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by

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Dedicated to Alex Hubert 14.5. 1938 – 16.2. 1999

Abstract

Ferromagnetic materials display a complex microstructure of domains, walls, Bloch lines and singular points on scales ranging from 100 μm down to a few nm. Understanding the formation and overall effects of these structures is crucial for key technological applications. At the same time the rich source of experimental data and the simple mathematical formulation makes the analysis of magnetic microstructure an excellent model problem to develop new mathematical tools for the understanding of multiscale problems, which are ubiquitous in science.

In this paper we describe some basic mathematical problems and report on recent analytical progress in three areas: rigorous scaling laws, branching and dimensionally reduced theories for thin films.

1 The energy functional

Ferromagnetic materials display a complex microstructure of magnetic domains, walls, Bloch lines and singular points ranging from 100 μm down to a few nm and beyond (see Fig. 1). Understanding the formation and the overall effects of these structures is crucial for a number of key technology applications. At the same time the rich source of experimental data and the simple mathematical formulation makes the analysis of magnetic microstructures an excellent model problem to develop new mathematical tools for the understanding of multiscale problems, which are ubiquitous in science.

Somewhat surprisingly the huge variety of magnetic structures can often be understood through minimisation of a simple energy functional, which only involves two material parameters. Let $\Omega \subset \mathbf{R}^3$ represent a magnetic body and let $m: \Omega \to \mathbf{R}^3$ denote its magnetisation (per unit volume). In non-dimensional variables the energy associated to such a magnetisation is given as [Br 63, HS 98]

$$E(m) = w^2 \int_\Omega |\nabla m|^2 + Q \int_\Omega \varphi(m) + \int_{\mathbf{R}^3} |\nabla u|^2 - 2 \int_\Omega h_{ext} \cdot m, \qquad (1.1)$$

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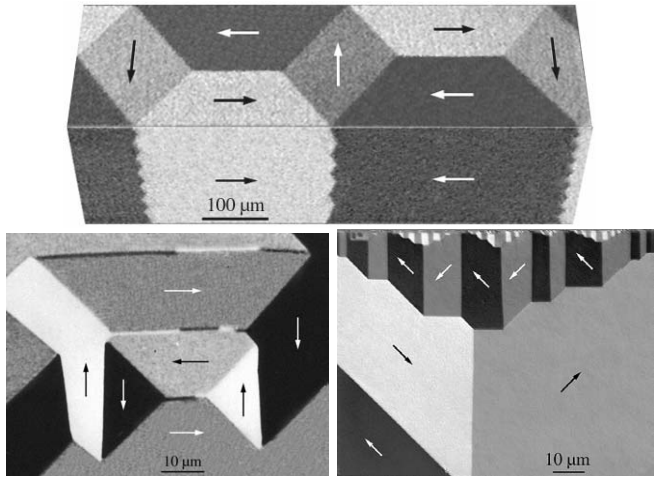

Figure 1: Top: Large domains on the surface of an iron whisker. Bottom left: Domains and domain walls (black and white lines) in a silicon-iron crystal - at much higher resolution vortex like patterns (Bloch lines) would become visible where the line switches from black to white. Bottom right: several generations of refinement near the boundary of an iron film; a related, but different, situation is discussed in Section 4 below. All pictures courtesy of R. Schäfer, IFW Dresden.

and the different terms are referred to as the exchange, anisotropy, magnetostatic and external field (or Zeeman) energy, respectively. The scalar variable u is the potential of the magnetic field h_{ind}, i.e. $h_{ind} = -\nabla u$, and is related to m by Maxwell's equations which read (in suitable units)

$$\operatorname{div}(-\nabla u + m) = 0 \quad \text{in } \mathbf{R}^3. \tag{1.2}$$

Here m is extended by zero outside Ω, and the equation is supposed to hold in the sense of distributions. Finally m satisfies the saturation condition

$$|m| = 1 \quad \text{in } \Omega. \tag{1.3}$$

To the naive mathematical eye the exchange energy $|\nabla m|^2$ is the highest order term which makes (1.1) – (1.3) a (nonlocal) lower order perturbation of the harmonic map problem. While this point of view is useful to understand some aspects (e.g. regularity of minimisers [Ca 97]) it does not provide much insight into the complexity of the observed magnetic microstructures. Indeed much of the microstructure formation is driven by the magnetostatic energy, and the exchange energy acts primarily as a limiting factor against infinite refinement. To get a better understanding of the energy functional it is useful to look at the different energy terms separately (see Fig. 2).

