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# A damage mechanics approach to stress softening and its application to rubber

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**Abstract** – We analyze stress softening phenomena within the framework of the 'generalized standard material' based on the notion of a 'normal dissipative mechanism'. We prove that the monotonicity properties of the 'yield function' governing such mechanism lead to local and global uniqueness of the response. Applications to oscillators with a single degree of freedom, whose anharmonic spring exhibits stress softening, are also presented. 

## 1. Introduction

Stress softening is a typical phenomenon observable for many materials during cyclic tension tests; when a given specimen is subjected to a load-unload-reload cycle, the stress accompanying a given stretch is always smaller during reloading than in the virgin loading path. If permanent set effects are negligible, stress softening can be described as a decay of elastic stiffness as the maximum strain (or stress) ever experienced by the material increases. In this paper we study stress softening phenomena of this kind from the point of view of damage mechanics.

Stress softening is particularly evident for specimens of (filled or unfilled) vulcanized rubber, [1] where it is known as Mullins effect. It has received considerable attention, not only in the early days of its discovery (Bouasse and Carrière, 1903; Holt, 1932), and the pioneering experimental and theoretical work of Mullins (Mullins, 1947; Mullins and Tobin, 1957), but also in recent times (see, e.g., (Johnson and Beatty, 1993) and the references quoted therein). Very detailed three-dimensional models have been proposed, studied and solved numerically (see, e.g., Govindjee and Simo, 1991; Lion, 1996; Miehe, 1995), but due to the limited availability of experimental data, calibration of the many material parameters involved is still problematic, and it is often difficult to assess the merits of each individual contribution. More recently, interest towards the development of simpler models, aiming at a conceptual understanding of the essentials of the phenomenon, has emerged. Models using a single scalar damage variable, either energy based (Ogden and Roxburgh, 1999), or based on the notion of maximum strain experienced by the material in its deformation history (Beatty and Krishnaswamy, 2000), have been proposed and compared with data from uniaxial experiments. Both competing models are formulated for more general, multiaxial states of stress and strain but, again, experimental results testing the

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[1] The actual response of a rubber specimen also shows other interesting features, such as permanent set, hysteresis in the unload-reload branch, healing, and rate effects. All these will be ignored in this paper, for the sake of simplicity.

predictive capabilities of the two approaches beyond the one-dimensional case are still missing. In particular, the question of whether a single scalar damage parameter can capture the complete experimental picture when complex stress or strain paths are imposed, or whether higher-dimensional parametrizations of damage are instead essential, seems open.

For the rest of the paper the Mullins effect will be used as a motivation, and we will focus on the response of an idealized specimen to one-dimensional tests. We show that stress softening phenomena of the type described above can be modelled within the theory of the “Generalized Standard Material” (GSM) (Germain, 1973; Germain et al., 1983; Halphen and Nguyen Quoc Son, 1975). This classical model rests upon the introduction of an internal damage variable, of its conjugate dissipation, of an elastic threshold (or yield function) $f$, and on an evolution law for damage (flow rule) arising from a postulate of maximal dissipation (normality rule). The GSM framework finds here an application of remarkable clarity, where the structural properties of the functions governing the response (in particular, the monotonicity of the yield function $f$ or, equivalently, the convexity of its primitive $F$) can be directly linked to the local and global uniqueness of the specimen response, i.e., to the well-posedness of the model.

We show that, if the experimental evidence to be matched is limited to uniaxial tests, then there is considerable freedom in the choice of the damage variable, and we use this freedom to our advantage. In fact, we prove the existence of a distinguished damage parameter, measuring the energy dissipated through damage (suitably normalized), which makes $F$ convex. Any reparametrization of damage preserving the convexity of $F$ leads to a well-posed model which defines the response of the specimen uniquely. Interestingly, although in the analysis of the model a parametrization of damage which is related to dissipation (somewhat similarly to (Ogden and Roxburgh, 1999)) proves to be the most effective, when considering the applications of Section 5, we find it more convenient to parametrize damage through the maximum strain ever experienced by the material, as in (Beatty and Krishnaswamy, 2000).

The paper is organized as follows. In Section 2 we describe the idealized response of a specimen exhibiting stress softening. In Section 3 we let a model based on a monotone yield function $f$ emerge naturally from the set of evolution rules presented in the previous section. Given the slightly pedagogical intent of these sections, we stage our discussion in the simplest possible nonlinear scenario: that of a specimen with piecewise linear response. This is done, however, without loss of generality because all of our results hold true for more realistic (more nonlinear and ‘rubber-like’) stress–strain characteristics such as the one schematically depicted in *figure 1*. Indeed, in Section 4, we prove that monotonicity of $f$ leads to the uniqueness of the specimen response without making use of the assumption of piecewise linearity. Finally, in Section 5, we briefly examine some concrete consequences of stress softening from a point of view closer to applications.

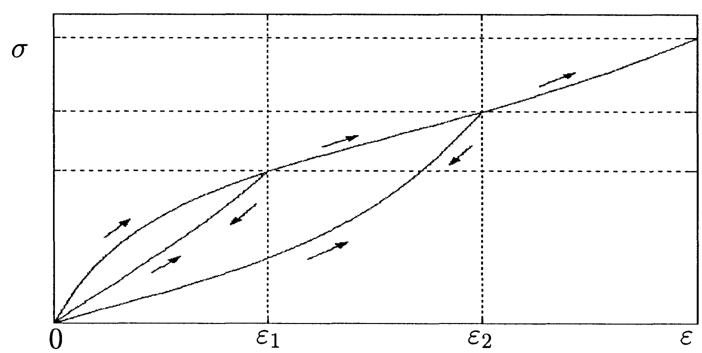


**Figure 1.** Schematic illustration of stress softening in a cyclic tension test. The arrows show the stress–strain curve during a strain controlled experiment with the sequence of strains $\varepsilon = 0, \varepsilon = \varepsilon_1, \varepsilon = 0, \varepsilon = \varepsilon_1, \varepsilon = \varepsilon_2, \varepsilon = 0, \varepsilon = \varepsilon_2, \varepsilon > \varepsilon_2$.

Since much of the technological interest for stress softening in rubbers comes from the automotive industry and seismic engineering, it seemed natural to explore it dynamically, and to consider a single-degree-of-freedom oscillator whose anharmonic spring exhibits stress softening. Even in this simple setting, nonlinearity poses severe challenges to a systematic understanding of the oscillatory response of such a system. For this reason, in Section 5 we restrict again attention to the model example of a specimen with piecewise linear response.

## 2. Stress softening in a model specimen with piecewise linear response

We consider an ideal specimen whose response under uniaxial extension (for definiteness, we will think of a strain controlled experiment) is the piecewise linear (PL) one shown in *figure 2*. Under a monotonically increasing loading, the virgin material exhibits the trilinear response $0 \to P_0 \to P_1 \to P$ (virgin curve) with respective elastic moduli[2] $E_0, E_2$ and $E_1$, with $0 < E_2 < E_1 < E_0$. If instead the virgin material is monotonically loaded only up to $P_\alpha$, an intermediate point along the segment $P_0 \to P_1$, and then unloaded and reloaded without trespassing $P_\alpha$, the behaviour is linearly elastic (in particular, no permanent deformations occur and the unloaded branch goes through the origin) with an elastic modulus $E_\alpha$ such that $E_1 < E_\alpha < E_0$. Upon loading past $P_\alpha$, the virgin curve is rejoined. Furthermore, a loading program with history such as the one in *figure 2*($a$) produces the stress–strain response of *figure 3*($b$). We emphasize that for every pair $\varepsilon_{\alpha_1}, \varepsilon_{\alpha_2}$ of inversion points of the imposed strain rates with $\varepsilon_{\alpha_1} < \varepsilon_{\alpha_2}$, the corresponding elastic moduli of the unloading branches are such that $E_{\alpha_1} > E_{\alpha_2}$. More formally: if $[0, 1] \ni \alpha \mapsto P_\alpha$ is any parametrization of the segment $P_0 \to P_1$, with $P_\alpha$ moving from $P_0$ to $P_1$ as $\alpha$ increases from 0 to 1, and if $E_\alpha$ is the elastic modulus of the unloading branch through $P_\alpha$, then $\alpha \mapsto E_\alpha$ is strictly decreasing. Finally, the specimen response is rate-independent.

The single most outstanding feature of the specimen response described above is the existence of (linearly) elastic unloading branches; their slopes decrease as the maximum loading level ever experienced by the virgin material increases (stress softening). Thus, we introduce a damage variable $\alpha \in [0, 1]$ ( $\alpha = 0$ denotes the virgin material, $\alpha = 1$ the fully damaged one) and we parametrize the 'fan' of unloading branches through their slopes, i.e., with the aid of a strictly decreasing map:

$$[0, 1] \ni \alpha \mapsto E_\alpha = E(\alpha) \in [E_1, E_0], \tag{2.1}$$

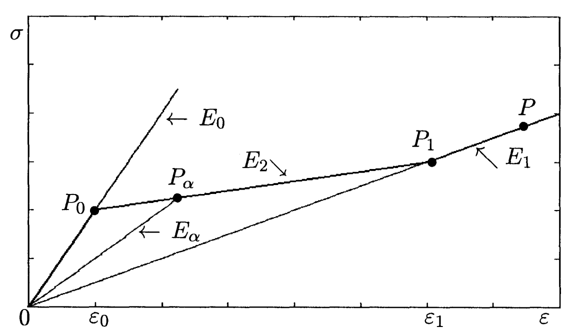


**Figure 2.** Piecewise linear (PL) stress–strain relation.

[2] All of our results are still valid in the limit case where the apparent modulus $E_2$ vanishes. In this case, however, consideration of a strain controlled experiment becomes essential.

