

Folding energetics in thin-film diaphragms

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We perform experiments to elucidate how the folding patterns of thin-film diaphragms subject to in-plane isotropic and anisotropic compressive strains depend on the shape, thickness and size of the diaphragms. We then use a constrained von Kármán model to relate the experimental results to the energetics of folding. We show that the differences between the isotropic and the anisotropic cases can be traced back to the structure of the membrane energy density function. In the isotropic case, we find foldings which satisfy the boundary conditions and minimize the membrane energy. In the anisotropic case, no such foldings exist, but we are able to construct sequences of increasingly fine foldings which satisfy the boundary conditions and whose membrane energies converge to the infimum. In both cases, we obtain solutions by allowing bending to select a preferred folding. The solutions compare well with the experimental observations.

Keywords: folding; elastic thin films; von Kármán plate theory; direct method of the calculus of variations

The study of folding patterns in thin films is a very active research field (Berdichevski & Truskinovski 1985; Pipkin 1986; Pogorelov 1988; Ortiz & Gioia 1994; Lobkovsky *et al.* 1995; Gioia & Ortiz 1997; Jin 1997; Chaieb *et al.* 1998; Cerda *et al.* 1999; Jin & Kohn 2000; Jin & Sternberg 2001). A recent motivation for this activity has been the proposal of a vast gamut of novel applications for relatively large, microfabricated thin-film diaphragms (Madou 1997). However, a thorough understanding of the mechanical behaviour of compressed thin-film diaphragms remains unavailable. A notable exception is the isotropically compressed case, which has been solved in the course of investigations into the mechanics of thin-film delamination (Gioia & Ortiz 1997; the one-dimensional version of this problem was solved by Berdichevski & Truskinovski (1985) in their pioneering paper). Our aim here is to study the anisotropic case, both experimentally and theoretically, and to compare it with the isotropic case. We also put forward new experimental results which verify our previous theoretical work on the isotropic case.

Consider a film of thickness h bonded to the flat surface of a substrate except over a domain Ω of characteristic size $d \ll h$; figure 1a. The portion Ω of the film