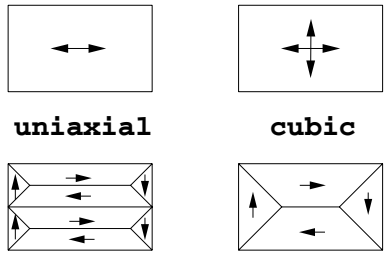

Figure 2: Competition of anisotropy and magnetostatic energy for a uniaxial material (left column) and a cubic material (right column). The preferred orientations are shown in the top row; the magnetostatic energy prefers alignment with the sides of the rectangle. For the uniaxial material this leads to the formation of small closure domains. Their size is limited by the exchange energy of the new horizontal phase boundaries.

- The anisotropy energy $\varphi(m)$ favours special directions of the magnetisation. Most materials have either uniaxial or cubic symmetry.
- The magnetostatic energy $|\nabla u|^2$ tries to eliminate the (distributional) divergence of m (see (1.2)). Written out separately for the interior and the boundary (with outer normal ν) of Ω this becomes

$$\operatorname{div} m = 0 \quad \text{in } \Omega, \qquad m \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

  This is known as the principle of 'pole avoidance'. It favours a magnetisation which is parallel to the boundary and in particular strongly disfavours uniform magnetisation of the sample. For uniaxial materials there is thus a competition between pole avoidance and low anisotropy energy, which drives the formation of fine scale structure (see Fig. 2).
- The exchange energy $|\nabla m|^2$ favours uniform or at least slowly varying magnetisation. It
  - sets a finest length scale (of order w) and influences the hierarchy of coarser scales
  - defines a wall energy (see Section 3 below) and
  - determines the inner structure of the walls.

The relative importance of the anisotropy energy and the magnetostatic energy is measured by the quality parameter Q, which varies over five orders of magnitude between different materials (see Table 1). Materials with a low values of Q are called soft, because the magnetisation is easy to rotate.

	Q	w
Permalloy	.00025	5 nm
α-Iron	.025	3.7 nm
$SmCo_5$	38	5.2 nm

Table 1: Material parameters for typical soft and hard materials

2 Basic mathematical questions

Over the last thirty years (and in some cases longer) the physics, engineering and materials science communities have gained an enormous amount of insight and intuition about magnetic microstructures through theory combined with large scale experimental and computational efforts. The recent book by Hubert and Schäfer [HS 98] gives a beautiful survey. Noteworthy achievements include

a) Constructions of plausible energy minimisers (usually with a few free parameters which are optimised analytically or numerically)

b) Large scale finite-element or boundary-element computations which give solutions to specific problems and allow one to compute 'phase diagrams' that show the switching between different prototype constructions (e.g. different types of walls) for different material parameters and specimen geometry.

c) Simplified theories, usually derived in a somewhat ad-hoc way by astutely neglecting (or greatly simplifying) certain contributions to the energy in special regimes. A typical example is phase theory, which applies to large bodies and corresponds to the limit $w = 0$.

From a mathematical point of view a) corresponds to *upper* bounds for the magnetic energy through subtle choices of test functions. Optimising over a certain number of constructions does not of course rule out that a radically different choice might yield a much lower energy. Regarding b) there have been remarkable successes but the nonlocal and nonconvex character of the energy in combination with the multitude of lengthscales involved still pose significant challenges. The reliable computation of a rather small piece of Permalloy (20nm x 1 μm x 2 μm) has been proposed as a benchmark problem and is near the limit of current capabilities [NIST, RZH 99]. For a rather critical view of numerical computation see also [Ah 99]. Regarding c) it would be desirable to have a framework in which simplified theories can be derived in a rigorous way (for phase theory this was done independently in [De 93] and [Ta 95]).

We thus see the following challenges for the mathematical analysis:

- Understanding the separation into multiple spatial scales.

  More precisely, one would like to know the typical lengthscales exhibited by the minimiser and how these depend on the material parameters

and the sample dimensions. Examples are the width and type of domain walls and the typical domain size in the basal plane of a strongly uniaxial ferromagnet (see e.g. [HS 98], pp. 401–406, and Section 4 below). This question can be partially answered by investigating how the minimal energy and its components (anisotropy, exchange and magnetostatic energy) depend on the material parameters and the sample dimensions. Of course, one expects different scalings in different regimes. The goal is thus to identify these scalings and their cross–overs.

Mathematically, this amounts to establishing *upper and lower* bounds for the minimal energy which exhibit the same scaling in the material parameters and sample dimensions.

- Making use of the separation of spatial scales to derive reduced theories with fewer lengthscales.

