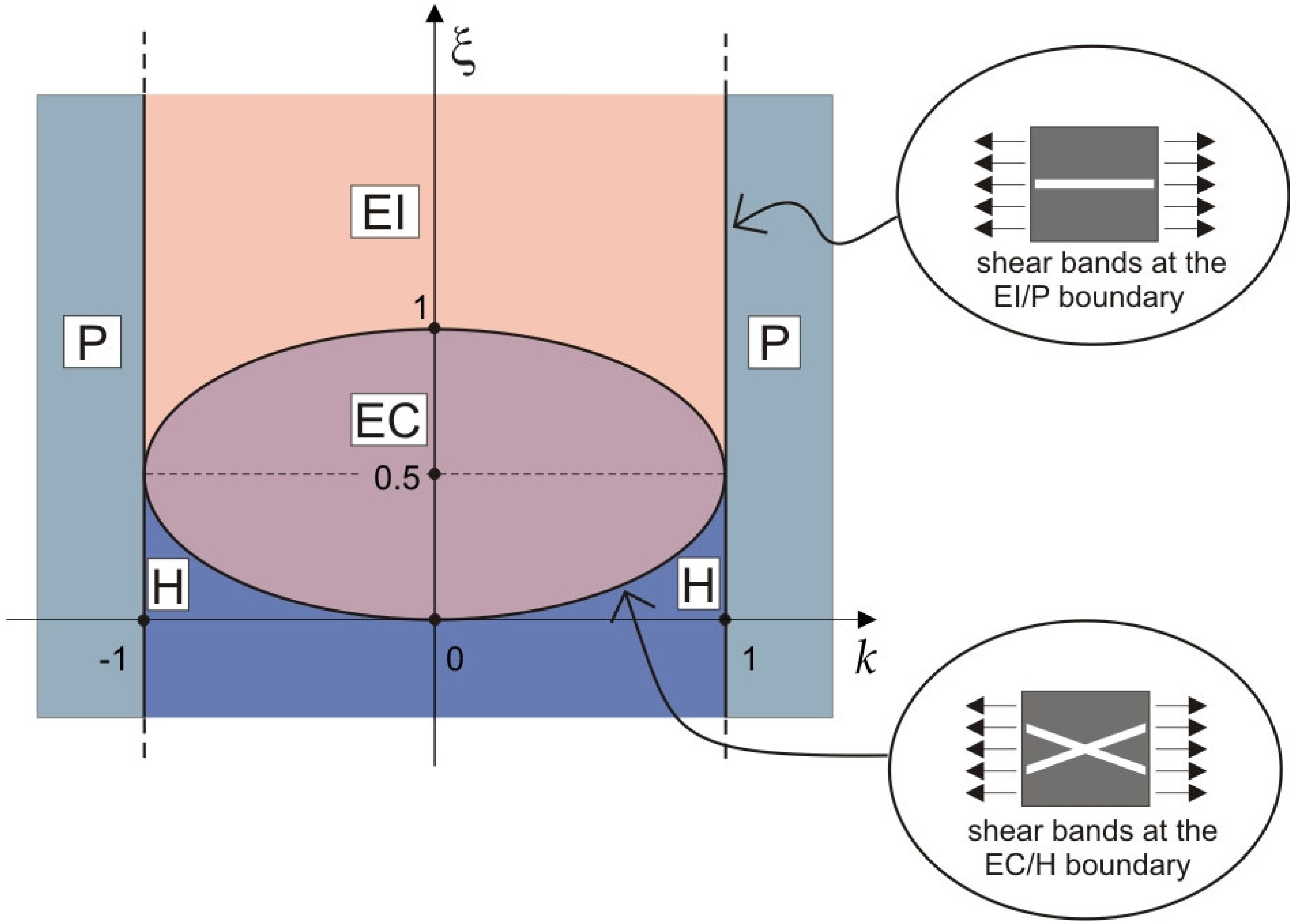
INCOMPRESSIBLE INCREMENTAL NONLINEAR ELASTICITY [2]



$x_1 - x_2$: Parallel and orthogonal to the principal axes of prestress

REGIME CLASSIFICATION

The homogeneous prestress state is stable within the elliptic regime

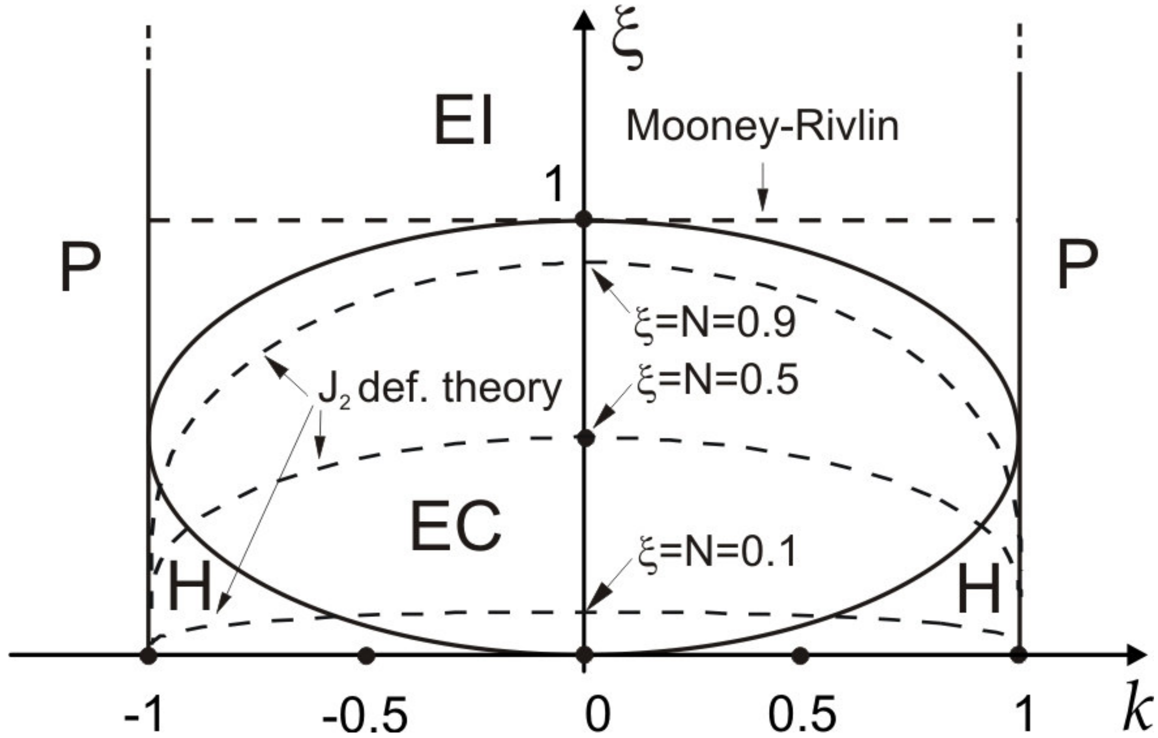


MATERIAL BEHAVIOUR

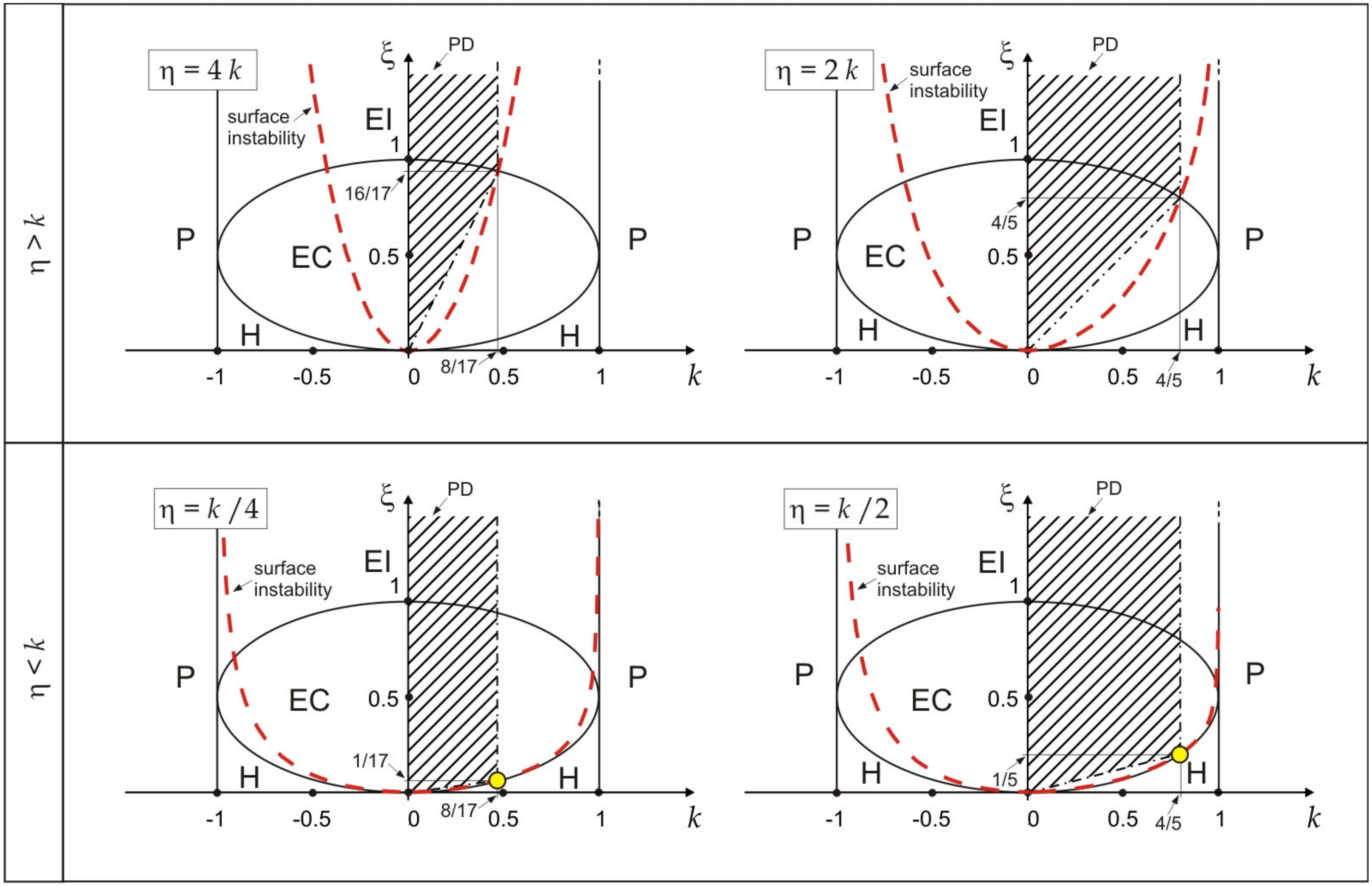
Material cases:

- *Mooney-Rivlin material* (to model isotropic rubber-like elastic material)

- *J_2-deformation theory of plasticity material* (to model the plastic branch of the constitutive response of ductile materials). N $\in$ [0,1) is the strain hardening parameter