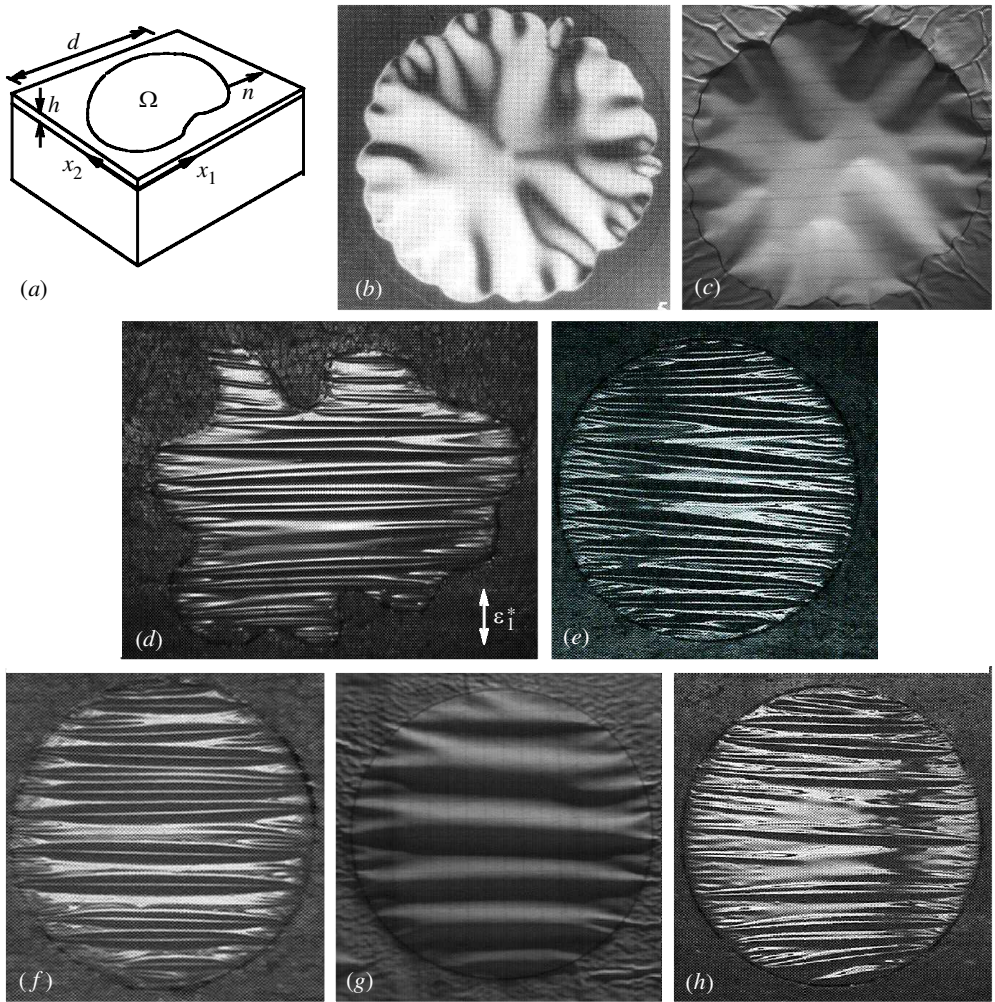


Figure 1. (a) Thin film/substrate system, diaphragm, and notation. Top view of folded diaphragms: (b) SiC film, $\varepsilon_1^* = \varepsilon_2^* = 0.011$, $d/h = 140$ (photograph by Argon *et al.* (1989); reprinted with permission of Chapman & Hall). (c) Paper film, $\varepsilon_1^* = \varepsilon_2^* = 0.040$, $d/h = 950$. The following are polymeric films (except when noted) with $\varepsilon_2^* = 0$: (d) $\varepsilon_1^* = 0.038$, $d/h = 5600$. (e) $\varepsilon_1^* = 0.035$, $d/h = 5600$. (f) $\varepsilon_1^* = 0.035$, $d/h = 2840$. (g) Paper film, $\varepsilon_1^* = 0.040$, $d/h = 1800$. (h) $\varepsilon_1^* = 0.019$, $d/h = 5600$.

is a *diaphragm*. If we apply a strain ε^* to the substrate, in the plane of the film, the diaphragm may deflect out of the substrate and fold. We study the conditions which ε^* must fulfil for this to occur, and the nature of the resulting folding.

We prepared diaphragms by gluing paper sheets and polymeric films ($h \sim 0.01$ to 0.1 mm) onto substrates (thickness *ca.* 30 mm) of the shape shown in figure 1a, made of a high-density Styrofoam.† We then compressed the substrates in two perpendicular directions, x_1 and x_2 , using screw-driven steel plates. When the applied strain

† A relatively high density (*ca.* 40 kg m^{-3}) is required for the foam to deform homogeneously. In

components are equal to each other, $\varepsilon_1^* = \varepsilon_2^*$, the strain is *isotropic*; otherwise it is *anisotropic*. (Note that the elasticity of the thin film is isotropic. Elastic isotropy should not be confused with strain isotropy.) Parts (b) and (c) of figure 1 show the folding of two equally shaped diaphragms subject to isotropic strains. The folding pattern is the same in both diaphragms; the number and spatial arrangement of the folds depend exclusively on the shape of the diaphragm, being independent of d/h and of the strain. A completely different situation obtains when the strains are anisotropic. Then, the folds are perpendicular to the direction of ε_1^* (where $\varepsilon_1^* > \varepsilon_2^*$ by convention), regardless of the shape of the diaphragm (figure 1d, e). The number of folds depends on d/h , however, see figure 1e–g; a comparison of parts (e) and (h) of figure 1 reveals that the number of folds depends also on the strain.