  Generally speaking, whenever there is a clear separation of lengthscales, there is hope that the original problem decomposes into a hierarchy of simpler problems. As a specific example consider the elimination of the domain wall length scale. The wall structure, given the state left and right of the wall, is determined by a microscopic variational problem, which defines a wall energy density. The expression for the wall energy density is an ingredient to a more macroscopic variational problem, which determines the location of the walls. One example of a regime where the derivation of such a hierarchy of simpler models is both realistic and desirable is that of soft thin films (see Section 5).

  Mathematically, the derivation of reduced theories amounts to identifying a suitable Γ–limit of the original variational problem. A first step is to establish sufficient compactness for the minimisers in the limit.

3 Scaling laws for domain walls

The competition of anisotropy, exchange and magnetostatic energy often leads to a separation of the specimen Ω into regions of slowly varying magnetisation (Weiss domains) which are separated by thin layers where the magnetisation changes rapidly (domain walls). Understanding the inner structure of such domain walls is one of the major research topics in micromagnetics (see [HS 98] and the references therein) and here we can only give a very brief introduction.

The simplest wall is the so called Bloch wall (which was actually first described by Landau and Lifshitz). Here the magnetisation only varies in one direction n and performs a 180 degree rotation in a plane perpendicular to n. Thus in an infinite body $\operatorname{div} m = 0$ so that the magnetostatic energy vanishes. Optimisation of the exchange and anisotropy energy yields a wall energy (per surface area) of $4w\sqrt{Q}$ and a wall thickness of order $w/\sqrt{Q}$ (with exponential decay beyond that scale).

In a thin film a Bloch wall with far field magnetisation in the film plane (chosen as the x_1, x_2 plane in the following) involves a large out-of-plane component

m_3 of the magnetisation. This gives a strong contribution to the distributional divergence of m at the top and bottom surface ('surface charges'), which is energetically very unfavourable in thin films (for a detailed mathematical discussion see e.g. [GJ 97], see also Table 2 below).

Hence Néel suggested already in the 40's a different construction, where the magnetisation rotates in the film plane. This completely avoids surface charges at the expense of a certain amount of in-plane divergence and leads to a much lower energy than the Bloch wall. In thin films the exchange energy also puts a high penalty on variations of m in the normal direction. For the purpose of the following discussion we will therefore consider a constrained theory in which we *assume* that

$$m = m(x_1, x_2), \quad m_3 = 0. \tag{3.4}$$

We make some remarks about the validity of these assumptions at the end of this section. Using (3.4) the magnetostatic energy can be computed using Fourier transform in the tangential variables, and if we assume that the typical length scale on which m varies in tangential directions is large compared to the film thickness the normalised energy of a film with basis $\omega \subset \mathbf{R}^2$ and thickness D becomes

$$I(m) = D^{-1}E(m) = w^2 \int_\omega |\nabla m|^2 + Q \int_\omega \varphi(m) + \frac{D}{2} ||\operatorname{div} m||^2_{H^{-1/2}}, \tag{3.5}$$

where the homogeneous $H^{-1/2}$ norm of a function f can be expressed in terms of its Fourier transform $\hat{f}(k) = \int e^{-2\pi i k\cdot y}\, dy$ as

$$||f||^2_{H^{-1/2}} = \int \frac{|\hat{f}|^2(k)}{2\pi|k|}\, dk$$

To study the energy of a Néel wall we consider a uniaxial material, i.e. $\varphi(m) = m_1^2$, and a strip $\omega = (-L, L) \times (0, 1)$ with boundary conditions

$$m(\pm L, x_2) = (0, \pm 1) \tag{3.6}$$

and periodic boundary conditions in x_2. A one dimensional ansatz yields an energy of order

$$\frac{D}{\ln \min(1/\tilde{Q}, L/w)}, \quad \text{with } \tilde{Q} = 4Q\frac{w^2}{D^2},$$

provided that $\tilde{Q} \ll 1$ and $w \ll L$. We ask whether this is essentially optimal or whether a genuinely two-dimensional wall (such as the commonly observed cross-tie walls [HS 98], p. 240, pp. 448–456) can give a qualitatively lower energy. For $Q = 0$ (i.e. an ideally soft material) we have the following result.

Theorem 3.1 (Upper and lower bounds for 2d Néel walls [DKMO 99b]) *There exist universal constants $0 < c_0 \le C_0 < \infty$ such that if $Q = 0$ and $w < L/2$ then*

$$c_0 \frac{D}{\ln L/w} \le \min_{(1.3),(3.6)} I(m) \le C_0 \frac{D}{\ln L/w}.$$

For one-dimensional walls much more precise results are available. We consider an infinite strip $\mathbf{R} \times (0,1)$ and to enforce the presence of a wall we add the constraint $m_1(0) = 1$.

