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Quasistatic Evolution Problems for Linearly Elastic–Perfectly Plastic Materials

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Abstract

The problem of quasistatic evolution in small strain associative elastoplasticity is studied in the framework of the variational theory for rate-independent processes. Existence of solutions is proved through the use of incremental variational problems in spaces of functions with bounded deformation. This approach provides a new approximation result for the solutions of the quasistatic evolution problem, which are shown to be absolutely continuous in time. Four equivalent formulations of the problem in rate form are derived. A strong formulation of the flow rule is obtained by introducing a precise definition of the stress on the singular set of the plastic strain.

1. Introduction

In this paper we study quasistatic evolution problems in small strain associative elastoplasticity. More precisely, we consider the case of a material that displays linear and isotropic elastic behavior, and whose plastic response is governed by the Prandtl-Reuss flow rule, without hardening (perfect plasticity). Perfect plasticity is a classical model in mechanics, which has played a crucial role in the understanding of irreversibility and nonlinearity associated with the emergence of plastic deformations. Its relevance is that of a conceptual tool. Quantitatively accurate predictions of the response of any given material typically require the resolution of other nonlinear phenomena that occur between the elastic and the plastic regime.

Usually, the problem is formulated as follows (in a domain $\Omega \subset \mathbb{R}^n$). The linearized strain Eu, defined as the symmetric part of the spatial gradient of the displacement u, is decomposed as the sum $Eu = e + p$, where e and p are the elastic and plastic strains. The stress σ is determined only by e, through the formula $\sigma = \mathbb{C}e$, where $\mathbb{C}$ is the elasticity tensor. It is constrained to lie in a prescribed subset $\mathbb{K}$, of the space $\mathbb{M}^{n\times n}_{sym}$ of $n\times n$ symmetric matrices, whose boundary $\partial\mathbb{K}$ is referred to as the yield surface.

Given a time dependent body force $f(t,x)$, the classical formulation of the quasistatic evolution problem in a time interval $[0,T]$ requires finding functions $u(t,x)$, $e(t,x)$, $p(t,x)$, $\sigma(t,x)$ that satisfy the following conditions for every $t \in [0,T]$ and every $x \in \Omega$:

(cf1) additive decomposition: $Eu(t,x) = e(t,x) + p(t,x)$,
(cf2) constitutive equation: $\sigma(t,x) = \mathbb{C}e(t,x)$,
(cf3) equilibrium: $-\mathrm{div}\,\sigma(t,x) = f(t,x)$,
(cf4) stress constraint: $\sigma(t,x) \in \mathbb{K}$,
(cf5) associative flow rule: $(\xi - \sigma(t,x)){:}\dot{p}(t,x) \leqq 0$ for every $\xi \in \mathbb{K}$,

where the colon denotes the scalar product between matrices. Condition (cf5) is also referred to as the maximum plastic work inequality (see [7]). The problem is supplemented by initial conditions at time $t = 0$ and by boundary conditions for $t \in [0,T]$, $x \in \partial\Omega$, of the form $u(t,x) = w(t,x)$ on a portion Γ_0 of the boundary, and $\sigma(t,x)\nu(x) = g(t,x)$ on the complementary portion Γ_1, where $\nu(x)$ is the outer unit normal to $\partial\Omega$, $w(t,x)$ is the prescribed displacement on Γ_0, and $g(t,x)$ is the prescribed surface force on Γ_1.

For clarity, we focus our discussion on the case where $\mathbb{K}$ is a cylinder of the form $\mathbb{K} = K + \mathbb{R}I$, where I is the identity matrix and K is a convex compact neighborhood of 0 in $\mathbb{M}^{n\times n}_D$, the space of trace free $n\times n$ symmetric matrices. This example results in yield criteria (often used for metals), which are insensitive to pressure, such as the ones of Tresca and von Mises (see, e.g. [14]). Thus condition (cf5) implies that $\dot{p}(t,x) \in \mathbb{M}^{n\times n}_D$ and it is not restrictive to assume that $p(t,x) \in \mathbb{M}^{n\times n}_D$.

Introducing the normal cone $N_K(\xi)$ to K at ξ, the support function

$$H(\xi) := \sup_{\zeta\in K} \xi{:}\zeta\,,$$

and the subdifferential $\partial H(\xi)$ of H at ξ, the flow rule (cf5) can be written in the equivalent forms (see, e.g. [7] & [10] (Chapter 4)):

(cf5′) normality: $\dot{p}(t,x) \in N_K(\sigma_D(t,x))$,
(cf5″) dissipation pseudo-potential formulation: $\sigma_D(t,x) \in \partial H(\dot{p}(t,x))$,
(cf5‴) maximal dissipation: $H(\dot{p}(t,x)) = \sigma_D(t,x){:}\dot{p}(t,x)$,

where $\sigma_D(t,x)$ denotes the deviator of $\sigma(t,x)$ (see Section 2.1).

In the engineering literature, quasistatic evolution problems of the type considered above are approximated numerically by solving a finite number of incremental variational problems (see [16, 24], and more recently [5, 18, 25]). The time interval $[0,T]$ is divided into k subintervals by means of points

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = T\,,$$

and the approximate solution u_k^i, e_k^i, p_k^i at time t_k^i is defined, inductively, as a minimizer of the incremental problem

$$\min_{(u,e,p)\in A(w(t_k^i))} \{\mathcal{Q}(e) + \mathcal{H}(p - p_k^{i-1}) - \langle \mathcal{L}(t_i^k)|u\rangle\}, \tag{1.1}$$

where

$$\mathcal{Q}(e) := \tfrac{1}{2}\int_{\Omega} \mathbb{C}e(x){:}e(x)\,dx\,,$$
$$\mathcal{H}(p) := \int_{\Omega} H(p(x))\,dx\,,$$
$$\langle \mathcal{L}(t)|u\rangle := \int_{\Omega} f(t,x)\,u(x)\,dx + \int_{\Gamma_1} g(t,x)\,u(x)\,d\mathcal{H}^{n-1}(x)\,, \tag{1.2}$$

$\mathcal{H}^{n-1}$ is the $(n-1)$ dimensional Hausdorff measure, and $A(w(t))$ is defined, at this stage of the discussion, as the set of triples (u, e, p), with $Eu(x) = e(x) + p(x)$ for every $x \in \Omega$, such that u satisfies the prescribed Dirichlet boundary condition at time t, i.e., $u(x) = w(t, x)$ for every $x \in \Gamma_0$. Finally, the stress at time t_k^i is obtained as $\sigma_k^i(x) := \mathbb{C}e_k^i(x)$.

Since $\mathcal{H}$ has linear growth, problem (1.1) has, in general, no solution in Sobolev spaces. This is very natural from the point of view of mechanics, owing to the phenomenon of strain localization. In the absence of hardening, solutions can develop shear bands, where shear deformation concentrates. Seen from a macroscopic perspective, shear bands can be thought of as sharp discontinuities of the displacement (slip surfaces). They cannot be resolved by Sobolev functions, but they find a natural mathematical representation if plastic deformations are allowed to take values in spaces of measures (see [28]).

These remarks lead naturally to a weak formulation of the problem, where the displacement u belongs to the space $BD(\Omega)$ of functions with bounded deformation, whose theory was developed in [17, 30, 13, 29], and the plastic strain p belongs to the space $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ of $\mathbb{M}^{n\times n}_D$- valued bounded Borel measures on $\Omega \cup \Gamma_0$.

In accordance to the theory of convex functions of measures developed in [9] and [29] (Chapter II, Section 4), we define the functional $\mathcal{H}(p)$ in the weak formulation of problem (1.1) as

$$\mathcal{H}(p) := \int_{\Omega\cup\Gamma_0} H(p/|p|)\,d|p|\,,$$

where $p/|p|$ is the Radon-Nikodym derivative of the measure p with respect to its variation $|p|$, while $A(w(t_k^i))$ is defined, here and henceforth, as the set of triples (u, e, p), with $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $p \in M_b(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_D)$, and $Eu = e + p$ on Ω, subject to the relaxed boundary condition $p = (w(t_k^i) - u) \odot \nu\, \mathcal{H}^{n-1}$ on Γ_0. In the last formula, $\odot$ denotes the symmetric tensor product.

Boundary conditions of this kind are typical in the variational theory of functionals with linear growth (see, e.g. [29, 8]). The mechanical interpretation of our condition on Γ_0 is that if the prescribed boundary displacement is not attained, a plastic slip develops at the boundary, which has a strength proportional to the difference between the prescribed and attained boundary displacements.

In the case $p_k^{i-1} = 0$, the weak formulation of problem (1.1) was studied in detail at the beginning of the 1980s (see [30, 2, 13, 29, 1]). With respect to this body of work, it is important to emphasize a change of perspective. The model we study

(Prandtl-Reuss plasticity) explicitly takes into account the history of plastic deformation. Setting $p_k^{i-1} = 0$ in (1.1) makes the problem oblivious to the accumulation of plastic strain. This is the so-called Hencky theory of plasticity, in which elastic unloading, following plastic loading, is incorrectly resolved (see [11, 28]).

However, we can rely on the results of the above mentioned papers to solve problem (1.1) in the general case (Theorem 3.3), provided a safe-load condition is satisfied. We then define the piecewise constant interpolations

$$u_k(t) := u_k^i, \quad e_k(t) := e_k^i, \quad p_k(t) := p_k^i, \quad \sigma_k(t) := \sigma_k^i,$$

where i is the largest integer such that $t_k^i \leqq t$.

The aim of this paper is to introduce a weak definition of continuous-time quasistatic evolution in the functional framework $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $p \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and to prove that, up to a subsequence, the discrete-time solutions $u_k(t)$, $e_k(t)$, $p_k(t)$, $\sigma_k(t)$, obtained by solving the weak formulations of problems (1.1), converge to a continuous-time solution $u(t)$, $e(t)$, $p(t)$, $\sigma(t)$, provided $\max_i (t_k^i - t_k^{i-1}) \to 0$ as $k \to \infty$.

Our definition fits the general scheme of continuous-time energy formulation of rate-independent processes developed in [22, 23, 19–21, 15]. Based on the work presented in these papers, for every time interval $[s, t]$ contained in $[0, T]$ we introduce the dissipation associated with $\mathcal{H}$, defined by

$$\mathcal{D}_{\mathcal{H}}(p; s, t) := \sup\Big\{ \sum_{j=1}^{N} \mathcal{H}(p(t_j) - p(t_{j-1})) : \\ s = t_0 \leqq t_1 \leqq \ldots \leqq t_N = t,\ N \in \mathbb{N} \Big\}.$$

The general definition proposed in [15] reads in our case as follows: a quasistatic evolution is a function $t \mapsto (u(t), e(t), p(t))$ from $[0, T]$ into $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ which satisfies the following conditions:

(qs1) global stability: for every $t \in [0, T]$ we have $(u(t), e(t), p(t)) \in A(w(t))$ and

$$\mathcal{Q}(e(t)) - \langle \mathcal{L}(t) | u(t) \rangle \leqq \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - \langle \mathcal{L}(t) | v \rangle$$

for every $(v, \eta, q) \in A(w(t))$;

(qs2) energy balance: the function $t \mapsto p(t)$ from $[0, T]$ into $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t \in [0, T]$

$$\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle \mathcal{L}(t) | u(t) \rangle \\ = \mathcal{Q}(e(0)) - \langle \mathcal{L}(0) | u(0) \rangle + \int_0^t \{ \langle \sigma(s) | E\dot{w}(s) \rangle - \langle \mathcal{L}(s) | \dot{w}(s) \rangle \}\, ds \\ - \int_0^t \langle \dot{\mathcal{L}}(s) | u(s) \rangle\, ds,$$

where $\sigma(t) := \mathbb{C}e(t)$, dots denote time derivatives, the first brackets $\langle\cdot|\cdot\rangle$ in the first integral denote the scalar product in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, while the other brackets $\langle\cdot|\cdot\rangle$ are defined as in (1.2).

The main result of our paper is the proof of the existence of a quasistatic evolution satisfying the prescribed initial conditions (Theorem 4.5), provided that a uniform safe-load condition is satisfied.

A different formulation of the problem in rate form was proposed in [12] and [28], where an existence result is proved by a visco-plastic approximation. It turns out that our definition is equivalent to the one considered in those papers (Theorem 6.1 and Remark 6.3). Therefore, the existence result is not new, but our proof is completely different and leads to a different approximation scheme for the solutions (Theorem 4.8). This different approximation scheme may prove useful in the construction and analysis of algorithms for the numerical solution of the problem. Moreover, it shows that this problem can be included in the general theory developed in [19, 15].

Our proof is obtained by considering the discrete time solutions $u_k(t)$, $e_k(t)$, $p_k(t)$, $\sigma_k(t)$, and by showing that they satisfy an approximate energy inequality (Lemma 4.6), which is similar to [15] (Theorem 4.1). This allows us to apply the generalization (Lemma 7.2) of the classical Helly Theorem proved in [15] (Theorem 3.2), and to extract a subsequence, independent of t and still denoted p_k, such that $p_k(t) \rightharpoonup p(t)$ weakly* in $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ for every $t \in [0, T]$.

Extracting a further subsequence, possibly depending on t, we may assume that $u_k(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega)$ and $e_k(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. We prove (Theorem 3.7) that $(u(t), e(t), p(t))$ satisfies the global stability condition (qs1). Since there exists at most one $(u, e) \in BD(\Omega)\times L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, such that $(u, e, p(t))$ satisfies (qs1) (Remark 3.9), we have $u_k(t) \rightharpoonup u(t)$ and $e_k(t) \rightharpoonup e(t)$ for the same subsequence (independent of t) for which $p_k(t) \rightharpoonup p(t)$.

One of the inequalities in the energy balance (qs2) is then proved by passing to the limit in the approximate energy inequality obtained for the discrete-time solutions, while the opposite inequality follows (Theorem 4.7) from the global stability, by adapting the proofs of [15] (Theorem 4.4) and [6] (Lemma 7.1).

The second part of our paper is devoted to the regularity of solutions and to the comparison of our definition of quasistatic evolution with other definitions in rate form. We prove (Theorem 5.2) that if the data of the problem are absolutely continuous functions of time, then for every quasistatic evolution the functions $t \mapsto u(t)$, $t \mapsto e(t)$, $t \mapsto p(t)$, and $t \mapsto \sigma(t)$ are absolutely continuous on $[0, T]$ with values in $BD(\Omega)$, $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, respectively. Moreover, we establish a pointwise estimate for the time derivatives of these functions which implies that if the data of the problem are Lipschitz continuous on $[0, T]$, then the same is true for $t \mapsto u(t)$, $t \mapsto e(t)$, $t \mapsto p(t)$, and $t \mapsto \sigma(t)$ (see Remark 5.4).

Similar arguments prove that $t \mapsto e(t)$ and $t \mapsto \sigma(t)$ are uniquely determined by their initial conditions (Theorem 5.9), while elementary examples in one dimension show that, in general, this is not true for $t \mapsto u(t)$ and $t \mapsto p(t)$ (see [28] (Section 2.1)).

The results on the regularity of solutions in time, which arise because of the convexity of the problem, appear to be unusual in the context of rate-independent evolution problems. They should be contrasted with the situation occurring in the quasistatic growth of brittle cracks (see [6]), where the regularity of solutions is much worse. The mechanical interpretation of this difference is that, while slip surfaces evolve continuosly in time if the data are continuous, cracks may evolve much less regularly.

The regularity results described above allow us to write the energy balance (qs2) as balance of powers (Proposition 5.6): for a.e. $t \in [0, T]$

$$\langle \sigma(t) | \dot{e}(t) \rangle + \mathcal{H}(\dot{p}(t)) = \langle \mathcal{L}(t) | \dot{u}(t) \rangle + \langle \sigma(t) | E\dot{w}(t) \rangle - \langle \mathcal{L}(t) | \dot{w}(t) \rangle .$$

We then show that our definition of quasistatic evolution is equivalent to four different sets of conditions, expressed in rate form (Theorems 6.1 and 6.4). The first can be interpreted as the weak formulation, in the spaces $BD(\Omega)$, $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ of the five conditions (cf1)–(cf5) considered in the classical presentation of the problem; the second takes into account the weak formulation of maximal dissipation (cf5‴); the third coincides with the definition considered in [28]; and the fourth (Theorem 6.4 and Remark 6.5) presents a strong formulation of the normality rule in either of the two forms (cf5′) and (cf5″). This last formulation requires a precise representative of $\sigma_D(t)$ defined $|\dot{p}(t)|$-a.e. on $\Omega \cup \Gamma_0$. If K is strictly convex, this representative is obtained as the limit of averages of $\sigma_D(t)$ (Theorem 6.6).

2. Notation and preliminary results

2.1. Mathematical preliminaries

Measures. The Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. Given a Borel set $B \subset \mathbb{R}^n$ and a finite dimensional Hilbert space X, $M_b(B; X)$ denotes the space of bounded Borel measures on B with values in X, endowed with the norm $\|\mu\|_1 := |\mu|(B)$, where $|\mu| \in M_b(B; \mathbb{R})$ is the variation of the measure μ. For every $\mu \in M_b(B; X)$ we consider the Lebesgue decomposition $\mu = \mu^a + \mu^s$, where μ^a is absolutely continuous and μ^s is singular with respect to the Lebesgue measure $\mathcal{L}^n$.

If $\mu^s = 0$, we always identify μ with its density with respect to the Lebesgue measure $\mathcal{L}^n$. In this way, $L^1(B; X)$ is regarded as a subspace of $M_b(B; X)$, with the induced norm. In particular, we have $\mu^a \in L^1(B; X)$ for every $\mu \in M_b(B; X)$. The indication of the space X is omitted when $X = \mathbb{R}$. The L^p norm, $1 \leqq p \leqq \infty$, is denoted by $\|\cdot\|_p$. The brackets $\langle \cdot | \cdot \rangle$ denote the duality product between conjugate L^p spaces, as well as between other pairs of spaces, according to the context.

If the relative topology of B is locally compact, by Riesz representation theorem (see e.g. [27] (Theorem 6.19)) $M_b(B; X)$ can be identified with the dual of $C_0(B; X)$, the space of continuous functions $\varphi : B \to X$ such that $\{|\varphi| \geqq \varepsilon\}$ is compact for every $\varepsilon > 0$. The weak* topology of $M_b(B; X)$ is defined using this duality.

Matrices. The space of *symmetric* $n\times n$ *matrices* is denoted by $\mathbb{M}^{n\times n}_{sym}$. It is endowed with the euclidean scalar product $\xi{:}\zeta := \mathrm{tr}(\xi\zeta) = \sum_{ij}\xi_{ij}\zeta_{ij}$ and with the corresponding euclidean norm $|\xi| := (\xi{:}\xi)^{1/2}$. The orthogonal complement of the subspace $\mathbb{R}I$ spanned by the identity matrix I is the subspace $\mathbb{M}^{n\times n}_D$ of all matrices of $\mathbb{M}^{n\times n}_{sym}$ with trace zero. For every $\xi \in \mathbb{M}^{n\times n}_{sym}$, the orthogonal projection of ξ on $\mathbb{R}I$ is $\frac{1}{n}\mathrm{tr}(\xi)I$, while the orthogonal projection on $\mathbb{M}^{n\times n}_D$ is the *deviator* ξ_D of ξ, so that we obtain the orthogonal decomposition

$$\xi = \xi_D + \frac{1}{n}(\mathrm{tr}\,\xi)I\,.$$

The *symmetrized-tensor product* $a\odot b$ of two vectors $a, b \in \mathbb{R}^n$ is the symmetric matrix with entries $(a_ib_j + a_jb_i)/2$. It is easy to see that $\mathrm{tr}(a\odot b) = a\cdot b$, the scalar product of a and b, and that $|a\odot b|^2 = \frac{1}{2}|a|^2|b|^2 + \frac{1}{2}(a\cdot b)^2$, so that $\frac{1}{\sqrt{2}}|a||b| \leqq |a\odot b| \leqq |a||b|$.

Functions with bounded deformation. Let U be an open set in $\mathbb{R}^n$. For every $u \in L^1(U;\mathbb{R}^n)$, let Eu be the $\mathbb{M}^{n\times n}_{sym}$-valued distribution on U, whose components are defined by $E_{ij}u = \frac{1}{2}(D_ju_i + D_iu_j)$. The space $BD(U)$ of functions with *bounded deformation* is the space of all $u \in L^1(U;\mathbb{R}^n)$, such that $Eu \in M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BD(U)$ is a Banach space with the norm

$$\|u\|_1 + \|Eu\|_1\,.$$

It is possible to prove that $BD(U)$ is the dual of a normed space (see [17, 30]). The weak* topology of $BD(U)$ is defined using this duality. A sequence u_k converges to u weakly* in $BD(U)$ if, and only if, $u_k \rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^n)$ and $Eu_k \rightharpoonup Eu$ weakly* in $M_b(U;\mathbb{M}^{n\times n}_{sym})$. Every bounded sequence in $BD(U)$ has a weakly* convergent subsequence. Moreover, if U is bounded and has Lipschitz boundary, every bounded sequence in $BD(U)$ has a subsequence which converges weakly in $L^{n/(n-1)}(U;\mathbb{R}^n)$ and strongly in $L^p(U;\mathbb{R}^n)$ for every $p < n/(n-1)$. For the general properties of $BD(U)$ we refer to [29].

In our problem, $u \in BD(U)$ represents the *displacement* of an elasto-plastic body and Eu is the corresponding linearized *strain*.

2.2. Mechanical preliminaries

The reference configuration. Throughout the paper Ω is a *bounded connected open set* in $\mathbb{R}^n$ with C^2 *boundary*. We suppose that the boundary $\partial\Omega$ is partitioned into two disjoint open sets Γ_0, Γ_1 and their common boundary $\partial\Gamma_0 = \partial\Gamma_1$ (topological notions refer here to the relative topology of $\partial\Omega$). We assume that

$$\Gamma_0 \neq \emptyset, \tag{2.1}$$

and that

$$\partial\Gamma_0 = \partial\Gamma_1 \quad \text{is } C^2 \text{ regular}, \tag{2.2}$$

i.e. for every $x \in \partial\Gamma_0 = \partial\Gamma_1$ there exists a C^2 diffeomorphism defined in an open neighborhood of x in $\mathbb{R}^n$ which maps $\partial\Omega$ to an $(n-1)$-dimensional plane and $\partial\Gamma_0 = \partial\Gamma_1$ to an $(n-2)$-dimensional plane.

On Γ_0 we will prescribe a Dirichlet boundary condition by assigning a function $w \in H^{1/2}(\Gamma_0; \mathbb{R}^n)$, or, equivalently, a function $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, whose trace on Γ_0 (also denoted by w) is the prescribed boundary value. The set Γ_1 will be the part of the boundary on which the traction is prescribed.

Every function $u \in BD(\Omega)$ has a *trace* on $\partial\Omega$, still denoted by u, which belongs to $L^1(\partial\Omega; \mathbb{R}^n)$. If $u_k, u \in BD(\Omega)$, $u_k \to u$ strongly in $L^1(\Omega; \mathbb{R}^n)$, and $\|Eu_k\|_1 \to \|Eu\|_1$, then $u_k \to u$ strongly in $L^1(\partial\Omega; \mathbb{R}^n)$ (see [29] (Chapter II, Theorem 3.1)). Moreover, there exists a constant $C > 0$, depending on Ω and Γ_0, such that

$$\|u\|_{1,\Omega} \leqq C\,\|u\|_{1,\Gamma_0} + C\,\|Eu\|_{1,\Omega} \tag{2.3}$$

(see [29] (Proposition 2.4 and Remark 2.5)).

We shall frequently use the space $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, which is the dual of $C_0(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$. The latter space can be identified with the space of functions in $C(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ vanishing on $\overline{\Gamma}_1$. The duality product is defined by

$$\langle \tau | \mu \rangle := \int_{\Omega\cup\Gamma_0} \tau{:}d\mu := \sum_{ij} \int_{\Omega\cup\Gamma_0} \tau_{ij}\, d\mu_{ij} \tag{2.4}$$

for every $\tau = (\tau_{ij}) \in C(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ and every $\mu = (\mu_{ij}) \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$.

The set of admissible stresses. Let K be a closed convex set in $\mathbb{M}^{n\times n}_D$, which will represent a constraint on the deviatoric part of the stress. Its boundary is interpreted as the *yield surface*. We assume that there exist two constants r_K and R_K, with $0 < r_K \leqq R_K < \infty$, such that

$$\{\xi \in \mathbb{M}^{n\times n}_D : |\xi| \leqq r_K\} \subset K \subset \{\xi \in \mathbb{M}^{n\times n}_D : |\xi| \leqq R_K\}\,. \tag{2.5}$$

It is convenient to introduce the convex set

$$\mathcal{K}_D(\Omega) := \{\tau \in L^2(\Omega; \mathbb{M}^{n\times n}_D) : \tau(x) \in K \ \text{ for a.e. } x \in \Omega\}\,.$$

The *set of admissible stresses* is defined by

$$\mathcal{K}(\Omega) := \{\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) : \sigma_D \in \mathcal{K}_D(\Omega)\}\,.$$

The *support function* $H{:}\mathbb{M}^{n\times n}_D \to [0, +\infty\,[$ of K is given by

$$H(\xi) := \sup_{\zeta\in K} \xi{:}\zeta\,. \tag{2.6}$$

It turns out that H is convex and positively homogeneous of degree one. In particular, it satisfies the triangle inequality

$$H(\xi + \zeta) \leqq H(\xi) + H(\zeta)\,.$$

From (2.5) it follows that

$$r_K|\xi| \leqq H(\xi) \leqq R_K|\xi| \tag{2.7}$$

for every $\xi \in \mathbb{M}^{n\times n}_D$.