For the analysis of these results we follow Gioia & Ortiz (1997) in (i) modelling the diaphragm as a von Kármán plate, and (ii) constraining the in-plane displacements of the diaphragm to remain null. As we shall show, this simple model allows for a straightforward interpretation of the overall structure of the folding patterns of thin-film diaphragms; however, the study of specific features of the folding patterns may require the consideration of the in-plane displacements (Lobkovsky *et al.* 1995; Cerda *et al.* 1999; Jin & Kohn 2000; Jin & Sternberg 2001). For the proposed model the energy density due to the bending of the thin film is

$$\phi^b = \frac{1}{12}Ch^2[(1 - \nu)(w_{,11}^2 + w_{,22}^2 + 2w_{,12}^2) + \nu(w_{,11} + w_{,22})^2], \tag{1.1}$$

and the energy density due to the membraneous stretching of the thin film is

$$\phi^m = \frac{1}{2}C[\frac{1}{4}(|\nabla w|^2 - 2(\varepsilon_1^* + \nu\varepsilon_2^*))^2 + (1 - \nu)w_{,2}^2(\varepsilon_1^* - \varepsilon_2^*)] + \phi_{\min}^m, \tag{1.2}$$

where $C = Eh/(1 - \nu^2)$ is the membraneous stiffness of the film; E is the Young’s modulus and ν is the Poisson’s ratio of the film; w is the out-of-plane deflection of the diaphragm; $\nabla w = (w_{,1}, w_{,2})$ is the gradient of w , where $w_{,\alpha} = \partial w/\partial x_\alpha$; $|\nabla w| = (w_{,1}^2 + w_{,2}^2)^{1/2}$ is the largest slope of the folded diaphragm at the given point; and we have defined $\phi_{\min}^m = C(1 - \nu^2)(\varepsilon_2^*)^2/2$. The total energy of the diaphragm is the sum of the bending and membraneous energies, $\Phi = \Phi^m + \Phi^b$, where Φ^m and Φ^b are the integrals over Ω of ϕ^m and ϕ^b , respectively. We seek to characterize the folding directly as an energy minimizer, not as a solution of the Euler equations associated with the energy functional $\Phi[w]$. In other words, we search a folding, described by $w(x_1, x_2)$, which minimizes $\Phi[w]$ subject to the boundary conditions $w = 0$ and $w_{,n} = 0$; figure 1a.

To outline a strategy of solution we start by noting that Φ^b and Φ^m are of order h^3 and h , respectively. This suggests that when h is small a promising first step towards finding a folding that minimizes Φ might be to find a folding that minimizes Φ^m . We shall see that there may be many foldings that minimize Φ^m , and that these foldings contain lines of slope discontinuity or *sharp folds*. The presence of sharp folds is physically admissible, because Φ^m represents the energy of a film devoid of bending stiffness, and such a film can sustain sharp folds. Therefore, we can interpret the first step in our strategy of solution in the following form: discard bending stiffness, and then find the foldings that minimize the energy. After having identified the foldings that minimize Φ^m , the next step is to add bending stiffness to the film. The effect

fact, low-density solid foams display deformation patterns which are analogous to the folding patterns of anisotropically compressed diaphragms; see Gioia *et al.* (2001).

of adding bending stiffness is that the sharp folds of the minimizing foldings become smooth. In the thin film limit of interest, $h \rightarrow 0$, the smoothing occurs over narrow bands aligned with the sharp folds. The bending energy is confined to these bands, and can be computed in the form of an energy per unit length of sharp fold or *line tension* (Modica 1987; Pogorelov 1988; Kohn & Müller 1994). After computing the line tension, it is possible to ascribe a value of sharp-fold energy to each of the foldings that minimize Φ^m . Among these foldings we can select the one with less sharp-fold energy (Gioia & Ortiz 1997; de Giorgi 1975; Sternberg 1988). Based on these ideas, we propose the following solution strategy: (i) find the foldings that minimize Φ^m , and then (ii) select among them the *preferred folding*, i.e. the one that contains the least sharp-fold energy.