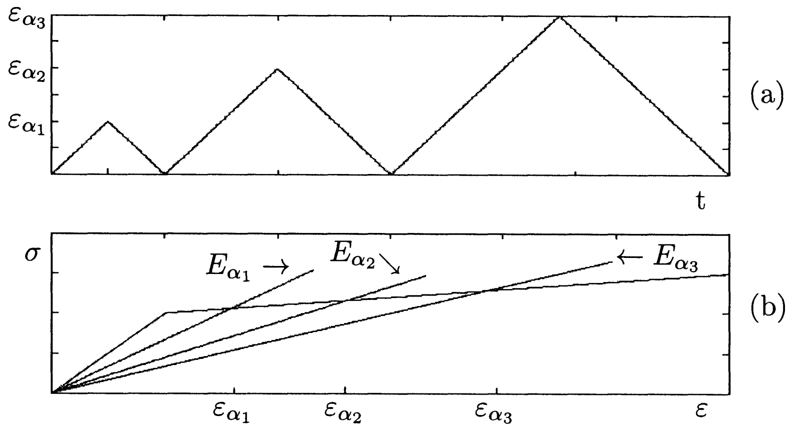


**Figure 3.** Loading program (a) and corresponding stress–strain response (b).

$$E'(\alpha) < 0 \Leftrightarrow \frac{\partial}{\partial \alpha} W(\varepsilon, \alpha) < 0. \tag{2.2}$$

Here $W(\varepsilon, \alpha) = 1/2E(\alpha)\varepsilon^2$ is the stored (elastic) energy corresponding to the state of the system associated with distortion $\varepsilon$ and damage $\alpha$. An example for (2.1), suitable for analyzing the interesting limiting case $E_0 \to \infty$ considered in (Johnson and Beatty, 1993), is:

$$E(\alpha) = \left( (1 - \alpha)\frac{1}{E_0} + \alpha \frac{1}{E_1} \right)^{-1}, \tag{2.3}$$

the harmonic mean[3] of $E_0$ and $E_1$. In spite of the fact that (2.3) arises from the 1-D homogenization of a two-phase material with elastic moduli $E_0$, $E_1$ and respective 'volume' fractions $(1 - \alpha)$, $\alpha$, see (Francfort and Marigo, 1991), we will refrain from attaching to (2.3) any special physical meaning. A convincing physical interpretation of the damage parameter $\alpha$ can only emerge from the identification of the microscopic mechanisms underlying the observed macroscopic response. This requires in turn a detailed 3-D model and a careful experimental validation, which are both beyond the scope of the present paper. Here, we will only be concerned with the specimen response as a whole (hence with a zero-dimensional model in the sense that no spatially variable quantity is ever introduced) and, at this level, the choice of one map among those which satisfy (2.1), (2.2) cannot be but arbitrary.

In order to link the parameter $\alpha$ with 'experimentally' measurable quantities, we need to exploit the other intrinsic feature of the specimen response: the existence of a virgin curve. This is described by an increasing function:

$$[0, \infty] \ni \varepsilon \mapsto \sigma = \sigma_v(\varepsilon), \tag{2.5}$$

$$\sigma_v'(\varepsilon) \geqslant 0, \tag{2.6}$$

[3] To analyze the limiting case $E_1 \to 0$, the arithmetic mean:

$$E(\alpha) = (1 - \alpha)E_0 + \alpha E_1 \tag{2.4}$$

could be used. All our results would still be valid in this case, with even simpler arguments. In fact, we will only rely on intrinsic properties of the specimen response, rather than on features which may depend on the choice of a specific parametrization such as (2.3) or (2.4).

giving the stress response to an imposed uniaxial strain history which is monotonically increasing. For the model specimen under consideration, the virgin curve is trilinear. A unique unloading branch passes through every accessible point $P$ in the $(\varepsilon, \sigma)$ plane, i.e. through every point below the virgin curve and above the unloading branch corresponding to $\alpha = 1$. This correspondence defines the map:

$$P = (\varepsilon, \sigma) \mapsto \alpha = \tilde{\alpha}(\varepsilon, \sigma) \in [0, 1], \tag{2.7}$$

which can be obtained by solving for $\alpha$ the expression:

$$E(\alpha) = \frac{\sigma}{\varepsilon}, \tag{2.8}$$

(recall that $E(\alpha)$ is invertible). In particular, by evaluating (2.7) along the virgin curve, we obtain a function of $\varepsilon$ alone:

$$\hat{\alpha}(\varepsilon) = \tilde{\alpha}\big(\varepsilon, \sigma_v(\varepsilon)\big), \tag{2.9}$$

which gives the state of damage in a specimen whose maximum previously experienced stretch is $\varepsilon$. For (2.3), we have:

$$\hat{\alpha}(\varepsilon) = \frac{E_0\varepsilon - \sigma_v(\varepsilon)}{E_0\sigma_v(\varepsilon)} \frac{E_0 E_1}{E_0 - E_1}. \tag{2.10}$$

Since:

$$\hat{\alpha}'(\varepsilon) \begin{cases} = 0 & \text{for } \varepsilon_0 < \varepsilon, \\ > 0 & \text{for } \varepsilon_0 \leqslant \varepsilon \leqslant \varepsilon_1, \\ = 0 & \text{for } \varepsilon_1 < \varepsilon, \end{cases} \tag{2.11}$$

where $\varepsilon_0$, $\varepsilon_1$ are defined in *figure 2*, we can define the strictly increasing function:

$$[0, 1] \ni \alpha \mapsto \hat{\varepsilon}(\alpha) \in [\varepsilon_0, \varepsilon_1], \tag{2.12}$$

$$\hat{\varepsilon}'(\alpha) > 0, \tag{2.13}$$

as the inverse of $\hat{\alpha}(\varepsilon)$ in the interval $[\varepsilon_0, \varepsilon_1]$. This enables us to evaluate the total energy dissipated during a loading history which has produced the state of damage $\alpha$ as:

$$K(\alpha) = \int_0^{\hat{\varepsilon}(\alpha)} \sigma_v(\eta)\,\mathrm{d}\eta - \frac{1}{2}E(\alpha)\hat{\varepsilon}^2(\alpha), \tag{2.14}$$

and its derivative:

$$K'(\alpha) = k(\alpha) = -\frac{1}{2}E'(\alpha)\hat{\varepsilon}^2(\alpha) > 0. \tag{2.15}$$

The energy $K(\alpha)$ is the one required to change the internal structure of the specimen (e.g., through breaking of internal bonds in the constitutive material) from the virgin state to the one corresponding to $\alpha$.

## 3. A closer look at the piecewise linear response

The goal of this section is to extract a model from the set of evolution rules which are encoded in the specimen response discussed above (in particular, see *figures 2* and *3*).

The state variables of the system under consideration are $\varepsilon$ and $\alpha$. Since the latter is an internal variable, the generalized 'force' conjugate to it is (Ziegler, 1983):

$$A(\varepsilon,\alpha) = -\frac{\partial W}{\partial \alpha}(\varepsilon,\alpha) = -\frac{1}{2}E'(\alpha)\varepsilon^2 \geqslant 0. \tag{3.1}$$

The expression $A = -\partial W/\partial\alpha$ has the physical meaning of dissipation per unit increase of damage, and its positivity [4] is not specific to the example (2.3).

The incremental response of our system is either purely elastic, or it entails an increase of damage. The second possibility, however, cannot occur if the point $(\varepsilon,\sigma)$, with $\sigma = E(\alpha)\varepsilon$, representing the state of the system in the $(\varepsilon,\sigma)$ plane is not on the virgin curve, i.e., if $\sigma < E(\alpha)\hat{\varepsilon}(\alpha)$. This is equivalent to:

$$f(\varepsilon,\alpha) := A(\varepsilon,\alpha) - k(\alpha) < 0. \tag{3.2}$$

Since, by definition, $\hat{\varepsilon}(\alpha)$ is the maximum strain ever experienced by a specimen whose damage state is $\alpha$, we always have:

$$f(\varepsilon,\alpha) \leqslant 0. \tag{3.3}$$

When (3.2) holds, the elastic energy $A(\varepsilon,\alpha)\,\mathrm{d}\alpha$ that can be released through a damage increase $\mathrm{d}\alpha$ (represented by the gray area in *figure 4*) is too small to actually drive the damage process, whose energy 'cost' (i.e., required dissipation) is $k(\alpha)\,\mathrm{d}\alpha$ (represented by the sum of gray and black area in the same figure). For damage to progress, it is necessary that:

$$f(\varepsilon,\alpha) = 0, \tag{3.4}$$

i.e., that the strain energy release rate $A\dot{\alpha}$ reaches the critical value $\kappa\dot{\alpha}$. Here $\dot{\alpha}$ is the rate of change of damage.