Theorem 3.2 (Finer asymptotics for 1d Néel walls [EM 99]) *For $\tilde{Q} \ll 1$ we have*

$$\min_{\substack{m_1(0)=1,(1.3)\\ m=m(x_1)}} I = \frac{D\pi}{2}\frac{1}{\ln 1/\tilde{Q}} + \mathcal{O}\left(\frac{(\ln\ln(1/\tilde{Q}))^2}{(\ln 1/\tilde{Q})^2}\right).$$

Slightly weaker estimates were obtained in Cervera's thesis [Ce 99]. Work in progress addresses the precise behaviour of the minimisers. Subtle formal asymptotics suggest that the minimisers have a logarithmic decay on a scale $D/\tilde{Q}$.

We finally briefly comment on the hypotheses $m_3 = 0$, $m = m(x_1, x_2)$. They are only justified for thin films, i.e. for films in which the ratio $\kappa = w/D$ is sufficiently large. Striking work of La Bonte and Hubert in the late 60's showed that for $\kappa \approx .05$ a new wall type with a genuinely three-dimensional structure has lower energy than the standard Néel wall and the cross-tie wall discussed above. Since then a number of other wall types have been discovered, see [HS 98], pp. 241–255, for a review and a numerically computed 'phase diagram' in the $D/w, Q$ plane (Figs. 3.79-3.81, pp. 251–253, *loc. cit.*). A rigorous analysis of the bifurcation pattern between different wall types is a major open problem.

4 Domain branching

Strongly uniaxial ferromagnets often show a complex domain pattern on a surface perpendicular to the easy axis (known as a basal plane), see Fig. 3 and [HS 98], pp. 330–336 (theory) and pp. 401–406 (experiment). The magnetic domains inside the ferromagnet are roughly parallel to the easy axis and refine towards the boundary. The possibility of such a refinement was first suggested by Lifshitz [Li 44], and in the 60's Hubert used a simplified one-dimensional energy to establish a 1/3 power law for the average domain spacing as a function of the distance from the basal plane, see [HS 98] *loc. cit.* for further discussion. Choksi et al. have recently studied the full three-dimensional energy

$$\tilde{E}(m) = \varepsilon \int_\Omega |\nabla m| + Q \int_\Omega m_2^2 + m_3^2 + \int_{\mathbf{R}^3} |\nabla u|^2 \tag{4.7}$$

in a domain $\Omega = (-L, L) \times (0,1)^2$. Here the exchange energy has been replaced by a wall energy $\varepsilon \int |\nabla m|$, where the integrand is understood as a Radon measure. This formulation allows for jumps of m across a surface and assigns them an energy proportional to the jump height and the surface area. In this way the energy of the walls is included without resolving the detailed wall structure.

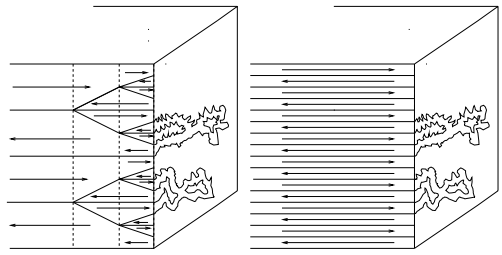

Figure 3: Branched and unbranched configurations; Theorem 4.1 shows that for small wall energy the unbranched configuration is always far from minimising, no matter how complex the structure in the basal plane is.

For the standard 180 degree Bloch wall one has $\varepsilon = 2w\sqrt{Q}$. As a further simplification we consider periodic boundary conditions in the x_2 and x_3 variables. We let

$$\gamma = \min(Q, 1), \quad E_0 = \min_{|m|=1} \tilde{E}(m)$$

and we define the minimal energy of unbranched configurations by

$$E_1 = \min\{\tilde{E}(m) : |m| = 1, m(x) = m(x_2, x_3)\}.$$

Theorem 4.1 (Domain branching [CKO 99]) *If $\varepsilon < \gamma \min(L, L^{-2})$ then*

$$c_0 \gamma^{1/3} \varepsilon^{2/3} L^{1/3} \leq E_0 \leq C_0 \gamma^{1/3} \varepsilon^{2/3} L^{1/3}, \tag{4.8}$$

$$c_0 \gamma^{1/2} \varepsilon^{1/2} L^{1/2} \leq E_1 \leq C_0 \gamma^{1/2} \varepsilon^{1/2} L^{1/2}. \tag{4.9}$$

In particular the energy of all unbranched configurations is much larger than E_0 if $\varepsilon \ll \gamma L$.