We try to minimize Φ^m by minimizing its integrand, the energy density ϕ^m . The value of ϕ^m at a point (x_1, x_2) is a function of the components $w_{,1}$ and $w_{,2}$ of the local gradient; see equation 1.2. We look for gradients $\nabla w = (w_{,1}, w_{,2})$ at which ϕ^m attains a minimum. For convenience we define a *compressive regime* $\varepsilon_1^* + \nu\varepsilon_2^* > 0$ and a *characteristic slope* $k = \sqrt{\varepsilon_1^* + \nu\varepsilon_2^*}$. When ε^* falls outside the compressive regime, there exists a single minimum of ϕ^m at $\nabla w = (0, 0)$. We conclude that the diaphragm remains flat outside the compressive regime, and proceed to focus our attention on the compressive regime. In the isotropic case, $\varepsilon_1^* = \varepsilon_2^*$, infinitely many minima of ϕ^m of value ϕ_{\min}^m exist in the compressive regime; they occur at $|\nabla w| = k$. In the anisotropic case, two minima of ϕ^m of value ϕ_{\min}^m exist in the compressive regime; they occur at $\nabla w = (\pm k, 0)$. Thus for a diaphragm in the compressive regime, subjected to isotropic or anisotropic strain, the infimum of the membrane energy is $\Phi_{\inf}^m = \phi_{\min}^m A_\Omega$, where A_Ω is the area of the diaphragm. For convenience we shall work with $\tilde{\varphi}^m = \phi^m - \phi_{\min}^m$, which implies $\tilde{\Phi}^m = \Phi^m - \Phi_{\inf}^m$, and $\tilde{\Phi}_{\inf}^m = 0$.

We study the isotropic case first. Consider a one-dimensional example, figure 2a. In this example, $w_{,2} = 0$ identically. Then, the minima of $\tilde{\varphi}^m$ occur at $w_{,1} = \pm k$, and we can construct a minimizer of $\tilde{\Phi}^m$ by covering Ω with any set of *simple roofs* of slopes $\pm k$ (such as R_1 , R_2 and R_3 in figure 2a), and then choosing the upper envelope of the set of simple roofs. Infinitely many such minimizers exist, all of which contain sharp folds, as expected. To minimize the energy associated with the sharp folds, we must minimize the number of sharp folds; this we effect by selecting the upper envelope of all the minimizers (figure 2a). The preferred folding coincides with the one obtained by Berdichevski & Truskinovski (1985). In two dimensions a preferred folding can be found analogously, as the upper envelope of all the possible coverings of Ω with conical roofs of slope k ; figure 2b (Gioia & Ortiz 1997). The preferred folding so selected does (i) minimize $\tilde{\Phi}^m$; (ii) depend exclusively on the shape of the diaphragm; and (iii) match the experimental observations, as shown by a comparison of figure 2c with parts (b) and (c) of figure 1.

Now we turn to the anisotropic case. Consider first the infinitely long diaphragm of figure 3a. For this diaphragm the preferred folding can be found as in the one-dimensional isotropic case, in the form of a single simple roof of slopes $w_{,1} = \pm k$, figure 3a. On the other hand, in the finite diaphragm of figure 3b a single simple roof violates the boundary conditions, $w(x_2 = 0) = w(x_2 = \frac{3}{2}) = 0$. To circumvent this problem we try the folding w_0 of figure 3b. On the triangular regions of figure 3b, which we call *closure domains*, $w_{,1} \neq \pm k$, and therefore $\tilde{\varphi}_\Delta^m > \tilde{\varphi}_{\min}^m$. It follows that w_0 is not a minimizer. In fact, calling $A_\Delta[w_0]$ the area of the closure domains, $\tilde{\Phi}^m[w_0] = \tilde{\varphi}_\Delta^m A_\Delta[w_0] > \tilde{\Phi}_{\inf}^m$. We can approach the infimum $\tilde{\Phi}_{\inf}^m = 0$, however, by using foldings