Let us now formulate the incremental form of the evolution problem precisely. Let $(\varepsilon,\alpha)$ be a state of the specimen, and let us impose a strain increment with rate $\dot{\varepsilon}$. We want to determine the unknown rates of

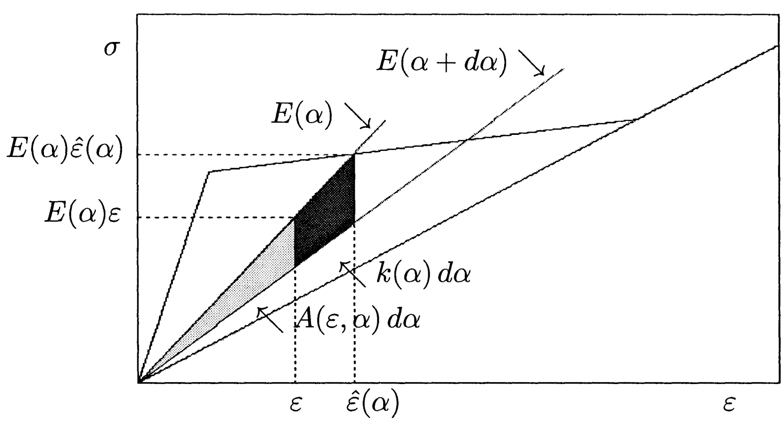


**Figure 4.** Damage only progresses when the dissipation $k(\alpha)\,\mathrm{d}\alpha$ associated with changes of internal structure (gray plus black area) can be supplied by the release of stored elastic energy $A(\varepsilon,\alpha)\,\mathrm{d}\alpha$ (gray area).

[4] This is rather a consequence of the second law of thermodynamics which, for a purely mechanical system, demands that the rate of change of the stored energy $W(\varepsilon,\alpha)$ never exceed the external power expended on the system (Gurtin, 2000).

increment $\dot{\alpha}$ and $\dot{\sigma}$. Since damage is irreversible:

$$\dot{\alpha} \geqslant 0, \tag{3.5}$$

and, in view of the discussion above, the equation:

$$\dot{\alpha} f(\varepsilon, \alpha) = 0, \tag{3.6}$$

is always satisfied. In fact, (3.6) is the Kuhn–Tucker optimality condition ensuring that $\dot{\alpha}$ is the solution of the constrained optimization problem:

$$\min_{\gamma \geqslant 0} -f(\varepsilon, \alpha)\gamma = -f(\varepsilon, \alpha)\dot{\alpha} = 0. \tag{3.7}$$

Here $\gamma$ is a virtual rate of damage increment, while $\dot{\alpha}$ is the one actually occurring. The proof of (3.7) is elementary. First, we rewrite (3.3) as:

$$-f(\varepsilon, \alpha)\gamma \geqslant 0, \quad \forall \gamma \geqslant 0, \tag{3.8}$$

an inequality that will play a crucial role in what follows and which we will refer to as the ‘admissibility condition’ for a state $(\varepsilon, \alpha)$. Then, for $f \neq 0$, (3.8) is minimized by $\gamma = \dot{\alpha} = 0$, while for a minimizer $\dot{\alpha}$ of (3.7) to be positive it is necessary that $f$ vanish. To determine $\dot{\alpha}$, one may use the fact that, if $\dot{\alpha}(t) > 0$ during a time interval, then (3.6) forces $f(\varepsilon(t), \alpha(t)) \equiv 0$ during the same time interval. This condition of ‘persistency’ (of the state point $(\varepsilon, \alpha)$ on the threshold $f = 0$) is usually written as:

$$f(\varepsilon, \alpha) = 0 \Rightarrow \dot{\alpha} \dot{f}(\varepsilon, \alpha) = 0. \tag{3.9}$$

For $\dot{\alpha} > 0$, this gives:

$$\dot{\alpha} = \frac{1}{H(\varepsilon, \alpha)} \frac{\partial f}{\partial \varepsilon}(\varepsilon, \alpha)\Big|_{f=0} \dot{\varepsilon}, \tag{3.10}$$

where we have introduced the hardening modulus:

$$H(\varepsilon, \alpha) = -\frac{\partial f}{\partial \alpha}(\varepsilon, \alpha)\Big|_{f=0} = -E'(\alpha)\hat{\varepsilon}(\alpha)\hat{\varepsilon}'(\alpha) > 0. \tag{3.11}$$

Since, however, $f = 0$ and $(\partial f/\partial \varepsilon)\dot{\varepsilon} \leqslant 0$ imply $\dot{\alpha} = 0$ (for otherwise $\dot{\alpha} > 0$ would occur simultaneously with $\dot{f} = -H\dot{\alpha} + (\partial f/\partial \varepsilon)\dot{\varepsilon} < 0$, against (3.9)), we can replace (3.10) with the more general:

$$\dot{\alpha} = \frac{1}{H(\varepsilon, \alpha)} \left( \frac{\partial f}{\partial \varepsilon}(\varepsilon, \alpha)\Big|_{f=0} \dot{\varepsilon} \right)^{+}, \tag{3.12}$$

where $g^{+} = \max\{0, g\}$ denotes the positive part of $g$. Even more generally:

$$\dot{\alpha} = \begin{cases} 0 & \text{if } f(\varepsilon, \alpha) < 0, \\ \frac{1}{H(\varepsilon, \alpha)} \left( \frac{\partial f}{\partial \varepsilon}(\varepsilon, \alpha)|_{f=0} \dot{\varepsilon} \right)^{+} & \text{if } f(\varepsilon, \alpha) = 0. \end{cases} \tag{3.13}$$

Finally, since $\partial f/\partial\varepsilon = -E'(\alpha)\varepsilon$, and using (3.11), we can write the last equation more explicitly:

$$\dot{\alpha} = \begin{cases} 0 & \text{if } f(\varepsilon,\alpha) < 0, \\ \frac{1}{\hat{\varepsilon}'(\alpha)}(\dot{\varepsilon})^{+} & \text{if } f(\varepsilon,\alpha) = 0. \end{cases} \tag{3.14}$$

To determine $\dot{\sigma}$, we let time appear in (2.8):

$$\sigma(t) = E(\alpha(t))\varepsilon(t), \tag{3.15}$$

and we differentiate with respect to $t$ to obtain:

$$\dot{\sigma}(t) = E(\alpha)\dot{\varepsilon} + E'(\alpha)\dot{\alpha}\varepsilon. \tag{3.16}$$

The geometric interpretation of the last equation in terms of a purely elastic trial $\mathrm{d}\sigma_{\text{trial}} = E(\alpha)\,\mathrm{d}\varepsilon$, $\mathrm{d}\alpha_{\text{trial}} = 0$, and of a correction step via a 'return mapping' (Simo and Hughes, 1998) projecting back the stress point on the updated threshold (the yield surface in stress space):

$$\mathcal{S}(\alpha + \mathrm{d}\alpha) = \{\sigma \in \mathcal{R}\colon\ f(E^{-1}(\alpha + \mathrm{d}\alpha)\sigma, \alpha + \mathrm{d}\alpha) = 0\}, \tag{3.17}$$

is given in *figure 5*.

The behaviour described so far fits in the framework of the so-called "Generalized Standard Medium" based on the notion of a "normal dissipative mechanism" (Germain, 1973; Germain et al., 1983; Halphen and Nguyen Quoc Son, 1975; Lemaitre and Chaboche, 1996; Marigo, 1981, 2000; Maugin, 1992); see also Remarks 1 and 2 below. In fact, for every $\alpha \in [0, 1]$ the dissipation rate:

$$\mathcal{D}_{\alpha}(\dot{\alpha}) = A_{\alpha}(\dot{\alpha})\dot{\alpha} = \begin{cases} +\infty & \text{if } \dot{\alpha} < 0, \\ 0 & \text{if } \dot{\alpha} = 0, \\ k(\alpha)\dot{\alpha} & \text{if } \dot{\alpha} > 0. \end{cases} \tag{3.18}$$

is a positively homogeneous function of degree one in $\dot{\alpha}$ satisfying the postulate of maximal dissipation: for a given $\dot{\alpha} \geqslant 0$:

$$\mathcal{D}_{\alpha}(\dot{\alpha}) = A_{\alpha}(\dot{\alpha})\dot{\alpha} = \max_{A \in \mathcal{A}(\alpha)} A\dot{\alpha} \geqslant 0, \tag{3.19}$$