The upper bound in (4.8) is achieved by a branched configuration with average domains size proportional to $d^{2/3}$, where d is the distance from the basal plane. The key point is the *lower* bound which assures that no other configuration gives a better scaling law for the energy. Its proof involves an interesting new interpolation inequality between BV, L^∞ and H^{-1} [CKO 99], Lemma 2.3.

An open mathematical question is whether the result can be extended to the full micromagnetic energy E instead of $\tilde{E}$ (with the correspondence $\varepsilon = 2w\sqrt{Q}$) and whether in a suitable asymptotic sense minimisation of E and $\tilde{E}$ are equivalent. In a much simpler one dimensional setting such a result was established in [AM 99] using the new concept of Young measures on micropatterns which allows one to take advantage of an asymptotic separation of scales.

5 Thin film limits

We wish to derive a limiting two-dimensional theory for thin films. We will consider a material with low anisotropy and include a weak applied field, lying in the film plane. Again we will focus on scaling laws and limits of the energy.

Physically, there are a number of small parameters: the film thickness D, its typical lateral dimension L, the Bloch line width w and the quality factor Q, which measures the strength of the anisotropy. We will focus on the limit of small aspect ratio $d = D/L$ and indicate below the conditions which are required on the other parameters. We suppose in particular that the applied field is proportional to the aspect ratio, $h_{ext} = dH_{ext}$ (this reflects the fact that the in-plane demagnetising factor is also approximately proportional to d for small d). We scale length by $1/L$ and energy by $1/L^3$ and we use the notation

$$\kappa = \frac{w}{D}, \quad x' = (x_1, x_2), \quad \operatorname{div}' m' = \partial_1 m_1 + \partial_2 m_2.$$

In the scaled variables the film domain is denoted by

$$\Omega_d = \omega \times (0, d), \quad \omega \subset \mathbf{R}^2.$$

Then the three dimensional problem and the two-dimensional limiting problem can be formulated as follows (as usual magnetisations are extended by zero outside their domain of definition for the computation of the distributional divergence).

Three-dimensional problem:
Admissible magnetisations

$$\mathcal{A}_d = \left\{ m \in L^2(\Omega_d; \mathbf{R}^3) : |m| = 1 \right\}. \tag{5.10}$$

Energy

$$\begin{aligned} E_d(m) &= \kappa^2 d^2 \int_{\Omega_d} |\nabla m|^2 + Q \int_{\Omega_d} \varphi(m) + \int_{\mathbf{R}^3} |\nabla u|^2 \\ &- 2d \int_{\Omega_d} H_{ext} \cdot m \qquad (H_{ext})_3 = 0, \end{aligned} \tag{5.11}$$

$$\Delta u = \operatorname{div} m. \tag{5.12}$$

Two-dimensional problem:
Admissible magnetisations

$$\mathcal{A}_0 = \left\{ m \in L^2(\omega; \mathbf{R}^3) : m_3 = 0, |m| \leq 1 \right\}. \tag{5.13}$$

Energy

$$E_0(m) = \int_{\mathbf{R}^3} |\nabla u|^2 - 2 \int_{\omega} H_{ext} \cdot m', \tag{5.14}$$

$$\Delta u = \operatorname{div}' m' \, \mathcal{H}^2 \llcorner \{x_3 = 0\}. \tag{5.15}$$

Order of E_d	relevant term
$d = \mathrm{vol}(\Omega)$	$\int_\Omega m_3^2$, [GJ 97]
$d^2 \ln \frac{1}{d}$	$[m' \cdot \nu'] = 0$ on domain walls and on $\partial\omega$
d^2	h_{ind}, H_{ext}, $\lvert m\rvert \le 1$
??	$\lvert m\rvert = 1$
$d^2 / \ln \frac{1}{d}$?	wall energy
$d^3 \ln \frac{1}{d}$	vortex energy (large literature)

Table 2: Conjectured and proven energy scalings in thin films in the regime $\kappa = \mathcal{O}(1)$, $Q \ll d \ll 1$

The last equation may be equivalently written as

$$\Delta u = 0 \quad \text{for } x_3 \neq 0, \qquad \left[\frac{\partial u}{\partial x_3}\right] = \mathrm{div}'\, m' \quad \text{on } x_3 = 0,$$

where $[f](x') = \lim_{\varepsilon \to 0} f(x', \varepsilon) - f(x', -\varepsilon)$ denotes the jump across the plane $\{x_3 = 0\}$.