SURFACE INSTABILITY



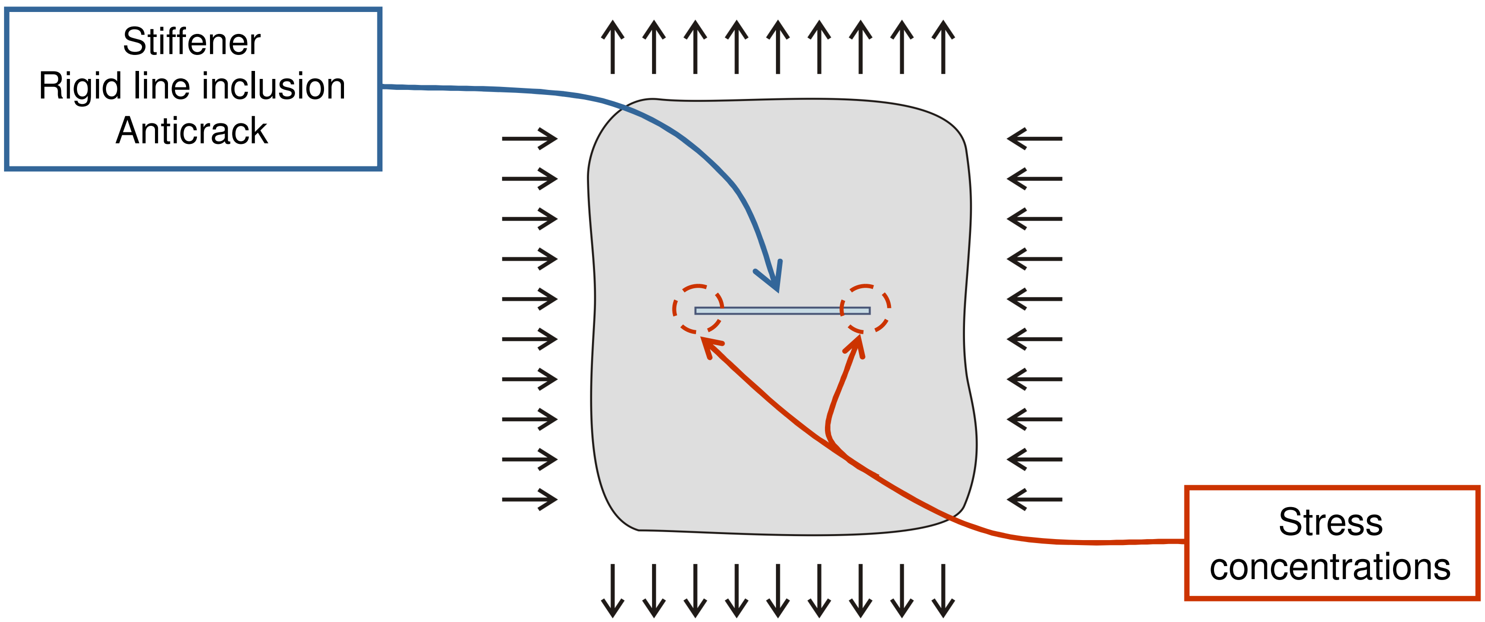
Even in presence of free-surfaces, we can reach the elliptic boundary choosing an appropriate value of $\eta/k<1$

STIFFENER

IS A STIFFENER JUST A MATHEMATICAL MODEL?

A thin, rigid inclusion model poses new problems:

- Is the inclusion sufficiently thin to induce a stress ‘singularity’?
- Is the inclusion bonding enough strong to prevent de-cohesion?
- Unlike cracks, stiffeners produce a singularity also for stresses parallel with respect to the inclusion. Is this true in practice?
- Is the hoop-stress criterion of fracture mechanics valid for a stiffener?



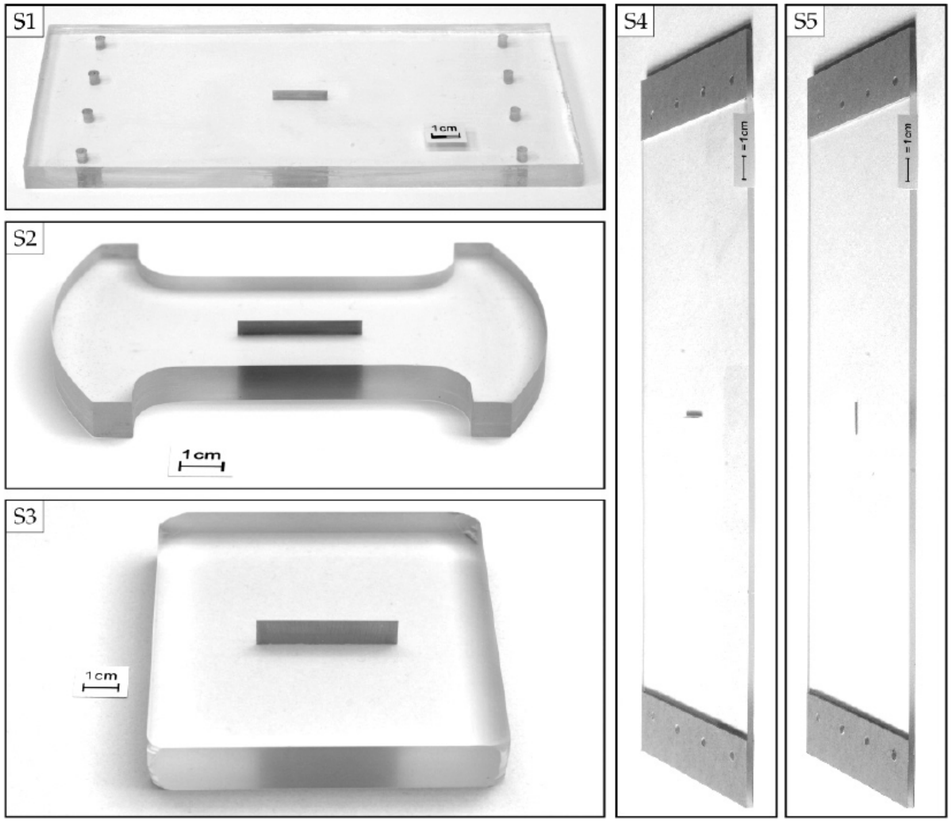
No answer until now…

… let’s perform our own experiments!!

EXPERIMENTS

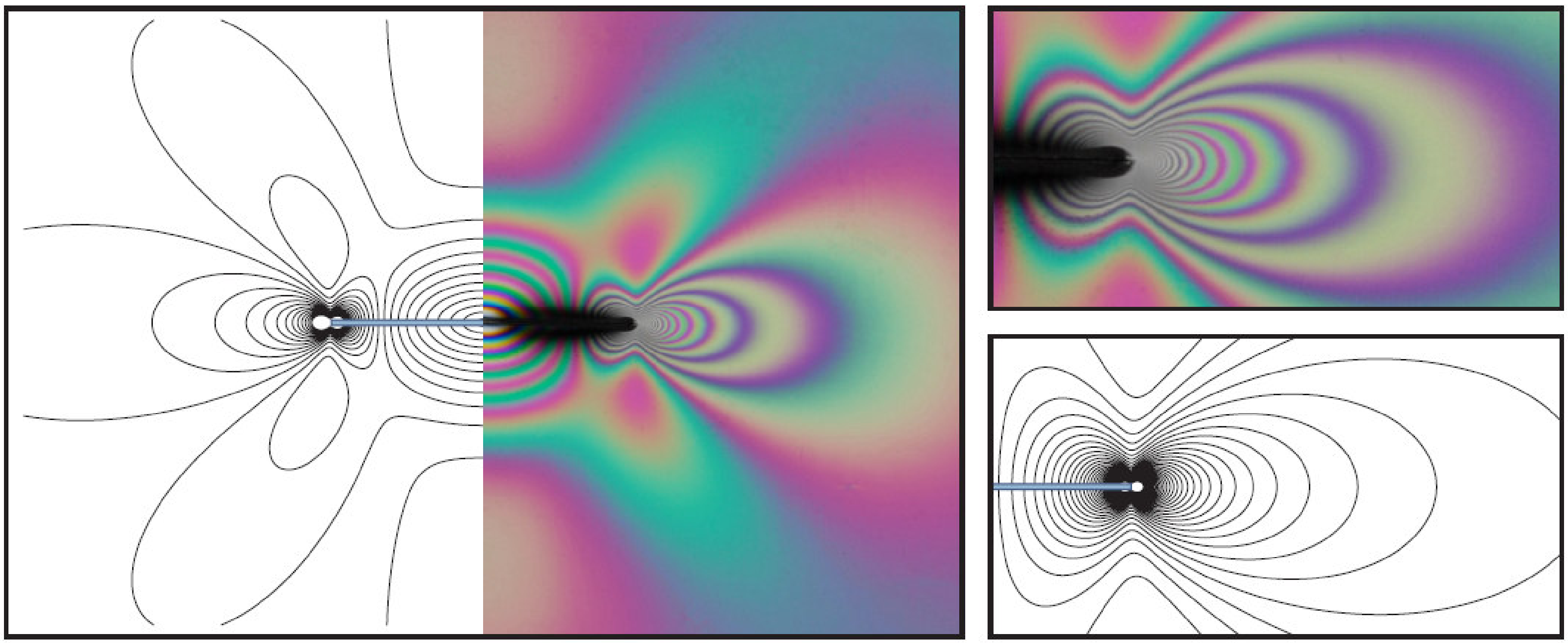
Matrix: transparent two-part epoxy resin

Inclusion: aluminum sheet (0.3 mm thick) with improved superficial rugosity



PHOTOELASTIC EXPERIMENTS MATCH ANALYTICAL RESULTS

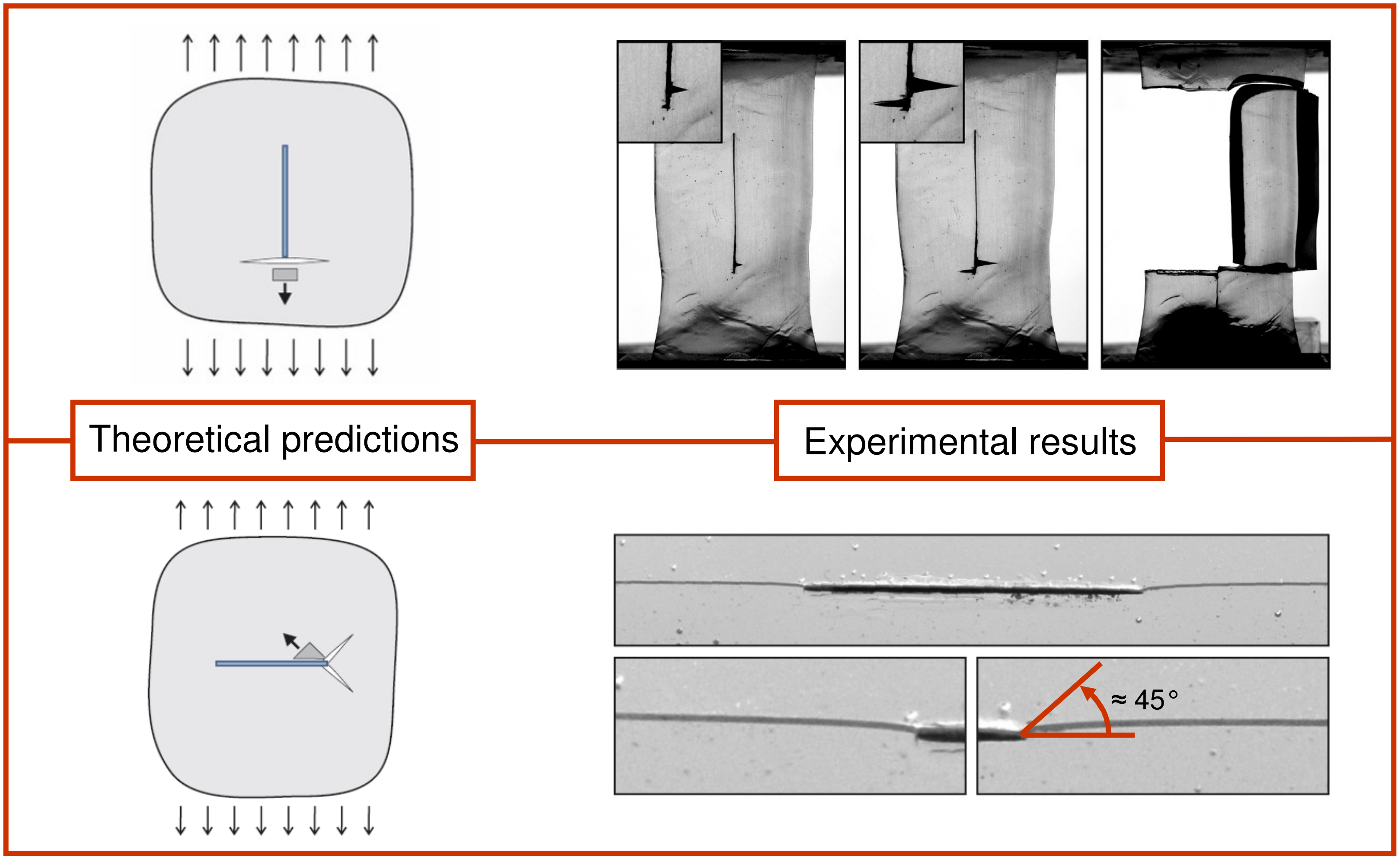
A stiff, thin inclusion has finite thickness and stiffness, and adhesion at the stiffener/matrix contact is necessarily imperfect but…



Comparison between analytical (black and white) and photoelastic (color) results

…STIFFENER IS A SOUND MODEL!!