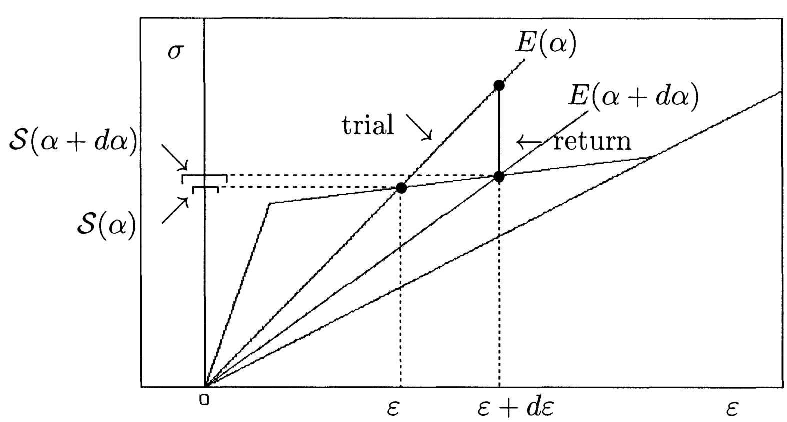


**Figure 5.** Geometric interpretation of elastic trial and return mapping.

where the admissible set $\mathcal{A}(\alpha)$ is defined by:

$$\mathcal{A}(\alpha) = \{A \in \mathcal{R}^+ \colon \; A - k(\alpha) \leqslant 0\}. \tag{3.20}$$

Indeed, the Kuhn–Tucker optimality condition associated with (3.19) is:

$$\big(A - k(\alpha)\big)\dot{\alpha} = 0, \tag{3.21}$$

and, in view of (3.2), this coincides with (3.6). Moreover, denoting by $I_{\mathcal{A}(\alpha)}$ the indicator function of $\mathcal{A}(\alpha)$, i.e., the function which vanishes on $\mathcal{A}(\alpha)$ and is $+\infty$ outside $\mathcal{A}(\alpha)$, we have that:

$$\mathcal{D}^*_\alpha = I_{\mathcal{A}(\alpha)}, \qquad A_\alpha(\dot{\alpha}) \in \partial\mathcal{D}_\alpha(\dot{\alpha}), \tag{3.22}$$

where $\mathcal{D}^*_\alpha$ and $\partial\mathcal{D}_\alpha(\dot{\alpha})$ are, respectively, the Legendre transform and the subgradient of the convex function $\mathcal{D}_\alpha$, see *figure 6* and (Rockafellar, 1970) for the relevant definitions. We close this section by observing that the uniqueness of the solution $(\dot{\alpha}, \dot{\sigma})$ of the incremental form of the evolution problem, which rests on the positivity of the hardening modulus $H(\varepsilon, \alpha)$, see (3.11), is in fact a consequence of the strict convexity of the function $\alpha \mapsto F(\varepsilon, \alpha)$, where:

$$F(\varepsilon, \alpha) = W(\varepsilon, \alpha) + K(\alpha), \tag{3.23}$$

at state points $(\varepsilon, \alpha)$ where $f(\varepsilon, \alpha) = 0$. Indeed:

$$\frac{\partial}{\partial \alpha}\big(W(\varepsilon, \alpha) + K(\alpha)\big) = -f(\varepsilon, \alpha), \tag{3.24}$$

and the inequality $\partial^2 F/\partial\alpha^2 > 0$ is equivalent to $\partial f/\partial\alpha < 0$ leading, in view of (3.11), to the positivity of $H$. This is an intrinsic property of the system, independent of the choice of damage parameter, which is equivalent to the fact that the yield surface in $\varepsilon$-space always expands when damage progresses, see the second formula in (3.11). If we insist that $F(\varepsilon, \cdot)$ be strictly convex not only for state points on the threshold but rather for all admissible states:

$$\frac{\partial^2}{\partial\alpha^2}F(\varepsilon, \alpha) > 0 \Leftrightarrow \frac{\partial}{\partial\alpha}f(\varepsilon, \alpha) < 0, \quad \forall(\varepsilon, \alpha)\colon \; f(\varepsilon, \alpha) \leqslant 0, \tag{3.25}$$

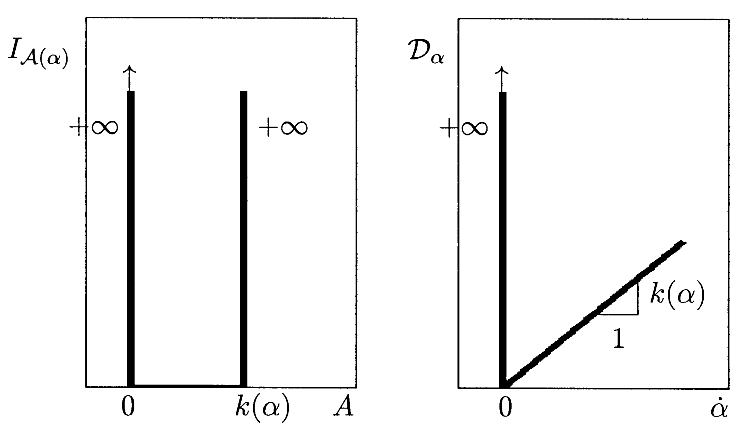


**Figure 6.** The Legendre transform of the dissipation rate $\mathcal{D}_\alpha(\dot{\alpha})$ is the indicator function of the admissible set $\mathcal{A}(\alpha)$. Note that $A = A(\varepsilon, \alpha) \leqslant k(\alpha)$ is equivalent to $f(\varepsilon, \alpha) \leqslant 0$.

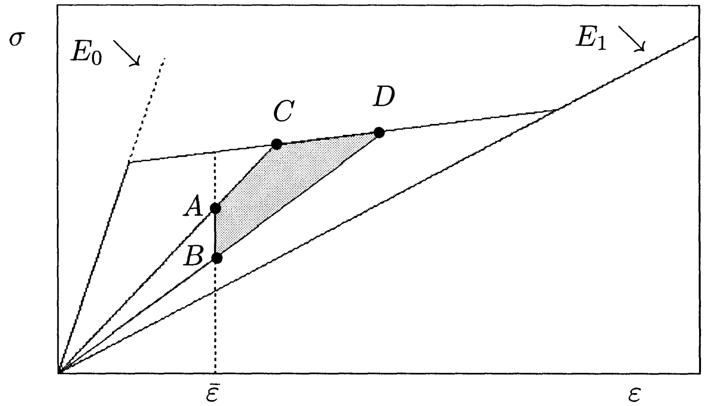


**Figure 7.** The direct path $AB$ is not experimentally accessible, but the path $ACDB$ leads from a state $(\bar{\varepsilon}, \alpha)$ to another $(\bar{\varepsilon}, \beta)$ with $\beta > \alpha$.

we then have that $f(\varepsilon, \cdot)$ is strictly decreasing and, in turn, that for every $(\varepsilon, \alpha)$ such that $f(\varepsilon, \alpha) \leqslant 0$ we have:

$$f(\varepsilon, \beta) \leqslant f(\varepsilon, \alpha) \leqslant 0, \quad \forall \beta \in [\alpha, 1]. \tag{3.26}$$

The last inequality is a sufficient (but not necessary) condition that every state $(\varepsilon, \beta)$ obtained from an admissible one (i.e., a state $(\varepsilon, \alpha)$ such that $f(\varepsilon, \alpha) \leqslant 0$ ) by a process which only entails an (admissible) increase of damage is always admissible, see *figure 7*. This is another intrinsic property of our system, holding true in spite of the fact that (3.25) (i.e., monotonicity of $f$ or, equivalently, convexity of $F$) may fail for some choices of damage parameter.[5] There exists, however, a distinguished parameter $\alpha$, proportional to the total energy dissipated through damage, which renders $F(\varepsilon, \alpha)$ a convex function of $\alpha$. The proofs of all the statements related to the convexity of $F$ are given in Remark 3 below. Section 4 will show that the convexity of $F(\varepsilon, \alpha)$ as a function of $\alpha$ plays a crucial role in the well-posedness of the model. Thus, it is natural to conclude that only parametrizations of damage that preserve the convexity of $F$ should be considered admissible.

*Remark* 1 (on 3-D generalizations of the theory): The general 3-D formulation describing a linearly elastic damageable material with a single scalar damage variable discussed in (Marigo, 1981) is at the very root of this work. The extension of our results to a non-linear, more rubber-like version of such a 3-D framework is rather straightforward. In fact, the pseudo-elastic model of (Ogden and Roxburgh, 1999) could be recast as an example of this kind of approach. In the absence of a compelling collection of 3-D experimental results, we will refrain from pursuing 3-D extensions of the 0-D model presented here. Our reasons are the following. As noted in the Introduction, the very first question to be addressed is the 'adequacy' of a formulation based on a single scalar damage variable. Let's assume this to be the case. Then, depending on the observed material behavior, it may happen that only a very specific parametrization of damage is compatible with the full set of experimental data, be it a notion of maximum experienced strain as suggested in (Beatty and Krishnaswamy, 2000), or rather some quantity related to the dissipated energy, as in (Ogden and Roxburgh, 1999). Even in this comparatively simple scenario, however, the question of the homogeneity of the specimen response (i.e., consideration of a fully 3-D, '$x$-dependent' model of the specimen response allowing for strain localization, fine-scale spatially inhomogeneous structures, etc.) will have to be addressed.

