QUASISTATIC EVOLUTION FOR CAM-CLAY PLASTICITY:
EXAMPLES OF SPATIALLY HOMOGENEOUS SOLUTIONS

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We study a quasistatic evolution problem for Cam-Clay plasticity under a special loading program which leads to spatially homogeneous solutions. Under some initial conditions, the solutions exhibit a softening behavior and time discontinuities. The behavior of the solutions at the jump times is studied by a viscous approximation.

Keywords: Cam-Clay plasticity; softening behavior; pressure-sensitive yield criteria; non-associative plasticity; quasistatic evolution; rate independent dissipative processes; viscosity approximation.

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1. Introduction

The modified Cam-Clay model has been introduced in the engineering literature on soil mechanics as a conceptual tool to understand the irreversible deformations experienced by fine grained soils (clays) upon loading, see Refs. 8–11. One of the interesting features of this model is that, depending on the loading conditions, the stress–strain response may exhibit a hardening or a softening behavior. Furthermore, it is an important example of nonassociative plasticity, for which a satisfactory mathematical theory is only partially developed.7

We restrict our attention to the spatially homogeneous case in dimension \(n\), with no volume forces. The system is driven by a time-dependent affine boundary condition \(w(t,x)\), whose symmetrized spatial gradient \(Ew(t,x)\) is independent of the space variable \(x\) and is denoted by \(\xi(t)\). In this situation, the displacement \(u(t,x)\) coincides with \(w(t,x)\) and the unknowns are the elastic part \(e(t)\) and the plastic part \(p(t)\) appearing in the additive decomposition of the strain \(Eu(t,x) = e(t) + p(t)\), as well as a scalar internal variable \(z(t)\), which describes the time evolving yield surface.
The stress $\sigma(t)$ is determined by the elastic part of the strain through the usual relation $\sigma(t) = C e(t)$, where $C$ is the tensor of elastic moduli.

One ingredient of the model is a closed convex cone $K \subset M_{\text{sym}}^{n \times n} \times [0, +\infty)$, where $M_{\text{sym}}^{n \times n}$ is the space of symmetric $n \times n$ matrices. It is assumed that $K$ contains the half-line $\{0\} \times [0, +\infty)$. The stress is constrained by the inclusion $\sigma(t) \in K(z(t))$, where for every $\zeta \in [0, +\infty)$ we define $K(\zeta) := \{\sigma \in M_{\text{sym}}^{n \times n} : (\sigma, \zeta) \in K\}$. The interior of $K(\zeta)$ is the elastic domain corresponding to the value $\zeta$ of the internal variable, while its boundary $\partial K(\zeta)$ is the yield surface. In the typical applications, $\partial K(\zeta)$ is a suitable ellipsoid in the space $M_{\text{sym}}^{n \times n}$.

The other ingredients of the model are the evolution laws for $p(t)$ and $z(t)$, resulting in the system
\[
\begin{align*}
e(t) + p(t) &= \xi(t), \quad \sigma(t) = C e(t) \in K(z(t)), \\
\dot{p}(t) &= N_{K(z(t))}(\sigma(t)), \\
\dot{z}(t) &= \text{tr}(\sigma(t)) \text{tr}(\dot{p}(t)),\\
\end{align*}
\]
(1.1)
where $N_{K(\zeta)}(\sigma)$ denotes the normal cone to $K(\zeta)$ at $\sigma$. The nonassociative nature of the problem is due to the fact that the equation for $\dot{z}$ in (1.1) does not depend on $K$. In view of the hypotheses on $K$, we have the monotonicity condition $\zeta_1 < \zeta_2 \Rightarrow K(\zeta_1) \subset K(\zeta_2)$. Therefore, if $\dot{z}(t) > 0$, the set $K(z(t))$ expands leading to a hardening response. On the contrary, if $\dot{z}(t) < 0$, the set $K(z(t))$ shrinks leading to a softening response. In the usual applications we have $\text{tr}(\sigma) \leq 0$ for every $\sigma \in K(\zeta)$, which reflects the compressive conditions typical of soil mechanics. Therefore, by the third line in (1.1), the hardening or softening behavior is determined only by the sign of $\text{tr}(\dot{p})$. An energetic approach to a class of rate-independent plasticity problems which present only a softening behavior has been proposed in Ref. 3.