For every $\mu \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ let $\mu/|\mu|$ be the Radon-Nikodym derivative of μ with respect to its variation $|\mu|$. Using the theory of convex functions of measures developed in [9], we introduce the nonnegative Radon measure $H(\mu) \in M_b(\Omega \cup \Gamma_0)$, defined by $H(\mu) := H(\mu/|\mu|)|\mu|$, i.e.

$$H(\mu)(B) := \int_B H(\mu/|\mu|)\, d|\mu| \tag{2.8}$$

for every Borel set $B \subset \Omega \cup \Gamma_0$. Finally, we consider the functional $\mathcal{H}: M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) \to \mathbb{R}$ defined by

$$\mathcal{H}(\mu) := H(\mu)(\Omega \cup \Gamma_0) = \int_{\Omega\cup\Gamma_0} H(\mu/|\mu|)\, d|\mu|\,. \tag{2.9}$$

Using [9] (Theorem 4) and [29] (Chapter II, Lemma 5.2) we can see that $H(\mu)$ coincides with the measure studied in [29] (Chapter II, Section 4). Hence

$$\mathcal{H}(\mu) = \sup\{\langle \tau|\mu\rangle : \tau \in C_0(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) \cap \mathcal{K}_D(\Omega)\}, \tag{2.10}$$

and $\mathcal{H}$ is lower semicontinuous on $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ with respect to weak* convergence. It follows from the properties of H that $\mathcal{H}$ satisfies the triangle inequality, i.e.

$$\mathcal{H}(\lambda + \mu) \leqq \mathcal{H}(\lambda) + \mathcal{H}(\mu) \tag{2.11}$$

for every $\lambda, \mu \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$.

The elasticity tensor. Let $\mathbb{C}$ be the *elasticity tensor*, considered as a symmetric positive definite linear operator $\mathbb{C}: \mathbb{M}^{n\times n}_{sym} \to \mathbb{M}^{n\times n}_{sym}$. We assume that the orthogonal subspaces $\mathbb{M}^{n\times n}_D$ and $\mathbb{R}I$ are invariant under $\mathbb{C}$. This assumption is equivalent to saying that there exist a symmetric positive definite linear operator $\mathbb{C}_D: \mathbb{M}^{n\times n}_D \to \mathbb{M}^{n\times n}_D$ and a constant $\kappa > 0$ such that

$$\mathbb{C}\xi := \mathbb{C}_D\xi_D + \kappa(\operatorname{tr}\xi)I \tag{2.12}$$

for every $\xi \in \mathbb{M}^{n\times n}_{sym}$. Note that when $\mathbb{C}$ is isotropic, we have $\mathbb{C}\xi = 2\mu\xi_D + \kappa(\operatorname{tr}\xi)I$, where $\mu > 0$ is the shear modulus and κ is the modulus of compression, so that our assumptions are satisfied.

Let $Q: \mathbb{M}^{n\times n}_{sym} \to [0, +\infty[$ be the quadratic form associated with $\mathbb{C}$, defined by

$$Q(\xi) := \frac{1}{2}\mathbb{C}\xi{:}\xi = \frac{1}{2}\mathbb{C}_D\xi_D{:}\xi_D + \frac{\kappa}{2}(\operatorname{tr}\xi)^2.$$

It turns out that there exist two constants $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$, with $0 < \alpha_{\mathbb{C}} \leqq \beta_{\mathbb{C}} < +\infty$, such that

$$\alpha_{\mathbb{C}}|\xi|^2 \leqq Q(\xi) \leqq \beta_{\mathbb{C}}|\xi|^2 \tag{2.13}$$

for every $\xi \in \mathbb{M}^{n\times n}_{sym}$. These inequalities imply

$$|\mathbb{C}\xi| \leqq 2\beta_{\mathbb{C}}|\xi|\,. \tag{2.14}$$

It is convenient to introduce the quadratic form $\mathcal{Q}:L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \to \mathbb{R}$ defined by

$$\mathcal{Q}(e) := \int_\Omega Q(e)\,dx$$

for every $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. It is well known that $\mathcal{Q}$ is lower semicontinuous on $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, with respect to weak convergence.

The prescribed boundary displacements. For every $t \in [0,T]$, we prescribe a *boundary displacement* $w(t)$ in the space $H^1(\mathbb{R}^n;\mathbb{R}^n)$. This choice is motivated by the fact that we do not want to impose "discontinuous" boundary data, so that, if the displacement develops sharp discontinuities, this is a result of energy minimization.

We also assume that

$$t \mapsto w(t) \text{ is absolutely continuous} \tag{2.15}$$

from $[0,T]$ into $H^1(\mathbb{R}^n;\mathbb{R}^n)$, so that the time derivative $t \mapsto \dot w(t)$ belongs to $L^1([0,T];H^1(\mathbb{R}^n;\mathbb{R}^n))$, and its strain $t \mapsto E\dot w(t)$ belongs to $L^1([0,T];L^2(\mathbb{R}^n;\mathbb{M}^{n\times n}_{sym}))$. For the main properties of absolutely continuous functions with values in reflexive Banach spaces we refer to [4] (Appendix).

Body and surface forces. For every $t \in [0,T]$ the *body force* $f(t)$ belongs to the space $L^n(\Omega;\mathbb{R}^n)$, and the *surface force* $g(t)$ acting on Γ_1 belongs to $L^\infty(\Gamma_1;\mathbb{R}^n)$. We assume that

$$t \mapsto f(t) \text{ and } t \mapsto g(t) \text{ are absolutely continuous} \tag{2.16}$$

from $[0,T]$ into $L^n(\Omega;\mathbb{R}^n)$ and $L^\infty(\Gamma_1;\mathbb{R}^n)$, respectively, so that the time derivative $t \mapsto \dot f(t)$ belongs to $L^1([0,T];L^n(\Omega;\mathbb{R}^n))$, the weak* limit

$$\dot g(t) := w^*\text{-}\lim_{s\to t}\frac{g(s)-g(t)}{s-t}$$

exists for a.e. $t \in [0,T]$, and $t \mapsto \|\dot g(t)\|_\infty$ belongs to $L^1([0,T])$ (see Theorem 7.1).

Throughout the paper we will also assume the following *uniform safe-load condition*: there exist a function $t \mapsto \varrho(t)$ from $[0,T]$ into $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and a constant $\alpha > 0$ such that for every $t \in [0,T]$

$$-\mathrm{div}\varrho(t) = f(t) \text{ a.e. on } \Omega\,, \qquad \big[\varrho(t)\nu\big] = g(t) \text{ on } \Gamma_1\,, \tag{2.17}$$

and

$$\varrho_D(t,x) + \xi \in K \tag{2.18}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{M}^{n\times n}_D$ with $|\xi| \leqq \alpha$. In these formulae $\varrho_D(t,x)$ denotes the value of $\varrho_D(t)$ at $x \in \Omega$, and the trace $\big[\varrho(t)\nu\big]$ of $\varrho(t)\nu$ on Γ_1 is interpreted in the sense of (2.24) below. We also assume that

$$t \mapsto \varrho(t) \text{ and } t \mapsto \varrho_D(t) \text{ are absolutely continuous} \tag{2.19}$$

from $[0,T]$ into $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$, respectively, so that the time derivative $t \mapsto \dot\varrho(t)$ belongs to $L^1([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$,

$$\frac{\varrho_D(s)-\varrho_D(t)}{s-t} \rightharpoonup \dot\varrho_D(t) \quad \text{as } s\to t \tag{2.20}$$

weakly* in $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$ for a.e. $t\in[0,T]$, and $t \mapsto \|\dot\varrho_D(t)\|_\infty$ belongs to $L^1([0,T])$ (see Theorem 7.1).

2.3. Stress and strain

Given a displacement $u \in BD(\Omega)$, and a boundary datum $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, the *elastic* and *plastic strains* $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ satisfy the equalities

$$Eu = e + p \qquad \text{in } \Omega\,, \tag{2.21}$$

$$p = (w-u)\odot\nu\,\mathcal{H}^{n-1} \qquad \text{on } \Gamma_0\,. \tag{2.22}$$

Therefore, we have $e = E^a u - p^a$ a.e. on Ω and $p^s = E^s u$ on Ω. Since $\operatorname{tr} p = 0$, it follows from (2.21) that $\operatorname{div} u = \operatorname{tr} e \in L^2(\Omega)$ and from (2.22) that $(w-u)\cdot\nu = 0$ $\mathcal{H}^{n-1}$-a.e. on Γ_0. The *stress* $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ is defined by

$$\sigma := \mathbb{C}e = \mathbb{C}_D e_D + \kappa \operatorname{tr} e\,.$$

The *stored elastic energy* is given by

$$\mathcal{Q}(e) = \int_\Omega Q(e)\,dx = \tfrac12\langle\sigma|e\rangle\,.$$

Given $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, the *set of admissible displacements and strains* for the boundary datum w on Γ_0 is denoted by $A(w)$: it is defined as the set of all triples (u,e,p), with $u \in BD(\Omega)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, satisfying (2.21) and (2.22).

We shall also use *the space* $\Pi_{\Gamma_0}(\Omega)$ *of admissible plastic strains*, defined as the set of all $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ for which there exist $u\in BD(\Omega)$, $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, and $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ satisfying (2.21) and (2.22), i.e. $(u,e,p) \in A(w)$.

We now prove a closure property for the multi-valued map $w \mapsto A(w)$.

Lemma 2.1. *Assume* (2.1) *and* (2.2). *Let* w_k *be a sequence in* $H^1(\mathbb{R}^n;\mathbb{R}^n)$ *and let* $(u_k,e_k,p_k) \in A(w_k)$. *Assume that* $u_k \rightharpoonup u_\infty$ *weakly* in* $BD(\Omega)$, $e_k \rightharpoonup e_\infty$ *weakly in* $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p_k \rightharpoonup p_\infty$ *weakly* in* $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, $w_k \rightharpoonup w_\infty$ *weakly in* $H^1(\mathbb{R}^n;\mathbb{R}^n)$. *Then* $(u_\infty,e_\infty,p_\infty) \in A(w_\infty)$.

Proof. Since Γ_0 is open in $\partial\Omega$, there exists a bounded open set U in $\mathbb{R}^n$, such that $\Gamma_0 = U \cap \partial\Omega$, and we define $\tilde{\Omega} := \Omega \cup U$.

For $k = 1, 2, \dots, \infty$ let $\tilde{u}_k \in BD(\tilde{\Omega})$ be defined by $\tilde{u}_k = u_k$ a.e. on Ω and $\tilde{u}_k = w_k$ a.e. on $U \setminus \Omega$. Then,

$$\begin{aligned} E\tilde{u}_k &= Eu_k && \text{on } \Omega\,, \\ E\tilde{u}_k &= (w_k - u_k) \odot \nu\, \mathcal{H}^{n-1} && \text{on } \Gamma_0\,, \\ E\tilde{u}_k &= Ew_k && \text{on } U \setminus \overline{\Omega}\,, \end{aligned} \tag{2.23}$$

(see e.g. [29] (Theorem 2.1 and Remark 2.3)). Since $w_k - u_k$ is bounded in $L^1(\Gamma_0; \mathbb{R}^n)$ by the continuity of the trace operator, the sequence $E\tilde{u}_k$ is bounded in $M_b(\tilde{\Omega}; \mathbb{M}^{n\times n}_{sym})$. As $\tilde{u}_k \to \tilde{u}_\infty$ weakly in $L^1(\tilde{\Omega}; \mathbb{R}^n)$, we conclude that $\tilde{u}_k \rightharpoonup \tilde{u}_\infty$ weakly* in $BD(\tilde{\Omega})$.

For $k = 1, 2, \dots, \infty$ let $\tilde{e}_k \in L^2(\tilde{\Omega}; \mathbb{M}^{n\times n}_{sym})$ be defined by $\tilde{e}_k = e_k$ a.e. on Ω and $\tilde{e}_k = Ew_k$ a.e. on $U\setminus\Omega$, and let $\tilde{p}_k \in M_b(\tilde{\Omega}; \mathbb{M}^{n\times n}_D)$ be defined by $\tilde{p}_k = p_k$ on $\Omega\cup\Gamma_0$ and $\tilde{p}_k = 0$ on $U \setminus \overline{\Omega}$. Then $\tilde{e}_k$ converges to $\tilde{e}_\infty$ weakly in $L^2(\tilde{\Omega}; \mathbb{M}^{n\times n}_{sym})$. Since the restrictions to $\Omega \cup \Gamma_0$ of functions in $C_0(\tilde{\Omega}; \mathbb{M}^{n\times n}_D)$ belong to $C_0(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, we also find that $\tilde{p}_k$ converges to $\tilde{p}_\infty$ weakly* in $M_b(\tilde{\Omega}; \mathbb{M}^{n\times n}_D)$.

As $(u_k, e_k, p_k) \in A(w_k)$ for $k < \infty$, by use of (2.23), we obtain $E\tilde{u}_k = \tilde{e}_k + \tilde{p}_k$ in $\tilde{\Omega}$. The convergence properties already proved for $(\tilde{u}_k, \tilde{e}_k, \tilde{p}_k)$ show that $E\tilde{u}_\infty = \tilde{e}_\infty + \tilde{p}_\infty$ in $\tilde{\Omega}$. Consequently, (2.23) for $k = \infty$ implies that $(u_\infty, e_\infty, p_\infty) \in A(w_\infty)$. □

The traces of the stress. If $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma \in L^2(\Omega; \mathbb{R}^n)$, then we can define a distribution $[\sigma, \nu]$ on $\partial\Omega$ by

$$\langle [\sigma\nu] | \psi \rangle_{\partial\Omega} := \langle \operatorname{div}\sigma | \psi \rangle + \langle \sigma | E\psi \rangle \tag{2.24}$$

for every $\psi \in H^1(\Omega; \mathbb{R}^n)$. It turns out that $[\sigma, \nu] \in H^{-1/2}(\partial\Omega; \mathbb{R}^n)$ (see e.g. [29] (Theorem 1.2, Chapter I)). We will consider the normal and tangential parts of $[\sigma\nu]$, defined by

$$[\sigma\nu]_\nu := ([\sigma\nu] \cdot \nu)\nu\,, \qquad [\sigma\nu]^\perp_\nu := [\sigma\nu] - ([\sigma\nu] \cdot \nu)\nu\,. \tag{2.25}$$

Since $\nu \in C^1(\partial\Omega; \mathbb{R}^n)$, we obtain $[\sigma\nu]_\nu, [\sigma\nu]^\perp_\nu \in H^{-1/2}(\partial\Omega; \mathbb{R}^n)$. If, in addition, $\sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$, then $[\sigma\nu]^\perp_\nu \in L^\infty(\partial\Omega; \mathbb{R}^n)$ and

$$\| [\sigma\nu]^\perp_\nu \|_{\infty,\partial\Omega} \leqq \frac{1}{\sqrt{2}} \|\sigma_D\|_\infty, \tag{2.26}$$

(see [13] (Lemma 2.4)).

Stress-strain duality. Let

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) : \operatorname{div}\sigma \in L^n(\Omega; \mathbb{R}^n),\ \sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)\}\,.$$

If $\sigma \in \Sigma(\Omega)$, then $\sigma \in L^r(\Omega; \mathbb{M}^{n\times n}_{sym})$ for every $r < \infty$ by [13] (Proposition 2.5). For every $u \in BD(\Omega)$ with $\operatorname{div} u \in L^{n/(n-1)}(\Omega)$, we define the distribution $[\sigma_D : E_D u]$ on Ω by

$$\langle [\sigma_D{:}E_D u] \,|\varphi\rangle := -\langle \operatorname{div}\sigma | \varphi\, u\rangle - \tfrac{1}{n}\langle \operatorname{tr}\sigma | \varphi \operatorname{div} u\rangle - \langle \sigma | u \odot \nabla\varphi\rangle \tag{2.27}$$

for every $\varphi \in C_c^\infty(\Omega)$. It is proved in [13] (Theorem 3.2) that $[\sigma_D{:}E_D u]$ is a bounded measure on Ω whose variation satisfies

$$|\,[\sigma_D{:}E_D u]\,| \leqq \|\sigma_D\|_\infty |E_D u| \qquad \text{in } \Omega\,. \tag{2.28}$$

Moreover

$$[\psi\sigma_D{:}E_D u] = \psi\,[\sigma_D{:}E_D u] \qquad \text{in } \Omega \tag{2.29}$$

for every $\psi \in C^1(\overline{\Omega})$, and

$$[\sigma_D{:}E_D u]^a = \sigma_D{:}E_D^a u \qquad\qquad \text{a.e. in } \Omega$$

(see [1] (Corollary 3.2)). We define the measure $\left[\sigma_D{:}E_D^s\right]$ on Ω by

$$\left[\sigma_D{:}E_D^s u\right] := [\sigma_D{:}E_D u]^s = [\sigma_D{:}E_D u] - \sigma_D{:}E_D^a u\,.$$

By (2.28), we have

$$|\left[\sigma_D{:}E_D^s u\right]| \leqq \|\sigma_D\|_\infty |E_D^s u| \qquad \text{in } \Omega\,. \tag{2.30}$$

This shows, in particular, that if $\hat\sigma$, $\hat u$ satisfy the same properties as σ, u, and $\sigma_D = \hat\sigma_D$ a.e. on Ω, $E_D^s u = E_D^s \hat u$ in Ω, then $\left[\sigma_D{:}E_D^s u\right] = \left[\hat\sigma_D{:}E_D^s \hat u\right]$ in Ω.

We define

$$\langle\sigma_D|E_D u\rangle := [\sigma_D{:}E_D u]\,(\Omega)\,, \qquad \langle\sigma_D|E_D^s u\rangle := \left[\sigma_D{:}E_D^s u\right](\Omega)\,,$$

so that $\langle\sigma_D|E_D u\rangle = \langle\sigma_D|E_D^a u\rangle + \langle\sigma_D|E_D^s u\rangle$. If $\sigma_k \rightharpoonup \sigma$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^n(\Omega;\mathbb{R}^n)$, and $(\sigma_k)_D$ is bounded in $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$, then $\sigma_k \rightharpoonup \sigma$ weakly in $L^r(\Omega;\mathbb{M}^{n\times n}_{sym})$ for every $r < +\infty$ (see [13] (Proposition 2.5)) and

$$\begin{aligned}\langle[(\sigma_k)_D{:}E_D u]\,|\varphi\rangle &\to \langle[\sigma_D{:}E_D u]\,|\varphi\rangle\,,\\ \langle\left[(\sigma_k)_D{:}E_D^s u\right]|\varphi\rangle &\to \langle\left[\sigma_D{:}E_D^s u\right]|\varphi\rangle\end{aligned} \tag{2.31}$$

for every $\varphi \in C(\overline{\Omega})$ (see [13] (Theorem 3.2), the proof of which gives the result also in the case of weak convergence).

We now define a duality between $\Sigma(\Omega)$ and $\Pi_{\Gamma_0}(\Omega)$. Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$, we fix $u \in BD(\Omega)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, satisfying (2.21) and (2.22). We then define a measure $[\sigma_D{:}p] \in M_b(\Omega\cup\Gamma_0)$ by setting

$$\begin{aligned}&[\sigma_D{:}p] := \sigma_D{:}p^a + \left[\sigma_D{:}E_D^s u\right] = [\sigma_D{:}E_D u] - \sigma_D{:}e_D &&\text{on } \Omega\,,\\ &[\sigma_D{:}p] := [\sigma\nu]_\nu^\perp \cdot (w-u)\,\mathcal{H}^{n-1} &&\text{on } \Gamma_0\,,\end{aligned}$$

so that

$$\langle[\sigma_D{:}p]\,|\varphi\rangle = \langle[\sigma_D{:}E_D u]\,|\varphi\rangle - \langle\sigma_D{:}e_D|\varphi\rangle + \langle[\sigma\nu]_\nu^\perp\,|\varphi(w-u)\rangle_{\Gamma_0} \tag{2.32}$$

for every $\varphi \in C(\overline{\Omega})$, where $\langle\cdot|\cdot\rangle_{\Gamma_0}$ denotes the duality pairing between $L^\infty(\Gamma_0;\mathbb{R}^n)$ and $L^1(\Gamma_0;\mathbb{R}^n)$. Using the previous remarks, it is easy to see that the measure

$[\sigma_D{:}p]$ does not depend on the choice of u, e, and w. It follows from the definition, and from (2.26) and (2.30), that

$$\begin{aligned} &[\sigma_D{:}p]^a = \sigma_D{:}p^a \quad \text{a.e. on } \Omega\,, \quad [\sigma_D{:}p]^s = \big[\sigma_D{:}E^s_D u\big] \quad \text{on } \Omega\cup\Gamma_0\,,\\ &|\,[\sigma_D{:}p]\,| \leqq \|\sigma_D\|_\infty |p| \text{ on } \Omega\cup\Gamma_0\,,\ |\,[\sigma_D{:}p]^s\,| \leqq \|\sigma_D\|_\infty |p^s| \text{ on } \Omega\cup\Gamma_0\,. \end{aligned} \tag{2.33}$$

Moreover, (2.29) implies that

$$[\psi\sigma_D{:}p] = \psi\,[\sigma_D{:}p] \qquad \text{in } \Omega\cup\Gamma_0 \tag{2.34}$$

for every $\psi \in C^1(\overline{\Omega})$. Using the definitions, we can deduce that

$$\langle[\sigma_D{:}p]\,|\varphi\rangle = \langle\varphi\,\sigma_D|p\rangle \tag{2.35}$$

for every $\sigma \in C^1(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ and every $\varphi \in C^1(\overline{\Omega})$, where the duality used in the right-hand side is defined in (2.4). Using the continuity properties given by (2.33), we can prove by approximation that (2.35) also holds for every $\sigma \in C(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ and every $\varphi \in C(\overline{\Omega})$. Therefore, for every $\sigma \in C(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ and every $p \in \Pi_{\Gamma_0}(\Omega)$ we obtain

$$[\sigma_D{:}p] = \sigma_D{:}p \qquad \text{on } \Omega\cup\Gamma_0\,, \tag{2.36}$$

where the right-hand side denotes the measure defined by

$$(\sigma_D{:}p)(B) := \int_B \sigma_D{:}dp := \sum_{ij}\int_B \sigma_{ij}\,dp_{ij} \tag{2.37}$$

for every Borel set $B \subset \Omega\cup\Gamma_0$.

If $\sigma_k \rightharpoonup \sigma$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^n(\Omega;\mathbb{R}^n)$, and $(\sigma_k)_D$ is bounded in $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$, then using (2.24)–(2.26) and (2.31), we obtain

$$\langle[(\sigma_k)_D{:}p]\,|\varphi\rangle \to \langle[\sigma_D{:}p]\,|\varphi\rangle \tag{2.38}$$

for every $\varphi \in C(\overline{\Omega})$.

Finally, for every $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$, we define

$$\begin{aligned} \langle\sigma_D|p\rangle &:= [\sigma_D{:}p]\,(\Omega\cup\Gamma_0)\\ &= \langle\sigma_D|p^a\rangle + \langle\sigma_D|E^s_D u\rangle + \langle[\sigma\nu]^\perp_\nu\,|w-u\rangle_{\Gamma_0}\\ &= \langle\sigma_D|E_D u\rangle - \langle\sigma_D|e_D\rangle + \langle[\sigma\nu]^\perp_\nu\,|w-u\rangle_{\Gamma_0}\,, \end{aligned} \tag{2.39}$$

where $u \in BD(\Omega)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$ satisfy (2.21) and (2.22).

We are now in a position to prove an integration by parts formula for stresses $\sigma \in \Sigma(\Omega)$ and displacements $u \in BD(\Omega)$, involving the elastic and plastic strains e and p.

Proposition 2.2 (Integration by parts). *Assume* (2.1) *and* (2.2). *Let* $\sigma \in \Sigma(\Omega)$, $f \in L^n(\Omega;\mathbb{R}^n)$, $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$, *and let* $(u, e, p) \in A(w)$, *with* $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$. *Assume that* $-\operatorname{div}\sigma = f$ *a.e. on* Ω *and* $[\sigma\nu] = g$ *on* Γ_1. *Then*

$$\langle \sigma_D | p\rangle + \langle \sigma | e - Ew\rangle = \langle f | u - w\rangle + \langle g | u - w\rangle_{\Gamma_1}\,, \tag{2.40}$$

where $\langle\cdot|\cdot\rangle_{\Gamma_1}$ *denotes the duality pairing between* $L^\infty(\Gamma_1;\mathbb{R}^n)$ *and* $L^1(\Gamma_1;\mathbb{R}^n)$. *Moreover,*

$$\begin{aligned}&\langle [\sigma_D{:}p]\,|\varphi\rangle + \langle \sigma{:}(e - Ew)|\varphi\rangle + \langle \sigma | (u-w)\odot\nabla\varphi\rangle\\ &\quad= \langle f | \varphi(u-w)\rangle + \langle g | \varphi(u-w)\rangle_{\Gamma_1}\end{aligned} \tag{2.41}$$

for every $\varphi \in C^1(\overline{\Omega})$.