STRANGE FAILURE MODES OF A STIFFENER IN A BRITTLE MATERIAL



THE HOOP STRESS CRITERION

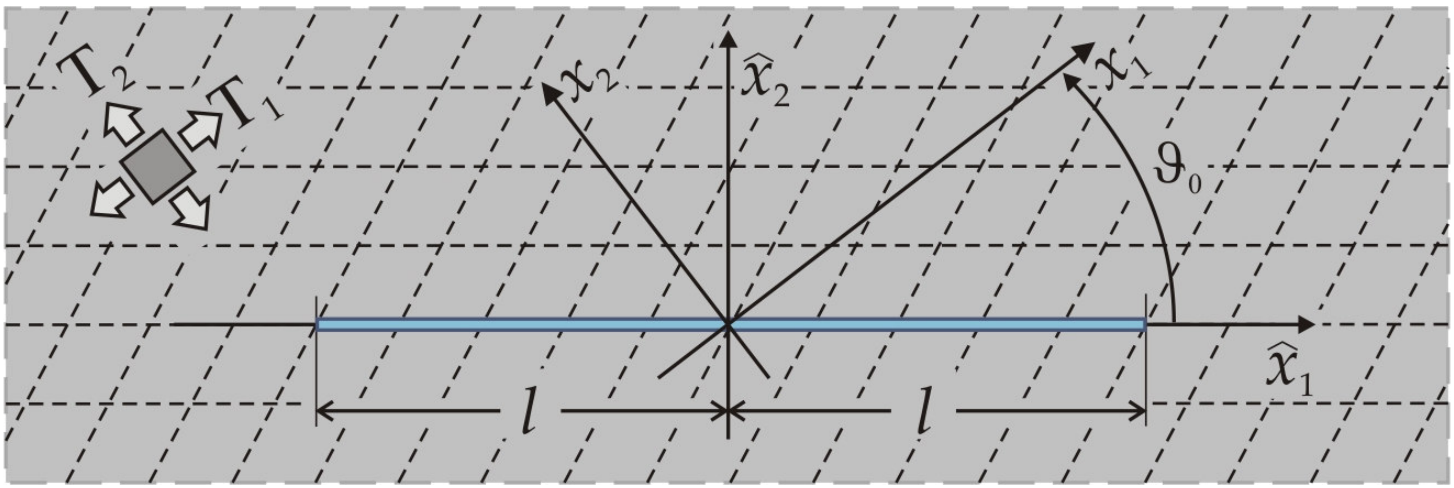
–TYPICAL OF FRACTURE MECHANICS–

IS NOT VALID!!

Now, theory !!

STIFFENER IN A PRESTRESSED MATERIAL

Reference systems
- $x_1 - x_2$: Parallel and orthogonal to the principal axes of prestress
- $\hat{x}_1 - \hat{x}_2$: Parallel and orthogonal to the “inclusion” line



ϑ_0 : Inclination of the “inclusion” with respect to the prestress axes

$2l$: Length of the “inclusion”

$$\hat{\boldsymbol{t}} = \widehat{\mathbb{G}}[\widehat{\nabla}\hat{\boldsymbol{v}}^{\mathrm{T}}] + \dot{p}\,\mathbf{I}$$

“ALIGNED” STIFFENER SOLUTION [1] – Perturbed problem $^{(o)}$

The boundary conditions of rigid line lead to two distinct **Riemann-Hilbert problems**:

EI $(\alpha = 0)$

$$F_1''^{+}(x_1) + F_1''^{-}(x_1) = -i\frac{2\, v_{2,2}^{(\infty)}}{\beta_1 - \beta_2} \qquad \forall\, |x_1| < l$$

EC $(\beta_1 = \beta_2)$

$$F_1''^{+}(x_1) + F_1''^{-}(x_1) = -\frac{v_{2,2}^{(\infty)}}{\alpha} \qquad \forall\, |x_1| < l$$

with solutions

$$F_1''(z_1) = -i\frac{v_{2,2}^{(\infty)}}{\beta_1 - \beta_2}\left(1 - \frac{z_1}{\sqrt{z_1^2 - l^2}}\right)$$

$$F_1''(z_1) = -\frac{v_{2,2}^{(\infty)}}{2\,\alpha}\left(1 - \frac{z_1}{\sqrt{z_1^2 - l^2}}\right)$$

“ALIGNED” STIFFENER SOLUTION [2] – EC Regime

$$\psi = \psi^{(\circ)} + \psi^{(\infty)} = -\frac{v_{2,2}^{(\infty)}}{4\,\alpha}\left\{\mathrm{Re}\left[\beta^2\frac{z_1^2 - z_2^2}{\alpha^2+\beta^2} - z_1\sqrt{z_1^2 - l^2} + z_2\sqrt{z_2^2 - l^2} + l^2 \ln\left(\frac{z_1 + \sqrt{z_1^2 - l^2}}{z_2 + \sqrt{z_2^2 - l^2}}\right)\right] + \mathrm{Im}\left[\alpha\beta\frac{z_1^2 + z_2^2}{\alpha^2+\beta^2}\right]\right\}$$

$$\dot{p} - \dot{p}^{(\infty)} = -\frac{\mu\, v_{2,2}^{(\infty)}}{2\,\alpha}\left\{-\alpha[2(1-k)\beta^2 + k]\,\mathrm{Re}\left[2 - \frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - \beta[2(1-k)\alpha^2 - k]\,\mathrm{Im}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right]\right\},$$

$$\dot{t}_{11} - \dot{p}^{(\infty)} = -\frac{\mu\, v_{2,2}^{(\infty)}}{2\,\alpha}\left\{(\beta\delta + \chi\alpha)\,\mathrm{Re}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} + \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - (\alpha\delta - \chi\beta)\,\mathrm{Im}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - 2\alpha[2(1-k)\beta^2 + k]\right\},$$

$$\dot{t}_{22} - \dot{p}^{(\infty)} = -\frac{\mu\, v_{2,2}^{(\infty)}}{2\,\alpha}\left\{(\beta\delta - \chi\alpha)\,\mathrm{Re}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} + \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - (\alpha\delta + \chi\beta)\,\mathrm{Im}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - 2\alpha[2(1-k)\beta^2 + k]\right\},$$

$$\dot{t}_{12} = -\frac{\mu\, v_{2,2}^{(\infty)}}{2\,\alpha}\left\{(\chi\beta^2 - \chi\alpha^2 + 2\alpha\beta\delta)\,\mathrm{Re}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - (\delta\alpha^2 - \delta\beta^2 + 2\alpha\beta\chi)\,\mathrm{Im}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} + \frac{z_2}{\sqrt{z_2^2 - l^2}}\right]\right\},$$

$$\dot{t}_{21} = -\frac{\mu\, v_{2,2}^{(\infty)}}{2\,\alpha}\left\{\chi\,\mathrm{Re}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - \delta\,\mathrm{Im}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} + \frac{z_2}{\sqrt{z_2^2 - l^2}}\right]\right\},$$

Coefficients depending on prestress

$$\chi = 2\xi - \eta, \quad \delta = 2(1-k)\alpha\beta$$

$$\dot{p}^{(\infty)} = \frac{\dot{t}_{11}^{(\infty)}(2\xi + k - \eta) + \dot{t}_{22}^{(\infty)}(2\xi - k - \eta)}{2(2\xi - \eta)}$$

When $z_j \rightarrow \pm l$ (stiffener tips) $\Rightarrow \dot{t}_{ij} \sim \dfrac{1}{\sqrt{r}}$

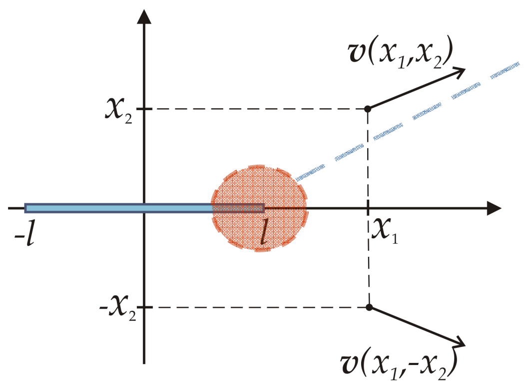
For a uniform incremental strain $v_{2,2}^{(\infty)}$ at infinity:

$$\dot{K}_{(\epsilon)\,I} = \lim_{r\rightarrow 0} 2\mu\sqrt{2\pi r}\, v_{2,2}(r, \vartheta = 0) = 2\,\mu\, v_{2,2}^{(\infty)}\sqrt{\pi l}$$

[independent of prestress (except μ), same value of incompressible classical elasticity]

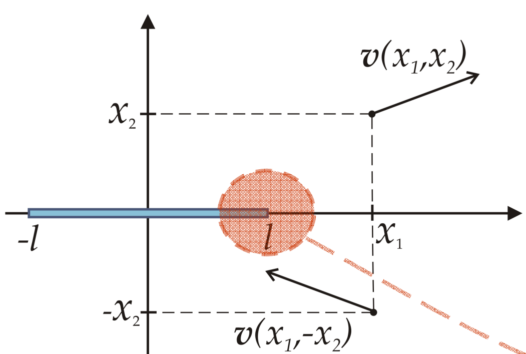
"ALIGNED" STIFFENER SOLUTION [3] – Mode I and Mode II asymptotics

When the stiffener is "aligned", the fields satisfy the **symmetry conditions** due to mode loading



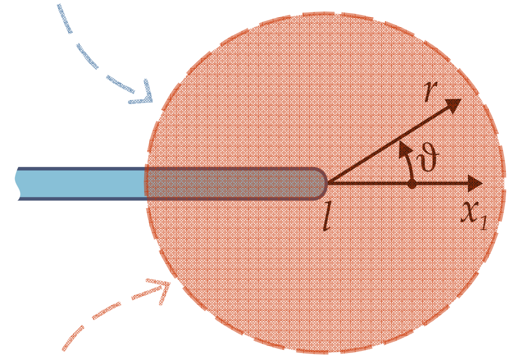
Mode I

$$
\begin{aligned}
v_1(x_1,x_2) &= v_1(x_1,-x_2) \\
v_2(x_1,x_2) &= -v_2(x_1,-x_2)
\end{aligned}
$$



Mode II

$$
\begin{aligned}
v_1(x_1,x_2) &= -v_1(x_1,-x_2) \\
v_2(x_1,x_2) &= v_2(x_1,-x_2)
\end{aligned}
$$