Note that the limit functional no longer involves ∇m and hence m can have jump discontinuities across an interface (these correspond to domain walls). The normal component of m, however, cannot jump across a smooth interface as long as E_0 is finite. If the normal component jumps then $\mathrm{div}\, m$ contains a line measure and the first term in (5.14) is infinite. Similarly we must have $m' \cdot \nu' = 0$ on $\partial\omega$ (in the sense of $H^{-1/2}(\partial\omega)$). A more detailed analysis shows that jumps in the normal component of m lead to an energy contribution of order $d^2 \ln(1/d)$ while the relevant competition between the external field and the induced field occurs at order d^2 (see Table 2).

To formulate our convergence result it is convenient to rescale the thickness variable and to define all functions on the fixed domain $\Omega_1 = \omega \times (0, 1)$. Let $m_d(x', x_3) = m(x', x_3/d)$, $\bar{m}(x') = \int_0^1 m(x', x_3) dx_3$ and define

$$\tilde{E}_d(m) = \begin{cases} E_d(m_d) & \text{if } m_d \in \mathcal{A}_d, \\ +\infty & \text{else} \end{cases}$$

$$\tilde{E}_0(m) = \begin{cases} E_0(\bar{m}) & \text{if } m(x', x_3) = \bar{m}(x') \text{ and } \bar{m} \in \mathcal{A}_0, \\ +\infty & \text{else} \end{cases}$$

We assert that minimisers of $\tilde{E}_d$ converge to minimisers of $\tilde{E}_0$. More generally we establish $\Gamma-$convergence.

Theorem 5.1 ([DKMO 99b]) *Let ω be open, simply connected and bounded with smooth boundary. Suppose that*

$$\frac{Q}{d} \to 0, \quad \kappa^2(d\ln(1/d)) \to 0, \quad \frac{\kappa^2}{d} \to \infty, \quad \text{as } d \to 0.$$

Then $d^{-2}\tilde{E}_d$ Γ-converges to $\tilde{E}_0$ with respect to the weak topology of $L^2(\Omega_1, \mathbf{R}^3)$.

Remark. As regards the assumptions on κ and d, a typical data set for a Permalloy film is $Q = .00025$, $D = 20nm$, $L = 1\mu m$, $w = 5nm$, whence $d = .02$, $\kappa = .25$. More generally one often has $Q \ll d \ll 1$, while $\kappa = \mathcal{O}(1)$, which is compatible with the hypotheses above.

Note that the three-dimensional problem and the limit problem involve different constraints on m. The limit problem has the additional constraints $m_3 = 0$ and $m(x', x_3) = \bar{m}(x')$ (hence we have established (3.4) rigorously at the current scaling of the energy). On the other hand the difficult nonconvex constraint $|m|^2 = 1$ has been replaced by the convex (even quadratic) constraint $|m|^2 \leq 1$. The limit problem is thus much easier to tackle both analytically and numerically. In particular all minimisers are solutions of the Euler-Lagrange equation

$$h_{ind} + h_{ext} = \lambda m, \tag{5.16}$$

$$\lambda(|m| - 1) = 0, \tag{5.17}$$

where $\lambda(x)$ are Lagrange multipliers and where $-h_{ind}$ is the tangential gradient $\nabla' u$ of the solution of (5.15) restricted to the plane $x_3 = 0$. From (5.16) and (5.17) one can immediately read off the qualitative behaviour of the solutions in dependence of the applied field $H_{ext} = He$, where e is a unit vector:

$$H = 0 \qquad \operatorname{div}' m' = 0 \qquad \text{'pole avoidance'}$$

$$0 < H < H_{crit}(e) \qquad h_{ext} + h_{ind} = 0 \qquad \text{'field expulsion'}$$

$$H > H_{crit}(e) \qquad \lambda > 0 \text{ somewhere} \qquad \text{'field penetration'}$$

Following work of van den Berg [vdB 86] for $H = 0$, Bryant and Suhl [BS 89] studied previously the second regime using physical reasoning through an electrostatic analogy. Their arguments, however, do not cover the regime of field expulsion (except in in very special cases), which arises naturally in our approach via energy minimisation.