Proof. By [13] (Theorem 3.2 and Propositions 3.3 and 3.4) we have

$$\begin{aligned}&\langle \operatorname{div}\sigma | \varphi\, v\rangle + \langle [\sigma_D{:}E_D v]\,|\varphi\rangle + \tfrac{1}{n}\langle \operatorname{tr}\sigma | \varphi \operatorname{div} v\rangle + \langle \sigma | v\odot\nabla\varphi\rangle\\ &\quad= \langle [\sigma\nu]_\nu^\perp\,|\varphi\, v\rangle_{\Gamma_0} + \langle g | \varphi\, v\rangle_{\Gamma_1}\end{aligned} \tag{2.42}$$

for every $\varphi \in C^1(\overline{\Omega})$ and every $v \in BD(\Omega)$ with $\operatorname{div} v \in L^2(\Omega)$ and $v\cdot\nu = 0$ $\mathcal{H}^{n-1}$-a.e. on Γ_0. By (2.32) we have

$$\begin{aligned}&\langle [\sigma_D{:}p]\,|\varphi\rangle + \langle \sigma{:}(e - Ew)|\varphi\rangle + \langle \sigma | (u-w)\odot\nabla\varphi\rangle\\ &\quad= \langle [\sigma_D{:}E_D(u-w)]\,|\varphi\rangle + \tfrac{1}{n}\langle \operatorname{tr}\sigma | \varphi \operatorname{div}(u-w)\rangle\\ &\qquad+\langle \sigma | (u-w)\odot\nabla\varphi\rangle - \langle [\sigma\nu]_\nu^\perp\,|\varphi(u-w)\rangle_{\Gamma_0}\,.\end{aligned} \tag{2.43}$$

If we apply (2.42) with $v = u - w$ we obtain

$$\begin{aligned}&\langle [\sigma_D{:}E_D(u-w)]\,|\varphi\rangle + \tfrac{1}{n}\langle \operatorname{tr}\sigma | \varphi \operatorname{div}(u-w)\rangle\\ &+\langle \sigma | (u-w)\odot\nabla\varphi\rangle - \langle [\sigma\nu]_\nu^\perp\,|\varphi(u-w)\rangle_{\Gamma_0}\\ &\quad= \langle f | \varphi(u-w)\rangle + \langle g | \varphi(u-w)\rangle_{\Gamma_1}\,.\end{aligned} \tag{2.44}$$

Equality (2.41) now follows from (2.43) and (2.44). To obtain (2.40) it is enough to take $\varphi = 1$ in (2.41). □

In order to show the connection between the duality (2.39) and the functional $\mathcal{H}$ defined in (2.9), we need the following approximation result.

Lemma 2.3. *Let* U *be a bounded open set in* $\mathbb{R}^n$ *with the segment property, let* $\mathbb{K}$ *be a closed convex subset of* $\mathbb{M}^{n\times n}_{sym}$, *and let* $\sigma \in L^r(U;\mathbb{M}^{n\times n}_{sym})$, $1 \leqq r < +\infty$, *with* $\operatorname{div}\sigma \in L^r(U;\mathbb{R}^n)$ *and* $\sigma(x) \in \mathbb{K}$ *for a.e.* $x \in U$. *There then exists a sequence* $\sigma_k \in C^\infty(\overline{U};\mathbb{M}^{n\times n}_{sym})$ *such that* $\sigma_k \to \sigma$ *strongly in* $L^r(U;\mathbb{M}^{n\times n}_{sym})$, $\operatorname{div}\sigma_k \to \operatorname{div}\sigma$ *strongly in* $L^r(U;\mathbb{R}^n)$, *and* $\sigma_k(x) \in \mathbb{K}$ *for every* $x \in \overline{U}$.

Proof. Since U is bounded and has the segment property, there exists a finite open cover (U_i), $i = 1, \dots, m$, of ∂U and a corresponding sequence of nonzero vectors y_i such that, if $x \in \overline{U} \cap U_i$ for some i, then $x + ty_i \in U$ for $0 < t < 1$. We set $U_0 := U$ and $y_0 := 0$. For $i = 0, \dots, m$ and $k = 1, 2, \dots$ the open set $U_k^i := \{x \in U_i : x + (1/k)y_i \in U\}$ contains $\overline{U} \cap U_i$. We define $\sigma_k^i(x) := \sigma(x + (1/k)y_i)$ for every $x \in U_k^i$. Let (V_i), $i = 0, \dots, m$, be an open cover of $\overline{U}$, such that $V_i \subset\subset U_i$ for every i. Since $\overline{U} \cap \overline{V}_i \subset U_k^i$ for every i and k, we can find a mollifier ψ_k^i of class $C_c^\infty(\mathbb{R}^n)$ such that the convolution $\sigma_k^i \star \psi_k^i$ is well defined in a neighborhood of $\overline{U} \cap \overline{V}_i$ and

$$
\begin{aligned}
\|\sigma_k^i \star \psi_k^i - \sigma_k^i\|_{r, U \cap V_i} &\leqq \frac{1}{k}, \\
\|\operatorname{div} \sigma_k^i \star \psi_k^i - \operatorname{div} \sigma_k^i\|_{r, U \cap V_i} &\leqq \frac{1}{k}.
\end{aligned} \tag{2.45}
$$

As $\mathbb{K}$ is closed and convex, we have $\sigma_k^i \star \psi_k^i(x) \in \mathbb{K}$ for every x in a neighborhood of $\overline{U} \cap \overline{V}_i$.

Let (φ_i), $i = 0, \dots, m$, be a C^∞ partition of unity for $\overline{U}$ subordinate to (V_i), and let

$$\sigma_k := \sum_{i=0}^{m} \varphi_i (\sigma_k^i \star \psi_k^i).$$

Then σ_k is of class C^∞ in a neighborhood of $\overline{U}$ and $\sigma_k(x) \in \mathbb{K}$ for every x in a neighborhood of $\overline{U}$. Since $\sigma_k^i \to \sigma$ strongly in $L^r(U \cap V_i; \mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div} \sigma_k^i \to \operatorname{div} \sigma$ strongly in $L^2(U \cap V_i; \mathbb{R}^n)$, from (2.45) and from the identity

$$\operatorname{div} \sigma := \sum_{i=0}^{m} (\varphi_i \operatorname{div} \sigma + \sigma \nabla \varphi_i),$$

we can deduce that $\sigma_k \to \sigma$ strongly in $L^r(U; \mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div} \sigma_k \to \operatorname{div} \sigma$ strongly in $L^r(U; \mathbb{R}^n)$. $\square$

The following proposition provides a variant of (2.10) expressed by using the duality (2.39).

Proposition 2.4. *Let $p \in \Pi_{\Gamma_0}(\Omega)$. Then*

$$H(p) \geqq [\sigma_D : p] \quad \text{on } \Omega \cup \Gamma_0 \tag{2.46}$$

for every $\sigma \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, and

$$\mathcal{H}(p) = \sup\{\langle \sigma_D | p \rangle : \sigma \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)\}. \tag{2.47}$$

Moreover, if $g \in L^\infty(\Gamma_1; \mathbb{R}^n)$ and there exists $\varrho \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ such that $[\varrho\nu] = g$ on Γ_1, then

$$\mathcal{H}(p) = \sup\{\langle \sigma_D | p \rangle : \sigma \in \Sigma(\Omega) \cap \mathcal{K}(\Omega),\ [\sigma\nu] = g \text{ on } \Gamma_1\}. \tag{2.48}$$

Proof. Let $\sigma \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$. To prove (2.46), it is sufficient to show that

$$\langle H(p)|\varphi\rangle \geqq \langle[\sigma_D{:}p]\,|\varphi\rangle \tag{2.49}$$

for every $\varphi \in C(\overline{\Omega})$ with $\varphi \geqq 0$ on $\overline{\Omega}$. By Lemma 2.3 there exists a sequence (σ_k) in $C^\infty(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}) \cap \mathcal{K}(\Omega)$, such that $\sigma_k \to \sigma$ strongly in $L^n(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma_k \to \operatorname{div}\sigma$ strongly in $L^n(\Omega; \mathbb{R}^n)$. By (2.6), (2.8), and (2.35) we obtain

$$\langle H(p)|\varphi\rangle \geqq \langle[(\sigma_k)_D{:}p]\,|\varphi\rangle\,,$$

and (2.49) follows from (2.38). This concludes the proof of (2.46).

From [29] (Chapter II, Section 4) we have

$$\mathcal{H}(p) = \sup\{\langle\sigma_D|p\rangle : \sigma \in C^\infty(\mathbb{R}^n; \mathbb{M}^{n\times n}_{sym}) \cap \mathcal{K}(\Omega),\ \operatorname{supp}\sigma \cap \Gamma_1 = \emptyset\}\,.$$

This equality, together with (2.35) and (2.46), implies (2.47) and (2.48) with $g = 0$.

Let $\phi \in C^\infty(\mathbb{R})$ be such that $0 \leqq \phi \leqq 1$, $\phi(s) = 0$ for $s \leqq 1$, and $\phi(s) = 1$ for $s \geqq 2$. For $\delta > 0$ we consider the function $\psi_\delta(x) := \phi(\frac{1}{\delta}\operatorname{dist}(x, \Gamma_1))$ defined for every $x \in \overline{\Omega}$. Let $\sigma \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ be such that $[\sigma\nu] = 0$ on Γ_1. Then, $\sigma_\delta := \psi_\delta\sigma + (1-\psi_\delta)\varrho \in \Sigma(\Omega)\cap\mathcal{K}(\Omega)$ and $[\sigma_\delta\nu] = g$ on Γ_1. Moreover, by (2.34) we have

$$\langle(\sigma_\delta)_D|p\rangle = \langle[\sigma_D{:}p]\,|\psi_\delta\rangle + \langle\big[\varrho_D{:}p\big]\,|1-\psi_\delta\rangle\,.$$

Since the right-hand side converges to $\langle\sigma_D|p\rangle$ as $\delta \to 0$, equality (2.48) follows from the equality already proved for $g = 0$ and from (2.46). □

3. The minimum problem

In this section we study in detail the minimum problem used in the incremental formulation of the quasistatic evolution. The data are the current values $p_0 \in \Pi_{\Gamma_0}(\Omega)$ of the plastic strain and the updated values $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, $f \in L^n(\Omega; \mathbb{R}^n)$, and $g \in L^\infty(\Gamma_1; \mathbb{R}^n)$, of the boundary displacement and of the body and surface loads. The total load $\mathcal{L} \in BD(\Omega)'$ is defined by

$$\langle\mathcal{L}|u\rangle := \langle f|u\rangle + \langle g|u\rangle_{\Gamma_1} \tag{3.1}$$

for every $u \in BD(\Omega)$. By solving the minimum problem

$$\min_{(u,e,p)\in A(w)} \{\mathcal{Q}(e) + \mathcal{H}(p - p_0) - \langle\mathcal{L}|u\rangle\}, \tag{3.2}$$

we get the updated values u, e, and p of displacement, elastic and plastic strain.

For the existence result we will assume the following safe-load condition: there exist $\varrho \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $\alpha > 0$ such that

$$-\operatorname{div}\varrho = f \ \text{ a.e. on } \Omega\,, \qquad \big[\varrho\nu\big] = g \text{ on } \Gamma_1\,, \tag{3.3}$$

and

$$\varrho_D(x) + \xi \in K \tag{3.4}$$

for a.e. $x \in \Omega$, and for every $\xi \in \mathbb{M}^{n\times n}_D$ with $|\xi| \leqq \alpha$.

3.1. Existence of a minimizer

We begin by proving two technical lemmas concerning the safe-load condition.

Lemma 3.1. *Let* $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $f \in L^n(\Omega;\mathbb{R}^n)$, $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$, *and let* $\mathcal{L}$ *be defined by* (3.1). *Assume* (2.1), (2.2), (3.3), *and* (3.4). *Then,*

$$\langle \mathcal{L}|u\rangle = \langle \varrho|e\rangle + \langle \varrho_D|p\rangle - \langle \varrho|Ew\rangle + \langle \mathcal{L}|w\rangle$$

for every $(u,e,p) \in A(w)$.

Proof. The result follows from the definition (2.39) of the duality product $\langle \varrho_D|p\rangle$, and from the integration by parts formula (2.40). □

The following lemma shows the coerciveness of the functional $\mathcal{H}(p) - \langle \varrho_D|p\rangle$.

Lemma 3.2. *Let* $f \in L^n(\Omega;\mathbb{R}^n)$, $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$, $\varrho \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, *and* $\alpha > 0$. *Assume* (2.1), (2.2), (3.3), *and* (3.4). *Then,*

$$\mathcal{H}(p) - \langle \varrho_D|p\rangle \geqq \alpha \|p\|_1 \tag{3.5}$$

for every $p \in \Pi_{\Gamma_0}(\Omega)$.

Proof. By Proposition 2.4 we have

$$\begin{aligned}\mathcal{H}(p) - \langle \varrho_D|p\rangle &= \sup\{\langle \sigma_D - \varrho_D|p\rangle : \sigma \in \Sigma(\Omega)\cap\mathcal{K}(\Omega)\}\\ &\geqq \sup\{\langle \tau_D|p\rangle : \tau \in \Sigma(\Omega),\ \|\tau_D\|_\infty \leqq \alpha\}.\end{aligned}$$

From (2.35) it follows that

$$\mathcal{H}(p) - \langle \varrho_D|p\rangle \geqq \sup\{\langle \tau_D|p\rangle : \tau \in C^\infty(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}),\ \|\tau_D\|_\infty \leqq \alpha\},$$

where the duality product in the right-hand side is defined by (2.4). The conclusion now follows from standard arguments in measure theory. □

We are now in a position to prove the existence of a solution to (3.2).

Theorem 3.3. *Let* $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $p_0 \in \Pi_{\Gamma_0}(\Omega)$, $f \in L^n(\Omega;\mathbb{R}^n)$, $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$, *and let* $\mathcal{L}$ *be defined by* (3.1). *Assume* (2.1), (2.2), (3.3), *and* (3.4). *Then, the minimum problem* (3.2) *has a solution.*

Proof. By Lemma 3.1 the minimum problem (3.2) is equivalent to

$$\min_{(u,e,p)\in A(w)} \{\mathcal{Q}(e) - \langle \varrho|e\rangle + \mathcal{H}(p-p_0) - \langle \varrho_D|p-p_0\rangle\}, \tag{3.6}$$

in the sense that these problems have the same solutions. Let $(u_k,e_k,p_k) \in A(w)$ be a minimizing sequence. By Lemma 3.2

$$\mathcal{H}(p_k - p_0) - \langle \varrho_D|p_k - p_0\rangle \geqq \alpha\|p_k - p_0\|_1,$$

while (2.13) gives

$$\mathcal{Q}(e_k) - \langle \varrho | e_k \rangle \geqq \frac{\alpha_{\mathbb{C}}}{2} \|e_k\|_2^2 - \frac{1}{2\alpha_{\mathbb{C}}} \|\varrho\|_2^2 .$$

Therefore, the sequences e_k and p_k are bounded in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and in $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, respectively. Since $Eu_k = e_k + p_k$ in Ω, it follows that Eu_k is bounded in M_b $(\Omega; \mathbb{M}^{n\times n}_{sym})$. Since $(w - u_k) \odot \nu \mathcal{H}^{n-1} = p_k$ is bounded in $M_b(\Gamma_0; \mathbb{M}^{n\times n}_D)$, the traces of u_k are bounded in $L^1(\Gamma_0; \mathbb{R}^n)$. Therefore, u_k is bounded in $BD(\Omega)$ by (2.3). Up to extracting a subsequence, we may assume that $u_k \rightharpoonup u$ weakly* in $BD(\Omega)$, $e_k \rightharpoonup e$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $p_k \rightharpoonup p$ weakly* in $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$. By Lemma 2.1 we obtain $(u, e, p) \in A(w)$. By lower semicontinuity

$$\mathcal{Q}(e) - \langle \varrho | e \rangle \leqq \liminf_{k\to\infty} \{\mathcal{Q}(e_k) - \langle \varrho | e_k \rangle\} . \tag{3.7}$$

To conclude we just need to show that

$$\mathcal{H}(p - p_0) - \langle \varrho_D | p - p_0 \rangle \leqq \liminf_{k\to\infty} \{\mathcal{H}(p_k - p_0) - \langle \varrho_D | p_k - p_0 \rangle\} . \tag{3.8}$$

To this aim, let $\phi \in C^\infty(\mathbb{R})$ be such that $0 \leqq \phi \leqq 1$, $\phi(s) = 0$ for $s \leqq 1$, and $\phi(s) = 1$ for $s \geqq 2$. Let $\delta > 0$ and $\psi_\delta(x) := \phi(\frac{1}{\delta}\mathrm{dist}(x, \Gamma_1))$ for every $x \in \overline{\Omega}$. Since the measure $H(p_k - p_0) - \big[\varrho_D{:}(p_k - p_0)\big]$ is nonnegative on $\Omega \cup \Gamma_0$ by (2.46),

$$\mathcal{H}(\psi_\delta(p_k - p_0)) - \big\langle \big[\varrho_D{:}(p_k - p_0)\big] \,|\, \psi_\delta \big\rangle \leqq \mathcal{H}(p_k - p_0) - \langle \varrho_D | p_k - p_0 \rangle \tag{3.9}$$

for every $\delta > 0$. The integration by parts formula (2.41) gives

$$\begin{aligned}\langle \big[\varrho_D{:}(p_k - p_0)\big] \,|\, \psi_\delta \rangle = &-\langle \varrho{:}(e_k - Ew) | \psi_\delta \rangle - \langle \varrho | (u_k - w) \odot \nabla \psi_\delta \rangle \\ &+ \langle f \,|\, \psi_\delta (u_k - w) \rangle - \big\langle \big[\varrho_D{:}p_0\big] \,|\, \psi_\delta \big\rangle .\end{aligned}$$

Passing to the limit as $k \to \infty$, and using (2.41) again, we deduce that

$$\big\langle \big[\varrho_D{:}(p - p_0)\big] \,|\, \psi_\delta \big\rangle = \lim_{k\to\infty} \big\langle \big[\varrho_D{:}(p_k - p_0)\big] \,|\, \psi_\delta \big\rangle . \tag{3.10}$$

By (3.9), (3.10), and the lower semicontinuity of $\mathcal{H}$, we obtain

$$\mathcal{H}(\psi_\delta(p - p_0)) - \big\langle \big[\varrho_D{:}(p - p_0)\big] \,|\, \psi_\delta \big\rangle \leqq \liminf_{k\to\infty} \{\mathcal{H}(p_k - p_0) - \langle \varrho_D | p_k - p_0 \rangle\} .$$

Passing to the limit as $\delta \to 0$, we finally obtain (3.8).

Since (u_k, e_k, p_k) is a minimizing sequence and $(u, e, p) \in A(w)$, by (3.7) and (3.8) we conclude that (u, e, p) is a minimizer of (3.6). □

3.2. The Euler conditions

We now derive the Euler conditions for a minimizer of (3.2) in the special case $p = p_0$.

Theorem 3.4. *Let $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $f \in L^n(\Omega;\mathbb{R}^n)$, $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$, and let $\mathcal{L}$ be defined by* (3.1). *Suppose that (u, e, p) is a solution of* (3.2) *with $p_0 = p$, and let $\sigma := \mathbb{C}e$. Then $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and*

$$-\mathcal{H}(q) \leqq \langle\sigma|\eta\rangle - \langle\mathcal{L}|v\rangle = \langle\sigma_D|\eta_D\rangle + \frac{1}{n}\langle\operatorname{tr}\sigma|\operatorname{div} v\rangle - \langle\mathcal{L}|v\rangle \leqq \mathcal{H}(-q) \tag{3.11}$$

for every $(v, \eta, q) \in A(0)$.

Proof. Let us fix $(v, \eta, q) \in A(0)$. For every $\varepsilon \in \mathbb{R}$, the triple $(u+\varepsilon v, e+\varepsilon\eta, p+\varepsilon q)$ belongs to $A(w)$, and hence

$$\mathcal{Q}(e+\varepsilon\eta) + \mathcal{H}(\varepsilon q) - \varepsilon\langle\mathcal{L}|v\rangle \geqq \mathcal{Q}(e) \qquad \text{for every } \varepsilon \in \mathbb{R}\,.$$

Using the positive homogeneity of H we obtain

$$\mathcal{Q}(e \pm \varepsilon\eta) + \varepsilon\mathcal{H}(\pm q) \mp \varepsilon\langle\mathcal{L}|v\rangle \geqq \mathcal{Q}(e) \qquad \text{for every } \varepsilon > 0\,.$$

Taking the derivative, with respect to ε at $\varepsilon = 0$, we get

$$\langle\sigma|\eta\rangle + \mathcal{H}(q) - \langle\mathcal{L}|v\rangle \geqq 0\,, \qquad -\langle\sigma|\eta\rangle + \mathcal{H}(-q) + \langle\mathcal{L}|v\rangle \geqq 0\,,$$

which implies (3.11). □

The following proposition shows that σ satisfies the Euler conditions obtained in Theorem 3.4 if, and only if, it satisfies the stress constraint and the equilibrium condition on Ω, as well as the boundary condition on Γ_1.

Proposition 3.5. *Assume* (2.1) *and* (2.2). *Let $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $f \in L^n(\Omega;\mathbb{R}^n)$, $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$, and let $\mathcal{L}$ be defined by* (3.1). *The following conditions are equivalent:*

(a) $-\mathcal{H}(q) \leqq \langle\sigma|\eta\rangle - \langle\mathcal{L}|v\rangle \leqq \mathcal{H}(-q)$ *for every $(v, \eta, q) \in A(0)$;*
(b) $\sigma \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\operatorname{div}\sigma = f$ *a.e. on Ω, and $[\sigma\nu] = g$ on Γ_1.*

Proof. Assume (a) and let $v \in H^1(\Omega;\mathbb{R}^n)$ with $v = 0$ $\mathcal{H}^{n-1}$ a.e. on Γ_0. Since the triple $(v, Ev, 0)$ belongs to $A(0)$, from (a) we obtain

$$\langle\sigma|Ev\rangle - \langle f|v\rangle - \langle g|v\rangle_{\Gamma_1} = 0\,. \tag{3.12}$$

Since this is true, in particular, for $v \in C^\infty_c(\Omega;\mathbb{R}^n)$, we conclude that $-\operatorname{div}\sigma = f$ on Ω, hence $\operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$. Using the distributional definition (2.24) of $[\sigma\nu]$, from (3.12) we also obtain $[\sigma\nu] = g$ on Γ_1.

Let $\eta \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$. Regarding $-\eta$ as an absolutely continuous measure on $\Omega \cup \Gamma_0$, the triple $(0, \eta, -\eta)$ belongs to $A(0)$, thus from (a) we obtain

$$-\mathcal{H}(-\eta) \leqq \langle \sigma_D | \eta \rangle \leqq \mathcal{H}(\eta)\,.$$

Let us fix $\xi \in \mathbb{M}^{n\times n}_D$. Since for every Borel set $B \subset \Omega$ we can take $\eta(x) = 1_B(x)\,\xi$, we deduce that

$$-H(-\xi) \leqq \sigma_D(x){:}\xi \leqq H(\xi) \qquad \text{for a.e. } x \in \Omega\,.$$

Therefore, $\sigma_D(x) \in \partial H(0)$ for a.e. $x \in \Omega$. As $\partial H(0) = K$ (see e.g. [26] (Corollary 23.5.3)), $\sigma_D(x) \in K$ for a.e. $x \in \Omega$, hence $\sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$ and $\sigma \in \mathcal{K}(\Omega)$.

Conversely, assume (b) and let $(v, \eta, q) \in A(0)$. By Proposition 2.4

$$-\mathcal{H}(-q) \leqq \langle \sigma_D | q \rangle \leqq \mathcal{H}(q)\,. \tag{3.13}$$

From the integration by parts formula (2.40) we get

$$\langle \sigma_D | q \rangle = -\langle \sigma | \eta \rangle + \langle f | v \rangle + \langle g | v \rangle_{\Gamma_1}\,,$$

so that (a) follows now from (3.13). □

The following theorem summarizes the results obtained thus far on the Euler conditions.

Theorem 3.6. *Assume* (2.1) *and* (2.2). *Let* $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, $f \in L^n(\Omega; \mathbb{R}^n)$, $g \in L^\infty(\Omega; \mathbb{R}^n)$, *let* $(u, e, p) \in A(w)$, *let* $\sigma := \mathbb{C}e$, *and let* $\mathcal{L}$ *be defined by* (3.1). *The following conditions are then equivalent:*

(a) (u, e, p) *is a solution of (3.2) with* $p_0 = p$*;*
(b) $-\mathcal{H}(q) \leqq \langle \sigma | \eta \rangle - \langle \mathcal{L} | v \rangle \leqq \mathcal{H}(-q)$ *for every* $(v, \eta, q) \in A(0)$*;*
(c) $\sigma \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\operatorname{div} \sigma = f$ *a.e. on* Ω, *and* $[\sigma \nu] = g$ *on* Γ_1.

Proof. The implication (a) $\Rightarrow$ (b) was proved in Theorem 3.4. The converse is true by convexity. The equivalence (b) $\Leftrightarrow$ (c) was proved in Proposition 3.5. □

Theorem 3.6 immediately gives a stability result with respect to weak convergence of the data.