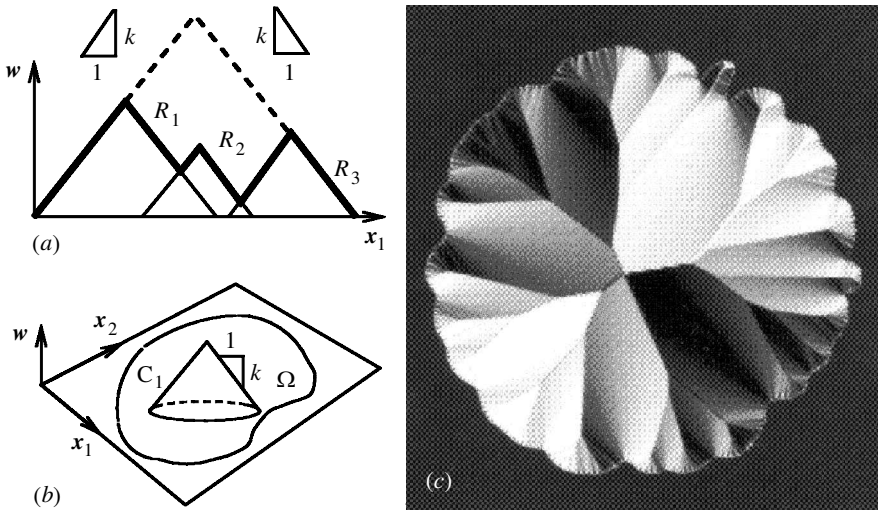


Figure 2. Analysis of the isotropic case. (a) Side view of a one-dimensional diaphragm. The domain of the diaphragm is covered with a set of three simple roofs of slopes $\pm k$; the upper envelope of the set (bold lines) is a minimizer of Φ^m . The preferred folding is the upper envelope of all the minimizers of Φ^m (dashed lines). (b) A two-dimensional diaphragm of domain Ω and an example of conical roof of slope k , denoted C_1 . (c) Top view of the preferred folding for the diaphragm shape of parts (b) and (c) of figure 1. The strain is isotropic with $\varepsilon_1^* = \varepsilon_2^* = 0.011$, and the corresponding characteristic slope is $k = 0.15$ ($\nu = 0.2$). Note that we obtain this folding analytically, using a simple construction. The sharp folds reproduce in surprising detail the experimentally observed folding pattern.

w_j with $j = 1, 2, \dots$, see, for example, figure 3c, for which $A_\Delta[w_j] \rightarrow 0$ and therefore $\tilde{\Phi}^m[w_j] = \tilde{\varphi}_\Delta^m A_\Delta[w_j] \rightarrow 0$ as $j \rightarrow \infty$. Thus, $\tilde{\Phi}^m$ can be made arbitrarily close to the infimum by allowing the diaphragm to become highly folded. The sequence w_j is called a minimizing sequence, and the associated foldings are called microstructures (Ball & James 1987; Kohn 1991).

For arbitrarily shaped diaphragms, we can construct minimizing sequences in an analogous way; figure 3d illustrates the procedure for the case of a circular diaphragm. All the microstructures in a minimizing sequence contain sharp folds and closure domains. As the microstructures become more folded, the sharp-fold energy increases, and the closure-domain energy decreases. In principle, we can identify a preferred microstructure for which the trade-off between closure-domain energy and sharp-fold energy is resolved in the least total energy. Thus, bending checks the infinite folding implied by a minimizing sequence (Ball & James 1987). A straightforward dimensional analysis reveals that the number of folds, N , in the preferred microstructure scales with $\tilde{\varphi}_\Delta^m d/T$, where T is the energy per unit length of sharp fold (i.e. the line tension). Setting $\varepsilon_2^* = 0$ for the sake of simplicity, it is easy to conclude that $\tilde{\varphi}_\Delta^m \propto E(\varepsilon_1^*)^2 h$; a more involved analysis leads to $T \propto E(\varepsilon_1^*)^{3/2} h^2$ (Gioia & Ortiz 1997). We are led to the same expression obtained by Pogorelov (1988),