*Remark* 2 (from Drucker–Ilyushin stability postulate to GSM): The critical energy release rate criterion (3.4) that we adopt here as the damage evolution law can be justified from a more primitive principle. Indeed, for

[5] An easy computation shows that (3.25) holds true for both parametrizations (2.3) and (2.4).

a brittle material, such an evolution law is actually a consequence of the Drucker–Ilyushin stability postulate requiring that the strain work:

$$\int_{\mathcal{C}} \sigma \cdot \mathrm{d}\varepsilon$$

performed in any strain cycle $\mathcal{C}$ be positive. Specifically, the following proposition is proved in (Marigo, 2000) (section 3.3).

PROPOSITION: *A brittle damaging material in the sense of* (Marigo, 2000) *satisfies the Drucker–Ilyushin inequality only if its damage parameter is an increasing function of time and if its yield surface is given by a critical energy release rate criterion, i.e.*:

$$\exists k(\alpha) > 0\colon\ f(\varepsilon, \alpha) = -\frac{1}{2} E'(\alpha)\varepsilon \cdot \varepsilon - k(\alpha).$$

Let us also emphasize that this statement holds true in any dimension.

*Remark* 3 (on the convexity of $F$): Let us first consider the case where the damage parameter is (up to a normalizing multiplicative constant) the dissipated energy. Then $F$ reads as:

$$\bar{F}(\varepsilon, \alpha) = \frac{1}{2} E(\alpha)\varepsilon^2 + K\alpha$$

and the yield function becomes:

$$\bar{f}(\varepsilon, \alpha) = -\frac{1}{2} E'(\alpha)\varepsilon^2 - K.$$

From (3.11), it follows that:

$$E''(\alpha) > 0 \tag{3.27}$$

which in turn implies that $\alpha \mapsto \bar{F}(\varepsilon, \alpha)$ is convex. Let us now perform an admissible reparametrization of damage:

$$\alpha = \phi(\beta) \quad \text{with } \phi' > 0.$$

Then the energy and the yield function read as:

$$\tilde{F}(\varepsilon, \beta) = \frac{1}{2} E\big(\phi(\beta)\big)\varepsilon^2 + K\phi(\beta),$$

$$\tilde{f}(\varepsilon, \beta) = -\left(\frac{1}{2} E'\big(\phi(\beta)\big)\varepsilon^2 + K\right)\phi'(\beta).$$

The hardening modulus is again positive. To prove it, we show that:

$$\tilde{f}_{,\beta}(\varepsilon, \beta) < 0, \quad \forall(\varepsilon, \beta)\colon\ \tilde{f}(\varepsilon, \beta) = 0.$$

Indeed, since:

$$\tilde{f}_{,\beta}(\varepsilon, \beta) = -\frac{1}{2} E''\left(\phi(\beta)\right)\varepsilon^2\big(\phi'(\beta)\big)^2 - \frac{\tilde{f}(\varepsilon, \beta)}{\phi'(\beta)}\phi''(\beta),$$

the result follows from (3.27). Note now that, since $\phi' > 0$, $\beta_0 < \beta$ implies $\alpha_0 = \phi(\beta_0) < \phi(\beta) = \alpha$. Therefore, if $(\varepsilon, \alpha_0)$ is admissible, then all states $(\varepsilon, \alpha = \phi(\beta))$ are admissible because $\bar{F}$ is convex in $\alpha$ and

$$\tilde{f}(\varepsilon, \beta) = -\left(\frac{1}{2}E'\big(\phi(\beta)\big)\varepsilon^2 + K\right)\phi'(\beta) \leqslant 0, \quad \forall \beta \geqslant \beta_0,$$

because the coefficient of $\phi'$ is non-negative. This shows that any process which only entails an increase of damage $\beta$ starting from the admissible value $\beta_0$ is admissible. Finally, since the sign of:

$$\tilde{F}_{,\beta\beta}(\varepsilon, \beta) = \frac{1}{2}E''\big(\phi(\beta)\big)\varepsilon^2\big(\phi'(\beta)\big)^2 + \frac{\tilde{f}(\varepsilon, \beta)}{\phi'(\beta)}\phi''(\beta)$$

depends on the sign of $\phi''(\beta)$, convexity of $\beta \mapsto \tilde{F}(\varepsilon, \beta)$ may fail.

## 4. Uniqueness theorems

The admissibility condition (3.8) for a state $(\varepsilon, \alpha)$:

$$-f(\varepsilon, \alpha)\gamma \geqslant 0, \quad \forall \gamma \geqslant 0, \tag{4.1}$$

the optimality property (3.7) of the actual damage rate $\dot{\alpha}$:

$$\min_{\gamma \geqslant 0} -f(\varepsilon, \alpha)\gamma = -f(\varepsilon, \alpha)\dot{\alpha} = 0, \tag{4.2}$$

and the strict convexity (3.25) of the function $\alpha \mapsto F(\varepsilon, \alpha)$, where:

$$\frac{\partial}{\partial \alpha}F(\varepsilon, \alpha) = -f(\varepsilon, \alpha), \tag{4.3}$$

imply that the response of a specimen to an arbitrarily imposed Lipschitz strain history $\varepsilon(t)$ is uniquely determined.

Our results will not be restricted to the case of PL response [6] discussed in Sections 2 and 3. On the other hand, we will need the following strengthened assumption of uniform convexity on $F$: There exists a positive constant $c$ such that, for all $(\varepsilon, \alpha)$:

$$\frac{\partial^2}{\partial \alpha^2}F(\varepsilon, \alpha) = -\frac{\partial}{\partial \alpha}f(\varepsilon, \alpha) \geqslant c > 0. \tag{4.4}$$

The function $F$ is, as before:

$$F(\varepsilon, \alpha) = W(\varepsilon, \alpha) + K(\alpha), \tag{4.5}$$

but the stored energy $W$ is no longer assumed quadratic in $\varepsilon$. Thus, the expression for the total dissipated energy $K(\alpha)$ becomes:

$$K(\alpha) = \int_0^{\hat{\varepsilon}(\alpha)} \sigma_v(\eta)\,\mathrm{d}\eta - W\big(\hat{\varepsilon}(\alpha), \alpha\big), \tag{4.6}$$

[6] In fact, they will be applicable to specimens exhibiting stress–strain response such as the one schematically depicted in *figure 1*.

where $\sigma_v(\varepsilon)$, $\hat{\varepsilon}(\alpha)$ (and hence $\hat{\sigma}(\alpha)$) are monotone increasing and have the same meaning as before. We will actually assume that $\hat{\varepsilon}'(\alpha)$ is bounded away from zero, so that (3.14) implies the existence of a constant $\tilde{c}$ such that:

$$\dot{\alpha} \leqslant \frac{1}{\hat{\varepsilon}'(\alpha)}|\dot{\varepsilon}| \leqslant \tilde{c}|\dot{\varepsilon}| \tag{4.7}$$

and, moreover, that all second derivatives of $f$ are uniformly bounded for the given Lipschitz history $\varepsilon(t)$ and $\alpha \in [0, 1]$, i.e.:

$$|f_{,\varepsilon\alpha}| \leqslant \bar{c}, \qquad |f_{,\alpha\alpha}| \leqslant \bar{c}, \tag{4.8}$$

etc., where $\bar{c}$ is some positive constant. Some other assumptions are needed to ensure that the model associated with (4.1)–(4.2) actually describe softening behavior (e.g., $W_{,\alpha\alpha} = \sigma_{,\alpha} < 0$), but these are not required for the uniqueness results presented in this section, which is self-contained.

The specimen's state of stress is given by:

$$\sigma(t) = \frac{\partial}{\partial\varepsilon} W\big(\varepsilon(t), \alpha(t)\big) \tag{4.9}$$

so that the response of the system to $\varepsilon(t)$ (or to $\dot{\varepsilon}$) is uniquely determined by $\alpha(t)$ (by $\dot{\alpha}$ in the incremental case).