Let's focus on the stiffener tip of coordinates $(l, 0)$

The complex variables z_j admit the polar representation →

$$z_j = x_1 + \Omega_j x_2 = x_1 + \alpha_j x_2 + i\beta_j x_2 = l + r_j e^{i\vartheta_j}$$

$$r_j = r\sqrt{(\cos\vartheta + \alpha_j \sin\vartheta)^2 + \beta_j^2 \sin^2\vartheta}\,, \quad \tan\vartheta_j = \frac{\beta_j \sin\vartheta}{\cos\vartheta + \alpha_j \sin\vartheta} \qquad (j = 1, \ldots, 4)$$

“ALIGNED” STIFFENER SOLUTION [4] – Mode I and Mode II asymptotics

Using principle of boundedness of energy, the asymptotic expansion of incremental fields result

$$\boldsymbol{v}(r,\vartheta) = \frac{\dot{K}_{(\epsilon)}}{\mu}\sqrt{\frac{r}{2\pi}}\,\boldsymbol{\omega}(\vartheta), \qquad \dot{\boldsymbol{t}}(r,\vartheta) = \frac{\dot{K}_{(\epsilon)}}{\sqrt{2\pi r}}\,\boldsymbol{\tau}(\vartheta), \qquad \dot{p}(r,\vartheta) = \frac{\dot{K}_{(\epsilon)}}{\sqrt{2\pi r}}\,\rho(\vartheta)$$

EI MODE I

$$\begin{aligned}
\omega_1(\vartheta) &= -2\sum_{n=1}^{2} b_n\beta_n\sqrt{g_n(\vartheta)+\cos\vartheta},\\
\omega_2(\vartheta) &= 2\sum_{n=1}^{2} b_n\sqrt{g_n(\vartheta)-\cos\vartheta},\\
\tau_{11}(\vartheta) &= -\sum_{n=1}^{2} b_n\varepsilon_n\beta_n\sqrt{g_n(\vartheta)+\cos\vartheta}/g_n(\vartheta),\\
\tau_{22}(\vartheta) &= \sum_{n=1}^{2} b_n\chi_n\beta_n\sqrt{g_n(\vartheta)+\cos\vartheta}/g_n(\vartheta),\\
\tau_{12}(\vartheta) &= -\sum_{n=1}^{2} b_n\chi_n\beta_n^2\sqrt{g_n(\vartheta)-\cos\vartheta}/g_n(\vartheta),\\
\tau_{21}(\vartheta) &= -\sum_{n=1}^{2} b_n\varepsilon_n\sqrt{g_n(\vartheta)-\cos\vartheta}/g_n(\vartheta),\\
\rho(\vartheta) &= \sum_{n=1}^{2} b_n\delta_n\beta_n\sqrt{g_n(\vartheta)+\cos\vartheta}/g_n(\vartheta),
\end{aligned}$$

ANGULAR FUNCTIONS

EI MODE II

$$\begin{aligned}
\omega_1(\vartheta) &= -2\sum_{n=1}^{2} a_n\beta_n\sqrt{g_n(\vartheta)-\cos\vartheta},\\
\omega_2(\vartheta) &= -2\sum_{n=1}^{2} a_n\sqrt{g_n(\vartheta)+\cos\vartheta},\\
\tau_{11}(\vartheta) &= \sum_{n=1}^{2} a_n\varepsilon_n\beta_n\sqrt{g_n(\vartheta)-\cos\vartheta}/g_n(\vartheta),\\
\tau_{22}(\vartheta) &= -\sum_{n=1}^{2} a_n\chi_n\beta_n\sqrt{g_n(\vartheta)-\cos\vartheta}/g_n(\vartheta),\\
\tau_{12}(\vartheta) &= -\sum_{n=1}^{2} a_n\chi_n\beta_n^2\sqrt{g_n(\vartheta)+\cos\vartheta}/g_n(\vartheta),\\
\tau_{21}(\vartheta) &= -\sum_{n=1}^{2} a_n\varepsilon_n\sqrt{g_n(\vartheta)+\cos\vartheta}/g_n(\vartheta),\\
\rho(\vartheta) &= -\sum_{n=1}^{2} a_n\delta_n\beta_n\sqrt{g_n(\vartheta)-\cos\vartheta}/g_n(\vartheta).
\end{aligned}$$

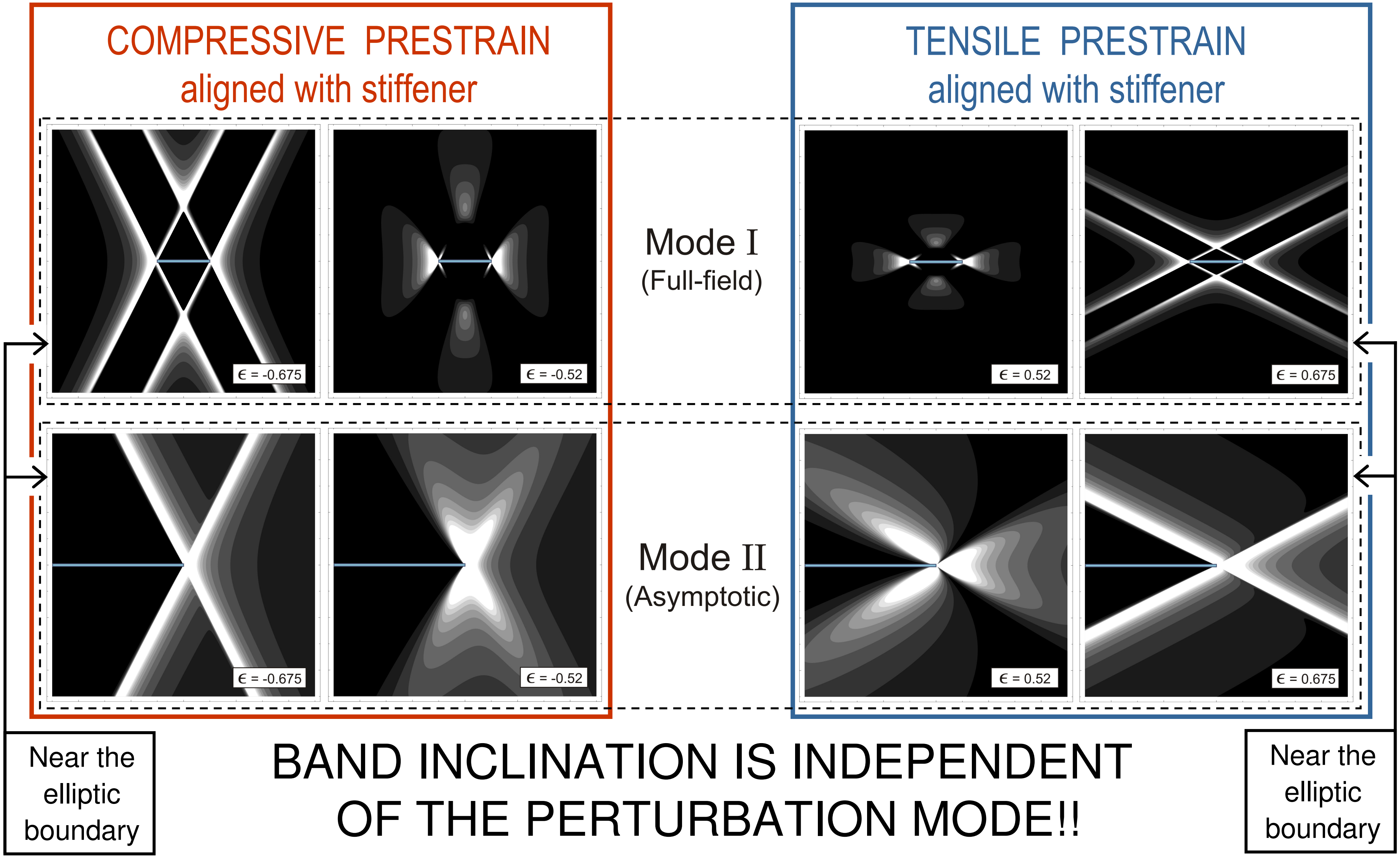
$$g_n(\vartheta) = \sqrt{\cos^2\vartheta + \beta_n^2\sin^2\vartheta} \qquad \left.\begin{array}{c} c_n(\vartheta) \\ s_n(\vartheta)\end{array}\right\} = \sqrt{g_n(\vartheta)\pm\cos\vartheta} \qquad \hat{c}_n(\vartheta) = \frac{c_n(\vartheta)}{g_n(\vartheta)}, \qquad \hat{s}_n(\vartheta) = \frac{s_n(\vartheta)}{g_n(\vartheta)}$$

Ok, it’s fine but…

…How are the mechanical fields near a stiffener ?

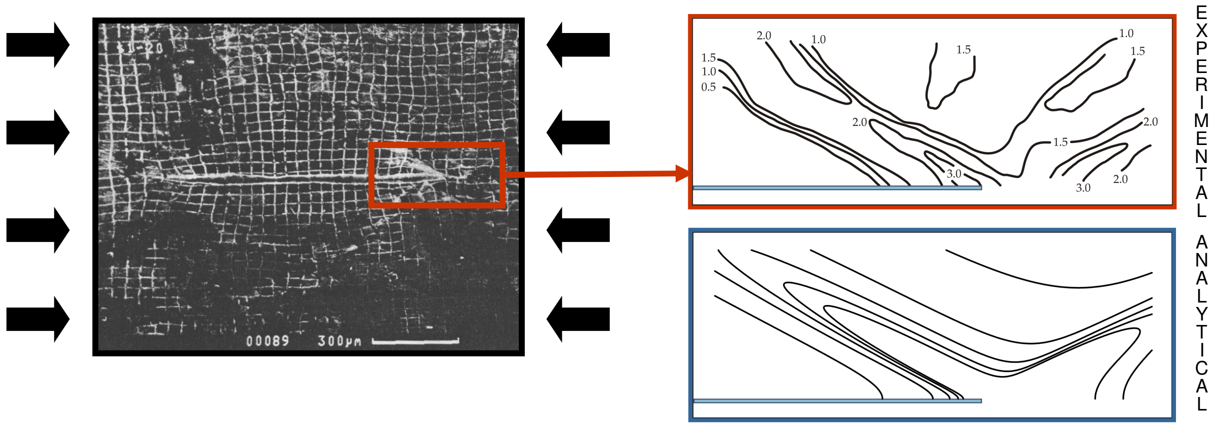
THEORY PREDICTS SHEAR BANDS TO EMERGE FROM STIFFENER TIPS

$|\text{dev}\,\dot{\epsilon}|^2 = const.$ *J_2-deformation theory* N=0.4

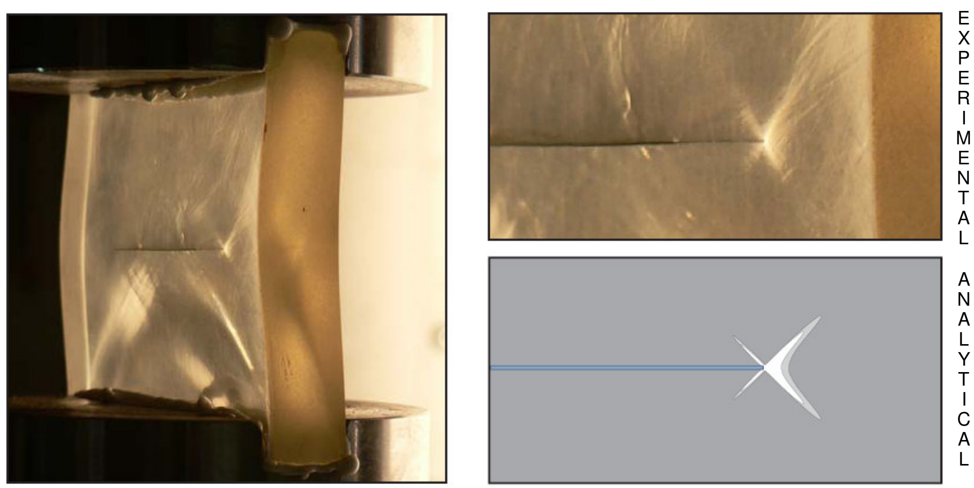


BAND INCLINATION IS INDEPENDENT OF THE PERTURBATION MODE!!