$$N \propto \frac{d}{h} (\varepsilon_1^*)^{1/2}. \tag{1.3}$$

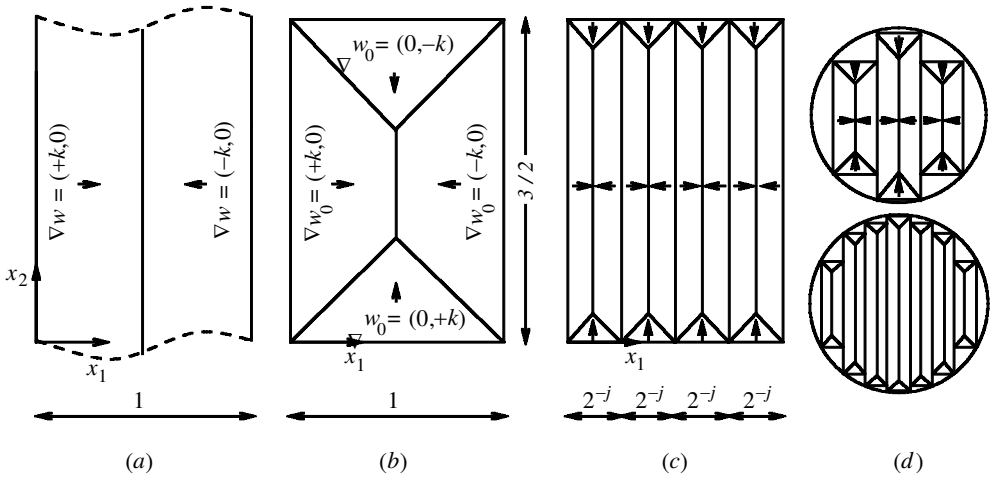


Figure 3. Analysis of the anisotropic case. (a) Top view of an infinitely long diaphragm with preferred folding. The preferred folding is a simple roof of slopes $w_{0,1} = \pm k$. The arrows represent the gradients of w . (b) Rectangular diaphragm with folding w_0 . (c) The same rectangular diaphragm with folding w_2 (i.e. with the microstructure $j = 2$ of the minimizing sequence w_j). (d) Two terms of a minimizing sequence for a circular diaphragm. Compare these foldings with those of figure 1e-h.

According to this expression, the number of folds for the diaphragms of figure 1d-h should be proportional to 32, 30, 16, 10 and 22, respectively. The observed numbers are 32, 28, 24, 12 and 22 (folds spanning the whole diaphragm). We conclude that the preferred microstructure of the minimizing sequence does (i) fail to minimize Φ^m ; and (ii) contain, in accord with the experimental evidence, folds which are perpendicular to the direction of ε_1^* , regardless of the shape of Ω , in a number which scales with d/h and ε^* in the form of equation (1.3).

To sum up, the folding of compressed diaphragms can be interpreted in terms of two operations: fold to release membrane energy, and then allow bending to select one among many possible foldings. In the isotropic case, infinitely many foldings exist which can accommodate the boundary conditions and simultaneously minimize Φ^m . Out of these foldings bending selects a preferred one. In the anisotropic case, no folding exists which can accommodate the boundary conditions and simultaneously minimize Φ^m . It is possible, however, to construct sequences of increasingly fine foldings, or microstructures, whose associated membrane energies converge to the infimum of Φ^m . Out of these foldings bending selects a preferred one.

The occurrence of microstructures is characteristic of diaphragm folding and many other problems governed by non-convex energy functionals. Among such problems we can instance solid-state phase transformation (Khachaturyan 1983), crystal plasticity (Ortiz & Repetto 1999), and ferromagnetism (DeSimone 1993), where microstructures have been documented in the form of twinning, dislocation cells, and magnetic domain structures, respectively. Other such problems are the stretching of solid foams (Gioia *et al.* 2001), and the self-assembling of polymer layers on patterned substrates (Böltau *et al.* 1998). Because experiments can be easily performed on compressed diaphragms, and a simple model appears to explain their behaviour well, the study of diaphragms may prove useful to gain insights into other systems. For example, a

90° rotation of the gradients in figure 3*b–d* leads to closed-flux vector fields of the type observed in ferromagnets, where magnetic poles are energetically penalized (DeSimone 1993). In this analogy, sharp folds correspond to ferromagnetic domain walls, and strain anisotropy corresponds to crystalline anisotropy. A comparison of the fold branching observable close to the boundaries of the diaphragms (see, for example, figure 1*g*), with the analogous phenomenon of domain branching in the vicinity of free surfaces in ferromagnets would enhance our understanding of the underlying energetics (Kohn & Müller 1992).

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