Theorem 3.7. *Assume* (2.1) *and* (2.2). *Let* w_k, f_k, g_k *be sequences in* $H^1(\mathbb{R}^n; \mathbb{R}^n)$, $L^n(\Omega; \mathbb{R}^n)$, $L^\infty(\Omega; \mathbb{R}^n)$, *respectively, let* $\mathcal{L}_k$ *be defined by* (3.1) *with* $f = f_k$ *and* $g = g_k$, *and let* $(u_k, e_k, p_k) \in A(w_k)$. *Assume that* $u_k \rightharpoonup u_\infty$ *weakly* in* $BD(\Omega)$, $e_k \rightharpoonup e_\infty$ *weakly in* $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $p_k \rightharpoonup p_\infty$ *weakly* in* $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, $w_k \rightharpoonup w_\infty$ *weakly in* $H^1(\mathbb{R}^n; \mathbb{R}^n)$, $f_k \rightharpoonup f_\infty$ *weakly in* $L^n(\Omega; \mathbb{R}^n)$, $g_k \rightharpoonup g_\infty$ *weakly* in* $L^\infty(\Omega; \mathbb{R}^n)$, *and let* $\mathcal{L}_\infty$ *be defined by* (3.1) *with* $f = f_\infty$ *and* $g = g_\infty$. *If*

$$\mathcal{Q}(e_k) - \langle \mathcal{L}_k | u_k \rangle \leqq \mathcal{Q}(\eta) + \mathcal{H}(q - p_k) - \langle \mathcal{L}_k | v \rangle \tag{3.14}$$

for every k *and every* $(v, \eta, q) \in A(w_k)$, *then* $(u_\infty, e_\infty, p_\infty) \in A(w_\infty)$ *and*

$$\mathcal{Q}(e_\infty) - \langle \mathcal{L}_\infty | u_\infty \rangle \leqq \mathcal{Q}(\eta) + \mathcal{H}(q - p_\infty) - \langle \mathcal{L}_\infty | v \rangle \tag{3.15}$$

for every $(v, \eta, q) \in A(w_\infty)$.

Proof. First, we note that $(u_\infty, e_\infty, p_\infty) \in A(w_\infty)$ by Lemma 2.1. Let $\sigma_k := \mathbb{C}e_k$ and $\sigma_\infty := \mathbb{C}e_\infty$. If (3.14) holds, then $u_k, e_k, p_k, w_k, f_k, g_k$ satisfy condition (a) of Theorem 3.6. By condition (c) of Theorem 3.6 we have $\sigma_k \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\operatorname{div} \sigma_k = f_k$ a.e. on Ω, and $[\sigma_k \nu] = g_k$ on Γ_1.

Since $e_k \rightharpoonup e_\infty$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $\sigma_k \rightharpoonup \sigma_\infty$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. As $\mathcal{K}(\Omega)$ is closed and convex in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, we deduce that $\sigma_\infty \in \mathcal{K}(\Omega)$. Since $-\operatorname{div} \sigma_k = f_k$ a.e. on Ω and $f_k \rightharpoonup f_\infty$ weakly in $L^n(\Omega; \mathbb{R}^n)$, $-\operatorname{div} \sigma_\infty = f_\infty$ a.e. on Ω. Hence, $\sigma_\infty \in \Sigma(\Omega)$. Moreover, from (2.24) it follows that $[\sigma_k \nu] \rightharpoonup [\sigma_\infty \nu]$ weakly in $H^{-1/2}(\partial\Omega; \mathbb{R}^n)$. As $[\sigma_k \nu] = g_k$ on Γ_1 and $g_k \rightharpoonup g_\infty$ weakly* in $L^\infty(\Omega; \mathbb{R}^n)$, we conclude that $[\sigma_\infty \nu] = g_\infty$ on Γ_1. Therefore, $u_\infty, e_\infty, p_\infty, w_\infty, f_\infty, g_\infty$ satisfy condition (c) of Theorem 3.6. Inequality (3.15) follows now from condition (a) of Theorem 3.6. □

3.3. Continuous dependence on the data

We complete our study of the solutions (u, e, p) of the minimum problem (3.2) for the special case $p = p_0$ by proving the continuous dependence, in the norm topology, of u and e on the data p_0, w, f, and g. In particular we prove a Hölder continuous dependence of e on p_0. If $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution according to Definition 4.2 below, this continuity estimate is not enough to deduce that $t \mapsto e(t)$ has bounded variation from the fact that $t \mapsto p(t)$ has bounded variation. However, it allows us to prove that $t \mapsto e(t)$ is continuous, except for a countable set of values of t, which is an important ingredient for an elementary proof of Theorem 4.7.

Theorem 3.8. *For $i = 1, 2$ let $w_i \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, $f_i \in L^n(\Omega; \mathbb{R}^n)$, $g_i \in L^\infty(\Gamma_1; \mathbb{R}^n)$, and let $\mathcal{L}_i$ be defined by* (3.1) *with $f = f_i$ and $g = g_i$. Suppose that (u_i, e_i, p_i) is a solution of* (3.2) *with $p_0 = p_i$, $w = w_i$, $\mathcal{L} = \mathcal{L}_i$, and let*

$$\begin{aligned}\omega_{12} := {}& \|p_2 - p_1\|_1 + \|p_2 - p_1\|_1^{1/2} \\ & + \|f_2 - f_1\|_n + \|g_2 - g_1\|_{\infty,\Gamma_1} + \|Ew_2 - Ew_1\|_2 .\end{aligned}$$

Then

$$\|e_2 - e_1\|_2 \leqq C_1\, \omega_{12}\,, \tag{3.16}$$

$$\|Eu_2 - Eu_1\|_1 \leqq C_2\, \omega_{12}\,, \tag{3.17}$$

$$\|u_2 - u_1\|_1 \leqq C_3\, (\omega_{12} + \|w_2 - w_1\|_2)\,, \tag{3.18}$$

where C_1, C_2, and C_3 are positive constants depending only on R_K, $\alpha_{\mathbb{C}}$, $\beta_{\mathbb{C}}$, Ω, and Γ_0.

Proof. Let $v := (u_2 - w_2) - (u_1 - w_1)$, $\eta := (e_2 - Ew_2) - (e_1 - Ew_1)$, and $q := p_2 - p_1$. Since $(v, \eta, q) \in AP(0)$, by Theorem 3.4 we obtain

$$\begin{aligned} -\mathcal{H}(p_2 - p_1) \leqq {}& \langle \mathbb{C}e_1 | \eta\rangle - \langle f_1 | v\rangle - \langle g_1 | v\rangle_{\Gamma_1}\,, \\ & \langle \mathbb{C}e_2 | \eta\rangle - \langle f_2 | v\rangle - \langle g_2 | v\rangle_{\Gamma_1} \leqq \mathcal{H}(p_1 - p_2)\,. \end{aligned}$$

Adding term by term, and using (2.7), we obtain

$$\langle \mathbb{C}(e_2 - e_1) | e_2 - e_1 \rangle \leqq \langle \mathbb{C}(e_2 - e_1) | Ew_2 - Ew_1 \rangle + \langle f_2 - f_1 | v \rangle$$
$$+ \langle g_2 - g_1 | v \rangle_{\Gamma_1} + 2\, R_K \| p_2 - p_1 \|_1 \, .$$

By (2.13) and (2.14), this implies

$$2\, \alpha_{\mathbb{C}} \| e_2 - e_1 \|_2^2 \leqq 2\, \beta_{\mathbb{C}} \| e_2 - e_1 \|_2 \, \| Ew_2 - Ew_1 \|_2$$
$$+ \| f_2 - f_1 \|_n \, \| v \|_{n/(n-1)}$$
$$+ \| g_2 - g_1 \|_{\infty, \Gamma_1} \| v \|_{1, \Gamma_1} + 2\, R_K \| p_2 - p_1 \|_1 \, . \quad (3.19)$$

Since the embedding of $BD(\Omega)$ into $L^{n/(n-1)}(\Omega; \mathbb{R}^n)$ is continuous, there exists a constant A_1, depending only on Ω, such that

$$\| v \|_{n/(n-1)} \leqq A_1 \, \| v \|_1 + A_1 \, \| Ev \|_1 \, . \quad (3.20)$$

By (2.3) there exists a constant $C > 0$, depending only on Ω and Γ_0, such that

$$\| v \|_1 \leqq C \, \| v \|_{1, \Gamma_0} + C \, \| Ev \|_1 \, . \quad (3.21)$$

Since $p_2 - p_1 = -v \odot \nu \, \mathcal{H}^{n-1}$ on Γ_0,

$$\| v \|_{1, \Gamma_0} \leqq \sqrt{2} \, \| p_2 - p_1 \|_1 \, . \quad (3.22)$$

As $Ev = (e_2 - e_1) + (p_2 - p_1) - (Ew_2 - Ew_1)$, by the Hölder inequality we obtain

$$\| Ev \|_1 \leqq \mathcal{L}^n(\Omega)^{1/2} \| e_2 - e_1 \|_2 + \| p_2 - p_1 \|_1$$
$$+ \mathcal{L}^n(\Omega)^{1/2} \| Ew_2 - Ew_1 \|_2 \, . \quad (3.23)$$

By (3.20)–(3.23) there exists a constant A_2, depending only on Ω and Γ_0, such that

$$\| v \|_{n/(n-1)} \leqq A_2 \, \| e_2 - e_1 \|_2 + A_2 \, \| p_2 - p_1 \|_1 + A_2 \, \| Ew_2 - Ew_1 \|_2 \, . \quad (3.24)$$

Since the trace operator is continuous from $BD(\Omega)$ into $L^1(\partial\Omega; \mathbb{R}^n)$, there exists a constant B_1, depending only on Ω, such that

$$\| v \|_{1, \Gamma_1} \leqq B_1 \, \| v \|_1 + B_1 \, \| Ev \|_1 \, .$$

From this inequality, and from (3.21)–(3.23), we deduce that there exists a constant B_2, depending only on Ω and Γ_0, such that

$$\| v \|_{1, \Gamma_1} \leqq B_2 \, \| e_2 - e_1 \|_2 + B_2 \, \| p_2 - p_1 \|_1 + B_2 \, \| Ew_2 - Ew_1 \|_2 \, . \quad (3.25)$$

Therefore (3.19), (3.24), and (3.25) imply that

$$2\, \alpha_{\mathbb{C}} \, \| e_2 - e_1 \|_2^2$$
$$\leqq 2\, \beta_{\mathbb{C}} \, \| e_2 - e_1 \|_2 \, \| Ew_2 - Ew_1 \|_2 + A_2 \, \| f_2 - f_1 \|_n \, \| e_2 - e_1 \|_2$$
$$+ A_2 \, \| f_2 - f_1 \|_n \, \| p_2 - p_1 \|_1 + A_2 \, \| f_2 - f_1 \|_n \, \| Ew_2 - Ew_1 \|_2$$
$$+ B_2 \, \| g_2 - g_1 \|_{\infty, \Gamma_1} \, \| e_2 - e_1 \|_2 + B_2 \, \| g_2 - g_1 \|_{\infty, \Gamma_1} \, \| p_2 - p_1 \|_1$$
$$+ B_2 \, \| g_2 - g_1 \|_{\infty, \Gamma_1} \, \| Ew_2 - Ew_1 \|_2 + 2\, R_K \| p_2 - p_1 \|_1 \, ,$$

which yields (3.16) by the Cauchy inequality.

As $Eu_i = e_i + p_i$ in Ω by (2.21), by the Hölder inequality we obtain

$$\|Eu_2 - Eu_1\|_1 \leqq \mathcal{L}^n(\Omega)^{1/2} \|e_2 - e_1\|_2 + \|p_2 - p_1\|_1 ,$$

so that (3.16) gives (3.17).

Since $p_2 - p_1 = [(w_2 - w_1) - (u_2 - u_1)] \odot \nu \, \mathcal{H}^{n-1}$ on Γ_0, we have

$$\|u_2 - u_1\|_{1,\Gamma_0} \leqq \|w_2 - w_1\|_{1,\Gamma_0} + \sqrt{2} \, \|p_2 - p_1\|_1 .$$

The continuity of the trace operator from $H^1(\Omega; \mathbb{R}^n)$ into $L^1(\partial\Omega; \mathbb{R}^n)$ implies that there exists a constant M, depending only on Ω, such that

$$\|u_2 - u_1\|_{1,\Gamma_0} \leqq M \, \|w_2 - w_1\|_2 + M \, \|Ew_2 - Ew_1\|_2 + \sqrt{2} \, \|p_2 - p_1\|_1 .$$

By (2.3) there exists a constant C, depending only on Ω and Γ_0, such that

$$\begin{aligned} \|u_2 - u_1\|_1 &\leqq C \, \|u_2 - u_1\|_{1,\Gamma_0} + C \, \|Eu_2 - Eu_1\|_1 \\ &\leqq C \, M \, \|w_2 - w_1\|_2 + C \, M \, \|Ew_2 - Ew_1\|_2 \\ &\quad + \sqrt{2} \, C \, \|p_2 - p_1\|_1 + C \, \|Eu_2 - Eu_1\|_1 . \end{aligned}$$

Inequality (3.18) now follows from (3.17). □

Remark 3.9. Theorem 3.8 implies that if (u_1, e_1, p_0) and (u_2, e_2, p_0) are solutions to problem (3.2) with the same w, f, and g, then $u_1 = u_2$ and $e_1 = e_2$ a.e. on Ω.

4. Quasistatic evolution

We now consider time-dependent boundary conditions $w(t)$ satisfying (2.15), as well as body and surface forces $f(t)$ and $g(t)$ satisfying the regularity assumption (2.16) and the uniform safe-load condition (2.17)–(2.19). For every $t \in [0, T]$ the total load $\mathcal{L}(t) \in BD(\Omega)'$ applied at time t is defined by

$$\langle \mathcal{L}(t) | u \rangle := \langle f(t) | u \rangle + \langle g(t) | u \rangle_{\Gamma_1} \tag{4.1}$$

for every $u \in BD(\Omega)$.

Remark 4.1. From (2.16) it follows that the weak* limit

$$\dot{\mathcal{L}}(t) := w^* - \lim_{s \to t} \frac{\mathcal{L}(s) - \mathcal{L}(t)}{s - t}$$

exists in $BD(\Omega)'$ for a.e. $t \in [0, T]$, and that

$$\langle \dot{\mathcal{L}}(t) | u \rangle = \langle \dot{f}(t) | u \rangle + \langle \dot{g}(t) | u \rangle_{\Gamma_1} \tag{4.2}$$

for every $u \in BD(\Omega)$. Therefore, the function $t \mapsto \langle \dot{\mathcal{L}}(t) | u(t) \rangle$ belongs to $L^1([0, T])$ whenever $t \mapsto u(t)$ belongs to $L^\infty([0, T]; BD(\Omega))$.

From (2.17)–(2.19) we find that $\dot\varrho(t) \in \Sigma(\Omega)$ for a.e. $t \in [0, T]$ and

$$-\mathrm{div}\,\dot\varrho(t) = \dot f(t) \text{ a.e. on } \Omega\,, \qquad \big[\dot\varrho(t)\nu\big] = \dot g(t) \text{ on } \Gamma_1\,.$$

Furthermore, owing to (2.38), we can prove that for every $p \in \Pi_{\Gamma_0}(\Omega)$, the function $s \mapsto \langle \varrho_D(s)|p\rangle$ is differentiable at each $t \in [0, T]$ where $\dot\varrho(t)$ exists and (2.20) holds with the derivative given by $\langle \dot\varrho_D(t)|p\rangle$. This implies that $t \mapsto \langle \dot\varrho_D(t)|p(t)\rangle$ is measurable for every simple function $t \mapsto p(t)$ from $[0, T]$ into $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ with $p(t) \in \Pi_{\Gamma_0}(\Omega)$ for a.e. $t \in [0, T]$. By approximation we conclude that $t \mapsto \langle \dot\varrho_D(t)|p(t)\rangle$ belongs to $L^1([0, T])$ whenever $t \mapsto p(t)$ belongs to $L^\infty([0, T]; M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D))$ and $p(t) \in \Pi_{\Gamma_0}(\Omega)$ for a.e. $t \in [0, T]$.

A function $p\colon [0, T] \to M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ will be regarded as a function defined on the time interval $[0, T]$ with values in the dual of the separable Banach space $C_0(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$. Therefore, for every $s, t \in [0, T]$ with $s \leqq t$ the total variation of p on $[s, t]$ is defined by

$$\mathcal{V}(p; s, t) = \sup\Big\{ \sum_{j=1}^{N} \|p(t_j) - p(t_{j-1})\|_1 : \\ s = t_0 \leqq t_1 \leqq \ldots \leqq t_N = t,\ N \in \mathbb{N}\Big\}.$$

By (2.10) all results proved in the Appendix with $X = M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, $Y = C_0(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, and $\mathcal{K} = \mathcal{K}_D(\Omega) \cap C_0(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ can be applied to $\mathcal{H}$. The $\mathcal{H}$-variation of p on $[s, t]$, which will play the role of the dissipation in the time interval $[s, t]$, is denoted $\mathcal{D}_{\mathcal{H}}(p; s, t)$ and is defined by

$$\mathcal{D}_{\mathcal{H}}(p; s, t) := \sup\Big\{ \sum_{j=1}^{N} \mathcal{H}(p(t_j) - p(t_{j-1})) : \\ s = t_0 \leqq t_1 \leqq \ldots \leqq t_N = t,\ N \in \mathbb{N}\Big\}.$$

4.1. Definition of quasistatic evolution

We are now in a position to introduce the following definition.

Definition 4.2. A *quasistatic evolution* is a function $t \mapsto (u(t), e(t), p(t))$ from $[0, T]$ into $BD(\Omega)\times L^2(\Omega; \mathbb{M}^{n\times n}_{sym})\times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ which satisfies the following conditions:

(qs1) *global stability*: for every $t \in [0, T]$ we have $(u(t), e(t), p(t)) \in A(w(t))$ and

$$\mathcal{Q}(e(t)) - \langle \mathcal{L}(t)|u(t)\rangle \leqq \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - \langle \mathcal{L}(t)|v\rangle \tag{4.3}$$

for every $(v, \eta, q) \in A(w(t))$;

(qs2) *energy balance*: the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t \in [0,T]$

$$\begin{aligned} &\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p;0,t) - \langle \mathcal{L}(t)|u(t)\rangle \\ &\quad = \mathcal{Q}(e(0)) - \langle \mathcal{L}(0)|u(0)\rangle + \int_0^t \{\langle \sigma(s)|E\dot{w}(s)\rangle - \langle \mathcal{L}(s)|\dot{w}(s)\rangle\}\,ds \\ &\qquad - \int_0^t \langle \dot{\mathcal{L}}(s)|u(s)\rangle\,ds\,, \end{aligned} \tag{4.4}$$

where $\sigma(t) := \mathbb{C}e(t)$.

Remark 4.3. Since the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ has bounded variation, it is bounded and the set of its discontinuity points (in the strong topology) is at most countable (see e.g. [4] (Lemma A.1)). By Theorem 3.8 the same properties hold for the functions $t \mapsto e(t)$ and $t \mapsto \sigma(t)$ from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and for the function $t \mapsto u(t)$ from $[0,T]$ into $BD(\Omega)$. Therefore, $t \mapsto e(t)$ and $t \mapsto \sigma(t)$ belong to $L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$, and $t \mapsto u(t)$ belongs to $L^\infty([0,T]; BD(\Omega))$. As $t \mapsto E\dot{w}(t)$ belongs to $L^1([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ and $t \mapsto \dot{w}(t)$ belongs to $L^1([0,T]; H^1(\mathbb{R}^n; \mathbb{R}^n))$, the integrals in the right-hand side of (4.4) are well defined (see Remark 4.1).

The following theorem gives an equivalent formulation of conditions (qs1) and (qs2), which uses the function $t \mapsto \varrho(t)$ introduced in the uniform safe-load condition of Section 2.2.

Theorem 4.4. *Assume* (2.1), (2.2), (2.7), (2.12), (2.13), *and* (2.15)–(2.19). *Then, a function* $t \mapsto (u(t), e(t), p(t))$ *from* $[0,T]$ *into* $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ *is a quasistatic evolution if, and only if, it satisfies the following conditions:*

(qs1′) *for every* $t \in [0,T]$ *we have* $(u(t), e(t), p(t)) \in A(w(t))$ *and*

$$\begin{aligned} &\mathcal{Q}(e(t)) - \langle \varrho(t)|e(t)\rangle \\ &\quad \leqq \mathcal{Q}(\eta) - \langle \varrho(t)|\eta\rangle + \mathcal{H}(q - p(t)) - \langle \varrho_D(t)|q - p(t)\rangle \end{aligned} \tag{4.5}$$

for every $(v, \eta, q) \in A(w(t))$;

(qs2′) *the function* $t \mapsto p(t)$ *from* $[0,T]$ *into* $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ *has bounded variation and for every* $t \in [0,T]$

$$\begin{aligned} &\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p;0,t) - \langle \varrho(t)|e(t) - Ew(t)\rangle - \langle \varrho_D(t)|p(t)\rangle \\ &\quad = \mathcal{Q}(e(0)) - \langle \varrho(0)|e(0) - Ew(0)\rangle - \langle \varrho_D(0)|p(0)\rangle \\ &\qquad + \int_0^t \langle \sigma(s)|E\dot{w}(s)\rangle\,ds \\ &\qquad - \int_0^t \{\langle \dot{\varrho}(s)|e(s) - Ew(s)\rangle + \langle \dot{\varrho}_D(s)|p(s)\rangle\}\,ds\,, \end{aligned} \tag{4.6}$$

where $\sigma(t) := \mathbb{C}e(t)$.

Proof. The equivalence of conditions (qs1) and (qs1′) follows from Lemma 3.1.

As the functions $t \mapsto f(t)$, $t \mapsto g(t)$, and $t \mapsto w(t)$ are absolutely continuous from $[0, T]$ into $L^n(\Omega; \mathbb{R}^n)$, $L^\infty(\Gamma_1; \mathbb{R}^n)$, and $H^1(\mathbb{R}^n; \mathbb{R}^n)$, respectively, the function $t \mapsto \langle \mathcal{L}(t)|w(t)\rangle$ is absolutely continuous on $[0, T]$ and its time derivative is given by $t \mapsto \langle \dot{\mathcal{L}}(t)|w(t)\rangle + \langle \mathcal{L}(t)|\dot{w}(t)\rangle$. It follows that

$$\int_0^t \{\langle \dot{\mathcal{L}}(s)|w(s)\rangle + \langle \mathcal{L}(s)|\dot{w}(s)\rangle\}\, ds = \langle \mathcal{L}(t)|w(t)\rangle - \langle \mathcal{L}(0)|w(0)\rangle\,. \tag{4.7}$$

By Lemma 3.1 we have

$$\langle \mathcal{L}(t)|v\rangle = \langle \varrho(t)|\eta - Ez\rangle + \langle \varrho_D(t)|q\rangle + \langle \mathcal{L}(t)|z\rangle \tag{4.8}$$

for every $t \in [0, T]$, $z \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, and $(v, \eta, q) \in A(z)$. Taking the derivative with respect to t (see Remark 4.1), we obtain

$$\langle \dot{\mathcal{L}}(t)|v\rangle = \langle \dot{\varrho}(t)|\eta - Ez\rangle + \langle \dot{\varrho}_D(t)|q\rangle + \langle \dot{\mathcal{L}}(t)|z\rangle$$

for a.e. $t \in [0, T]$, for every $z \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, and every $(v, \eta, q) \in A(z)$.

If conditions (qs1) or (qs1′) hold, then by Remark 4.3 the function $t \mapsto (u(t), e(t), p(t))$ belongs to $L^\infty([0, T]; BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D))$. As $(u(t), e(t), p(t)) \in A(w(t))$ for every $t \in [0, T]$,

$$\langle \dot{\mathcal{L}}(t)|u(t)\rangle = \langle \dot{\varrho}(t)|e(t) - Ew(t)\rangle + \langle \dot{\varrho}_D(t)|p(t)\rangle + \langle \dot{\mathcal{L}}(t)|w(t)\rangle$$

for a.e. $t \in [0, T]$. Therefore, (4.7) implies that

$$\begin{aligned}\int_0^t \{\langle \dot{\mathcal{L}}(s)|u(s)\rangle &+ \langle \mathcal{L}(s)|\dot{w}(s)\rangle\}\, ds \\ &= \langle \mathcal{L}(t)|w(t)\rangle - \langle \mathcal{L}(0)|w(0)\rangle \\ &\quad + \int_0^t \{\langle \dot{\varrho}(s)|e(s) - Ew(s)\rangle + \langle \dot{\varrho}_D(s)|p(s)\rangle\}\, ds\,.\end{aligned}$$

The equivalence of conditions (qs2) and (qs2′) now follows from this equality and (4.8). □

4.2. The existence result

The following theorem is the main result of the paper.

Theorem 4.5. *Assume* (2.1), (2.2), (2.7), (2.12), (2.13), *and* (2.15)–(2.19). *Let* $(u_0, e_0, p_0) \in A(w(0))$ *satisfy the stability condition*

$$\mathcal{Q}(e_0) - \langle \mathcal{L}(0)|u_0\rangle \leqq \mathcal{Q}(e) + \mathcal{H}(p - p_0) - \langle \mathcal{L}(0)|v\rangle \tag{4.9}$$

for every $(u, e, p) \in A(w(0))$. *Then, there exists a quasistatic evolution* $t \mapsto (u(t), e(t), p(t))$ *such that* $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$.

Theorem 4.5 will be proved by a time discretization process. First, fix a sequence of subdivisions $(t_k^i)_{0\leqq i\leqq k}$ of the interval $[0, T]$, with

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = T\,, \tag{4.10}$$

$$\lim_{k\to\infty} \max_{1\leqq i\leqq k} (t_k^i - t_k^{i-1}) = 0. \tag{4.11}$$

For $i = 0, \ldots, k$, we set $w_k^i := w(t_k^i)$, $f_k^i := f(t_k^i)$, $g_k^i := g(t_k^i)$, $\mathcal{L}_k^i := \mathcal{L}(t_k^i)$, and $\varrho_k^i := \varrho(t_k^i)$.