EXPERIMENTAL EVIDENCE [1]



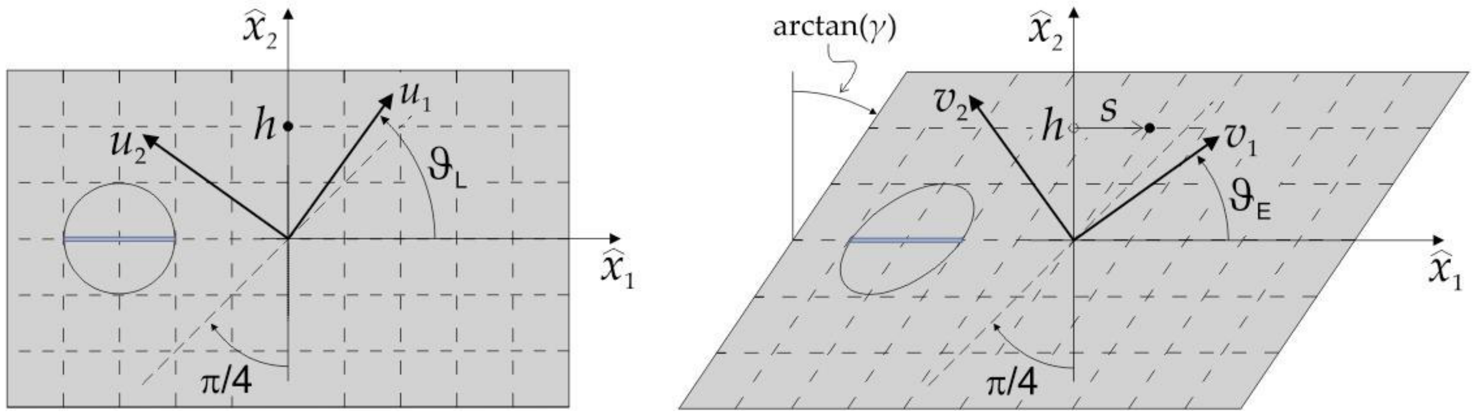
Cu-W laminates subject to large strain (Öztürk et al., 1991)



Compression test on an epoxy resin matrix
containing an aluminum lamellar inclusion

“INCLINED” STIFFENER SOLUTION [1]

Under **finite simple shear deformation**, stiffener does not perturb the homogeneous stress state



Let’s consider a uniform incremental Mode I perturbation, the problem can be solved in the “inclined” reference system. The problem has central symmetry properties!!

$$\begin{cases} \hat{v}_{1,1}(\hat{x}_1, 0) = 0, & \\ \hat{v}_{2,1}(\hat{x}_1, 0) = \omega_S, & \forall\, |\hat{x}_1| < l \\ [\![\hat{t}_{22}(\hat{x}_1, 0)]\!] = 0, & \end{cases}$$

"INCLINED" STIFFENER SOLUTION [2]

Similarly to the "aligned" problem the stream function can be sought in the form

$$\widehat{\psi}^{\circ}(\hat{x}_1, \hat{x}_2) = \frac{\hat{v}_{2,2}^{\infty}}{2} \sum_{j=1}^{2} \mathrm{Re}\left\{ D_j \left[\hat{z}_j^{\,2} - \hat{z}_j \sqrt{\hat{z}_j^{\,2} - l^2} + l^2 \ln\left(\hat{z}_j + \sqrt{\hat{z}_j^{\,2} - l^2} \right) \right] \right\}$$

Applying the boundary conditions we obtain a linear problem for the complex constants D_1, D_2:

$$\begin{bmatrix} \mathrm{Re}[W_1] & -\mathrm{Im}[W_1] & \mathrm{Re}[W_2] & -\mathrm{Im}[W_2] \\ \mathrm{Im}[W_1] & \mathrm{Re}[W_1] & \mathrm{Im}[W_2] & \mathrm{Re}[W_2] \\ 0 & 1 & 0 & 1 \\ -c_{21} & c_{11} & -c_{22} & c_{12} \end{bmatrix} \begin{bmatrix} \mathrm{Re}[D_1] \\ \mathrm{Im}[D_1] \\ \mathrm{Re}[D_2] \\ \mathrm{Im}[D_2] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

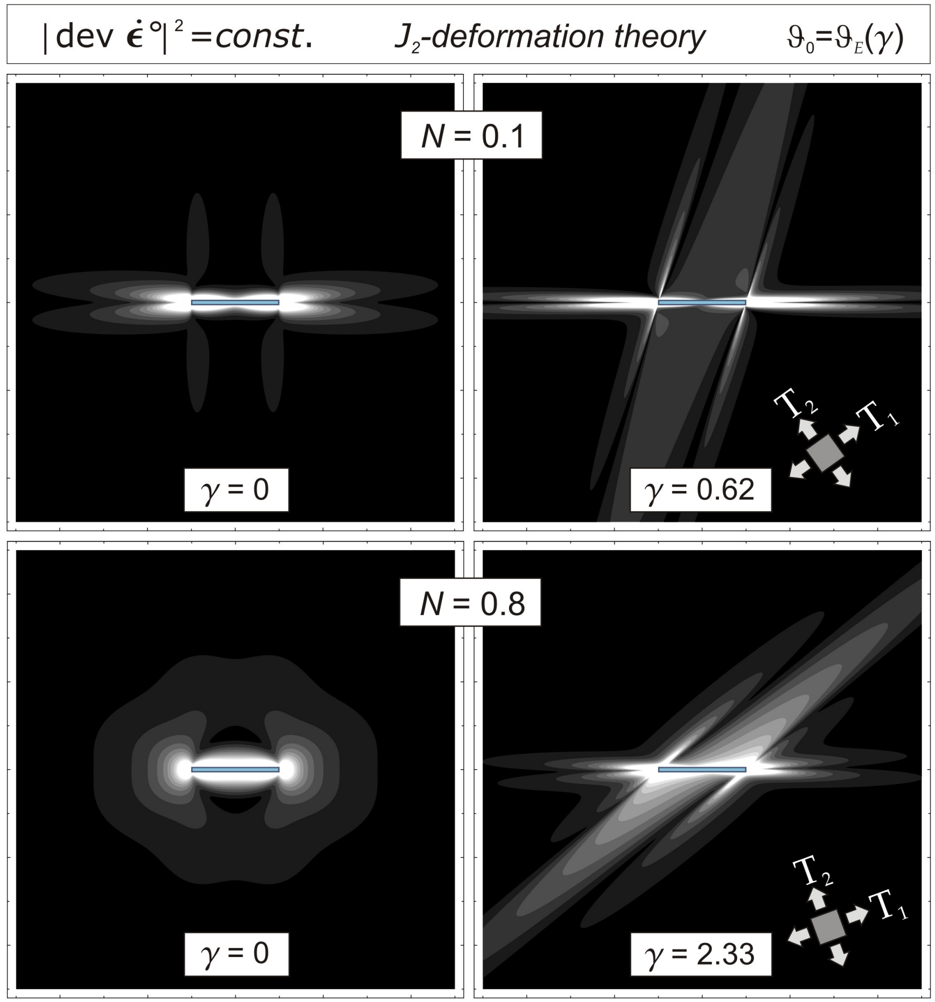
where:

$$\begin{aligned} 2\,\mu\, c_{1j} = \; & \widehat{\mathbb{G}}_{1112} - \widehat{\mathbb{G}}_{1222} - \mathrm{Re}[W_j] \left[\widehat{\mathbb{G}}_{1111} - 2\widehat{\mathbb{G}}_{1122} - \widehat{\mathbb{G}}_{1221} + \widehat{\mathbb{G}}_{2222} + \mathrm{Re}[W_j] \left(2\widehat{\mathbb{G}}_{1121} - 2\widehat{\mathbb{G}}_{2122} + \mathrm{Re}[W_j]\widehat{\mathbb{G}}_{2121} \right) \right] \\ & + \mathrm{Im}[W_j]^2 \left(2\widehat{\mathbb{G}}_{1121} - 2\widehat{\mathbb{G}}_{2122} + 3\,\mathrm{Re}[W_j]\widehat{\mathbb{G}}_{2121} \right), \\ 2\,\mu\, c_{2j} = \; & \mathrm{Im}[W_j] \left[\widehat{\mathbb{G}}_{1111} - 2\widehat{\mathbb{G}}_{1122} - \widehat{\mathbb{G}}_{1221} + \widehat{\mathbb{G}}_{2222} + \mathrm{Re}[W_j] \left(4\widehat{\mathbb{G}}_{1121} - 4\widehat{\mathbb{G}}_{2122} + 3\,\mathrm{Re}[W_j]\widehat{\mathbb{G}}_{2121} \right) - \mathrm{Im}[W_j]^2 \widehat{\mathbb{G}}_{2121} \right] \end{aligned}$$

"INCLINED" STIFFENER SOLUTION [3]

For the "inclined" stiffener the axial symmetry is lost…

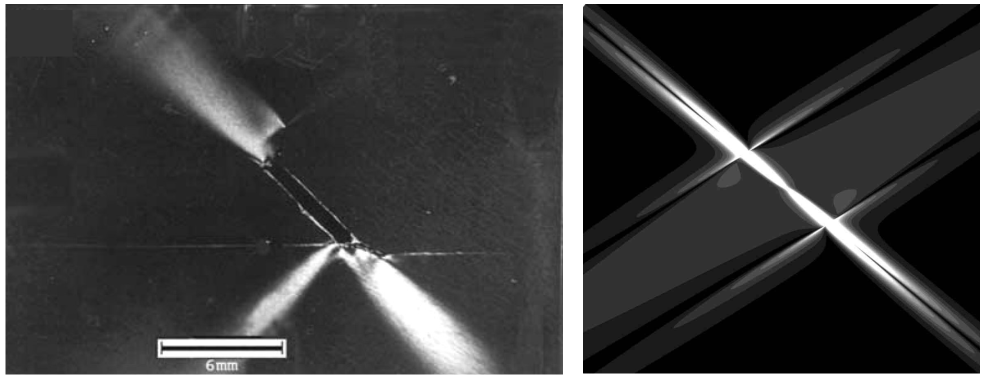
Far from the elliptic boundary



Near the elliptic boundary

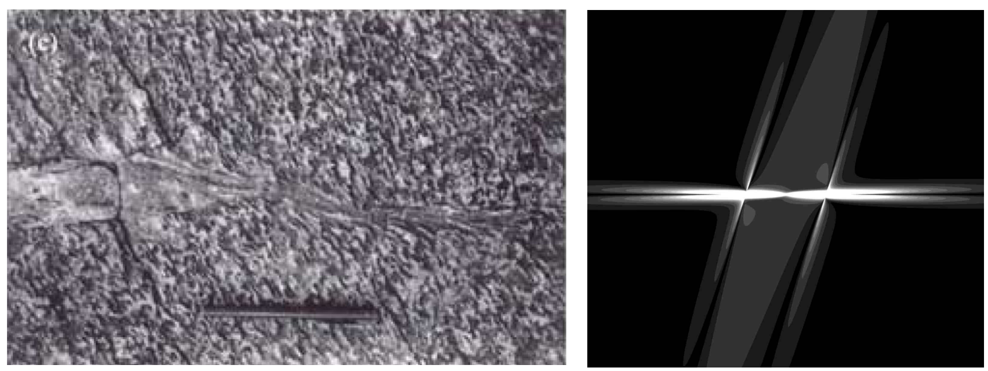
THE SHEAR BAND CLOSEST TO THE STIFFENER LINE IS PRIVILEGED!!