To deal with the instabilities of the softening regime, we propose a viscosity approximation \cite{2,4} to (1.1). Denoting the minimal distance projection of $\sigma$ onto $K(\zeta)$ by $\pi_{K(\zeta)}(\sigma)$, for every $\varepsilon > 0$ we consider the unconstrained system
\[
\begin{align*}
e_\varepsilon(t) + p_\varepsilon(t) &= \xi(t), \quad \sigma_\varepsilon(t) = C e_\varepsilon(t), \\
\dot{p}_\varepsilon(t) &= N_{K(\varepsilon(z(t)))}^\varepsilon(\sigma_\varepsilon(t)), \\
\dot{z}_\varepsilon(t) &= \text{tr}(\pi_{K(z(t))}(\sigma_\varepsilon(t))) \text{tr}(\dot{p}_\varepsilon(t)),\\
\end{align*}
\]
(1.2)
where $N_{K(\zeta)}^\varepsilon(\sigma) := \frac{1}{\varepsilon} (\sigma - \pi_{K(\zeta)}(\sigma))$ is the usual approximation of the normal to $K(\zeta)$. A viscosity solution $(e(t), p(t), \sigma(t), z(t))$ to (1.1) is defined as a left continuous map which, for almost every time $t$, is the pointwise limit of a sequence $(e_\varepsilon(t), p_\varepsilon(t), \sigma_\varepsilon(t), z_\varepsilon(t))$ of solutions of (1.2).

In this paper, we study in detail the case where $C e = e$ for every $e \in M_{\text{sym}}^{n \times n}$, so that $\sigma(t) = e(t)$ and $\sigma_\varepsilon(t) = e_\varepsilon(t)$. Moreover, we assume that $K(\zeta)$ is the closed ball centered at $-\frac{1}{n} \zeta I$ with radius $\frac{1}{\sqrt{n}} \zeta$, namely,
\[
K(\zeta) = \left\{ \sigma \in M_{\text{sym}}^{n \times n} : \left| \sigma + \frac{1}{n} \zeta I \right| \leq \frac{1}{\sqrt{n}} \zeta \right\},
\]
(1.3)
where $I$ is the identity matrix in $M_{\text{sym}}^{n \times n}$. The fact that all the elements in the interior of $K(\zeta)$ are negative definite reflects the fact that the material can only sustain compressive stresses.

Given a constant $a_0 > 0$, and a matrix $e_0 \in M_{\text{sym}}^{n \times n}$ with $\text{tr}(e_0) = 0$ and $|e_0| = 1$, we consider the special loading path

$$\xi(t) = -a_0 \frac{1}{n} I + t \frac{1}{\sqrt{n}} e_0,$$

and the initial conditions $e_{\zeta}(0) = e(0) = -a_0 \frac{1}{n} I$ and $z_{\zeta}(0) = z(0) = z_0$. Then $e_{\zeta}(t)$ and $e(t)$ have the form

$$e_{\zeta}(t) = -\frac{1}{n} x_{\zeta}(t) I + \frac{1}{\sqrt{n}} y_{\zeta}(t) e_0 \quad \text{and} \quad e(t) = -\frac{1}{n} x(t) I + \frac{1}{\sqrt{n}} y(t) e_0,$$

for suitable scalar function $x_{\zeta}(t), y_{\zeta}(t), x(t), y(t)$ satisfying $x_{\zeta}(0) = x(0) = a_0$ and $y_{\zeta}(0) = y(0) = 0$, while the constraint $\sigma(t) \in K(z(t))$ becomes

$$\sqrt{(x(t) - z(t))^2 + y(t)^2} \leq z(t).$$

Since the initial condition must satisfy this constraint, we assume that $0 \leq a_0 \leq 2z_0$. Then the solution is given by

$$x_{\zeta}(t) = x(t) = a_0, \quad y_{\zeta}(t) = y(t) = t, \quad z_{\zeta}(t) = z(t) = z_0,$$

in the interval $[0, t_0]$, where $t_0$ satisfies $\sqrt{(a_0 - z_0)^2 + t_0^2} = z_0$. This corresponds to the elastic regime (see Fig. 1).

![Fig. 1. The elastic regime. The thick line segment is the trajectory of $(x(t), y(t))$ in the time interval $[0, t_0]$. The circle represents the yield surface in the $(x, y)$ plane, which remains constant in this time interval.](image-url)
After time \( t_0 \) the solution exhibits a plastic behavior. To study the solution for \( t > t_0 \) we introduce polar coordinates

\[
\begin{align*}
  x_c(t) - z_c(t) &= \rho_c(t) \cos \theta_c(t), \\
  y_c(t) &= \rho_c(t) \sin \theta_c(t),
\end{align*}
\]

(1.6)

with \( \rho_c(t) > 0 \) and \( \rho(t) > 0 \) and we consider the angle \( \theta_0 \in [0, \pi] \) (see Fig. 1) such that

\[
a_0 = z_0 + z_0 \cos \theta_0 \quad \text{and} \quad t_0 = z_0 \sin \theta_0.
\]

To study the instabilities due to softening, it is convenient to introduce a fast time \( s := \frac{1}{\varepsilon} t \). By contrast, the standard time \( t \) will be called slow time. In certain time intervals the problem has no singularities and the evolution can be studied using the slow time. The limit system in this case is called the system of the slow dynamics and is studied in Sec. 3. It is used to describe the limit behavior in the hardening regime (Sec. 5.1) and in some cases of softening (Secs. 5.2 and 5.3).