For every k, we define u_k^i, e_k^i, and p_k^i by induction. We set $(u_k^0, e_k^0, p_k^0) := (u_0, e_0, p_0)$, which by assumption belongs to $A(w(0))$. For $i = 1, \ldots, k$ we define (u_k^i, e_k^i, p_k^i) as a solution to the incremental problem

$$\min_{(u,e,p)\in A(w_k^i)} \{\mathcal{Q}(e) + \mathcal{H}(p - p_k^{i-1}) - \langle \mathcal{L}_k^i | u\rangle\}\,. \tag{4.12}$$

The existence of a solution to this problem is proved in Theorem 3.3. We recall that by Lemma 3.1 the minimum problem (4.12) is equivalent to

$$\min_{(u,e,p)\in A(w_k^i)} \{\mathcal{Q}(e) - \langle \varrho_k^i | e\rangle + \mathcal{H}(p - p_k^{i-1}) - \langle (\varrho_k^i)_D | p - p_k^{i-1}\rangle\}\,. \tag{4.13}$$

Moreover, by the triangle inequality (2.11), the triple (u_k^i, e_k^i, p_k^i) is also a solution of the problem

$$\min_{(u,e,p)\in A(w_k^i)} \{\mathcal{Q}(e) + \mathcal{H}(p - p_k^i) - \langle \mathcal{L}_k^i | u\rangle\}\,. \tag{4.14}$$

For $i = 0, \ldots, k$ we set $\sigma_k^i := \mathbb{C}e_k^i$, and for every $t \in [0, T]$ we define the piecewise constant interpolations

$$\begin{gathered} u_k(t) := u_k^i\,, \quad e_k(t) := e_k^i\,, \quad p_k(t) := p_k^i\,, \quad \sigma_k(t) := \sigma_k^i\,, \\ w_k(t) := w_k^i\,, \quad f_k(t) := f_k^i\,, \quad g_k(t) := g_k^i\,, \\ \mathcal{L}_k(t) := \mathcal{L}_k^i\,, \quad \varrho_k(t) := \varrho_k^i\,, \end{gathered} \tag{4.15}$$

where i is the largest integer such that $t_k^i \leqq t$. By definition, $(u_k(t), e_k(t), p_k(t)) \in A(w_k(t))$, and by (4.14) we have

$$\mathcal{Q}(e_k(t)) - \langle \mathcal{L}_k(t) | u_k(t)\rangle \leqq \mathcal{Q}(\eta) + \mathcal{H}(q - p_k(t)) - \langle \mathcal{L}_k(t) | v\rangle \tag{4.16}$$

for every $(v, \eta, q) \in A(w_k(t))$.

4.3. The discrete energy inequality

We now derive an energy estimate for the solutions of the incremental problems. Note that a remainder δ_k is needed because the integral terms which appear in the right-hand side of (4.17) provide only an approximate value of the work done by the external forces.

Lemma 4.6. *There exists a sequence $\delta_k \to 0^+$ such that for every k and every $t \in [0, T]$*

$$\begin{aligned}
&\mathcal{Q}(e_k(t)) - \langle \varrho_k(t) | e_k(t) - Ew_k(t) \rangle \\
&\qquad + \sum_{0 < t_k^r \leqq t} \{ \mathcal{H}(p_k^r - p_k^{r-1}) - \langle \varrho_D(t_k^r) | p_k^r - p_k^{r-1} \rangle \} \\
&\leqq \mathcal{Q}(e_0) - \langle \varrho(0) | e_0 - Ew(0) \rangle - \int_0^{t_k^i} \langle \dot{\varrho}(s) | e_k(s) - Ew_k(s) \rangle \, ds \\
&\qquad + \int_0^{t_k^i} \langle \sigma_k(s) | E\dot{w}(s) \rangle \, ds + \delta_k \,, \qquad (4.17)
\end{aligned}$$

where i is the largest integer such that $t_k^i \leqq t$.

The integrals in the right-hand side of (4.17) can be written as

$$\int_0^{t_k^i} \langle \dot{\varrho}(s) | e_k(s) - Ew_k(s) \rangle \, ds = \sum_{j=1}^{i} \langle \varrho_k^j - \varrho_k^{j-1} | e_k^{j-1} - Ew_k^{j-1} \rangle \,,$$

$$\int_0^{t_k^i} \langle \sigma_k(s) | E\dot{w}(s) \rangle \, ds = \sum_{j=1}^{i} \langle \sigma_k^{j-1} | Ew_k^j - Ew_k^{j-1} \rangle \,,$$

where the sums involve only the values of $\varrho(t)$ and $w(t)$ at the discretization points t_k^j. This is the main difference between inequality (4.17) and those considered in [15] (Theorem 4.1).

Proof of Lemma 4.6. We have to prove that there exists a sequence $\delta_k \to 0^+$ such that

$$\begin{aligned}
&\mathcal{Q}(e_k^i) - \langle \varrho_k^i | e_k^i - Ew_k^i \rangle \\
&\qquad + \sum_{r=1}^{i} \{ \mathcal{H}(p_k^r - p_k^{r-1}) - \langle (\varrho_k^r)_D | p_k^r - p_k^{r-1} \rangle \} \\
&\leqq \mathcal{Q}(e_0) - \langle \varrho(0) | e_0 - Ew(0) \rangle - \int_0^{t_k^i} \langle \dot{\varrho}(s) | e_k(s) - Ew_k(s) \rangle \, ds \\
&\qquad + \int_0^{t_k^i} \langle \sigma_k(s) | E\dot{w}(s) \rangle \, ds + \delta_k \qquad (4.18)
\end{aligned}$$

for every k and every $i = 1, \dots, k$.

Let us fix an integer r with $1 \leqq r \leqq i$, and let $v := u_k^{r-1} - w_k^{r-1} + w_k^r$ and $\eta := e_k^{r-1} - Ew_k^{r-1} + Ew_k^r$. Since $(v, \eta, p_k^{r-1}) \in A(w_k^r)$, by the minimality condition (4.13) we obtain

$$\begin{aligned}
&\mathcal{Q}(e_k^r) - \langle \varrho_k^r | e_k^r \rangle + \mathcal{H}(p_k^r - p_k^{r-1}) - \langle (\varrho_k^r)_D | p_k^r - p_k^{r-1} \rangle \\
&\quad \leqq \mathcal{Q}(e_k^{r-1} + Ew_k^r - Ew_k^{r-1}) - \langle \varrho_k^r | e_k^{r-1} + Ew_k^r - Ew_k^{r-1} \rangle \,, \quad (4.19)
\end{aligned}$$

where the quadratic form in the right-hand side can be developed as

$$\begin{aligned}&\mathcal{Q}(e_k^{r-1} + Ew_k^r - Ew_k^{r-1})\\&\quad = \mathcal{Q}(e_k^{r-1}) + \langle \sigma_k^{r-1} | Ew_k^r - Ew_k^{r-1} \rangle + \mathcal{Q}(Ew_k^r - Ew_k^{r-1})\,. \end{aligned} \tag{4.20}$$

From the absolute continuity of w with respect to t we obtain

$$w_k^r - w_k^{r-1} = \int_{t_k^{r-1}}^{t_k^r} \dot{w}(t)\,dt\,,$$

where we use a Bochner integral of a function with values in $H^1(\mathbb{R}^n;\mathbb{R}^n)$. This implies

$$Ew_k^r - Ew_k^{r-1} = \int_{t_k^{r-1}}^{t_k^r} E\dot{w}(t)\,dt\,, \tag{4.21}$$

where we use a Bochner integral of a function with values in $L^2(\mathbb{R}^n;\mathbb{M}^{n\times n}_{sym})$. By (2.13) and (4.21)

$$\mathcal{Q}(Ew_k^r - Ew_k^{r-1}) \leqq \beta_{\mathbb{C}}\Big(\int_{t_k^{r-1}}^{t_k^r} \|E\dot{w}(t)\|_2\,dt\Big)^2. \tag{4.22}$$

From the absolute continuity of ϱ with respect to t we obtain

$$\begin{aligned}&\langle \varrho_k^r | e_k^{r-1} - Ew_k^{r-1}\rangle\\&\quad = \langle \varrho_k^{r-1} | e_k^{r-1} - Ew_k^{r-1}\rangle + \int_{t_k^{r-1}}^{t_k^r} \langle \dot{\varrho}(t) | e_k^{r-1} - Ew_k^{r-1}\rangle\,dt\,.\end{aligned} \tag{4.23}$$

By (4.19)–(4.23) we obtain

$$\begin{aligned}&\mathcal{Q}(e_k^r) - \langle \varrho_k^r | e_k^r - Ew_k^r\rangle + \mathcal{H}(p_k^r - p_k^{r-1}) - \langle (\varrho_k^r)_D | p_k^r - p_k^{r-1}\rangle\\
&\quad\leqq \mathcal{Q}(e_k^{r-1}) - \langle \varrho_k^{r-1} | e_k^{r-1} - Ew_k^{r-1}\rangle - \int_{t_k^{r-1}}^{t_k^r} \langle \dot{\varrho}(t) | e_k^{r-1} - Ew_k^{r-1}\rangle\,dt\\
&\qquad + \int_{t_k^{r-1}}^{t_k^r} \langle \sigma_k^{r-1} | E\dot{w}(t)\rangle\,dt + \beta_{\mathbb{C}}\Big(\int_{t_k^{r-1}}^{t_k^r} \|E\dot{w}(t)\|_2\,dt\Big)^2\\
&\quad\leqq \mathcal{Q}(e_k^{r-1}) - \langle \varrho_k^{r-1} | e_k^{r-1} - Ew_k^{r-1}\rangle - \int_{t_k^{r-1}}^{t_k^r} \langle \dot{\varrho}(t) | e_k^{r-1} - Ew_k^{r-1}\rangle\,dt\\
&\qquad + \int_{t_k^{r-1}}^{t_k^r} \langle \sigma_k^{r-1} | E\dot{w}(t)\rangle\,dt + \omega_k \int_{t_k^{r-1}}^{t_k^r} \|E\dot{w}(t)\|_2\,dt\,,\end{aligned} \tag{4.24}$$

where

$$\omega_k := \beta_{\mathbb{C}} \max_{1\leqq r\leqq k} \int_{t_k^{r-1}}^{t_k^r} \|E\dot{w}(t)\|_2\,dt \;\to 0,$$

by the absolute continuity of the integral. Now iterating inequality (4.24) for $1 \leqq r \leqq i$, we get (4.18) with $\delta_k := \omega_k \int_0^T \|E\dot{w}(t)\|_2\,dt$. □

4.4. Proof of the existence theorem

We are now in a position to prove Theorem 4.5.

Proof of Theorem 4.5. Let us fix a sequence of subdivisions $(t_k^i)_{0\leqq i\leqq k}$ of the interval $[0,T]$ satisfying (4.10) and (4.11). For every k let (u_k^i, e_k^i, p_k^i), $i=1,\dots,k$, be defined inductively as solutions of the discrete problems (4.12), with $(u_k^0, e_k^0, p_k^0) = (u_0, e_0, p_0)$, and let $u_k(t)$, $e_k(t)$, $p_k(t)$, $\sigma_k(t)$, $w_k(t)$, $f_k(t)$, $g_k(t)$, $\mathcal{L}_k(t)$, $\varrho_k(t)$ be defined by (4.15).

Let us prove that there exists a constant C, depending only on the constants $\alpha_{\mathbb{C}}$, $\beta_{\mathbb{C}}$, and α, and on the functions e_0, $t\mapsto w(t)$, and $t\mapsto \varrho(t)$, such that

$$\sup_{t\in[0,T]} \|e_k(t)\|_2 \leqq C \qquad \text{and} \qquad \mathcal{V}(p_k; 0, T) \leqq C \tag{4.25}$$

for every k. As $t\mapsto w(t)$ and $t\mapsto\varrho(t)$ are absolutely continuous with values in $H^1(\mathbb{R}^n;\mathbb{R}^n)$ and $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, respectively, the functions $t\mapsto\|Ew(t)\|_2$ and $t\mapsto\|\varrho(t)\|_2$ are bounded on $[0,T]$, and the functions $t\mapsto\|E\dot{w}(t)\|_2$ and $t\mapsto\|\dot\varrho(t)\|_2$ are integrable on $[0,T]$. This fact, together with (2.13), (2.14), (3.5), and (4.17), implies that

$$\begin{aligned}
&\alpha_{\mathbb{C}}\|e_k(t)\|_2^2 - \sup_{t\in[0,T]}\|\varrho(t)\|_2 \sup_{t\in[0,T]}\|Ew(t)\|_2 + \alpha \sum_{0<t_k^r\leqq t}\|p_k^r - p_k^{r-1}\|_1 \\
&\quad \leqq \sup_{t\in[0,T]}\|e_k(t)\|_2\Big(\int_0^T\|\dot\varrho(s)\|_2\,ds + 2\beta_{\mathbb{C}}\int_0^T\|E\dot w(s)\|_2\,ds + \sup_{t\in[0,T]}\|\varrho(t)\|_2\Big) \\
&\qquad + \sup_{t\in[0,T]}\|Ew(t)\|_2\int_0^T\|\dot\varrho(s)\|_2\,ds + \beta_{\mathbb{C}}\|e_0\|_2^2 + \|\varrho(0)\|_2\|e_0\|_2 \\
&\qquad + \|\varrho(0)\|_2\|Ew(0)\|_2 + \delta_k
\end{aligned} \tag{4.26}$$

for every k and every $t\in[0,T]$. The former inequality in (4.25) can now be obtained by using the Cauchy inequality. As for the latter, by (4.26) and the first inequality in (4.25), we deduce that

$$\sum_{0<t_k^r\leqq t}\|p_k^r - p_k^{r-1}\|_1 \leqq C \tag{4.27}$$

for every k and every $t\in[0,T]$. Since $t\mapsto p_k(t)$ is constant on the intervals $\big[t_k^{r-1}, t_k^r\big[$, the estimate (4.27) is equivalent to the second inequality in (4.25).

By the generalized version of the classical Helly theorem given in Lemma 7.2, there exist a subsequence, still denoted p_k, and a function, $p\colon[0,T]\to M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, with bounded variation on $[0,T]$, such that $p_k(t)\rightharpoonup p(t)$ weakly* in $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ for every $t\in[0,T]$.

Since by (4.25), $\|e_k(t)\|_2\leqq C$ and $\|p_k(t)\|_1\leqq C$ for every k and every t, following the same argument used in the proof of Theorem 3.3, we can deduce that $u_k(t)$ is bounded in $BD(\Omega)$ uniformly with respect to k and t. Let us fix $t\in[0,T]$. There exist an increasing sequence k_j (possibly depending on t) and two functions

$u(t) \in BD(\Omega)$ and $e(t) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ such that $u_{k_j}(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega)$ and $e_{k_j}(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. By (4.16), we can apply Theorem 3.7 and we find that the triple $(u(t), e(t), p(t))$ is a solution of the minimum problem

$$\min_{(v,\eta,q)\in A(w(t))} \{\mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - \langle \mathcal{L}(t)|v\rangle\}\,. \tag{4.28}$$

By Remark 3.9 there exists a unique $(u, e) \in BD(\Omega)\times L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ such that $(u, e, p(t))$ is a solution to (4.28). Therefore, the convergence result holds for the whole sequence, i.e., $u_k(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega)$ and $e_k(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$.

Let us now show that the function $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution, satisfying $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$. The initial condition is fulfilled, since $u_k(0) = u_0$, $e_k(0) = e_0$, $p_k(0) = p_0$ for every k. In (4.28) we already proved that $(u(t), e(t), p(t))$ satisfies (4.3) for every $t \in [0, T]$.

It now remains to prove the energy balance (4.4), or equivalently (4.6). By Theorem 4.7, proved below, it is enough to establish the energy inequality

$$\begin{aligned}
&\mathcal{Q}(e(t)) - \langle \varrho(t)|e(t) - Ew(t)\rangle + \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle \varrho_D(t)|p(t)\rangle \\
&\leqq \mathcal{Q}(e(0)) - \langle \varrho(0)|e(0) - Ew(0)\rangle - \langle \varrho_D(0)|p(0)\rangle \\
&\quad + \int_0^t \langle \sigma(s)|E\dot{w}(s)\rangle\, ds \\
&\quad - \int_0^t \{\langle \dot{\varrho}(s)|e(s) - Ew(s)\rangle + \langle \dot{\varrho}_D(s)|p(s)\rangle\}\, ds\,.
\end{aligned} \tag{4.29}$$

Let us fix $t \in [0, T]$. As in the proof of Theorem 3.3, let $\delta > 0$ and $\psi_\delta(x) := \phi(\frac{1}{\delta}\mathrm{dist}(x, \Gamma_1))$ for every $x \in \overline{\Omega}$, where $\phi \in C^\infty(\mathbb{R})$, $0 \leqq \phi \leqq 1$, $\phi(s) = 0$ for $s \leqq 1$, and $\phi(s) = 1$ for $s \geqq 2$. Since the measure $H(p^r_k - p^{r-1}_k) - \big[\varrho_D(t^r_k):(p^r_k - p^{r-1}_k)\big]$ is nonnegative on $\Omega \cup \Gamma_0$ by (2.46), we obtain

$$\begin{aligned}
&\mathcal{H}(\psi_\delta(p^r_k - p^{r-1}_k)) - \langle [\varrho_D(t^r_k):(p^r_k - p^{r-1}_k)]|\psi_\delta\rangle \\
&\quad \leqq \mathcal{H}(p^r_k - p^{r-1}_k) - \langle \varrho_D(t^r_k)|p^r_k - p^{r-1}_k\rangle
\end{aligned} \tag{4.30}$$

for every $r = 1, \ldots, i$. Since $t \mapsto p_k(t)$ is constant on the intervals $\big[t^{r-1}_k, t^r_k\big[$, we have

$$\mathcal{D}_{\mathcal{H}}(\psi_\delta p_k; 0, t) \leqq \sum_{0<t^r_k\leqq t} \mathcal{H}(\psi_\delta(p^r_k - p^{r-1}_k))\,,$$

so that the lower semicontinuity of the dissipation (see (7.2)) gives

$$\mathcal{D}_{\mathcal{H}}(\psi_\delta p; 0, t) \leqq \liminf_{k\to\infty} \sum_{0<t^r_k\leqq t} \mathcal{H}(\psi_\delta(p^r_k - p^{r-1}_k))\,. \tag{4.31}$$

It is convenient to write:

$$\sum_{r=1}^{i}\langle[\varrho_D(t_k^r):(p_k^r - p_k^{r-1})]|\psi_\delta\rangle$$
$$= -\sum_{r=1}^{i}\langle[(\varrho_D(t_k^r) - \varrho_D(t_k^{r-1})):p_k^{r-1}]|\psi_\delta\rangle + \langle[\varrho_D(t_k^i):p_k^i]|\psi_\delta\rangle$$
$$-\left\langle\left[\varrho_D(0):p_0\right]|\psi_\delta\right\rangle . \tag{4.32}$$

Since $t \mapsto \varrho(t)$ and $t \mapsto f(t)$ are absolutely continuous from $[0, T]$ into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $L^n(\Omega; \mathbb{R}^n)$, respectively, by (2.41) we obtain

$$\sum_{r=1}^{i}\left\langle\left[(\varrho_D(t_k^r) - \varrho_D(t_k^{r-1})):p_k^{r-1}\right]|\psi_\delta\right\rangle$$
$$= \int_0^{t_k^i}\langle \dot f(s)|\psi_\delta(u_k(s) - w_k(s))\rangle\, ds - \int_0^{t_k^i}\langle\dot\varrho(s)|\psi_\delta(e_k(s) - Ew_k(s))\rangle\, ds$$
$$- \int_0^{t_k^i}\langle\dot\varrho(s)|(u_k(s) - w_k(s)) \odot \nabla\psi_\delta\rangle\, ds .$$

Passing to the limit as $k \to \infty$ and using (2.41) again, we obtain

$$\lim_{k\to\infty}\sum_{r=1}^{i}\left\langle\left[(\varrho_D(t_k^r) - \varrho_D(t_k^{r-1})):p_k^{r-1}\right]|\psi_\delta\right\rangle = \int_0^t\left\langle\left[\dot\varrho_D(s):p(s)\right]|\psi_\delta\right\rangle ds . \tag{4.33}$$

Analogously, we can show that

$$\lim_{k\to\infty}\left\langle\left[\varrho_D(t_k^i):p_k^i\right]|\psi_\delta\right\rangle = \left\langle\left[\varrho_D(t):p(t)\right]|\psi_\delta\right\rangle . \tag{4.34}$$

Combining (4.30)–(4.34), we obtain

$$\mathcal{D}_{\mathcal{H}}(\psi_\delta p; 0, t) - \left\langle\left[\varrho_D(t):p(t)\right]|\psi_\delta\right\rangle$$
$$+\left\langle\left[\varrho_D(0):p(0)\right]|\psi_\delta\right\rangle + \int_0^t\left\langle\left[\dot\varrho_D(s):p(s)\right]|\psi_\delta\right\rangle ds$$
$$\leqq \liminf_{k\to\infty}\sum_{r=1}^{i}\{\mathcal{H}(p_k^r - p_k^{r-1}) - \langle\varrho_D(t_k^r)|p_k^r - p_k^{r-1}\rangle\},$$

and passing to the limit as $\delta \to 0^+$, we conclude that

$$\mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle\varrho_D(t)|p(t)\rangle + \langle\varrho_D(0)|p(0)\rangle + \int_0^t\langle\dot\varrho_D(s)|p(s)\rangle\, ds$$
$$\leqq \liminf_{k\to\infty}\sum_{r=1}^{i}\{\mathcal{H}(p_k^r - p_k^{r-1}) - \langle\varrho_D(t_k^r)|p_k^r - p_k^{r-1}\rangle\} . \tag{4.35}$$

For every $s \in [0,t]$ we have $\sigma_k(s) = \mathbb{C}e_k(s) \rightharpoonup \mathbb{C}e(s) = \sigma(s)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. As $\sigma_k(s)$ is bounded in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ uniformly with respect to k and s, we can pass to the limit in (4.17) as $k \to \infty$ and we obtain (4.29) from (4.35) and from the lower semicontinuity of $\mathcal{Q}$. □

As in [15] (Theorem 4.4) and [6] (Lemma 7.1), the energy inequality (4.29) together with the global stability (qs1′) imply the exact energy balance (qs2′).