We begin by proving the uniqueness of the solution to the discrete incremental problem. The Kuhn–Tucker condition (4.2) will lead us to a variational formulation of this problem, with a unique solution due to the strict convexity of $F(\varepsilon, \cdot)$. To proceed, note that in view of (4.3) we can rewrite (4.2) as:

$$0 = \frac{\partial}{\partial\alpha} F\big(\varepsilon(t), \alpha(t)\big)\dot{\alpha}\Delta t \leqslant \frac{\partial}{\partial\alpha} F\big(\varepsilon(t), \alpha(t)\big)\gamma\Delta t, \quad \forall\gamma, \Delta t \geqslant 0. \tag{4.10}$$

Let $x_i = x(t)$ and $x_{i-1} = x(t - \Delta t)$ be the values of the generic quantity $x$ at two consecutive time steps. It follows from (4.10), modulo an error which is infinitesimal of higher order with respect to $\Delta t$, that:

$$F(\varepsilon_i, \alpha_{i-1} + \dot{\alpha}\Delta t) - F(\varepsilon_i, \alpha_{i-1}) \leqslant F(\varepsilon_i, \alpha_{i-1} + \gamma\Delta t) - F(\varepsilon_i, \alpha_{i-1}), \tag{4.11}$$

and in turn, by subtracting the constant $K(\alpha_{i-1})$ on both sides, that:

$$F(\varepsilon_i, \alpha_i) - K(\alpha_{i-1}) \leqslant F(\varepsilon_i, \alpha) - K(\alpha_{i-1}), \quad \forall\alpha \in [\alpha_{i-1}, 1]. \tag{4.12}$$

Equivalently, defining the incremental dissipation $D$ as:

$$D(\alpha, \beta) = \begin{cases} K(\beta) - K(\alpha) & \text{if } \beta \geqslant \alpha, \\ +\infty & \text{if } \beta < \alpha, \end{cases} \tag{4.13}$$

we have that:

$$W(\varepsilon_i, \alpha_i) + D(\alpha_{i-1}, \alpha_i) \leqslant W(\varepsilon_i, \alpha) + D(\alpha_{i-1}, \alpha) \tag{4.14}$$

so that $\alpha_i$ is the unique solution of the convex minimization problem:

$$\min_{\alpha\in[0,1]} W(\varepsilon_i, \alpha) + D(\alpha_{i-1}, \alpha). \tag{4.15}$$

This concludes the proof.

We now turn to the proof of the global uniqueness of the solution $\alpha(t)$ to the evolution problem. We follow an argument due to Mielke (1999). First, observe that (4.1) and (4.2) imply that:

$$-f(\varepsilon,\alpha)(\dot{\alpha}-\gamma) \geqslant 0, \quad \forall \gamma \geqslant 0. \tag{4.16}$$

Moreover, (4.4) implies that, for every $\varepsilon$, $-f(\varepsilon,\cdot)$ defines a monotone operator:

$$-(f(\varepsilon,\alpha_1)-f(\varepsilon,\alpha_2))(\alpha_1-\alpha_2) \geqslant c(\alpha_1-\alpha_2)^2 \geqslant 0, \quad \forall \alpha_1,\alpha_2 \in [0,1]. \tag{4.17}$$

Now let $\alpha_1(t)$, $\alpha_2(t)$ be two solutions of the evolution problem under a given $\varepsilon(t)$ satisfying the same initial conditions, say, at time $t=0$, and define:

$$e(t) = -\big(f\big(\varepsilon(t),\alpha_1(t)\big) - f\big(\varepsilon(t),\alpha_2(t)\big)\big)\big(\alpha_1(t)-\alpha_2(t)\big). \tag{4.18}$$

Clearly, (4.17) implies:

$$e(t) \geqslant c\big(\alpha_1(t)-\alpha_2(t)\big)^2 \tag{4.19}$$

while since $(\varepsilon(t),\alpha_1(t))$, $(\varepsilon(t),\alpha_2(t))$ are admissible and $\dot{\alpha}_1(t)$, $\dot{\alpha}_2(t) \geqslant 0$, it follows from (4.16) that:

$$-f\big(\varepsilon(t),\alpha_1(t)\big)\big(\dot{\alpha}_1(t)-\dot{\alpha}_2(t)\big) \leqslant 0,$$

$$-f\big(\varepsilon(t),\alpha_2(t)\big)\big(\dot{\alpha}_2(t)-\dot{\alpha}_1(t)\big) \leqslant 0.$$

These last inequalities, together with the observation that, in view of (4.8):

$$\dot{\varepsilon}\big(f_{,\varepsilon}(\varepsilon,\alpha_1)-f_{,\varepsilon}(\varepsilon,\alpha_2)\big)(\alpha_1-\alpha_2) \leqslant \bar{c}|\dot{\varepsilon}|(\alpha_1-\alpha_2)^2,$$

$$\dot{\alpha}_1\big(f(\varepsilon,\alpha_1)-f(\varepsilon,\alpha_2)+f_{,\alpha}(\varepsilon,\alpha_1)(\alpha_2-\alpha_1)\big) \leqslant \bar{c}|\dot{\alpha}_1|(\alpha_1-\alpha_2)^2,$$

$$\dot{\alpha}_2\big(f(\varepsilon,\alpha_2)-f(\varepsilon,\alpha_1)+f_{,\alpha}(\varepsilon,\alpha_2)(\alpha_1-\alpha_2)\big) \leqslant \bar{c}|\dot{\alpha}_2|(\alpha_1-\alpha_2)^2,$$

show that:

$$\begin{aligned}
\dot{e}(t) &= -2\big(f(\varepsilon,\alpha_1)-f(\varepsilon,\alpha_2)\big)(\dot{\alpha}_1-\dot{\alpha}_2) \\
&\quad - \dot{\varepsilon}\big(f_{,\varepsilon}(\varepsilon,\alpha_1)-f_{,\varepsilon}(\varepsilon,\alpha_2)\big)(\alpha_1-\alpha_2) \\
&\quad + \dot{\alpha}_1\big(f(\varepsilon,\alpha_1)-f(\varepsilon,\alpha_2)+f_{,\alpha}(\varepsilon,\alpha_1)(\alpha_2-\alpha_1)\big) \\
&\quad + \dot{\alpha}_2\big(f(\varepsilon,\alpha_2)-f(\varepsilon,\alpha_1)+f_{,\alpha}(\varepsilon,\alpha_2)(\alpha_1-\alpha_2)\big) \\
&\leqslant \bar{c}\big(\big|\dot{\varepsilon}(t)\big|+\dot{\alpha}_1(t)+\dot{\alpha}_2(t)\big)\big(\alpha_1(t)-\alpha_2(t)\big)^2.
\end{aligned}$$

Note that the first summand in the expression of $\dot{e}$ has been dropped in view of its sign, see (4.16). Using (4.19) and the boundedness of $\dot{\varepsilon}$, $\dot{\alpha}_1$, $\dot{\alpha}_2$, see (4.7), we conclude that:

$$\dot{e}(t) \leqslant Ce(t) \tag{4.20}$$

for some constant $C$. But this implies that $e(t) \leqslant e(0)\exp(Ct) = 0$ because $e(0) = 0$ and, in turn, that $\alpha_1(t)-\alpha_2(t) \equiv 0$ in view of (4.19). This concludes the proof.

## 5. Application to oscillators with a single degree of freedom

In this section we study the oscillations of a spring-damper-mass system where the spring exhibits the PL response described in Section 2, both in tension and in compression. Our aim is to compare the behavior of a spring that can suffer damage, which for brevity we shall refer to as piecewise linear, damageable (PLD) spring, with that of a standard linear spring.

The actual response of the PLD spring depends on the level of damage, which can be parametrized by the maximum strain $\varepsilon_m$ ever experienced by the material. Let $\varepsilon_0$ be the strain at which damage starts, and let $\varepsilon_1$ be the strain at which damage is complete. Given the one-to-one relation between damage and maximum strain (2.9), we characterize the response of the specimen by the map $(\varepsilon, \varepsilon_m) \mapsto \sigma(\varepsilon, \varepsilon_m)$ given by:

$$\sigma(\varepsilon, \varepsilon_m) = \begin{cases} \sigma_v(\varepsilon) & |\varepsilon| \geqslant \varepsilon_m, \\ \sigma_d(\varepsilon, \varepsilon_m) & |\varepsilon| < \varepsilon_m. \end{cases} \tag{5.1}$$

The virgin response $\sigma_v$ is given by:

$$\sigma_v(\varepsilon) = \begin{cases} E_0\varepsilon & |\varepsilon| \leqslant \varepsilon_0, \\ \operatorname{sgn}(\varepsilon)(E_0 - E_2)\varepsilon_0 + E_2\varepsilon & \varepsilon_0 \leqslant |\varepsilon| \leqslant \varepsilon_1, \\ E_1\varepsilon & |\varepsilon| \geqslant \varepsilon_1 \end{cases} \tag{5.2}$$

with $E_2 = (E_1\varepsilon_1 - E_0\varepsilon_0)/(\varepsilon_1 - \varepsilon_0)$, while:

$$\sigma_d(\varepsilon, \varepsilon_m) = \frac{\sigma_v(\varepsilon_m)}{\varepsilon_m}\varepsilon, \quad |\varepsilon| \leqslant \varepsilon_m. \tag{5.3}$$

The response (5.1) is characterized by four parameters: $\varepsilon_0, \varepsilon_1$ represent thresholds for the damage mechanism; $E_0, E_1$ are the initial and final stiffness. Choosing for the parameters the values[7] $E_0 = 2$, $E_1 = 1$, $\varepsilon_0 = 1$, $\varepsilon_1 = 3$, we obtain the stress–strain diagram of *figure 8*.