The limit functional has, however, one deficiency. While it determines $\operatorname{div}' m'$ and hence h_{ind} uniquely, it usually does not determine m uniquely. This can already be seen in the simplest case $h_{ext} = 0$ and $\omega =$ unit disc. In this case the only conditions on m' are that it be divergence free and of length less than one. One possible solution is the vortex field $m'(x) = (-x_2, x_1)/|x'|$ another is $m' = 0$. As we shall see in the next section a strong degree of nonuniqueness persists even if we reimpose the constraint $|m'| = 1$.

The reason for this is that E_0 only captures the leading order term in an asymptotic expansion of the full energy functional (see Table 2). It misses in particular the wall energy. Including these higher order terms one may be able to recover the constraint $|m| = 1$ and derive a further selection criterion based on a limiting wall energy. Underpinning these ideas by a rigorous analysis is a very challenging problem, see the following two sections for a more detailed discussion.

On the other hand nonuniqueness of minimisers of E_0 provides an interesting insight. The quantities $\operatorname{div}' m'$ (and hence h_{ind}) are robust in the sense that they are not much affected by a slight perturbation of the original energy. The full field m' by contrast is not robust and can be easily changed by lower order perturbations of the energy. Experimentally one often observes a ripple-like fine structure of the magnetisation field. It is an interesting question whether this phenomenon may be related to a lack of robustness of m'.

Empirically, we found that the following simple procedure yields a robust and efficient numerical scheme, whose results agree well with experiment (and even predicted new experimental results) [DKMO 99b]. We use the notation $\nabla^\perp u = (-\partial_2 u, \partial_1 u)$. Clearly $\operatorname{div}' \nabla^\perp u = 0$.

- Step 1: Compute a minimiser m_0 of the convex energy E_0.
- Step 2: Determine $m = m_0 + \nabla^\perp u$ by computing the viscosity solution u of
$$|m_0 + \nabla^\perp u| = 1.$$
This can be done very efficiently by a fast marching method [Se 99].

In this way the original difficult nonconvex problem is split into two subproblems which can be solved efficiently and reliably. An interesting open problem is whether the selection of the viscosity solution agrees with 'energy-based' selection criteria (see Conjecture 6.2).

6 The van den Berg construction

Let us consider in more detail a thin film without external field, $H_{ext} = 0$. Van den Berg [vdB 86] argued on physical grounds that as $d \to 0$ the limiting magnetisation m should be independent of x_3 and should be determined by solving the following problem:

$$m_3 = 0, \qquad |m| = 1 \quad \text{in } \omega \tag{6.18}$$

$$\operatorname{div}' m' = 0 \quad \text{in } \omega, \qquad m' \cdot \nu' = 0 \quad \text{on } \partial\omega. \tag{6.19}$$

The latter condition is equivalent to the existence of a stream function v, with $m' = \nabla^\perp v$. One thus obtains the eikonal equation

$$|\nabla v| = 1 \quad \text{in } \omega, \qquad v = \text{const} \quad \text{on } \partial\omega. \tag{6.20}$$

Van den Berg proposes to solve this equation by the method of characteristics and graphically constructs explicit solutions for a variety of film geometries. The characteristics typically intersect and thus ∇v exhibits jump discontinuities on a singular set. This singular set, which is typically one-dimensional, corresponds to domain walls (whose thickness has been shrunk to zero in taking the limit $d \to 0$). This construction does not give unique solutions; indeed there is a large flexibility in introducing additional singularities. Van den Berg argues that by including a suitable wall energy, which penalises the jumps on the singular set, one should be able to select a physically preferred solution.

Leaving aside the details of the geometric construction by characteristics one can easily understand non-uniqueness as follows. A typical solution of (6.20) is given by the distance function $v(x) = \mathrm{dist}(x, \mathbf{R}^2 \setminus \omega)$, and for any compact subset $\gamma \subset \omega$ of measure zero one obtains another solution by $v_\gamma = \mathrm{dist}(x, \mathbf{R}^2 \setminus (\omega \setminus \gamma))$.

Comparing van den Berg's problem (6.18), (6.19) with minimisation of our limit functional E_0, we see that the two agree except for the fact that we only require $|m'| \leq 1$ while van den Berg requires equality. Mathematically his approach may thus be summarised in the following two conjectures.

Conjecture 6.1 (Compactness) *A subsequence of minimisers m_d of $\tilde{E}_d$ converges strongly in L^2 and the limit m satisfies $|m| = 1$.*

Remark. By strict convexity of the L^2 norm weak convergence and the condition $|m| = 1$ together are equivalent to strong convergence.

Conjecture 6.2 (Selection) *A higher order expansion of the energy selects those solutions m of the eikonal equation which minimise a suitable wall energy computed along the singular set of m.*

Both conjectures are wide open. Compactness has recently been established for a related problem, which we now describe.