Theorem 4.7. *Assume* (2.1), (2.2), (2.7), (2.12), (2.13), *and* (2.15)–(2.19). *Let* $t \mapsto (u(t), e(t), p(t))$ *be a function from* $[0,T]$ *into* $BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ *satisfying the stability condition* (qs1′) *in Theorem* 4.4. *Assume that* $t \mapsto p(t)$ *from* $[0,T]$ *into* $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ *has bounded variation. Then for every* $t\in[0,T]$

$$\begin{aligned}
&\mathcal{Q}(e(t)) - \langle \varrho(t)|e(t)-Ew(t)\rangle + \mathcal{D}_{\mathcal{H}}(p;0,t) - \langle \varrho_D(t)|p(t)\rangle \\
&\quad\geqq \mathcal{Q}(e(0)) - \langle \varrho(0)|e(0)-Ew(0)\rangle - \langle \varrho_D(0)|p(0)\rangle + \int_0^t \langle \sigma(s)|E\dot w(s)\rangle\,ds \\
&\qquad - \int_0^t \{\langle \dot\varrho(s)|e(s)-Ew(s)\rangle + \langle \dot\varrho_D(s)|p(s)\rangle\}\,ds\,,
\end{aligned} \tag{4.36}$$

where $\sigma(t) := \mathbb{C}e(t)$. *If, in addition,* (4.29) *is satisfied, then the exact energy balance* (qs2′) *holds.*

Proof. Let us fix $t\in(0,T]$ and let $(s^i_k)_{0\leqq i\leqq k}$ be a sequence of subdivisions of the interval $[0,t]$ satisfying

$$0 = s^0_k < s^1_k < \cdots < s^{k-1}_k < s^k_k = t\,, \tag{4.37}$$

$$\lim_{k\to\infty}\ \max_{1\leqq i\leqq k} (s^i_k - s^{i-1}_k) = 0\,. \tag{4.38}$$

For every $i=1,\dots,k$ let $v := u(s^i_k) - w(s^i_k) + w(s^{i-1}_k)$ and $\eta := e(s^i_k) - Ew(s^i_k) + Ew(s^{i-1}_k)$. Since $(v,\eta,p(s^i_k)) \in A(w(s^{i-1}_k))$, by the global stability (4.5), we have

$$\begin{aligned}
\mathcal{Q}(e(s^{i-1}_k)) - \langle \varrho(s^{i-1}_k)|e(s^{i-1}_k)\rangle \leqq\ & \mathcal{Q}(e(s^i_k) - (Ew(s^i_k) - Ew(s^{i-1}_k))) \\
& - \langle \varrho(s^{i-1}_k)|e(s^i_k) - (Ew(s^i_k) - Ew(s^{i-1}_k))\rangle \\
& + \mathcal{H}(p(s^i_k) - p(s^{i-1}_k)) \\
& - \langle \varrho_D(s^{i-1}_k)|p(s^i_k) - p(s^{i-1}_k)\rangle\,.
\end{aligned} \tag{4.39}$$

The first term in the right-hand side can be written as

$$\begin{aligned}
&\mathcal{Q}(e(s^i_k) - (Ew(s^i_k) - Ew(s^{i-1}_k))) \\
&\quad = \mathcal{Q}(e(s^i_k)) - \langle \sigma(s^i_k)|Ew(s^i_k) - Ew(s^{i-1}_k)\rangle + \mathcal{Q}(Ew(s^i_k) - Ew(s^{i-1}_k))\,.
\end{aligned}$$

Now, following the same argument used in both (4.23) and the proof of the last inequality in (4.24), from the previous equality and from (4.39), we find that there exists a sequence $\omega_k \to 0^+$ such that

$$\begin{aligned}
&\mathcal{Q}(e(s_k^{i-1})) - \langle \varrho(s_k^{i-1}) | e(s_k^{i-1}) - Ew(s_k^{i-1}) \rangle - \langle \varrho_D(s_k^{i-1}) | p(s_k^{i-1}) \rangle \\
&\quad \leqq \mathcal{Q}(e(s_k^i)) + \mathcal{H}(p(s_k^i) - p(s_k^{i-1})) - \langle \varrho(s_k^i) | e(s_k^i) - Ew(s_k^i) \rangle \\
&\quad\quad - \langle \varrho_D(s_k^i) | p(s_k^i) \rangle + \int_0^t \langle \dot{\varrho}(s) | e(s_k^i) - Ew(s_k^i) \rangle \, ds + \int_0^t \langle \dot{\varrho}_D(s) | p(s_k^i) \rangle \, ds \\
&\quad\quad - \int_{s_k^{i-1}}^{s_k^i} \langle \sigma(s_k^i) | E\dot{w}(s) \rangle \, ds + \omega_k \int_{s_k^{i-1}}^{s_k^i} \| E\dot{w}(s) \|_2 \, ds \, .
\end{aligned}$$

On $[0, t]$ we define the piecewise constant functions

$$\overline{e}_k(s) := e(s_k^i) \, , \quad E\overline{w}_k(s) := Ew(s_k^i) \, , \quad \overline{p}_k(s) := p(s_k^i) \, , \quad \overline{\sigma}_k(s) := \sigma(s_k^i) \, ,$$

where i is the smallest index such that $s \leqq s_k^i$. Since $\sum_i \mathcal{H}(p(s_k^i) - p(s_k^{i-1})) \leqq \mathcal{D}_{\mathcal{H}}(p; 0, t)$, iterating the last inequality for $1 \leqq i \leqq k$ we obtain

$$\begin{aligned}
&\mathcal{Q}(e(0)) - \langle \varrho(0) | e(0) - Ew(0) \rangle - \langle \varrho_D(0) | p(0) \rangle \\
&\quad \leqq \mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle \varrho(t) | e(t) - Ew(t) \rangle - \langle \varrho_D(t) | p(t) \rangle \\
&\quad\quad + \int_0^t \langle \dot{\varrho}(s) | \overline{e}_k(s) - E\overline{w}_k(s) \rangle \, ds + \int_0^t \langle \dot{\varrho}_D(s) | \overline{p}_k(s) \rangle \, ds \\
&\quad\quad - \int_0^t \langle \overline{\sigma}_k(s) | E\dot{w}(s) \rangle \, ds + \delta_k \, ,
\end{aligned} \tag{4.40}$$

where $\delta_k := \omega_k \int_0^T \| E\dot{w}(s) \|_2 \, ds$. By Remark 4.3, the set of discontinuity points of the functions $s \mapsto p(s)$, $s \mapsto e(s)$, and $s \mapsto \sigma(s)$ is at most countable, and $\|\overline{p}_k(s)\|_1$, $\|\overline{e}_k(s)\|_2$, and $\|\overline{\sigma}_k(s)\|_2$ are bounded uniformly with respect to s and k. Therefore, (4.38) implies that $\overline{p}_k(s) \to p(s)$ strongly in $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, $\overline{e}_k(s) \to e(s)$ and $\overline{\sigma}_k(s) \to \sigma(s)$ strongly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ for a.e. $s \in [0, t]$. Now, (4.36) follows from (4.40) by the dominated convergence theorem. □

4.5. Convergence of the approximate solutions

For every k, let (u_k^i, e_k^i, p_k^i), $i = 1, \dots, k$, be defined inductively as solutions of the discrete problems (4.12), starting from $(u_k^0, e_k^0, p_k^0) = (u_0, e_0, p_0)$. Let $u_k(t)$, $e_k(t)$, $p_k(t)$, $\sigma_k(t)$ be defined by (4.15). Also let $t \mapsto (u(t), e(t), p(t))$ be a quasistatic evolution. Assume that

$$p_k(t) \rightharpoonup p(t) \qquad \text{weakly* in } M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) \tag{4.41}$$

for every $t \in [0, T]$. The following theorem shows, in particular, that stresses and elastic strains of the approximate solutions converge strongly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$.

Theorem 4.8. *Under the hypothesis of Theorem* 4.5*, assume that the plastic strain of the approximate solutions satisfies* (4.41)*. Then,* $e_k(t) \to e(t)$ *and* $\sigma_k(t) \to \sigma(t)$ *strongly in* $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$*. Moreover,*

$$\lim_{k\to\infty} \sum_{0<t_k^r \leqq t} \{\mathcal{H}(p_k^r - p_k^{r-1}) - \langle \varrho_D(t_k^r) | p_k^r - p_k^{r-1}\rangle\}$$

$$= \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle \varrho_D(t) | p(t)\rangle + \langle \varrho_D(0) | p(0)\rangle + \int_0^t \langle \dot{\varrho}_D(s) | p(s)\rangle \, ds \tag{4.42}$$

for every $t \in [0, T]$.

Proof. By the discrete energy inequality (4.17) for every $t \in [0, T]$ we have

$$\begin{aligned} &\mathcal{Q}(e_k(t)) + \sum_{0<t_k^r \leqq t} \{\mathcal{H}(p_k^r - p_k^{r-1}) - \langle \varrho_D(t_k^r) | p_k^r - p_k^{r-1}\rangle\} \\ &\quad \leqq \mathcal{Q}(e_0) - \langle \varrho(0) | e_0 - Ew(0)\rangle + \langle \varrho_k(t) | e_k(t) - Ew_k(t)\rangle \\ &\qquad - \int_0^{t_k^i} \langle \dot{\varrho}(s) | e_k(s) - Ew_k(s)\rangle \, ds + \int_0^{t_k^i} \langle \sigma_k(s) | E\dot{w}(s)\rangle \, ds + \delta_k \,, \end{aligned} \tag{4.43}$$

where $\delta_k \to 0$ and i is the largest integer such that $t_k^i \leqq t$. By the energy balance (4.6) we also have

$$\begin{aligned} &\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle \varrho_D(t) | p(t)\rangle + \langle \varrho_D(0) | p(0)\rangle + \int_0^t \langle \dot{\varrho}_D(s) | p(s)\rangle \, ds \\ &\quad = \mathcal{Q}(e_0) - \langle \varrho(0) | e_0 - Ew(0)\rangle + \langle \varrho(t) | e(t) - Ew(t)\rangle \\ &\qquad - \int_0^t \langle \dot{\varrho}(s) | e(s) - Ew(s)\rangle \, ds + \int_0^t \langle \sigma(s) | E\dot{w}(s)\rangle \, ds \,. \end{aligned} \tag{4.44}$$

In the proof of Theorem 4.5 we found that $e_k(t) \rightharpoonup e(t)$ and $\sigma_k(t) \rightharpoonup \sigma(t)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and that $\|e_k(t)\|_2$ and $\|\sigma_k(t)\|_2$ are bounded uniformly with respect to t and k. Moreover, $\varrho_k(t) \to \varrho(t)$ and $Ew_k(t) \to Ew(t)$ strongly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. Therefore, the right-hand side of (4.43) converges to the right-hand side of (4.44), implying

$$\begin{aligned} &\limsup_{k\to\infty} \Big\{ \mathcal{Q}(e_k(t)) + \sum_{0<t_k^r \leqq t} \{\mathcal{H}(p_k^r - p_k^{r-1}) - \langle \varrho_D(t_k^r) | p_k^r - p_k^{r-1}\rangle\} \Big\} \\ &\quad \leqq \mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle \varrho_D(t) | p(t)\rangle + \langle \varrho_D(0) | p(0)\rangle \\ &\qquad + \int_0^t \langle \dot{\varrho}_D(s) | p(s)\rangle \, ds \,. \end{aligned}$$

By the lower semicontinuity of $\mathcal{Q}$ and by (4.35), we obtain (4.42) and

$$\mathcal{Q}(e_k(t)) \to \mathcal{Q}(e(t)) \,,$$

which gives the strong convergence of the strains $e_k(t)$, and consequently of the stresses $\sigma_k(t) = \mathbb{C}e_k(t)$. □

5. Regularity and uniqueness results

In this section we prove that every quasistatic evolution $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous with respect to time, and that the functions $t \mapsto e(t)$ and $t \mapsto \sigma(t)$ are determined uniquely by their initial conditions.

5.1. Regularity

For the general properties of absolutely continuous functions with values in Banach spaces we refer to [4] (Appendix) for the reflexive case, and to the Appendix of the present paper for the case of the dual of a separable Banach space.

If $t \mapsto q(t)$ and $t \mapsto v(t)$ are absolutely continuous from $[0, T]$ into $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ and $BD(\Omega)$, respectively, we define

$$\dot{q}(t) := w^* - \lim_{s\to t} \frac{q(s) - q(t)}{s - t}, \qquad \dot{v}(t) := w^* - \lim_{s\to t} \frac{v(s) - v(t)}{s - t}. \quad (5.1)$$

By Theorem 7.1 $\dot{q}(t)$ and $\dot{v}(t)$ are defined for a.e. $t \in [0, T]$, the function $t \mapsto \mathcal{H}(\dot{q}(t))$ is measurable, and

$$\mathcal{D}_{\mathcal{H}}(q; 0, t) = \int_0^t \mathcal{H}(\dot{q}(s))\, ds \quad (5.2)$$

for every $t \in [0, T]$.

Remark 5.1. If we apply (7.4) to the absolutely continuous function $t \mapsto q(t)$, with $X = M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, $Y = C_0(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, and $\mathcal{K} = \{\varphi \in C_0(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) : \|\varphi\|_\infty \leqq 1\}$, for a.e. $t \in [0, T]$ we obtain

$$\|\dot{q}(t)\|_1 = \lim_{s\to t} \left\| \frac{q(s) - q(t)}{s - t} \right\|_1 . \quad (5.3)$$

By the definition of weak* convergence in $BD(\Omega)$, it follows from (5.1) that for a.e. $t \in [0, T]$ we have $(v(s) - v(t))/(s - t) \to \dot{v}(t)$ strongly in $L^1(\Omega; \mathbb{R}^n)$ and $(Ev(s) - Ev(t))/(s - t) \rightharpoonup E\dot{v}(t)$ weakly* in $M_b(\Omega; \mathbb{M}^{n\times n}_{sym})$ as $s \to t$. If we apply (7.4) to the absolutely continuous function $t \mapsto Ev(t)$, with $X = M_b(\Omega; \mathbb{M}^{n\times n}_{sym})$, $Y = C_0(\Omega; \mathbb{M}^{n\times n}_{sym})$, and $\mathcal{K} = \{\varphi \in C_0(\Omega; \mathbb{M}^{n\times n}_{sym}) : \|\varphi\|_\infty \leqq 1\}$, for a.e. $t \in [0, T]$ we obtain

$$\|E\dot{v}(t)\|_1 = \lim_{s\to t} \left\| \frac{Ev(s) - Ev(t)}{s - t} \right\|_1 .$$

The above implies that for a.e. $t \in [0, T]$, the trace of $\dot{v}(t)$ is the strong limit in $L^1(\partial\Omega; \mathbb{R}^n)$ of the traces of $(v(s) - v(t))/(s - t)$ as $s \to t$ (see [29] (Chapter II, Theorem 3.1)). In other words, the time derivative of the trace of $v(t)$ is the trace of the time derivative of $v(t)$. Therefore, using (4.1) and (4.2), we can prove by a standard argument that

$$\frac{d}{dt}\langle \mathcal{L}(t) | v(t)\rangle = \langle \dot{\mathcal{L}}(t) | v(t)\rangle + \langle \mathcal{L}(t) | \dot{v}(t)\rangle \quad (5.4)$$

for a.e. $t \in [0, T]$.

The next proposition deals with the absolute continuity of the functions $t \mapsto e(t)$, $t \mapsto p(t)$, and $t \mapsto u(t)$ from $[0, T]$ into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, and $BD(\Omega)$, respectively.

Theorem 5.2. *Assume* (2.1), (2.2), (2.7), (2.12), (2.13), *and* (2.15)–(2.19). *Let $t \mapsto (u(t), e(t), p(t))$ be a quasistatic evolution. Then, the functions $t \mapsto e(t)$, $t \mapsto p(t)$, and $t \mapsto u(t)$ are absolutely continuous from $[0, T]$ into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, and $BD(\Omega)$, respectively. Moreover, for a.e. $t \in [0, T]$ we have*

$$\|\dot e(t)\|_2 \leqq C_1(\|\dot\varrho(t)\|_2 + \|\dot\varrho_D(t)\|_\infty + \|E\dot w(t)\|_2)\,, \tag{5.5}$$

$$\|\dot p(t)\|_1 \leqq C_2(\|\dot\varrho(t)\|_2 + \|\dot\varrho_D(t)\|_\infty + \|E\dot w(t)\|_2)\,, \tag{5.6}$$

$$\|E\dot u(t)\|_1 \leqq C_3(\|\dot\varrho(t)\|_2 + \|\dot\varrho_D(t)\|_\infty + \|E\dot w(t)\|_2)\,, \tag{5.7}$$

$$\|\dot u(t)\|_1 \leqq C_4(\|\dot\varrho(t)\|_2 + \|\dot\varrho_D(t)\|_\infty + \|E\dot w(t)\|_2 + \|\dot w(t)\|_2)\,, \tag{5.8}$$

where C_1 and C_2 are positive constants depending on R_K, $\alpha_{\mathbb{C}}$, $\beta_{\mathbb{C}}$, α, $\sup_t \|\varrho(t)\|_2$, $\sup_t \|e(t)\|_2$, and $\sup_t \|p(t)\|_1$, while C_3 depends also on Ω, and C_4 also on Ω and Γ_0.

Proof. Since $\mathcal{H}(p(t_2) - p(t_1)) \leqq \mathcal{D}_{\mathcal{H}}(p; t_1, t_2)$, by the energy equality (4.6) we obtain after an integration by parts,

$$\begin{aligned}
&\tfrac12\langle\sigma(t_2)|e(t_2)\rangle - \tfrac12\langle\sigma(t_1)|e(t_1)\rangle + \mathcal{H}(p(t_2) - p(t_1))\\
&\leqq \langle\varrho(t_2)|e(t_2)\rangle - \langle\varrho(t_1)|e(t_1)\rangle + \langle\varrho_D(t_2)|p(t_2)\rangle - \langle\varrho_D(t_1)|p(t_1)\rangle\\
&\quad - \int_{t_1}^{t_2} \{\langle\dot\varrho(s)|e(s)\rangle + \langle\dot\varrho_D(s)|p(s)\rangle - \langle\sigma(s) - \varrho(s)|E\dot w(s)\rangle\}\,ds
\end{aligned} \tag{5.9}$$

for every $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Now consider the functions $v := u(t_2) - u(t_1) - (w(t_2) - w(t_1))$, $\eta := e(t_2) - e(t_1) - (Ew(t_2) - Ew(t_1))$, and the measure $q := p(t_2) - p(t_1)$. Since $(v, \eta, q) \in A(0)$ and $(u(t_1), e(t_1), p(t_1))$ is a solution of the minimum problem (3.2) with $p_0 = p(t_1)$ and $\mathcal{L} = \mathcal{L}(t_1)$, by Theorem 3.4 and Lemma 3.1 we obtain

$$\begin{aligned}
&-\langle\sigma(t_1)|e(t_2) - e(t_1)\rangle + \langle\varrho(t_1)|e(t_2) - e(t_1)\rangle + \langle\varrho_D(t_1)|p(t_2) - p(t_1)\rangle\\
&+\langle\sigma(t_1) - \varrho(t_1)|Ew(t_2) - Ew(t_1)\rangle \leqq \mathcal{H}(p(t_2) - p(t_1))\,,
\end{aligned}$$

so that (5.9) implies

$$\begin{aligned}
&\tfrac12\langle\sigma(t_2)|e(t_2)\rangle - \tfrac12\langle\sigma(t_1)|e(t_1)\rangle - \langle\sigma(t_1)|e(t_2) - e(t_1)\rangle\\
&\leqq \langle\varrho(t_2) - \varrho(t_1)|e(t_2)\rangle + \langle\varrho_D(t_2) - \varrho_D(t_1)|p(t_2)\rangle\\
&\quad -\langle\sigma(t_1) - \varrho(t_1)|Ew(t_2) - Ew(t_1)\rangle - \int_{t_1}^{t_2}\langle\dot\varrho(s)|e(s)\rangle\,ds\\
&\quad + \int_{t_1}^{t_2}\langle\dot\varrho_D(s)|p(s)\rangle\,ds - \int_{t_1}^{t_2}\langle\sigma(s) - \varrho(s)|E\dot w(s)\rangle\,ds\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}&\tfrac{1}{2}\langle \mathbb{C}(e(t_2)-e(t_1))|e(t_2)-e(t_1)\rangle\\&\quad\leqq \int_{t_1}^{t_2}\langle \sigma(s)-\sigma(t_1)|E\dot{w}(s)\rangle\,ds+\int_{t_1}^{t_2}\langle \dot{\varrho}(s)|e(t_2)-e(s)\rangle\,ds\\&\qquad+\int_{t_1}^{t_2}\langle \dot{\varrho}_D(s)|p(t_2)-p(s)\rangle\,ds-\int_{t_1}^{t_2}\langle \varrho(s)-\varrho(t_1)|E\dot{w}(s)\rangle\,ds\,.\end{aligned}$$

By (2.13) and (2.14), we obtain

$$\begin{aligned}\alpha_{\mathbb{C}}\|e(t_2)-e(t_1)\|_2^2\leqq 2\beta_{\mathbb{C}}&\int_{t_1}^{t_2}\|e(s)-e(t_1)\|_2\,\|E\dot{w}(s)\|_2\,ds\\&+\int_{t_1}^{t_2}\|\dot{\varrho}(s)\|_2\,\|e(t_2)-e(s)\|_2\,ds\\&+\int_{t_1}^{t_2}\|\dot{\varrho}_D(s)\|_\infty\,\|p(t_2)-p(s)\|_1\,ds\\&+\int_{t_1}^{t_2}\|\varrho(s)-\varrho(t_1)\|_2\,\|E\dot{w}(s)\|_2\,ds\,. \qquad (5.10)\end{aligned}$$

By Lemma 3.2, for every $t_1\leqq s\leqq t_2$, we have

$$\alpha\|p(t_2)-p(s)\|_1\leqq \mathcal{H}(p(t_2)-p(s))-\langle \varrho_D(t_2)|p(t_2)-p(s)\rangle\,,$$

therefore, inequality (5.9) with $t_1=s$ implies

$$\begin{aligned}\alpha\|p(t_2)-p(s)\|_1\leqq{}&\tfrac{1}{2}\langle\sigma(s)|e(s)\rangle-\tfrac{1}{2}\langle\sigma(t_2)|e(t_2)\rangle+\langle\varrho(t_2)|e(t_2)-e(s)\rangle\\&+\langle\varrho(t_2)-\varrho(s)|e(s)\rangle+\langle\varrho_D(t_2)-\varrho_D(s)|p(s)\rangle\\&-\int_s^{t_2}\{\langle\dot{\varrho}(t)|e(t)\rangle+\langle\dot{\varrho}_D(t)|p(t)\rangle\}\,dt\\&+\int_s^{t_2}\langle\sigma(t)-\varrho(t)|E\dot{w}(t)\rangle\,dt\,.\end{aligned}$$

We observe that $\sup_t\|\varrho(t)\|_2$, $\sup_t\|\varrho_D(t)\|_\infty$, $\sup_t\|e(t)\|_2$, and $\sup_t\|p(t)\|_1$ are finite (see Remark 4.3 for $e(t)$). In the rest of the proof, C will denote a positive constant, with a value that can change from line to line, depending on these suprema and on the constants $\alpha_{\mathbb{C}}$, $\beta_{\mathbb{C}}$, α. The previous inequality implies that

$$\begin{aligned}&\|p(t_2)-p(s)\|_1\\&\quad\leqq C(\|e(t_2)-e(s)\|_2+\|\varrho(t_2)-\varrho(s)\|_2+\|\varrho_D(t_2)-\varrho_D(s)\|_\infty)\\&\qquad+C\int_s^{t_2}\{\|\dot{\varrho}(t)\|_2+\|\dot{\varrho}_D(t)\|_\infty+\|E\dot{w}(t)\|_2\}\,dt\,.\end{aligned}$$

Therefore, for every $t_1 \leqq s \leqq t_2$,

$$\|p(t_2) - p(s)\|_1 \leqq C\,\|e(t_2) - e(s)\|_2 \\ +C \int_{t_1}^{t_2} \{\|\dot{\varrho}(t)\|_2 + \|\dot{\varrho}_D(t)\|_\infty + \|E\dot{w}(t)\|_2\}\,dt\,. \quad (5.11)$$

By (5.10) and (5.11), using $\|e(t_2) - e(s)\|_2 \leqq \|e(t_2) - e(t_1)\|_2 + \|e(s) - e(t_1)\|_2$, we deduce that

$$\|e(t_2) - e(t_1)\|_2^2 \\ \leqq C\,\|e(t_2) - e(t_1)\|_2 \int_{t_1}^{t_2} \{\|\dot{\varrho}(s)\|_2 + \|\dot{\varrho}_D(s)\|_\infty\}\,ds \\ +C \int_{t_1}^{t_2} \{\|\dot{\varrho}(s)\|_2 + \|\dot{\varrho}_D(s)\|_\infty + \|E\dot{w}(s)\|_2\}\,\|e(s) - e(t_1)\|_2\,ds \\ +C\Big(\int_{t_1}^{t_2} \{\|\dot{\varrho}(s)\|_2 + \|\dot{\varrho}_D(s)\|_\infty + \|E\dot{w}(s)\|_2\}\,ds\Big)^2.$$

By the Cauchy inequality,

$$\|e(t_2) - e(t_1)\|_2^2 \leqq \int_{t_1}^{t_2} \psi(s)\,\|e(s) - e(t_1)\|_2\,ds + \Big(\int_{t_1}^{t_2} \psi(s)\,ds\Big)^2,$$

where

$$\psi(s) := C(\|\dot{\varrho}(s)\|_2 + \|\dot{\varrho}_D(s)\|_\infty + \|E\dot{w}(s)\|_2)\,.$$

We can now apply a version of the Gronwall inequality, proved in Lemma 5.3 below, which gives

$$\|e(t_2) - e(t_1)\|_2 \leqq \frac{3}{2} \int_{t_1}^{t_2} \psi(s)\,ds \\ \leqq C \int_{t_1}^{t_2} \{\|\dot{\varrho}(s)\|_2 + \|\dot{\varrho}_D(s)\|_\infty + \|E\dot{w}(s)\|_2\}\,ds\,.$$

This implies that $t \mapsto e(t)$ is absolutely continuous from $[0, T]$ into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and that $\dot{e}(t)$ satisfies (5.5).

Using the absolute continuity of $t \mapsto e(t)$ and (5.5), inequality (5.11) with $s = t_1$ yields the absolute continuity of $t \mapsto p(t)$ and (5.6).

From the decomposition $Eu(t) = e(t) + p(t)$, it follows that $t \mapsto Eu(t)$ is absolutely continuous from $[0, T]$ into $M_b(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $E\dot{u}(t) = \dot{e}(t) + \dot{p}(t)$ for a.e. $t \in [0, T]$. Inequality (5.7) is an easy consequence of this decomposition. It now remains to prove that $t \mapsto u(t)$ is absolutely continuous from $[0, T]$ into $L^1(\Omega; \mathbb{R}^n)$, and satisfies (5.8). By (2.3), there exists a constant $C > 0$, depending on Ω and Γ_0, such that

$$\|u(t_2) - u(t_1)\|_1 \leqq C\,\|u(t_2) - u(t_1)\|_{1,\Gamma_0} + C\,\|Eu(t_2) - Eu(t_1)\|_1\,. \quad (5.12)$$

Using (2.22) and the continuity of the trace operator from $H^1(\Omega; \mathbb{R}^n)$ into $L^1(\partial\Omega; \mathbb{R}^n)$, we obtain that there exists a constant M, depending on Ω, such that

$$\|u(t_2) - u(t_1)\|_{1,\Gamma_0} \leqq \sqrt{2}\,\|p(t_2) - p(t_1)\|_1 + M\|w(t_2) - w(t_1)\|_2 + M\|Ew(t_2) - Ew(t_1)\|_2\,. \tag{5.13}$$

As $t \mapsto w(t)$, $t \mapsto Eu(t)$, and $t \mapsto p(t)$ are absolutely continuous from $[0, T]$ into $H^1(\Omega; \mathbb{R}^n)$, $M_b(\Omega; \mathbb{M}^{n\times n}_{sym})$, and $M_b(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_D)$, respectively, inequalities (5.12) and (5.13) imply that $t \mapsto u(t)$ is absolutely continuous from $[0, T]$ into $L^1(\Omega; \mathbb{R}^n)$ and (5.8) is satisfied. □

The version of the Gronwall inequality contained in the following lemma allows us to conclude the proof of Theorem 5.2.