We consider a spring-damper-mass system connected to a driven support undergoing harmonic motion; let $t \mapsto x(t)$ be the displacement of the mass, $t \mapsto y(t) = A\sin(\omega_f t)$ the (imposed) motion of the support, and let $c$ and $m$ be the constitutive parameters for the damper and the mass; moreover, let the elastic force in the spring be described by the map $\varepsilon \mapsto \sigma_e(\varepsilon, \varepsilon_m)$, with $\varepsilon$ the strain of the spring and $\varepsilon_m$ the maximum strain ever

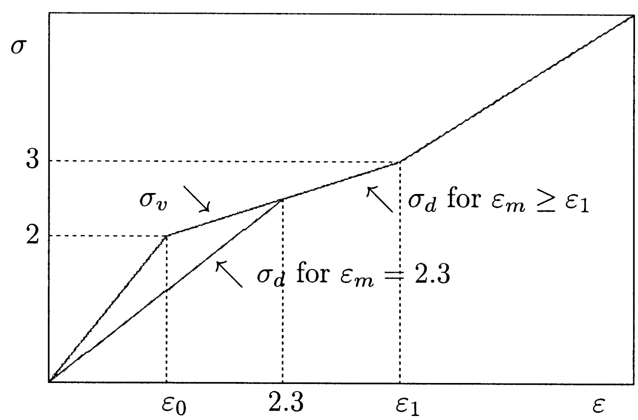


**Figure 8.** Response of the PLD spring.

[7] All physical quantities have been non-dimensionalized through the choice of a suitable system of units.

attained. Then, the equation of motion of the system is:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + \sigma_e\big(x - y, (x - y)_m\big) = 0. \tag{5.4}$$

Two different springs are considered: a (standard) linear elastic one, for which $\sigma_e(\varepsilon, \varepsilon_m) = k_l\varepsilon$; and a PLD spring with $\sigma_e$ given by (5.1). In the case of the linear elastic spring we denote by $x_l$ the response of the system, and rewrite (5.4) as:

$$\ddot{x}_l + 2\zeta\omega\dot{x}_l + \omega^2 x_l = 2\zeta\omega\dot{y} + \omega^2 y, \tag{5.5}$$

where $\omega$ and $\zeta$, defined by:

$$\omega = \sqrt{(k_l/m)}, \qquad \zeta = \frac{c}{2\sqrt{k_l m}}, \tag{5.6}$$

represent the natural frequency and the damping coefficient of the system, respectively. For the linear equation (5.5) we have an explicit representation of the solution:

$$x_l(t) = g(\zeta, \omega_f/\omega)A\sin(\omega_f t - \phi), \tag{5.7}$$

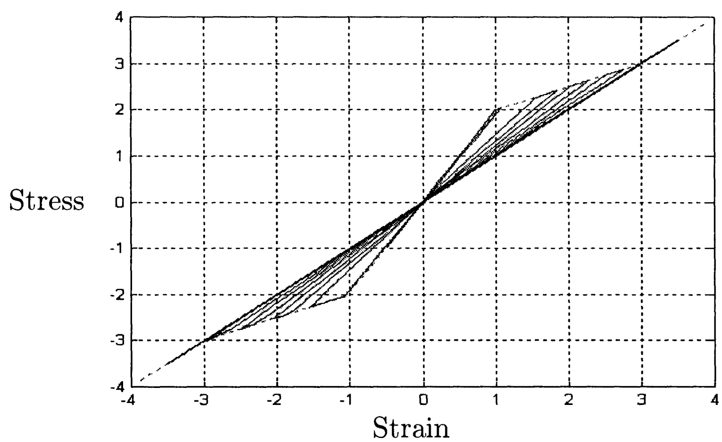


**Figure 9.** Load history for the PL spring; $c = .28$, $\omega_f = 0.7\sqrt{E_0/m}$.

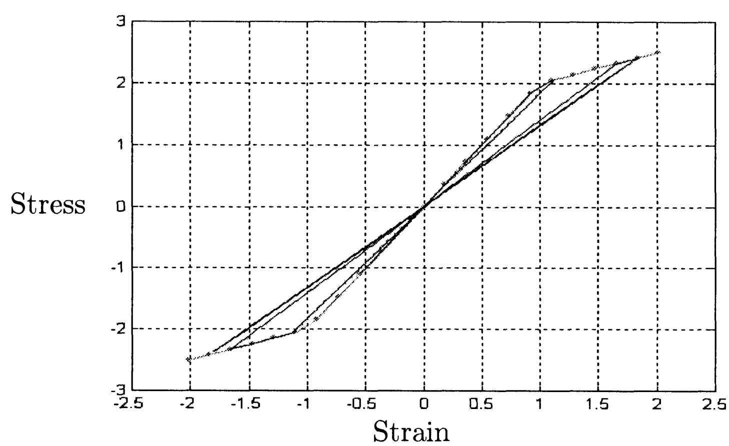


**Figure 10.** Load history for the PLD spring; $c = .70$, $\omega_f = \sqrt{E_0/m}$.

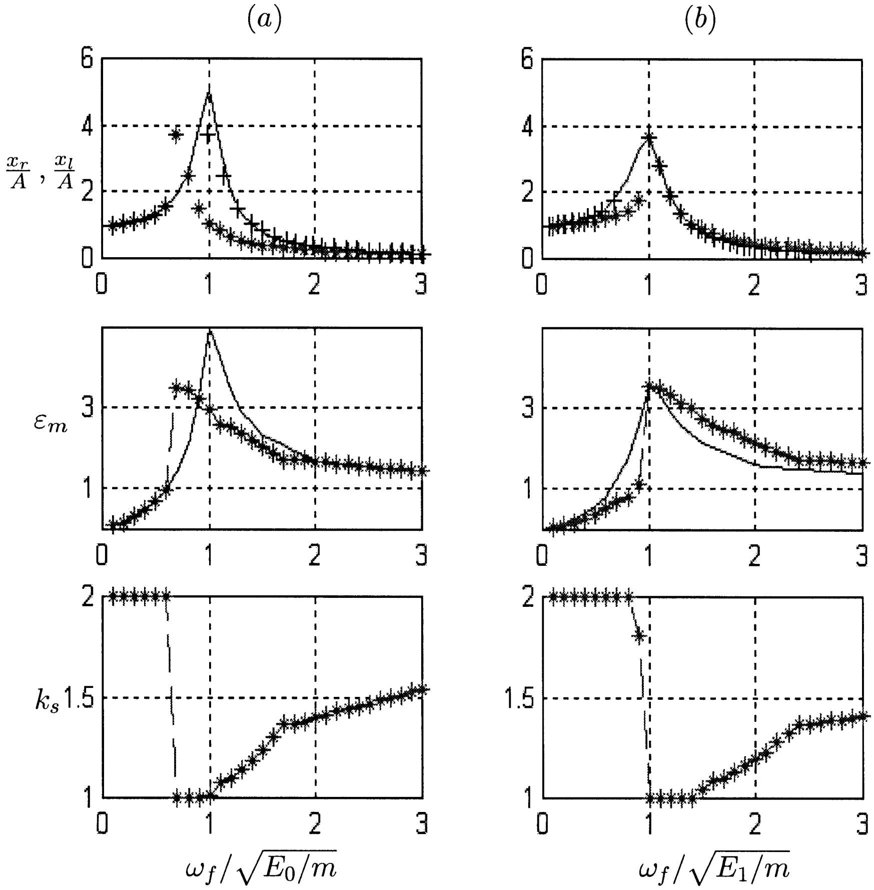


**Figure 11.** Frequency response for PLD system; $A = 1, c = 0.28$.

with the amplification factor $g(\zeta, \omega_f/\omega)$ given by:

$$g(\zeta, \omega_f/\omega) = \left[1 + \left(2\zeta\frac{\omega_f}{\omega}\right)^2\right]^{1/2} \left[\left(1 - \left(\frac{\omega_f}{\omega}\right)^2\right)^2 + \left(2\zeta\frac{\omega_f}{\omega}\right)^2\right]^{-1/2}. \tag{5.8}$$

For the PLD spring, equation (5.4) is solved numerically using the ODE solver of Matlab and we denote by $x_r$ the computed response of the system.

The behavior of the system with PLD spring strongly depends on the damage that can occur during the transient phase. This is enhanced both by low values of damping $c$ and by large amplitudes $A$ of the motion of the support. For example, set $A = 1$ and let $c$ and $\omega_f$ vary. Then, the butterfly pattern of *figure 9* shows that during the transient phase the material is completely damaged and, at steady state, the oscillations take place along the curve $\sigma = E_1\varepsilon$. For the case of *figure 10*, the material is damaged during the transient phase, but not completely, and we have oscillations on an intermediate line. In order to analyze the behavior of the spring-damper-mass system, we fix once and for all the value of $m$ and $c$.

Then, we compare the response of a system with a PLD spring (PLD system) with those of two systems with standard linear springs of different stiffnesses, $k_l = E_0$ ($L_0$ system) and $k_l = E_1$ ($L_1$ system). These correspond to the stiffnesses of the virgin and the completely damaged PLD spring, respectively. For $m = 1$, $c = .28$, the

$L_0$ system has natural frequency $\omega = 1.4142$ and damping coefficient $\zeta = .1$, while for the $L_1$ system $\omega = 1$ and $\zeta = 1.14$.