7 Compactness

Consider a function $v : \omega \subset \mathbf{R}^2 \to \mathbf{R}$. In connection with work on liquid crystals Aviles and Giga [AG 87] have studied the functional

$$E_\varepsilon(v) = \varepsilon \int_\omega |\nabla^2 v|^2 \, dx + \frac{1}{\varepsilon} \int_\omega (1 - |\nabla v|^2)^2 \, dx \tag{7.21}$$

for $\varepsilon \ll 1$. Together with the inequality $a + b \geq 2\sqrt{ab}$ the following result ensures that sequences with bounded energy are relatively compact as $\varepsilon \to 0$.

Theorem 7.1 (Compactness for the AG functional [ADM 99, DKMO 99a]) *Let $\omega \subset \mathbf{R}^2$ be open and suppose that*

$$||1 - |\nabla v_k|^2||_{L^2} \to 0,$$

$$||\nabla^2 v_k||_{L^2} \, ||1 - |\nabla v_k|^2||_{L^2} \quad \textit{is bounded.}$$

Then the sequence

$$\{\nabla v_k\} \quad \textit{is relatively compact in } L^2(\omega).$$

Our proof uses the method of compensated compactness [Ta 79] and a suitable choice of 'entropies', i.e. nonlinear expressions $\Phi(m)$ whose divergence can be controlled under the hypothesis of the theorem.

A major unsolved problem is to determine the $\Gamma-$limit of E_ε. The conjectured form is [AG 87, OG 94]

$$E_0 = \begin{cases} \int_S \frac{1}{3}\,[\nabla v]^3 \; d\mathcal{H}^1 & \text{if } |\nabla v| = 1 \text{ a.e.} \\ & \text{and } \mathcal{H}^1(S) < \infty, \\ +\infty & \text{else,} \end{cases}$$

where S is the discontinuity set of ∇v.

From the work of Jin and Kohn [JK 99] we know, roughly speaking, that this form is correct if v only jumps across a single flat interface. The general case is completely open. In fact it is not even known whether the $\Gamma-$limit is a local functional of v. A necessary condition for E_0 to be a $\Gamma-$limit is that it be lower semicontinuous. Aviles and Giga showed that this is indeed the case under the additional restriction $\nabla v \in \mathrm{BV}$ and that the exponent 3 plays a particular role: lower semicontinuity fails for any exponent $\beta > 3$ (see [AG 99] for the details).

Let us briefly comment on the similarities and differences between the AG functional and the micromagnetic energy. To simplify the discussion assume from the beginning that $Q = 0$ and $m(x', x_3) = m(x')$. If we assume, as in Section 3, that m varies on a length scale large compared to the film thickness ('wide walls'), we can approximate the magnetostatic energy and we obtain in dimensionless variables

$$E_d(m) = \kappa^2 d^3 \int_\omega |\nabla m|^2 \, dx + d \int_\omega m_3^2 \, dx + \frac{d^2}{2} ||\mathrm{div}'\, m'||^2_{H^{-1/2}}. \tag{7.22}$$

The usual Bloch wall construction yields a test function with $\mathrm{div}'\, m' = 0$ and shows that the minimum of the normalised energy $E_d/(\kappa d^2)$ is uniformly bounded. Setting $\varepsilon = \kappa d$ we obtain

$$\frac{E_d(m)}{\kappa d^2} = \varepsilon \int_\omega |\nabla m|^2 \, dx + \frac{1}{\varepsilon} \int_\omega m_3^2 \, dx + \frac{1}{2\kappa} ||\mathrm{div}'\, m'||^2_{H^{-1/2}}. \tag{7.23}$$

Hence for $\kappa \ll 1$ the expression $\mathrm{div}'\, m'$ must be small so that $m' \sim \nabla^\perp v$. Moreover $m_3^2 = 1 - |m'|^2$. Thus the compactness result stated above 'almost' gives compactness of sequences for which the normalised energy (7.23) is bounded. There are two difficulties, however. First, for $\kappa \ll 1$ the walls are actually narrow compared to the film thickness, so the correct approximation for the magnetostatic energy is $d||\mathrm{div}'\, m'||^2_{H^{-1}} + ||m_3||^2_{H^{-1/2}}$. Secondly, and more seriously, the assumption $m(x', x_3) = m(x')$ is no longer justified when $\kappa < .05$, see the discussion at the end of Section 3. Hence compactness for the full micromagnetic energy without additional constraints remains a major challenge.

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