EXPERIMENTAL EVIDENCE [2]



ANALYTICAL

Compression test on a PMMA block
containing a thin steel inclusion
(Misra and Mandal, 2007)



ANALYTICAL

Localization of a single shear band
at the tip of a quartz vein in a naturally deformed rocks
(Misra and Mandal, 2007)

A strange question:

Can a rigid inclusion grow (or reduce its length)?

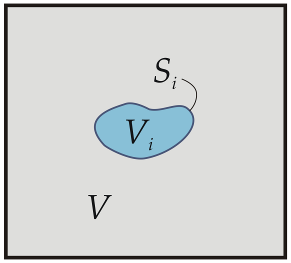
Problem related to:

Damage diffusion: a material is present within a damaged one. Due to damage growth, it progressively reduces its volume;

Phase tranformation: two phases (one more stiffer than other) of a material, growth or reduction would be related to a progression or regression of a phase transformation.

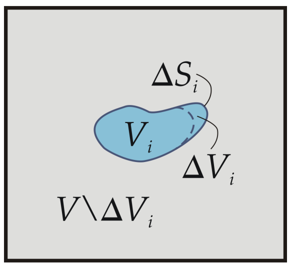
RIGID INCLUSION GROWTH (OR REDUCTION?)

Let's consider two incremental boundary value problems differing only in the sizes of rigid inclusion



$$\dot{P}^0 = \int_V \phi(\nabla \mathbf{v}^0)dV - \int_{S_\sigma} \dot{\boldsymbol{\sigma}}^0 \cdot \mathbf{v}^0 dS$$

$$\ominus$$



$$\dot{P}^0 + \Delta\dot{P} = \int_{V\backslash \Delta V_i} \phi(\nabla \mathbf{v}^0 + \nabla \tilde{\mathbf{v}})dV - \int_{S_\sigma} \dot{\boldsymbol{\sigma}}^0 \cdot (\mathbf{v}^0 + \tilde{\mathbf{v}})dS$$

$$\ominus$$

Incremental potential energy decrease for rigid inclusion growth

$$-\Delta\dot{P} = -\int_{\Delta V_i} \phi(\nabla \mathbf{v}^0)dV + \frac{1}{2}\int_{\Delta S_i} \tilde{\mathbf{t}}^T \mathbf{n} \cdot \mathbf{v}^0 dS < 0$$

for positive definite $\mathbb{G}$

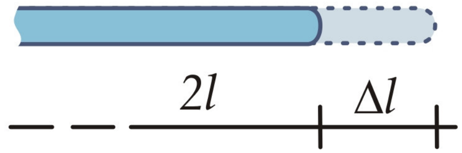
RIGID INCLUSION REDUCTION IS ALWAYS PREDICTED!!

(Void problem: $-\Delta\dot{P} = \int_{\Delta V_i} \phi(\nabla \mathbf{v}^0)dV - \frac{1}{2}\int_{\Delta S_i} \mathbf{t}^{0T} \mathbf{n} \cdot \tilde{\mathbf{v}} dS > 0$ Favorite growth)

INCREMENTAL ENERGY RELEASE RATE FOR A STIFFENER

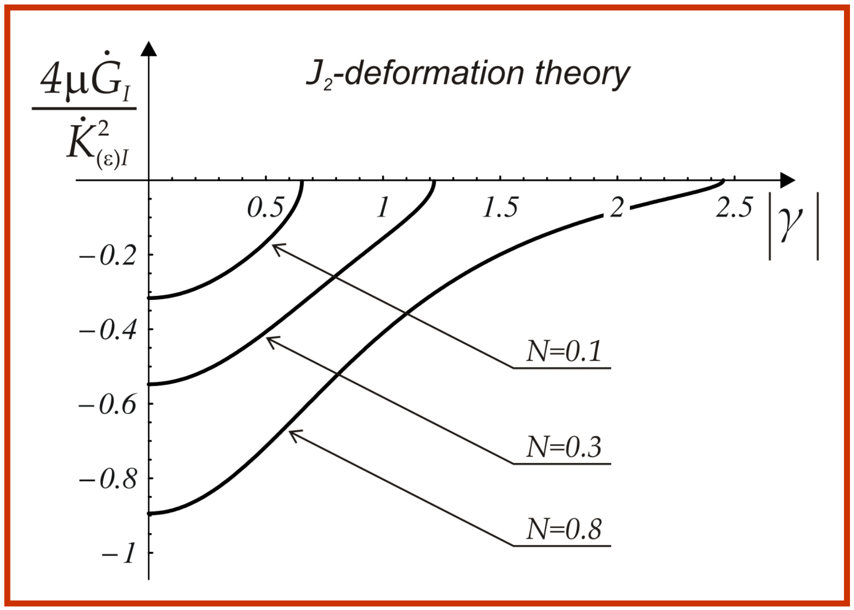
In the case of a stiffener, we consider the limit of the length increase $\Delta l \to 0$

$$\dot{G} = -\frac{d\dot{P}}{dl} = -\lim_{\Delta l \to 0} \frac{1}{2\Delta l} \int_0^{\Delta l} [[\hat{\dot{t}}_{2i}(\Delta l - r, \pi)]] \hat{\dot{v}}_i(r, 0)\, dr$$



The incremental energy release rate for an incremental Mode I becomes

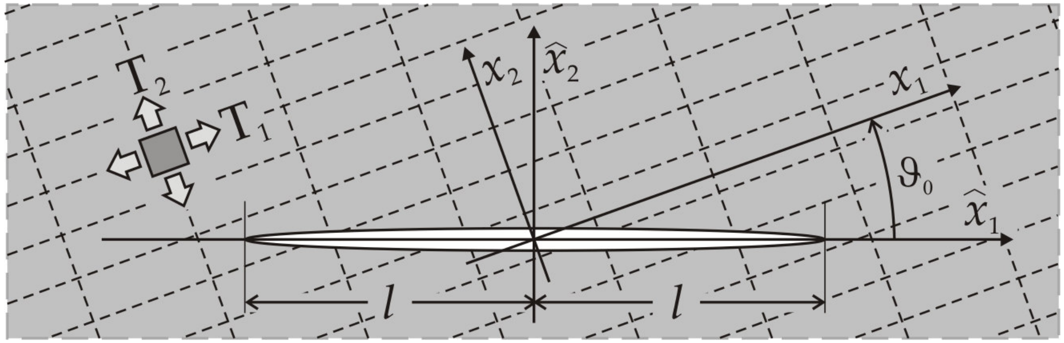
$$\dot{G}_I = -\frac{\dot{K}^2_{(\epsilon)I}}{8\mu^2} \left\{ \left[\widehat{\mathbb{G}}_{2121} + 2\Gamma(\widehat{\mathbb{G}}_{2221} - \widehat{\mathbb{G}}_{2111}) \right] \mathrm{Im}[W_1^2 D_1 + W_2^2 D_2] - \Gamma \widehat{\mathbb{G}}_{2121} \mathrm{Im}[W_1^3 D_1 + W_2^3 D_2] \right\}$$



STIFFENER REDUCTION IS INHIBITED AT THE ELLIPTIC BOUNDARY

CRACK

CRACK IN A PRESTRESSED MATERIAL



Traction free boundary conditions

$$\begin{cases} \hat{t}_{22}(\hat{x}_1, 0^{\pm}) = 0, \\ \hat{t}_{21}(\hat{x}_1, 0^{\pm}) = 0, \end{cases} \quad \forall\, |\hat{x}_1| < l$$

Similarly to the "inclined" stiffener problem the stream function can be sought in the form

$$\widehat{\psi}^{\circ}_M(\hat{x}_1, \hat{x}_2) = \frac{\hat{t}^{\infty}_{2n}}{2\mu} \sum_{j=1}^{2} \mathrm{Re}\left\{ A_j^M \left[\hat{z}_j^{\,2} - \hat{z}_j\sqrt{\hat{z}_j^{\,2} - l^2} + l^2 \ln\left(\hat{z}_j + \sqrt{\hat{z}_j^{\,2} - l^2} \right) \right] \right\}$$

where n=2 and M=1 for Mode I, and n=1 and M=2 for Mode II.

Applying the boundary conditions we obtain a linear problem for the complex constants A_1^M, A_2^M:

$$\begin{bmatrix} c_{11} & c_{21} & c_{12} & c_{22} \\ -c_{21} & c_{11} & -c_{22} & c_{12} \\ c_{31} & c_{41} & c_{32} & c_{42} \\ -c_{41} & c_{31} & -c_{42} & c_{32} \end{bmatrix} \begin{bmatrix} \mathrm{Re}[A_1^M] \\ \mathrm{Im}[A_1^M] \\ \mathrm{Re}[A_2^M] \\ \mathrm{Im}[A_2^M] \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{for Mode I}} \text{ or } \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\text{for Mode II}}$$

INCREMENTAL ENERGY RELEASE RATE FOR A CRACK

Generalizing Rice (1968), we obtain the incremental potential energy decrease for void growth

$$-\Delta \dot{P} = \int_{\Delta V_i} \phi(\nabla \mathbf{v}^0) dV - \frac{1}{2} \int_{\Delta S_i^*} \boldsymbol{n} \cdot \boldsymbol{t}^0 \tilde{\boldsymbol{v}} dS > 0$$

for positive definite $\mathbb{G}$

In the case of a crack, we consider the limit of the length increase $\Delta l \rightarrow 0$

$$\dot{G} = -\frac{d\dot{P}}{dl} = \lim_{\Delta l \rightarrow 0} \frac{1}{2\Delta l} \int_0^{\Delta l} \hat{t}_{2i}(r,0)[[\hat{v}_i(\Delta l - r, \pi)]]\, dr,$$

which for a generic incremental loading becomes

$$\dot{G} = -\dot{K}_I^2 \frac{\mathrm{Im}\left[A_1^I + A_2^I\right]}{4\mu} + \dot{K}_{II}^2 \frac{\mathrm{Im}\left[W_1 A_1^{II} + W_2 A_2^{II}\right]}{4\mu} + \dot{K}_I \dot{K}_{II} \frac{\mathrm{Im}\left[W_1 A_1^I + W_2 A_2^I - A_1^{II} - A_2^{II}\right]}{4\mu}$$

where $\dot{K}_I = \lim_{r \rightarrow 0} \sqrt{2\pi r}\, \hat{t}_{22}(r, \vartheta = 0), \quad \dot{K}_{II} = \lim_{r \rightarrow 0} \sqrt{2\pi r}\, \hat{t}_{21}(r, \vartheta = 0)$

represent the incremental stress intensity factors under Mode I and II loading, respectively.