Lemma 5.3. *Let* $\phi\colon [0, T] \to [0, +\infty[$ *be a bounded measurable function and let* $\psi\colon [0, T] \to [0, +\infty[$ *be an integrable function. Suppose that*

$$\phi(t)^2 \leqq \int_0^t \phi(s)\,\psi(s)\,ds + \Big(\int_0^t \psi(s)\,ds\Big)^2 \tag{5.14}$$

for every $t \in [0, T]$. *Then,*

$$\phi(t) \leqq \frac{3}{2}\int_0^t \psi(s)\,ds$$

for every $t \in [0, T]$.

Proof. Let us fix $t_0 \in [0, T]$ and let $\gamma_0 := (\int_0^{t_0} \psi(s)\,ds)^2$. For every $t \in [0, t_0]$ we define $V(t) := \int_0^t \phi(s)\,\psi(s)\,ds$. Thus V is absolutely continuous on $[0, t_0]$, $\phi(t)^2 \leqq V(t) + \gamma_0$ for every $t \in [0, t_0]$, and $\dot V(t) \leqq \psi(t)(V(t) + \gamma_0)^{1/2}$ for a.e. $t \in [0, t_0]$. Integrating between 0 and t_0, we get $2(V(t_0) + \gamma_0)^{1/2} \leqq 2\gamma_0^{1/2} + \int_0^{t_0} \psi(s)\,ds = 3\int_0^{t_0} \psi(s)\,ds$. By (5.14) we have $\phi(t_0) \leqq (V(t_0) + \gamma_0)^{1/2}$, so that the previous inequality gives $2\,\phi(t_0) \leqq 3\int_0^{t_0} \psi(s)\,ds$. □

Remark 5.4. Estimates (5.5)–(5.8) imply that if $t \mapsto w(t)$, $t \mapsto \varrho(t)$, and $t \mapsto \varrho_D(t)$ are Lipschitz continuous from $[0, T]$ into the spaces $H^1(\mathbb{R}^n; \mathbb{R}^n)$, $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and $L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$, respectively, then the functions $t \mapsto u(t)$, $t \mapsto e(t)$, $t \mapsto p(t)$ are Lipschitz continuous from $[0, T]$ into the spaces $BD(\Omega)$, $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, respectively.

The following lemma will be crucial in the rest of the paper.

Lemma 5.5. *Assume* (2.1), (2.2), *and* (2.15). *Let* $t \mapsto u(t)$, $t \mapsto e(t)$, $t \mapsto p(t)$ *be absolutely continuous functions from* $[0, T]$ *into* $BD(\Omega)$, $L^2(\Omega; \mathbb{R}^n)$, *and* $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, *respectively. Assume that* $(u(t), e(t), p(t)) \in A(w(t))$ *for every* $t \in [0, T]$. *Then,* $(\dot u(t), \dot e(t), \dot p(t)) \in A(\dot w(t))$ *for a.e.* $t \in [0, T]$.

Proof. It is enough to apply Lemma 2.1 to the difference quotients. □

Owing to the following proposition, we can differentiate the energy balance (4.4) and obtain a balance of powers: the rate of change of stored energy plus the rate of plastic dissipation equals the power of external forces.

Proposition 5.6. *Assume* (2.1), (2.2), (2.7), (2.12), (2.13), *and* (2.15)–(2.19). *Let* $t \mapsto (u(t), e(t), p(t))$ *be an absolutely continuous function from* $[0,T]$ *into* $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ *and let* $\sigma(t) := \mathbb{C}e(t)$. *The following conditions are then equivalent:*

(a) *for every* $t \in [0,T]$

$$\begin{aligned}&\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p;0,t) - \langle \mathcal{L}(t)|u(t)\rangle \\ &\quad = \mathcal{Q}(e(0)) - \langle \mathcal{L}(0)|u(0)\rangle + \int_0^t \{\langle \sigma(s)|E\dot{w}(s)\rangle - \langle \mathcal{L}(s)|\dot{w}(s)\rangle\}\,ds \\ &\qquad - \int_0^t \langle \dot{\mathcal{L}}(s)|u(s)\rangle\,ds\,;\end{aligned}$$

(b) *for a.e.* $t \in [0,T]$

$$\langle \sigma(t)|\dot{e}(t)\rangle + \mathcal{H}(\dot{p}(t)) = \langle \sigma(t)|E\dot{w}(t)\rangle - \langle \mathcal{L}(t)|\dot{w}(t)\rangle + \langle \mathcal{L}(t)|\dot{u}(t)\rangle\,;$$

(c) *for a.e.* $t \in [0,T]$

$$\langle \sigma(t) - \varrho(t)|\dot{e}(t)\rangle + \mathcal{H}(\dot{p}(t)) = \langle \varrho_D(t)|\dot{p}(t)\rangle + \langle \sigma(t) - \varrho(t)|E\dot{w}(t)\rangle\,;$$

(d) *for every* $t \in [0,T]$

$$\begin{aligned}&\mathcal{Q}(e(t)) + \int_0^t \{\mathcal{H}(\dot{p}(s)) - \langle \varrho_D(s)|\dot{p}(s)\rangle\}\,ds \\ &\quad = \mathcal{Q}(e(0)) + \int_0^t \{\langle \varrho(s)|\dot{e}(s)\rangle + \langle \sigma(s) - \varrho(s)|E\dot{w}(s)\rangle\}\,ds\,.\end{aligned}$$

Proof. Using (5.2) and (5.4) we obtain (b) by differentiating (a), and (a) by integrating (b). Similarly, we obtain (c) by differentiating (d), and (d) by integrating (c). The equivalence between (b) and (c) follows from Lemmas 3.1 and 5.5. □

Condition (d) of Proposition 5.6 allows us to prove an estimate of the quantities $\sup_t \|e(t)\|_2$ and $\sup_t \|p(t)\|_1$ in terms of the data of the problem.

Proposition 5.7. *Assume* (2.1), (2.2), (2.7), (2.12), (2.13), *and* (2.15)–(2.19). *Let* $t \mapsto (u(t), e(t), p(t))$ *be a quasistatic evolution. Then,*

$$\begin{aligned}\sup_{t\in[0,T]} \|e(t)\|_2 \leqq C_1 \Big\{ &\|e(0)\|_2 + \sup_{t\in[0,T]} \|\varrho(t)\|_2 + \int_0^T \|\dot{\varrho}(t)\|_2\,dt \\ &+ \int_0^T \|E\dot{w}(t)\|_2\,dt \Big\},\end{aligned} \tag{5.15}$$

and

$$\sup_{t\in[0,T]} \|p(t)\|_1 \leqq \|p(0)\|_1 + C_2 \Big\{ \|e(0)\|_2^2 + \sup_{t\in[0,T]} \|\varrho(t)\|_2^2 + \Big(\int_0^T \|\dot\varrho(t)\|_2 \, dt \Big)^2 + \Big(\int_0^T \|E\dot w(t)\|_2 \, dt \Big)^2 \Big\}, \tag{5.16}$$

where C_1 is a positive constant depending only on $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$, while C_2 depends also on α.

Proof. By Theorem 5.2, the function $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous from $[0, T]$ into $BD(\Omega)\times L^2(\Omega; \mathbb{M}^{n\times n}_{sym})\times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$. Since $t \mapsto (u(t), e(t), p(t))$ satisfies (qs2) in Definition 4.2, it satisfies conditions (a) and (d) of Proposition 5.6. After an integration by parts, we obtain from (d)

$$\mathcal{Q}(e(t)) + \int_0^t \{\mathcal{H}(\dot p(s)) - \langle \varrho_D(s)|\dot p(s)\rangle\} \, ds - \langle \varrho(t)|e(t)\rangle$$
$$= \mathcal{Q}(e(0)) + \int_0^t \{\langle \sigma(s) - \varrho(s)|E\dot w(s)\rangle - \langle \dot\varrho(s)|e(s)\rangle\} \, ds - \langle \varrho(0)|e(0)\rangle \, .$$

By (2.13), (2.14), and (3.5) for every $t \in [0, T]$ we have

$$\alpha_{\mathbb{C}} \|e(t)\|_2^2 + \alpha \int_0^t \|\dot p(s)\|_1 \, ds$$
$$\leqq \beta_{\mathbb{C}} \|e(0)\|_2^2 + 2 \sup_{t\in[0,T]} \|\varrho(t)\|_2 \sup_{t\in[0,T]} \|e(t)\|_2$$
$$+ \sup_{t\in[0,T]} \|e(t)\|_2 \int_0^T \{2\, \beta_{\mathbb{C}} \|E\dot w(s))\|_2 + \|\dot\varrho(s)\|_2\} \, ds$$
$$+ \sup_{t\in[0,T]} \|\varrho(t)\|_2 \int_0^T \|E\dot w(s))\|_2 \, ds \, ,$$

which implies (5.15) and (5.16) by the Cauchy inequality. □

Remark 5.8. Let $t \mapsto (u(t), e(t), p(t))$ be a quasistatic evolution. By Proposition 5.7, estimates (5.5)–(5.8) are satisfied with constants $C_1, \dots, C_4$ depending only on the data of the problem. More precisely, C_1 and C_2 depend on $R_K, \alpha_{\mathbb{C}}, \beta_{\mathbb{C}}$, α, $\sup_t \|\varrho(t)\|_2$, $\int_0^T \|\dot\varrho(t)\|_2 \, dt$, $\int_0^T \|E\dot w(t)\|_2 \, dt$, $\|e(0)\|_2$, and $\|p(0)\|_1$, while C_3 also depends on Ω, and C_4 also depends on Ω and Γ_0.

5.2. Uniqueness of stress and elastic strain

We now prove that $t \mapsto e(t)$ (and, consequently, $t \mapsto \sigma(t)$) is determined uniquely by its initial condition.

Theorem 5.9. *Assume* (2.1), (2.2), (2.7), (2.12), (2.13), *and* (2.15)–(2.19). *Let* $t \mapsto (u(t), e(t), p(t))$ *and* $t \mapsto (v(t), \eta(t), q(t))$ *be two quasistatic evolutions, corresponding to the same data* $t \mapsto w(t)$, $t \mapsto f(t)$, *and* $t \mapsto g(t)$, *and let* $\sigma(t) := \mathbb{C}e(t)$ *and* $\tau(t) := \mathbb{C}\eta(t)$. *If* $e(0) = \eta(0)$, *then* $e(t) = \eta(t)$ *for every* $t \in [0, T]$. *Equivalently, if* $\sigma(0) = \tau(0)$, *then* $\sigma(t) = \tau(t)$ *for every* $t \in [0, T]$.

Proof. By Theorem 5.2 the functions $t \mapsto (u(t), e(t), p(t))$ and $t \mapsto (v(t), \eta(t), q(t))$ are absolutely continuous. By condition (c) of Proposition 5.6 we have

$$\langle \sigma(t) - \varrho(t) | \dot{e}(t) - E\dot{w}(t) \rangle + \mathcal{H}(\dot{p}(t)) = \langle \varrho_D(t) | \dot{p}(t) \rangle \,, \tag{5.17}$$

$$\langle \tau(t) - \varrho(t) | \dot{\eta}(t) - E\dot{w}(t) \rangle + \mathcal{H}(\dot{q}(t)) = \langle \varrho_D(t) | \dot{q}(t) \rangle \,. \tag{5.18}$$

From the global stability condition (4.3) and from Theorem 3.6, it follows that for every $t \in [0, T]$ we have $\tau(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\mathrm{div}\, \tau(t) = f(t)$ a.e. on Ω, and $[\tau(t)\nu] = g(t)$ on Γ_1. By Lemma 5.5 we have $(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in A(\dot{w}(t))$ for a.e. $t \in [0, T]$. Therefore, from Proposition 2.4 we have $\mathcal{H}(\dot{p}(t)) \geqq \langle \tau_D(t) | \dot{p}(t) \rangle$. By (5.17), this implies

$$\langle \sigma(t) - \varrho(t) | \dot{e}(t) - E\dot{w}(t) \rangle + \langle \big[\tau_D(t) - \varrho_D(t)\big] \, | \dot{p}(t) \rangle \leqq 0 \,.$$

As $\mathrm{div}(\tau(t) - \varrho(t)) = 0$ a.e. on Ω and $\big[(\tau(t) - \varrho(t))\nu\big] = 0$ on Γ_1 by (2.17) and Theorem 3.6, this inequality is equivalent to

$$\langle \sigma(t) - \tau(t) | \dot{e}(t) - E\dot{w}(t) \rangle \leqq 0,$$

in view of the integration by parts formula (2.40). Analogously, from (5.18) we obtain

$$\langle \tau(t) - \sigma(t) | \dot{\eta}(t) - E\dot{w}(t) \rangle \leqq 0 \,.$$

Summing these two inequalities we find that

$$\langle \mathbb{C}(e(t) - \eta(t)) | \dot{e}(t) - \dot{\eta}(t) \rangle \leqq 0 \,;$$

hence,

$$\frac{d}{dt} \langle \mathbb{C}(e(t) - \eta(t)) | e(t) - \eta(t) \rangle \leqq 0 \,.$$

If $e(0) = \eta(0)$, then $\langle \mathbb{C}(e(0) - \eta(0)) | e(0) - \eta(0) \rangle = 0$, thus for every $t \in [0, T]$ $\langle \mathbb{C}(e(t) - \eta(t)) | e(t) - \eta(t) \rangle \leqq 0$, which is equivalent to $e(t) = \eta(t)$ by (2.13). □

6. Equivalent formulations in rate form

Let $t \mapsto (u(t), e(t), p(t))$ be a quasistatic evolution. Suppose, for a moment, that $\dot{p}(t) \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$, and denote the values of $\dot{p}(t)$ and $\sigma_D(t)$ at $x \in \Omega$ by $\dot{p}(t, x)$ and $\sigma_D(t, x)$, respectively. We recall that the normal cone $N_K(\xi_0)$ to K at $\xi_0 \in \mathbb{M}^{n\times n}_D$ is defined in the following way: if $\xi_0 \in K$, then $N_K(\xi_0)$ is the set of matrices $\zeta \in \mathbb{M}^{n\times n}_D$, such that $\zeta{:}(\xi - \xi_0) \leqq 0$ for every $\xi \in K$; if $\xi_0 \notin K$, then $N_K(\xi_0) := \emptyset$. In this section, we want to prove that

$$\dot{p}(t, x) \in N_K(\sigma_D(t, x)) \qquad \text{for a.e. } x \in \Omega \,, \tag{6.1}$$

which represents the classical formulation of the flow rule.

6.1. Weak formulation

By the definition of N_K, it is easy to see that (6.1) is equivalent to

$$\langle \sigma_D(t) - \tau_D | \dot{p}(t) \rangle \geqq 0 \tag{6.2}$$

for every $\tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ with $[\tau\nu] = g(t)$ on Γ_1. Indeed, the fact that (6.1) implies (6.2) is straightforward. To prove the converse implication, it is enough to consider test functions of the form $\tau = \varphi \xi + (1-\varphi)\sigma$, with $\varphi \in C_c^\infty(\Omega)$, $0 \leqq \varphi \leqq 1$, and $\xi \in K$.

Note that the variational inequality (6.2) makes sense, even if $\dot{p}(t)$ is only a measure, since in any case $\dot{p}(t) \in \Pi_{\Gamma_0}(\Omega)$ by Theorem 5.2 and Lemma 5.5. Thus the duality product $\langle \sigma_D(t) - \tau_D | \dot{p}(t) \rangle$ well defined by (2.39). We will regard (6.2) as the weak formulation of inclusion (6.1) when $\dot{p}(t) \in M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n\times n})$.

The following theorem collects three different sets of conditions, including (6.2), and expressed in terms of the time derivatives $\dot{p}(t)$, $\dot{e}(t)$, and $\dot{u}(t)$, which are equivalent to the conditions considered in Definition 4.2.

Theorem 6.1. *Assume* (2.1), (2.2), (2.7), (2.12), (2.13), *and* (2.15)–(2.19). *Let* $t \mapsto (u(t), e(t), p(t))$ *be a function from* $[0,T]$ *into* $BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n\times n}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n\times n})$ *and let* $\sigma(t) := \mathbb{C}e(t)$. *The following conditions are then equivalent:*

(a) $t \mapsto (u(t), e(t), p(t))$ *is a quasistatic evolution;*

(b) $t \mapsto (u(t), e(t), p(t))$ *is absolutely continuous and*

- (b1) *for every* $t \in [0,T]$ *we have* $(u(t), e(t), p(t)) \in A(w(t))$, $\sigma(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\operatorname{div}\sigma(t) = f(t)$ *a.e. on* Ω, *and* $[\sigma(t)\nu] = g(t)$ *on* Γ_1,
- (b2) *for a.e.* $t \in [0,T]$ *we have*

$$\mathcal{H}(\dot{p}(t)) = \langle \sigma_D(t) | \dot{p}(t) \rangle\,;$$

(c) $t \mapsto (u(t), e(t), p(t))$ *is absolutely continuous and*

- (c1) *for every* $t \in [0,T]$ *we have* $(u(t), e(t), p(t)) \in A(w(t))$, $\sigma(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\operatorname{div}\sigma(t) = f(t)$ *a.e. on* Ω, *and* $[\sigma(t)\nu] = g(t)$ *on* Γ_1,
- (c2) *for a.e.* $t \in [0,T]$ *we have*

$$\langle \sigma_D(t) - \tau_D | \dot{p}(t) \rangle \geqq 0$$

for every $\tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ *with* $[\tau\nu] = g(t)$ *on* Γ_1;

(d) $t \mapsto (u(t), e(t))$ *is absolutely continuous and*

- (d1) *for every* $t \in [0,T]$ *we have* $\sigma(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\operatorname{div}\sigma(t) = f(t)$ *a.e. on* Ω, *and* $[\sigma(t)\nu] = g(t)$ *on* Γ_1,
- (d2) *for a.e.* $t \in [0,T]$ *we have*

$$\langle \tau - \sigma(t) | \dot{e}(t) \rangle + \langle \operatorname{div}\tau - \operatorname{div}\sigma(t) | \dot{u}(t) \rangle \geqq \langle [(\tau - \sigma(t))\nu] \,|\, \dot{w}(t) \rangle_{\partial\Omega}$$

for every $\tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ *with* $[\tau\nu] = g(t)$ *on* Γ_1, *where* $\langle\cdot|\cdot\rangle_{\partial\Omega}$ *denotes the duality pairing between* $H^{-1/2}(\partial\Omega; \mathbb{R}^n)$ *and* $H^{1/2}(\partial\Omega; \mathbb{R}^n)$,
- (d3) *for every* $t \in [0,T]$ $p(t) = Eu(t) - e(t)$ *on* Ω *and* $p(t) = (w(t) - u(t)) \odot \nu\, \mathcal{H}^{n-1}$ *on* Γ_0.

Note that in conditions (b) and (c) the duality products $\langle \sigma_D(t) | \dot p(t)\rangle$ and $\langle \sigma_D(t) - \tau_D | \dot p(t)\rangle$ are well defined by (2.39), since $\dot p(t) \in \Pi_{\Gamma_0}(\Omega)$ by Lemma 5.5, and $\sigma(t)$, $\tau \in \Sigma(\Omega)$.

Proof of Theorem 6.1. We first prove that (a) $\Leftrightarrow$ (b). We already proved, in Theorem 5.2, that every quasistatic evolution is absolutely continuous. Moreover, Theorem 3.6 shows that (b1) is equivalent to the global stability condition (qs1) of Definition 4.2. By Proposition 5.6, it only remains to prove that, for an absolutely continuous function $t \mapsto (u(t), e(t), p(t))$ satisfying either (b1) or (qs1), condition (b2) is equivalent to the balance of powers

$$\langle \sigma(t) | \dot e(t)\rangle + \mathcal{H}(\dot p(t)) = \langle \sigma(t) | E\dot w(t)\rangle - \langle \mathcal{L}(t) | \dot w(t)\rangle + \langle \mathcal{L}(t) | \dot u(t)\rangle \tag{6.3}$$

for a.e. $t \in [0, T]$. Since $(\dot u(t), \dot e(t), \dot p(t)) \in A(\dot w(t))$ for a.e. $t \in [0, T]$ by Lemma 5.5, condition (b2) is equivalent to (6.3) in view of the integration by parts formula (2.40).

We now prove that (b) $\Leftrightarrow$ (c). It is enough to show that, if (b1) is satisfied, then (b2) $\Leftrightarrow$ (c2). Condition (c2) is equivalent to

$$\langle \sigma_D(t) | \dot p(t)\rangle \geqq \sup\{\langle \tau_D | \dot p(t)\rangle : \tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega),\ [\tau\nu] = g(t) \text{ on } \Gamma_1\}.$$

Since $\sigma(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ and $[\sigma(t)\nu] = g(t)$ on Γ_1 by (b1), the opposite inequality is trivial, thus (c2) is equivalent to

$$\langle \sigma_D(t) | \dot p(t)\rangle = \sup\{\langle \tau_D | \dot p(t)\rangle : \tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega),\ [\tau\nu] = g(t) \text{ on } \Gamma_1\}.$$

This last condition is equivalent to (b2) by Proposition 2.4.

Finally, we prove that (c) $\Leftrightarrow$ (d). We observe first that (d3) and the absolute continuity of $t \mapsto (u(t), e(t))$ imply that $t \mapsto p(t)$ is also absolutely continuous and $(u(t), e(t), p(t)) \in A(w(t))$ for every $t \in [0, T]$. It remains to prove that if (c1) is satisfied, then (c2) $\Leftrightarrow$ (d2).

By (2.24) we have

$$\langle [(\tau - \sigma(t))\nu] \,|\, \dot w(t)\rangle_{\partial\Omega} = \langle \operatorname{div}\tau - \operatorname{div}\sigma(t) | \dot w(t)\rangle + \langle \tau - \sigma(t) | E\dot w(t)\rangle .$$

Therefore, (d2) is equivalent to

$$\langle \tau - \sigma(t) | \dot e(t) - E\dot w(t)\rangle + \langle \operatorname{div}\tau - \operatorname{div}\sigma(t) | \dot u(t) - \dot w(t)\rangle \geqq 0 . \tag{6.4}$$

Since $(\dot u(t), \dot e(t), \dot p(t)) \in A(\dot w(t))$ for a.e. $t \in [0, T]$ by Lemma 5.5, and $[(\tau - \sigma(t))\nu] = 0$ on Γ_1, condition (c2) is equivalent to (6.4) as a consequence of the integration by parts formula (2.40). □

Remark 6.2. By Proposition 2.4, the measure $H(\dot p(t)) - [\sigma_D(t){:}\dot p(t)]$ is nonnegative on $\Omega \cup \Gamma_0$, so that (b2) of Theorem 6.1 implies

$$H(\dot p(t)) = [\sigma_D(t){:}\dot p(t)] \qquad \text{on } \Omega \cup \Gamma_0 . \tag{6.5}$$

Remark 6.3. Condition (d) of Theorem 6.1 is the weak formulation of the quasistatic evolution problem for perfectly plastic materials, proposed in [12] in a slightly different form, and analysed in [28].

6.2. Strong formulation and precise definition of the stress

Let us return to the classical formulation (6.1) of the flow rule, which makes sense if $\dot p(t) \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$. It can be written in the equivalent form

$$\frac{\dot p(t,x)}{|\dot p(t,x)|} \in N_K(\sigma_D(t,x)) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \{|\dot p(t)| > 0\}.$$

When $\dot p(t) \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, we can consider the Radon-Nikodym derivative $\dot p(t)/|\dot p(t)|$ of $\dot p(t)$ with respect to its variation $|\dot p(t)|$, which is a function defined $|\dot p(t)|$- a.e. on $\Omega \cup \Gamma_0$. We note that

$$\frac{\dot p(t)}{|\dot p(t)|}(x) = \frac{\dot p(t,x)}{|\dot p(t,x)|} \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \{|\dot p(t)| > 0\},$$

when $\dot p(t) \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$. It is tempting to consider the inclusion

$$\frac{\dot p(t)}{|\dot p(t)|}(x) \in N_K(\sigma_D(t,x)), \tag{6.6}$$

as a pointwise formulation of the flow rule in the general case $\dot p(t) \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$. There is, however, a problem owing to the fact that the left-hand side of (6.6) is defined $|\dot p(t)|$- a.e. on $\Omega \cup \Gamma_0$, while the right-hand side is defined only $\mathcal{L}^n$-a.e. on Ω. This difficulty is overcome in Theorem 6.4 below, by introducing a precise representative $\hat\sigma_D(t,x)$ of $\sigma_D(t,x)$, defined almost everywhere with respect to the measure $\mu(t) := \mathcal{L}^n + |\dot p(t)|$. A delicate point in the choice of this representative is the fact that it must also satisfy an integration by parts formula (see Remark 6.5). If K is strictly convex, this representative is essentially unique and can be obtained, in Ω, as the limit of the averages of σ_D (see Theorem 6.6).