In *figures 11* and *12* we compare the frequency response of the PLD system with those of systems $L_0$ (left column) and $L_1$ (right column), for two different values of the amplitude $A$ of the imposed motion. In this comparison, the steady-state stiffness of the PLD system, $k_s = \sigma_v(\varepsilon_m)/\varepsilon_m$, will play an important role. We plot:

- the steady-state amplification ratio for the PLD system, $x_r/A$, (star mark) and for the linear systems, $x_l/A$ (continuous line). The graph with plus mark is the plot of $x_r/A$ against the ratio $\omega_f/\sqrt{k_s/m}$;
- the maximum deformation $\varepsilon_m$ in the PLD system, $(\max(x_r - y)$, star mark) and in the linear systems $(\max(x_l - y)$, continuous line);
- the steady-state stiffness $k_s$ in the PLD system.

In analyzing the plots, it is worth keeping in mind that damage takes place for $\varepsilon_m \in (1, 3)$, and that the values $k_e = 1, 2$ signal a totally damaged or undamaged spring, respectively. To understand *figure 11*$(a)$, note that for $0.6\sqrt{E_0/m} < \omega_f$ we have $\varepsilon_m > \varepsilon_0$, i.e., damage occurs. Thus, the PLD system is, at steady state, softer than the $L_0$ one: this explains the forward shift in the resonance peak (star mark). Plotting the amplitude of each steady-state motion against the a-priori unknown frequency ratio $\omega_f/\sqrt{k_s/m}$ (plus mark), the familiar shape of the graph of $g(\zeta, \omega_f/\omega)$ (continuous line) is recovered. The resonance peak is, however, lower for the PLD spring, in this case, energy is dissipated not only in the damper but also through the damage process. Similar

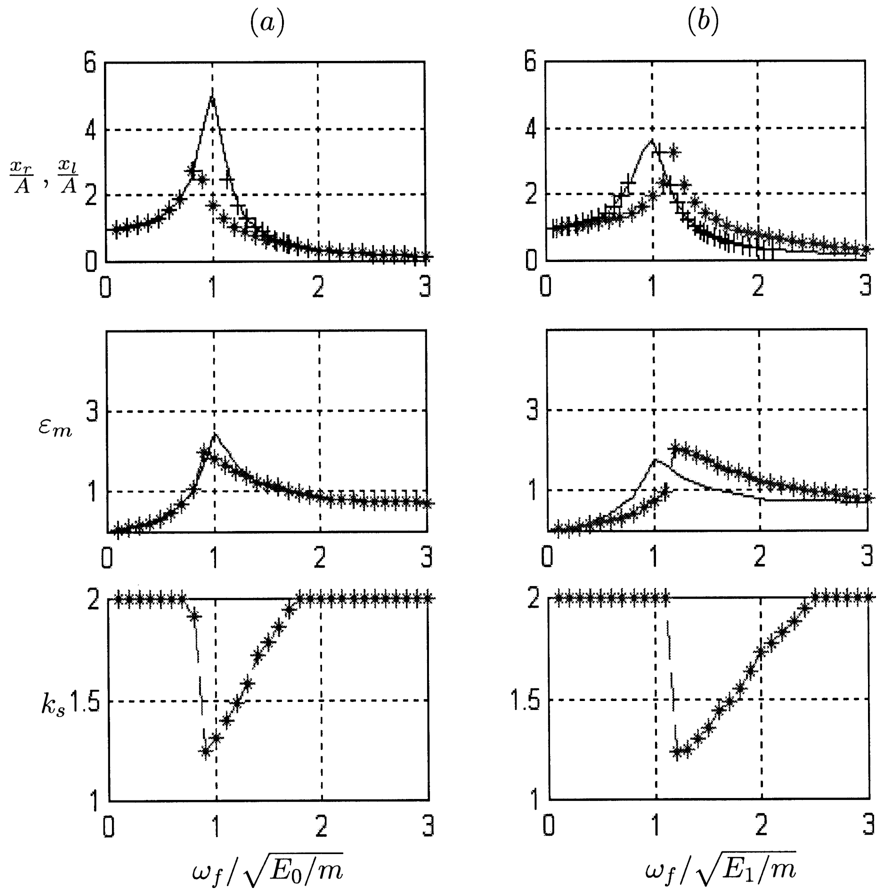


**Figure 12.** Frequency response for PLD system; $A = 0.5$, $c = 0.28$.

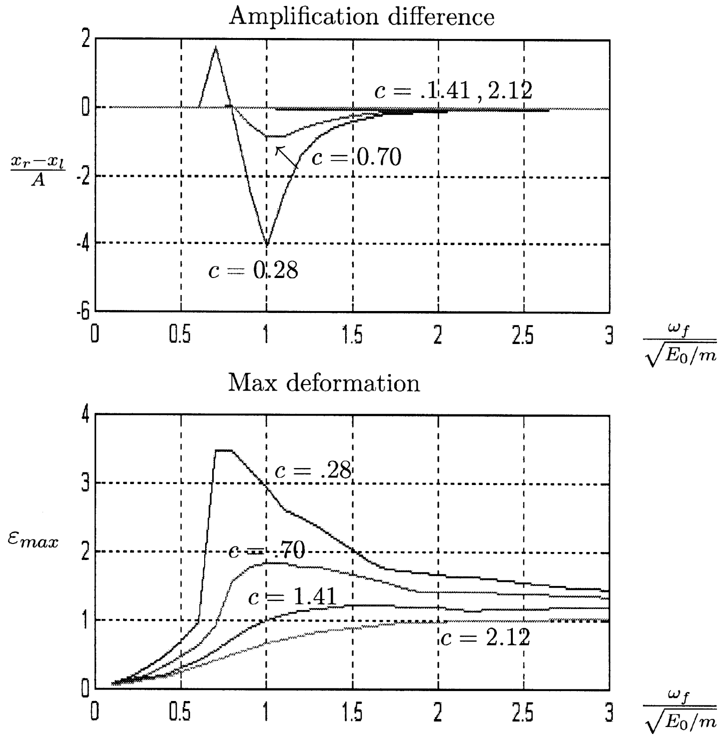


**Figure 13.** Frequency response for PLD system; $A = 1$, $c = 0.28$, 0.70, 1.41, 2.12.

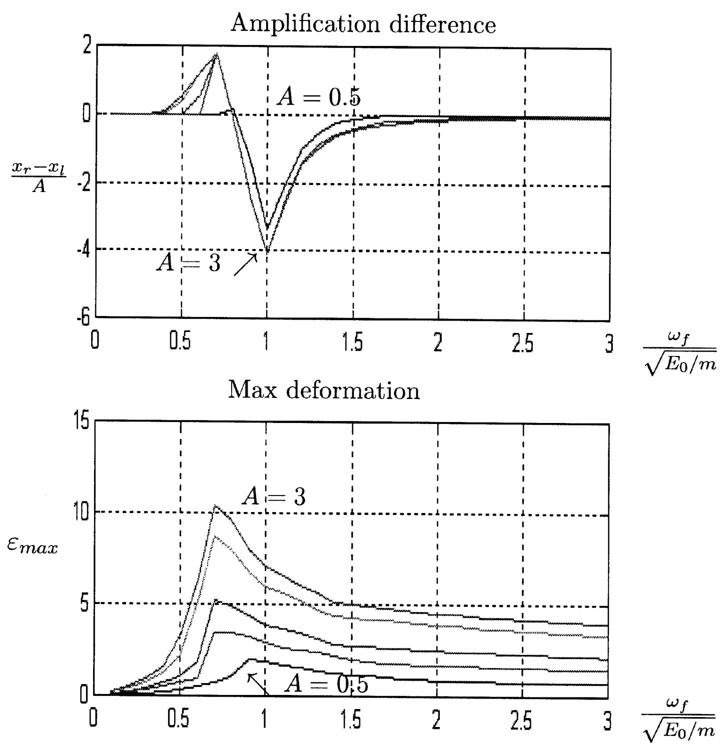


**Figure 14.** Frequency response for PLD system; $c = 0.28$, $A = 0.5$, 1, 1.5, 2, 2.5, 3.

arguments explain the delay in the resonance peak in *figure 12*($b$), where the PLD system is compared with the softer $L_1$ system. This point of view is, in fact, closer in spirit to the choice of design parameter often used in seismic engineering applications, where the higher stiffness of the virgin material is disregarded (Kelly, 1993).

Finally, we examine the effect of the damping coefficient and of the amplitude of the motion of the support. We display:

- the steady-state difference of amplification ratio $x_r/A - x_l/A$;
- the maximum deformation in the PLD system $\max(x_r - y)$;

as functions of the frequency ratio $\omega_f/\sqrt{E_0/m}$. Here, the linear system used to define $x_l$ is $L_0$. In *figure 13*, $A$ is fixed and the curves correspond to different values of $c$. Conversely, in *figure 14* the damping $c$ is fixed and the curves correspond to different values of $A$.

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