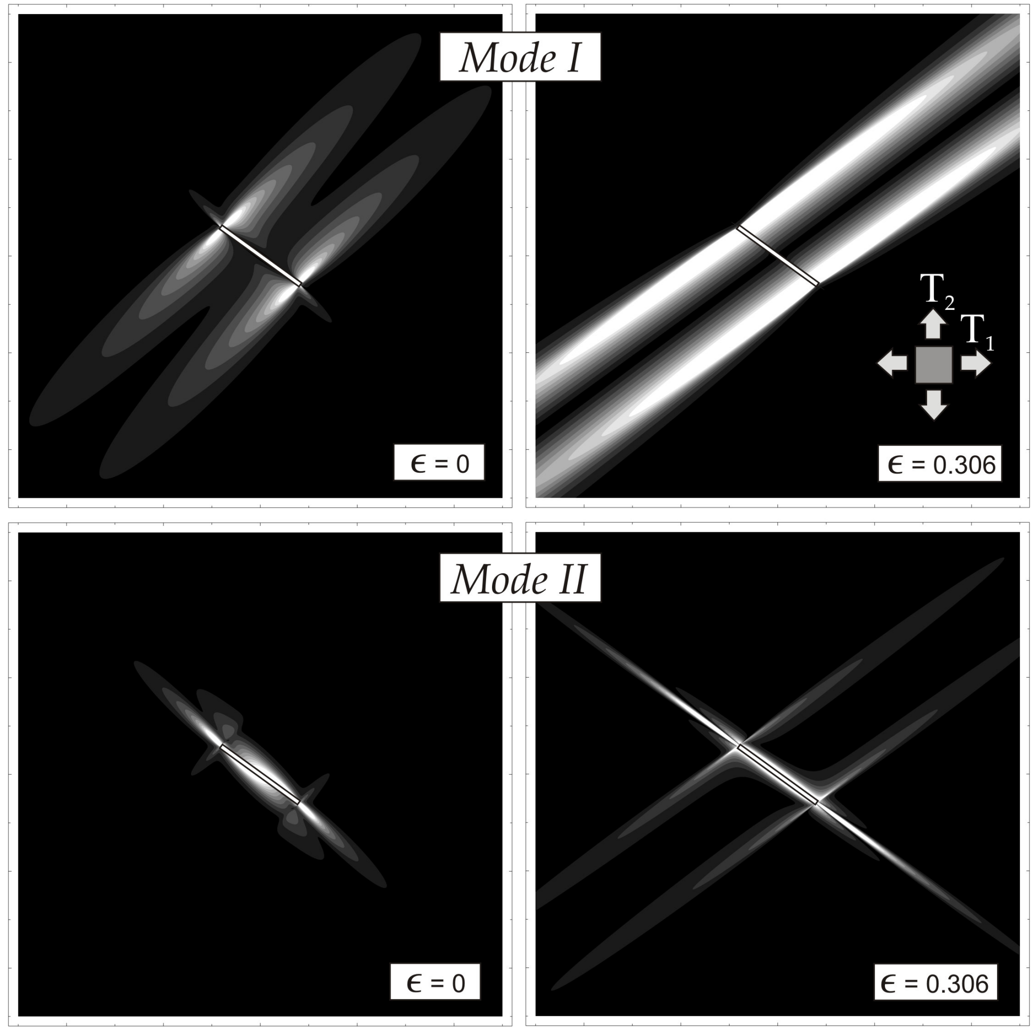
For uniform loadings: $\dot{K}_I = \hat{t}_{22}^{\infty} \sqrt{\pi l}, \quad \dot{K}_{II} = \hat{t}_{21}^{\infty} \sqrt{\pi l}$

same values of the classical elasticity case

SHEAR BANDS INTERACTING WITH A CRACK

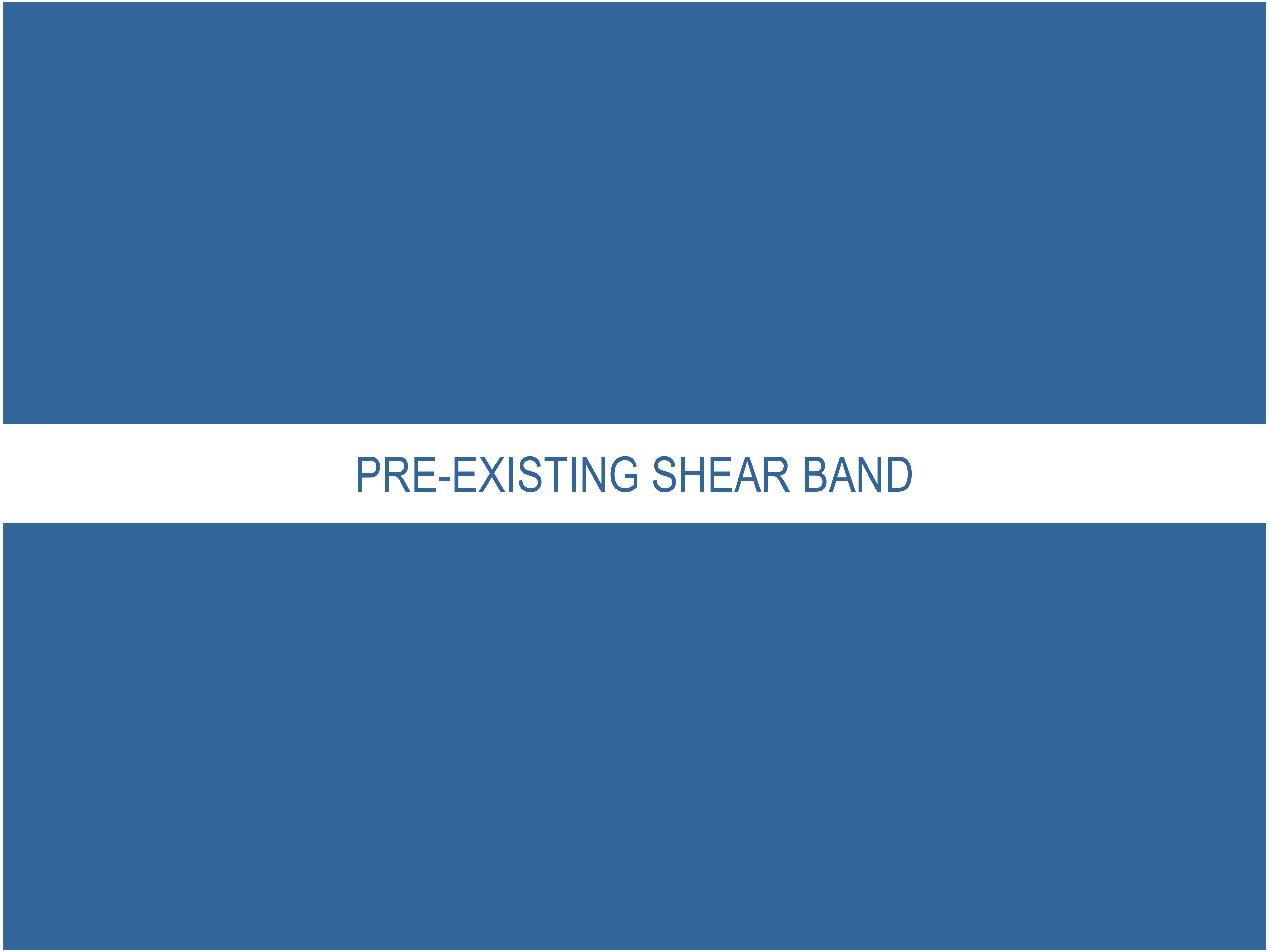
$|\text{dev}\,\dot{\boldsymbol{\epsilon}}^{\circ}|^2 = const.$ J_2-deformation theory $N=0.1$ $\vartheta_0=\vartheta^{SB}=35.95°$

Far from the elliptic boundary



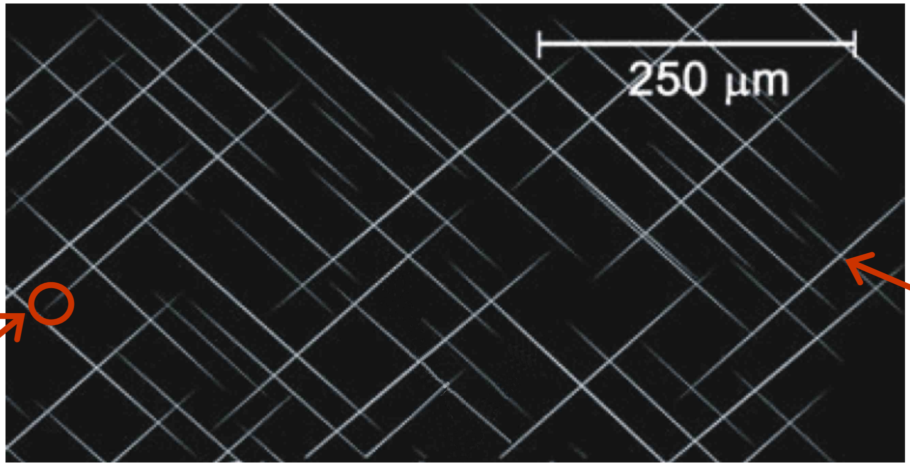
Near the elliptic boundary

DEFORMATION BECOMES HIGHLY FOCUSSED AND ALIGNED PARALLEL TO THE SHEAR BAND CONJUGATE DIRECTIONS


PRE-EXISTING SHEAR BAND

OPEN PROBLEMS ON SHEAR BAND

Shear band tip



Rectlinear, Mode II growth

Shear bands in polystyrene (Anand & Spitzig, 1982)

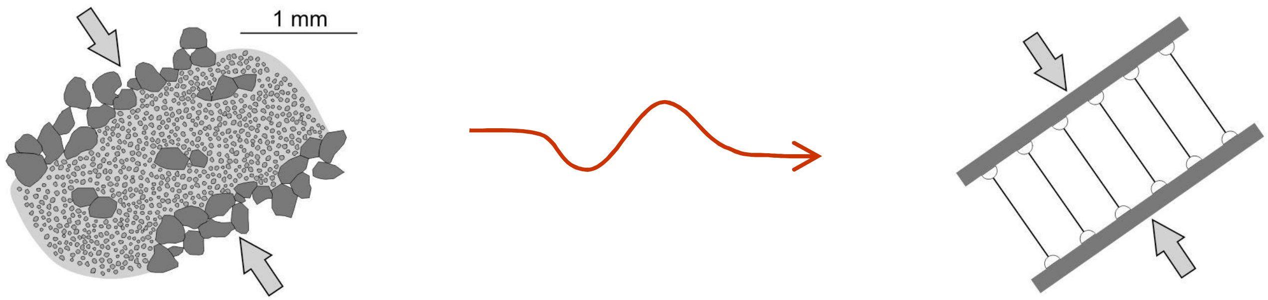
There are **still many, important open problems** about shear bands:

- How is the stress state near a shear band?
- Does it involve a stress concentration (‘singularity’)?
- Why, unlike cracks, do shear bands grow rectilinearly and for ‘long distances’ under Mode II loading?
- Why are shear bands a preferred failure mode for ductile materials?