Theorem 6.4. *Assume* (2.1), (2.2), (2.7), (2.12), (2.13), *and* (2.15)–(2.19). *Let* $t \mapsto (u(t), e(t), p(t))$ *be a function from* $[0,T]$ *into* $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$*, let* $\sigma(t) := \mathbb{C}e(t)$*, and let* $\mu(t) := \mathcal{L}^n + |\dot p(t)|$*. Then* $t \mapsto (u(t), e(t), p(t))$ *is a quasistatic evolution if, and only if,*

(e) $t \mapsto (u(t), e(t), p(t))$ *is absolutely continuous and*

(e1) *for every* $t \in [0,T]$ *we have* $(u(t), e(t), p(t)) \in A(w(t))$*,* $\sigma(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$*,* $-\operatorname{div}\sigma(t) = f(t)$ *a.e. on* Ω*, and* $[\sigma(t)\nu] = g(t)$ *on* Γ_1*,*

(e2) *for a.e.* $t \in [0,T]$ *there exists* $\hat\sigma_D(t) \in L^\infty_{\mu(t)}(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ *such that*

$$\hat\sigma_D(t) = \sigma_D(t) \qquad \mathcal{L}^n\text{-a.e. on } \Omega, \tag{6.7}$$

$$[\sigma_D(t){:}\dot p(t)] = \Big(\hat\sigma_D(t){:}\frac{\dot p(t)}{|\dot p(t)|}\Big)|\dot p(t)| \quad \text{on } \Omega \cup \Gamma_0, \tag{6.8}$$

$$\frac{\dot p(t)}{|\dot p(t)|}(x) \in N_K(\hat\sigma_D(t,x)) \qquad \text{for } |\dot p(t)|\text{-a.e. } x \in \Omega \cup \Gamma_0, \tag{6.9}$$

where $\hat\sigma_D(t,x)$ *denotes the value of* $\hat\sigma_D(t)$ *at the point* x.

Remark 6.5. Assume that $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous. If (e1) holds, then we can prove using (2.41) that condition (6.8) of Theorem 6.4 is equivalent to the following integration by parts formula: for every $\varphi \in C^1(\overline{\Omega})$ we have

$$\begin{aligned}&\langle \varphi\, \hat\sigma_D(t) | \dot p(t)\rangle \\ &= \langle \sigma(t) | \varphi\,(\dot e(t) - E\dot w(t))\rangle - \langle\, \sigma(t) | (\dot u(t) - \dot w(t)) \odot \nabla\varphi\rangle \\ &\quad + \langle f(t) | \varphi\,(\dot u(t) - \dot w(t))\rangle + \langle g(t) | \varphi\,(\dot u(t) - \dot w(t))\rangle_{\Gamma_1}\,,\end{aligned}$$

where the duality product in the left-hand side is defined by (2.4).

As $\dot p(t)/|\dot p(t)| = 1$ $|\dot p(t)|$-a.e. on $\Omega \cup \Gamma_0$, and $N_K(\xi) = \{0\}$ if ξ is in the interior of K, we can deduce from (6.9) that for a.e. $t \in [0, T]$

$$\hat\sigma_D(t, x) \in \partial K \qquad \text{for } |\dot p(t)|\text{- a.e. } x \in \Omega \cup \Gamma_0\,. \tag{6.10}$$

Using [26] (Theorem 23.5), we can prove that condition (6.9) is equivalent to

$$\hat\sigma_D(t, x) \in \partial H\left(\frac{\dot p(t)}{|\dot p(t)|}(x)\right) \qquad \text{for } |\dot p(t)|\text{- a.e. } x \in \Omega \cup \Gamma_0\,. \tag{6.11}$$

Since ∂H is positively homogeneous of degree 0, this is equivalent to the fact that both of the following inclusions are satisfied:

$$\hat\sigma_D(t, x) \in \partial H(\dot p^a(t)(x)) \qquad \text{for } \mathcal{L}^n\text{- a.e. } x \in \{|\dot p^a(t)| > 0\}\,, \tag{6.12}$$

$$\hat\sigma_D(t, x) \in \partial H\left(\tfrac{\dot p(t)}{|\dot p(t)|}(x)\right) \qquad \text{for } |\dot p^s(t)|\text{- a.e. } x \in \Omega \cup \Gamma_0\,. \tag{6.13}$$

Conditions (6.12) and (6.13) are the measure theoretic version of the formulation (cf5″), based on a dissipation pseudo-potential (see the Introduction). Our terminology is based on [7], where a further formulation is examined, based on the notion of a maximally responsive multivalued map G, and expressed by a relation of the form $\sigma_D(t, x) \in G(\dot p(t, x))$. Since it is shown in [7] that compatibility with the maximum plastic work inequality (cf5) leads to $G = \partial H$, we will not present a separate analysis of the formulation based on maximal responsiveness.

Proof of Theorem 6.4. Assume that $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution. Then $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous by Theorem 5.2 and condition (e1) is satisfied by Theorem 6.1.

Let $A(t) \subset \Omega$ and $B(t) \subset \Omega \cup \Gamma_0$ be two disjoint Borel sets, such that $A(t) \cup B(t) = \Omega \cup \Gamma_0$ and $|\dot p^s(t)|(A(t)) = \mathcal{L}^n(B(t)) = 0$. We define

$$\begin{aligned}\hat\sigma_D(t, x) &:= \sigma_D(t, x) && \text{for } \mathcal{L}^n\text{- a.e. } x \in A(t)\,,\\ \hat\sigma_D(t, x) &:= \partial_0 H\left(\frac{\dot p(t)}{|\dot p(t)|}(x)\right) && \text{for } |\dot p^s(t)|\text{- a.e. } x \in B(t)\,,\end{aligned}$$

where $\partial_0 H(\xi)$ denotes the element of $\partial H(\xi)$ with minimum norm. Equation (6.7) then follows from the definition of $\hat\sigma_D(t)$ on $A(t)$, and (6.13) follows from the definition of $\hat\sigma_D(t)$ on $B(t)$. To prove (6.12), it is enough to show that

$$\sigma_D(t, x) \in \partial H(\dot p^a(t)(x)) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \{|\dot p^a(t)| > 0\}\,. \tag{6.14}$$

Taking the absolutely continuous parts in (6.5), We obtain that $H(\dot{p}^a(t)) = \sigma_D(t){:}\dot{p}^a(t)$ $\mathcal{L}^n$-a.e. on Ω. Since for $\mathcal{L}^n$- a.e. $x \in \Omega$ we have $\sigma_D(t,x) \in K = \partial H(0)$ (see e.g. [26] (Corollary 23.5.3)), we obtain $\sigma_D(t,x){:}\xi \leqq H(\xi)$ for every $\xi \in \mathbb{M}^{n\times n}_D$. Therefore, for $\mathcal{L}^n$- a.e. $x \in \Omega$ we have $\sigma_D(t,x){:}(\xi - \dot{p}^a(t)) \leqq H(\xi) - H(\dot{p}^a(t)(x))$ for every $\xi \in \mathbb{M}^{n\times n}_D$, which implies (6.14).

To prove (6.8), we begin by proving the equality on $A(t)$. Since $|\dot{p}^s(t)| = 0$ on $A(t)$, we have $[\sigma_D(t){:}\dot{p}(t)] = \sigma_D(t){:}\dot{p}^a(t)$ on $A(t)$ by (2.33). As $\hat{\sigma}_D(t) = \sigma_D(t)$ $\mathcal{L}^n$- a.e. on $A(t)$ and $\dot{p}(t) = \dot{p}^a(t)$ on $A(t)$, we conclude that

$$[\sigma_D(t){:}\dot{p}(t)] = \sigma_D(t){:}\dot{p}^a(t) = \left(\hat{\sigma}_D(t){:}\frac{\dot{p}(t)}{|\dot{p}(t)|}\right)|\dot{p}(t)| \qquad \text{on } A(t)\,.$$

To prove the equality on $B(t)$, we rely on (6.5). Using the definition (2.8) of $H(\dot{p}(t))$, the proof of (6.8) will be complete if we show that

$$H\left(\frac{\dot{p}(t)}{|\dot{p}(t)|}\right) = \hat{\sigma}_D(t){:}\frac{\dot{p}(t)}{|\dot{p}(t)|} \qquad |\dot{p}(t)|\text{-a.e. on } B(t)\,.$$

But this equality follows from the definition of $\hat{\sigma}_D(t)$ on $B(t)$, using the Euler identity

$$H(\xi) = \zeta{:}\xi \qquad \text{for every } \xi \in \mathbb{M}^{n\times n}_D \text{ and every } \zeta \in \partial H(\xi)\,.$$

This concludes the proof of (e2).

Conversely, assume (e). By (6.11), again using the Euler identity, for a.e. $t \in [0,T]$ we obtain

$$H\left(\frac{\dot{p}(t)}{|\dot{p}(t)|}\right) = \hat{\sigma}_D(t){:}\frac{\dot{p}(t)}{|\dot{p}(t)|} \qquad |\dot{p}(t)|\text{-a.e. on } \Omega\cup\Gamma_0\,.$$

From the definition (2.8) of the measure $H(\dot{p}(t))$, and from (6.8), we deduce that $\mathcal{H}(\dot{p}(t)) = \langle\sigma_D(t)|\dot{p}(t)\rangle$ for a.e. $t \in [0,T]$. Therefore, $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution by Theorem 6.1. □

For every $r > 0$ and every $t \in [0,T]$ we consider the function $\sigma^r(t) \in C(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, defined by

$$\sigma^r(t,x) := \frac{1}{\mathcal{L}^n(B(x,r)\cap\Omega)}\int_{B(x,r)\cap\Omega}\sigma(t,y)\,dy\,. \tag{6.15}$$

Since K is convex, we have $\sigma^r(t,x) \in K$ for every $x \in \Omega$.

If K is strictly convex, i.e. $s\,\xi_1 + (1-s)\,\xi_2$ is an interior point of K for every $0 < s < 1$ and every pair of distinct points $\xi_1, \xi_2 \in K$, then H is differentiable at all points $\xi \neq 0$ (see e.g. [26] (Corollary 23.5.4 and Theorem 25.1)) and we keep the notation $\partial H(\xi)$ for the gradient. Under this hypothesis, for a.e. $t \in [0,T]$ the function $\hat{\sigma}_D(t)$ is uniquely determined $\mu(t)$- a.e. on $\Omega\cup\Gamma_0$ by (6.7) and (6.11) as

$$\hat{\sigma}_D(t) = \sigma_D(t) \qquad \mathcal{L}^n\text{-a.e. on } \Omega\,, \tag{6.16}$$

$$\hat{\sigma}_D(t) = \partial H\left(\frac{\dot{p}(t)}{|\dot{p}(t)|}\right) \qquad |\dot{p}(t)|\text{-a.e. on } \Omega\cup\Gamma_0\,. \tag{6.17}$$

The following theorem shows that under the same hypothesis, $\hat{\sigma}_D(t)$ can be obtained in Ω as the limit of $\sigma^r_D(t)$ as $r \to 0$. This confirms the intrinsic character of the precise representative introduced in Theorem 6.4.

Theorem 6.6. *Assume* (2.1), (2.2), (2.7), (2.12), (2.13), *and* (2.15)–(2.19). *Assume in addition that K is strictly convex. Let $t \mapsto (u(t), e(t), p(t))$ be a quasistatic evolution, let $\mu(t) := \mathcal{L}^n + |\dot{p}(t)|$, let $\sigma(t) := \mathbb{C}e(t)$, and let $\sigma^r(t)$ and $\hat{\sigma}_D(t)$ be defined by* (6.15) *and* (6.17). *Thus $\sigma^r_D(t) \to \hat{\sigma}_D(t)$ strongly in $L^1_{\mu(t)}(\Omega; \mathbb{M}^{n\times n}_D)$ for a.e. $t \in [0, T]$.*

Proof. This proof is inspired by the proof of [1] (Theorem 3.7). Since $\sigma^r_D(t)$ converge to $\sigma_D(t)$ strongly in $L^1(\Omega; \mathbb{M}^{n\times n}_D)$ and $\|\sigma^r_D(t)\|_\infty$ is bounded uniformly with respect to r, it suffices to prove that $\sigma^r_D(t) \to \hat{\sigma}_D(t)$ strongly in $L^1_{|\dot{p}(t)|}(U; \mathbb{M}^{n\times n}_D)$ for every open set $U \subset\subset \Omega$. Let us fix U. Since $\sigma^r(t) \to \sigma(t)$ strongly in $L^2(U; \mathbb{M}^{n\times n}_{sym})$, $\operatorname{div}\sigma^r(t) \to \operatorname{div}\sigma(t)$ strongly in $L^n(U; \mathbb{R}^n)$, and $\sigma^r_D(t)$ is bounded in $L^\infty(U; \mathbb{M}^{n\times n}_D)$, by (2.38) we have

$$\langle\big[\sigma^r_D(t){:}\dot{p}(t)\big]\,|\varphi\rangle \to \langle[\sigma_D(t){:}\dot{p}(t)]\,|\varphi\rangle \tag{6.18}$$

for every $\varphi \in C_0(U)$ and for a.e. $t \in [0, T]$. By (2.36) we also have $\big[\sigma^r_D(t){:}\dot{p}(t)\big] = \sigma^r_D(t){:}\dot{p}(t)$ on U, where the right-hand side is defined by (2.37). By (6.5) we also have $[\sigma_D(t){:}\dot{p}(t)] = H(\dot{p}(t))$ on U. Therefore, the definition (2.8) of $H(\dot{p}(t))$ and (6.18), together with the boundedness of $\sigma^r_D(t)$, imply that

$$\sigma^r_D(t){:}\frac{\dot{p}(t)}{|\dot{p}(t)|} \rightharpoonup H\left(\frac{\dot{p}(t)}{|\dot{p}(t)|}\right) \qquad \text{weakly* in } L^\infty_{|\dot{p}(t)|}(U) \tag{6.19}$$

for a.e. $t \in [0, T]$.

Let us fix $t \in [0, T]$ such that (6.10), (6.17), and (6.19) hold. Since $\sigma^r_D(t)$ is bounded in $L^\infty_{|\dot{p}(t)|}(U; \mathbb{M}^{n\times n}_D)$, there exists a sequence $r_j \to 0$ such that $\sigma^{r_j}_D(t) \rightharpoonup \sigma^*$ for some $\sigma^* \in L^\infty_{|\dot{p}(t)|}(U; \mathbb{M}^{n\times n}_D)$. From (6.19) we deduce that

$$\sigma^*{:}\frac{\dot{p}(t)}{|\dot{p}(t)|} = H\left(\frac{\dot{p}(t)}{|\dot{p}(t)|}\right) \qquad |\dot{p}(t)|\text{-a.e. on } U\,. \tag{6.20}$$

Let us now fix $\xi \in \mathbb{M}^{n\times n}_D$. Since $\sigma^{r_j}_D(t, x) \in K = \partial H(0)$ for every $x \in U$, we have $\sigma^{r_j}_D(t){:}\xi \leqq H(\xi)$ $|\dot{p}(t)|$- a.e. on U. As $\sigma^{r_j}_D(t){:}\xi \rightharpoonup \sigma^*{:}\xi$ weakly* in $L^\infty_{|\dot{p}(t)|}(U)$, we have also $\sigma^*{:}\xi \leqq H(\xi)$ $|\dot{p}(t)|$- a.e. on U. Taking (6.20) into account, we obtain

$$\sigma^*{:}\left(\xi - \frac{\dot{p}(t)}{|\dot{p}(t)|}\right) \leqq H(\xi) - H\Big(\frac{\dot{p}(t)}{|\dot{p}(t)|}\Big) \qquad |\dot{p}(t)|\text{-a.e. on } U\,.$$

In view of the differentiability properties of H, this implies

$$\sigma^* = \partial H\left(\frac{\dot{p}(t)}{|\dot{p}(t)|}\right) \qquad |\dot{p}(t)|\text{-a.e. on } U\,.$$

By (6.17) we deduce that $\sigma^* = \hat{\sigma}_D(t)$ $|\dot{p}(t)|$- a.e. on U. Since the limit does not depend on the sequence r_j, we conclude that

$$\sigma_D^r(t) \rightharpoonup \hat{\sigma}_D(t) \qquad \text{weakly* in } L^\infty_{|\dot{p}(t)|}(U; \mathbb{M}^{n\times n}_D)\,. \tag{6.21}$$

As $\hat{\sigma}_D(t,x) \in \partial K$ for $|\dot{p}(t)|$-a.e. $x \in U$ by Remark 6.5 and $\sigma_D^r(t,x) \in K$ for every $x \in U$, the strict convexity of K can be used to improve the weak* convergence in (6.21), and to obtain strong convergence in $L^1_{|\dot{p}(t)|}(U; \mathbb{M}^{n\times n}_D)$ (see e.g. [31]). □

7. Appendix

Let X be the dual of a separable Banach space Y. Let $\mathcal{K}$ be a bounded closed convex subset of Y, containing the origin as an interior point, and let $\mathcal{H}: X \to \mathbb{R}$ be its support function, defined by

$$\mathcal{H}(x) := \sup_{y\in\mathcal{K}} \langle x|y\rangle .$$

Since $\mathcal{K}$ is a bounded neighborhood of the origin, there exist two constants $\alpha_{\mathcal{H}}$ and $\beta_{\mathcal{H}}$, with $0 < \alpha_{\mathcal{H}} \leqq \beta_{\mathcal{H}} < +\infty$, such that

$$\alpha_{\mathcal{H}} \|x\|_X \leqq \mathcal{H}(x) \leqq \beta_{\mathcal{H}} \|x\|_X \qquad \text{for every } x \in X\,. \tag{7.1}$$

Given $f: [0,T] \to X$ and $a, b \in [0,T]$ with $a \leqq b$, we denote the total variation of f on $[a,b]$ by

$$\mathcal{V}(f; a, b) := \sup\left\{ \sum_{i=1}^N \|f(t_i) - f(t_{i-1})\|_X : \quad a = t_0 \leqq t_1 \leqq \ldots \leqq t_N = b,\ N \in \mathbb{N} \right\},$$

and we define the $\mathcal{H}$-variation of f on $[a,b]$ as

$$\mathcal{V}_{\mathcal{H}}(f; a, b) := \sup\left\{ \sum_{i=1}^N \mathcal{H}(f(t_i) - f(t_{i-1})) : \quad a = t_0 \leqq t_1 \leqq \ldots \leqq t_N = b,\ N \in \mathbb{N} \right\}.$$

From (7.1), it follows that

$$\alpha_{\mathcal{H}} \mathcal{V}(f; a, b) \leqq \mathcal{V}_{\mathcal{H}}(f; a, b) \leqq \beta_{\mathcal{H}} \mathcal{V}(f; a, b)\,.$$

Since $\mathcal{H}$ is weakly* lower semicontinuous, we have

$$\mathcal{V}_{\mathcal{H}}(f; a, b) \leqq \liminf_{k\to\infty} \mathcal{V}_{\mathcal{H}}(f_k; a, b) \tag{7.2}$$

whenever $f_k(t) \rightharpoonup f(t)$ weakly* for every $t \in [a,b]$.

We now prove a theorem about weak* derivatives of absolutely continuous functions with values in X and their relationships with the notion of $\mathcal{H}$-variation.

Theorem 7.1. *Let $f\colon [0,T] \to X$ be an absolutely continuous function. Then the weak*-limit*

$$\dot f(t) := w^* - \lim_{s\to t} \frac{f(s)-f(t)}{s-t} \tag{7.3}$$

exists for a.e. $t \in [0,T]$, and

$$\mathcal{H}(\dot f(t)) = \lim_{s\to t} \mathcal{H}\left(\frac{f(s)-f(t)}{s-t}\right) \tag{7.4}$$

for a.e. $t \in [0,T]$. Moreover, the function $t \mapsto \mathcal{H}(\dot f(t))$ is measurable and

$$\mathcal{V}_{\mathcal{H}}(f;a,b) = \int_a^b \mathcal{H}(\dot f(t))\,dt \tag{7.5}$$

for every $a, b \in [0,T]$ with $a \leqq b$.

Proof. Let F be the linear span over $\mathbb{Q}$ of a countable dense set in Y. For every $y \in F$ the map $t \mapsto \langle f(t)|y\rangle$ is absolutely continuous on $[0,T]$. Therefore, there exists a set N_y of measure zero, such that the limit

$$D_y(t) := \lim_{s\to t} \frac{\langle f(s)-f(t)|y\rangle}{s-t}$$

exists for every $t \in [0,T]\backslash N_y$. Let $\mathcal{V}(t) := \mathcal{V}(f;0,t)$. Since the function $t \mapsto \mathcal{V}(t)$ is non-decreasing, it is differentiable for every $t \in [0,T]\backslash M$, where M is a set of measure zero. Let N be the union of M with the sets N_y for $y \in F$. Then, $\mathcal{L}^1(N) = 0$, the derivative $D_y(t)$ exists for every $y \in F$ and every $t \in [0,T]\backslash N$, and

$$|D_y(t)| = \lim_{s\to t} \frac{|\langle f(s)-f(t)|y\rangle|}{|s-t|} \leqq \dot{\mathcal{V}}(t)\|y\|_Y \tag{7.6}$$

for every $y \in F$ and every $t \in [0,T]\backslash N$. Now, for $t \in [0,T]\backslash N$, consider the linear map $y \in F \mapsto D_y(t)$. This map is continuous by (7.6). Therefore there exists a vector in X, which we call $\dot f(t)$, such that

$$D_y(t) = \langle \dot f(t)|y\rangle$$

for every $y \in F$. Using the density of F and (7.6) it is easy to show that the vector $\dot f(t)$ satisfies

$$\langle \dot f(t)|y\rangle = \lim_{s\to t} \frac{\langle f(s)-f(t)|y\rangle}{s-t}$$

for every $y \in Y$ and every $t \in [0,T]\backslash N$, thus (7.3) is satisfied.

We note that the function $t \mapsto \mathcal{H}(\dot f(t))$ is measurable, since the map $t \to \langle \dot f(t)|y\rangle$ is measurable for every $y \in Y$ and $\mathcal{H}(\dot f(t)) = \sup_{y\in\mathcal{K}_0}\langle \dot f(t)|y\rangle$, where $\mathcal{K}_0$ is a countable dense subset of $\mathcal{K}$. Moreover, if $a = t_0 \leqq t_1 \leqq \ldots \leqq t_{N-1} \leqq t_N = b$ is a subdivision of $[a,b]$, then

$$\langle f(t_i) - f(t_{i-1})|y\rangle = \int_{t_{i-1}}^{t_i} \langle \dot f(t)|y\rangle\,dt \leqq \int_{t_{i-1}}^{t_i} \mathcal{H}(\dot f(t))\,dt$$

for every $1 \leqq i \leqq N$ and every $y \in \mathcal{K}$. Hence

$$\mathcal{H}(f(t_i) - f(t_{i-1})) \leqq \int_{t_{i-1}}^{t_i} \mathcal{H}(\dot{f}(t))\, dt$$

for every $1 \leqq i \leqq N$. Summing over i and taking the supremum over all subdivisions, we obtain

$$\mathcal{V}_{\mathcal{H}}(f; a, b) \leqq \int_a^b \mathcal{H}(\dot{f}(t))\, dt\,. \tag{7.7}$$

To show the converse inequality, we note that the function $t \mapsto \mathcal{V}_{\mathcal{H}}(f; 0, t)$ is non-decreasing. Therefore, it is differentiable for a.e. $t \in [0, T]$ and

$$\int_a^b \frac{d}{dt}\mathcal{V}_{\mathcal{H}}(f; 0, t)\, dt \leqq \mathcal{V}_{\mathcal{H}}(f; a, b)\,. \tag{7.8}$$

Let $t_0 \in [0, T]$ be a point where both f and $\mathcal{V}_{\mathcal{H}}(f; 0, \cdot)$ are differentiable. Since $\mathcal{H}$ is positively homogeneous of degree 1, we have

$$\mathcal{H}\left(\frac{f(t) - f(t_0)}{t - t_0}\right) \leqq \frac{\mathcal{V}_{\mathcal{H}}(f; 0, t) - \mathcal{V}_{\mathcal{H}}(f; 0, t_0)}{t - t_0}$$

for every $t \neq t_0$. Passing to the limit as $t \to t_0$ and using the weak*-lower semicontinuity of $\mathcal{H}$, we get

$$\begin{aligned}\mathcal{H}(\dot{f}(t_0)) &\leqq \liminf_{t\to t_0} \mathcal{H}\left(\frac{f(t) - f(t_0)}{t - t_0}\right)\\ &\leqq \limsup_{t\to t_0} \mathcal{H}\left(\frac{f(t) - f(t_0)}{t - t_0}\right)\\ &\leqq \frac{d}{dt}\mathcal{V}_{\mathcal{H}}(f; 0, t)\Big|_{t=t_0}\end{aligned}$$

for a.e. $t_0 \in [0, T]$. We now integrate the first and the last term in the previous inequality from a to b and we obtain (7.5) and (7.4) from (7.7) and (7.8). □

We conclude this appendix with a lemma that generalizes the classical Helly Theorem for real valued functions with uniformly bounded variation, as well as its extension to reflexive separable Banach spaces (see e.g. [3] (Chapter 1, Theorem 3.5)).

Lemma 7.2. *Let $f_k\colon [0, T] \to X$ be a sequence of functions such that $f_k(0)$ and $\mathcal{V}(f_k; 0, T)$ are bounded uniformly with respect to k. Then, there exist a subsequence, still denoted f_k, and a function $f\colon [0, T] \to X$ with bounded variation on $[0, T]$, such that $f_k(t) \rightharpoonup f(t)$ weakly* for every $t \in [0, T]$.*

Proof. It is enough to apply [15] (Theorem 3.2) with $\mathcal{Y} = X$, $\mathcal{R}(t) = \mathcal{V}(t)$ equal to the corresponding unit ball, and $\mathcal{T}$ equal to the weak* topology. □

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