SHEAR BAND MODEL

Deformation band observed in dry sandstone
(Sulem and Ouffroukh, 2006)

Zero-thickness hinged quadrilateral model



The introduced **shear band model** is equivalent to the following boundary conditions:

- Null incremental nominal shearing tractions

$$\hat{t}_{21}(\hat{x}_1, 0^{\pm}) = 0, \quad \forall |\hat{x}_1| < l$$

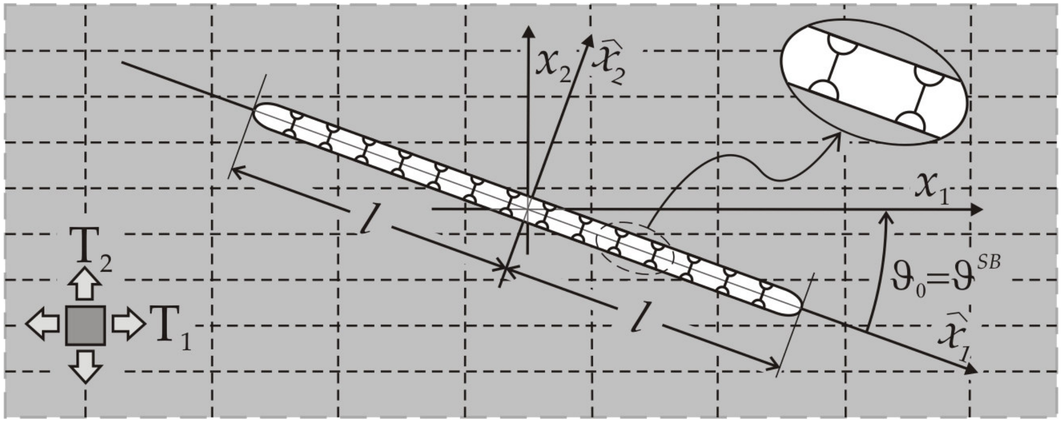
- Continuity of the incremental nominal normal traction

$$[[\hat{t}_{22}(\hat{x}_1, 0)]] = 0, \quad \forall |\hat{x}_1| < l$$

- Continuity of normal incremental displacement

$$[[\hat{v}_2(\hat{x}_1, 0)]] = 0, \quad \forall |\hat{x}_1| < l$$

SHEAR BAND PROBLEM [1]



Shear band is **neutral** under Mode I loading

Similarly to the stiffener problem we use the stream function in the form

$$\widehat{\psi}^{\circ}(\hat{x}_1,\hat{x}_2)=\frac{\hat{t}_{21}^{\infty}}{2\mu}\sum_{j=1}^{2}\mathrm{Re}\left\{B_j^{II}\left[\hat{z}_j^{\,2}-\hat{z}_j\sqrt{\hat{z}_j^{\,2}-l^2}+l^2\ln\left(\hat{z}_j+\sqrt{\hat{z}_j^{\,2}-l^2}\right)\right]\right\}$$

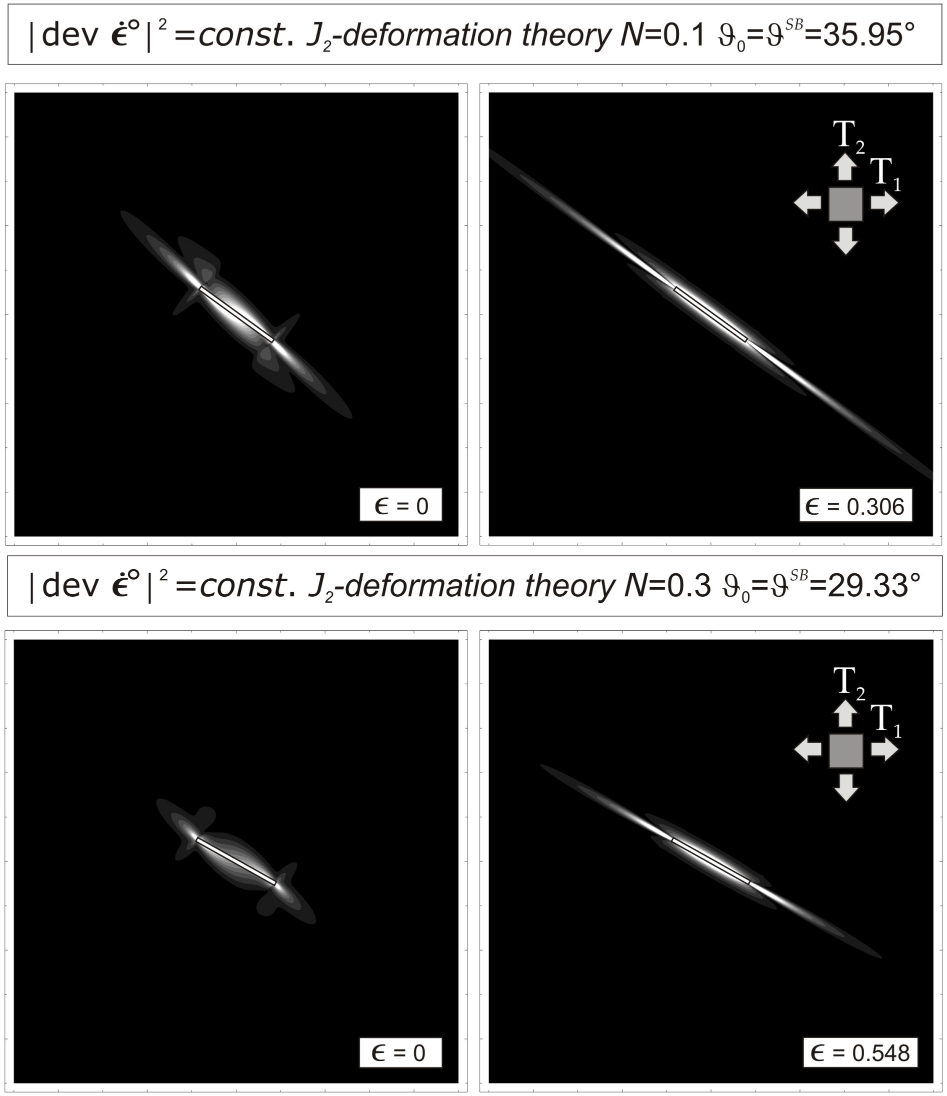
Applying the shear bands boundary conditions we obtain the linear problem for B_1^{II}, B_2^{II} :

$$\begin{bmatrix} -c_{21} & c_{11} & -c_{22} & c_{12} \\ c_{31} & c_{41} & c_{32} & c_{42} \\ -c_{41} & c_{31} & -c_{42} & c_{32} \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mathrm{Re}[B_1^{II}] \\ \mathrm{Im}[B_1^{II}] \\ \mathrm{Re}[B_2^{II}] \\ \mathrm{Im}[B_2^{II}] \end{bmatrix}=\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

THE SOLUTION SHOWS A STRESS SQUARE ROOT SINGULARITY AT THE SHEAR BAND TIPS

SHEAR BAND PROBLEM [2]

Far from the elliptic boundary



Near the elliptic boundary

DEFORMATION IS FOCUSSED ONLY ALONG THE PRE-EXISTING SHEAR BAND DIRECTION!!

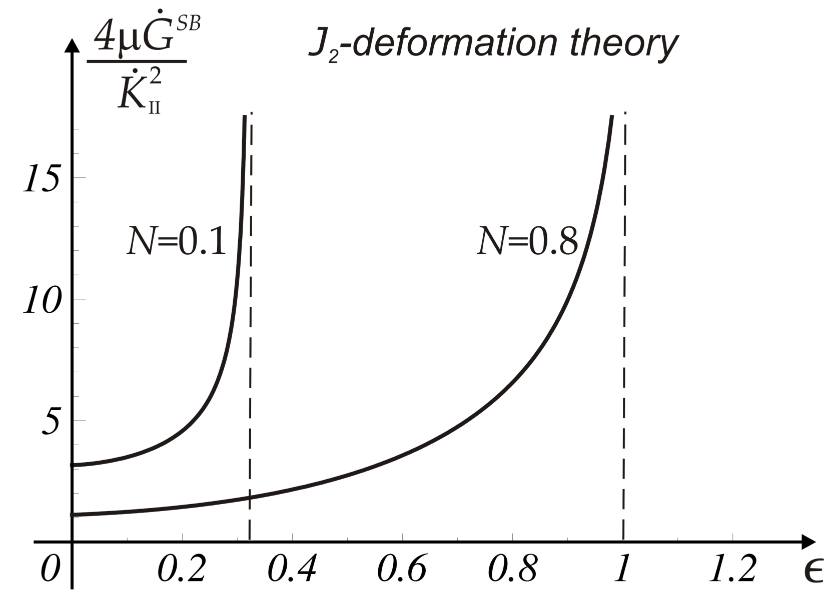
THE CONJUGATE SHEAR BAND DIRECTION REMAINS INACTIVE!!

SHEAR BAND PROBLEM [3]

Incremental energy release rate for an infinitesimal shear band advance under incremental Mode II loading

$$\dot{G}^{SB} = \dot{K}_{II}^2 \frac{\mathrm{Im}\left[W_1 B_1^{II} + W_2 B_2^{II}\right]}{4\mu}$$

The incremental energy release rate blows up to infinity when the elliptic boundary is approached.



THE GROWTH BECOMES UNRESTRAINABLE!!

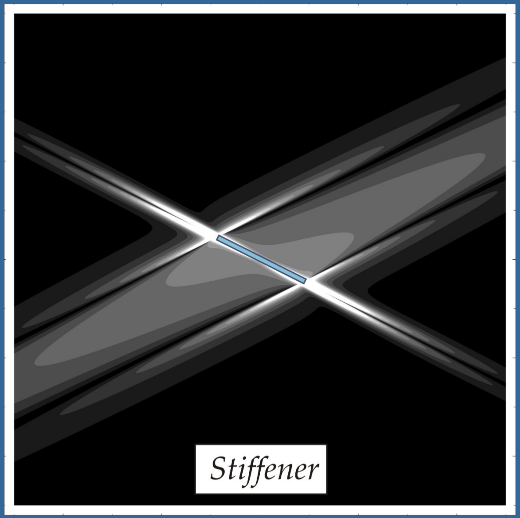
SHEAR BANDS ARE PREFERENTIAL FAILURE MODES!!

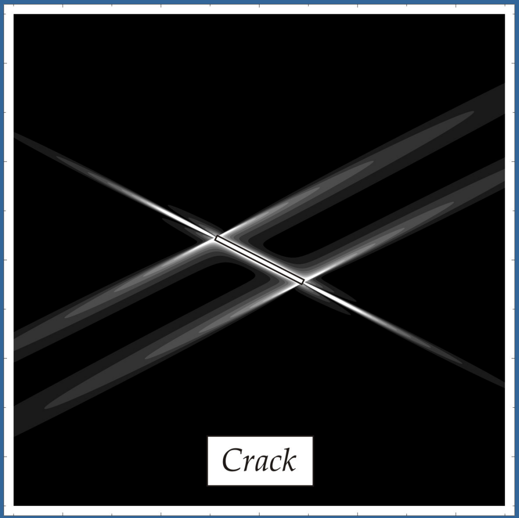
CONCLUSIONS

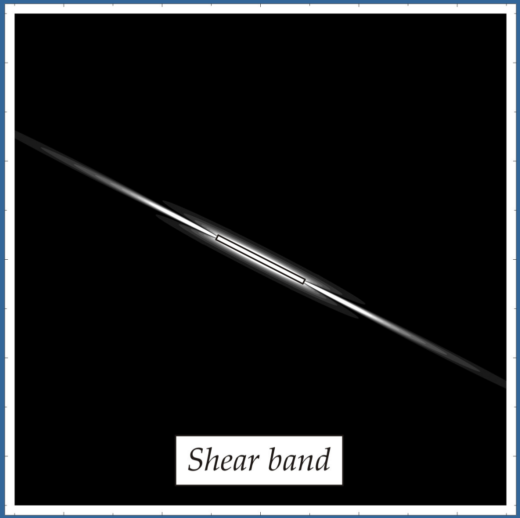
Analytical solutions for nonlinear elastic materials homogeneously prestressed containing dilute suspension of thin defects are obtained.

In particular, these solutions **show** that:

- The presence of defects strongly promotes shear band formation;
- Shear bands emerge from the defect tips;
- When the symmetry is lost, one shear band direction results to be privileged;
- Shear bands are preferential failure modes.







THANK YOU FOR THE ATTENTION

The main results presented in this thesis have been summarized in:

- F. Dal Corso, D. Bigoni and M. Gei, The stress concentration near a rigid line inclusion in a prestressed, elastic material. Part I Full-field solution and asymptotics.
Journal of the Mechanics and Physics of Solids, 2008, **56**, 815–838.

- D. Bigoni, F. Dal Corso and M. Gei, The stress concentration near a rigid line inclusion in a prestressed, elastic material. Part II Implications on shear band nucleation, growth and energy release rate.
Journal of the Mechanics and Physics of Solids, 2008, **56**, 839–857.

- D. Bigoni and F. Dal Corso, The unrestrainable growth of a shear band in a prestressed material.
Proceedings of the Royal Society A, 2008, **464**, 2365-2390.

- F. Dal Corso and D. Bigoni, Interactions between shear bands and rigid lamellar inclusions in a ductile metal matrix.
Proceedings of the Royal Society A 2009, **465**, 143-163.