When singular behavior occurs in the softening regime, we use the fast time \( s \). The corresponding limit system is called the system of the fast dynamics and is studied in Sec. 6. It is formally obtained by rescaling time in (1.2) according to \( s = \frac{t}{\varepsilon} \) and is used to determine the transfer map at a jump point \( t_1 \geq t_0 \), defined as the map

\[
(\rho(t_1 -), \theta(t_1 -), z(t_1 -)) \mapsto (\rho(t_1 +), \theta(t_1 +), z(t_1 +))
\]

where + and − refer to left and right limit, respectively (see Fig. 2). More precisely, the right limit \( (\rho(t_1 +), \theta(t_1 +), z(t_1 +)) \) is given by the asymptotic value for \( s \to +\infty \) of the solution \( (\rho^f(s), \theta^f(s), z^f(s)) \) of the system of the fast dynamics (6.30) whose limit as \( s \to -\infty \) is given by \( (\rho(t_1 -), \theta(t_1 -), z(t_1 -)) \).

The behavior of the system in the plastic regime depends on the initial condition \( (\theta_0, z_0) \) at time \( t_0 \) given in (1.7). If \( 0 \leq \theta_0 < \frac{\pi}{2} \), then we are in the hardening regime. The viscosity solution \( (\rho(t), \theta(t), z(t)) \) is continuous in time, is the uniform limit of the viscosity approximations \( (\rho_c(t), \theta_c(t), z_c(t)) \) on compact sets, and satisfies

\[
\rho(t) = z(t) \quad \text{for} \quad t \in [t_0, +\infty),
\]

\[
\dot{\rho}(t) > 0 \quad \text{and} \quad \dot{\theta}(t) > 0 \quad \text{for} \quad t \in [t_0, +\infty),
\]

\[
\lim_{t \to +\infty} \rho(t) < +\infty \quad \text{and} \quad \lim_{t \to +\infty} \theta(t) = \frac{\pi}{2}.
\]

If \( \frac{\pi}{2} < \theta_0 \leq \pi \), then we are in the softening regime and the viscosity solution \( (\rho(t), \theta(t), z(t)) \) may be discontinuous at a time \( t_1 \geq t_0 \) depending on the initial conditions \( (\theta_0, z_0) \). The jump at the discontinuity time is determined by the transfer map considered above and satisfies the inequalities \( 0 < \rho(t_1 +) = z(t_1 +) < \rho(t_1 -) = z(t_1 -) \) and \( \frac{\pi}{2} < \theta(t_1 +) < \theta(t_1 -) \).

Three possible behaviors occur, according to the phase diagram illustrated in Fig. 3. A crucial role is played by the separation line \( z = z_0(\theta) \), whose explicit formula is given by (3.4), by the critical line \( z = r_c(\theta) \), described in (3.6), and by the critical point \( (z_c, \theta_c) \) where the two lines meet, given explicitly in (3.3) and (3.5).
Fig. 2. Transfer map in the \((\theta, \rho)\) plane. The solid rectilinear grid is transformed into the dotted curvilinear grid, the solid thick line is transformed into the dashed thick line, and the dotted line remains fixed.

Fig. 3. Phase diagram in the \((\theta, \rho)\) plane. Dark gray region (including the thick line): initial data \((\theta_0, z_0)\) of the plastic regime with continuous evolution. Light gray region: initial data with discontinuity time \(t_1 > t_0\). White region: initial data with discontinuity time \(t_1 = t_0\).
(a) If either $\frac{\pi}{2} < \theta_0 \leq \theta_c$ and $z_0 \leq z_s(\theta_0)$, or $\theta_c \leq \theta_0 \leq \pi$ and $z_0 \leq r_c(\theta_0)$, then the viscosity solution $(\rho(t), \theta(t), z(t))$ is continuous in time (see Figs. 4–6), is the uniform limit of the viscosity approximations $(\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t))$ on every compact subset of $[t_0, +\infty)$, and satisfies

$$
\rho(t) = z(t) \quad \text{for } t \in [t_0, +\infty),
$$

$$
\dot{\rho}(t) < 0 \quad \text{and} \quad \dot{\theta}(t) < 0 \quad \text{for } t \in [t_0, +\infty),
$$

$$
\lim_{t \to +\infty} \rho(t) > 0 \quad \text{and} \quad \lim_{t \to +\infty} \theta(t) = \frac{\pi}{2}.
$$

(b) If either $\frac{\pi}{2} < \theta_0 \leq \theta_c$ and $z_0 > z_s(\theta_0)$, or $\theta_c < \theta_0 < \pi$ and $z_0 \geq z_s(\theta_0)$, then the viscosity solution $(\rho(t), \theta(t), z(t))$ is discontinuous at $t = t_0$. Moreover the solution $(\rho(t), \theta(t), z(t))$ is the uniform limit of the viscosity approximations $(\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t))$ on every compact subset of $(t_0, +\infty)$. It satisfies

$$
\rho(t) = z(t) \quad \text{for } t \in (t_0, +\infty),
$$

$$
\dot{\rho}(t) < 0 \quad \text{and} \quad \dot{\theta}(t) < 0 \quad \text{for } t \in (t_0, +\infty),
$$

$$
\lim_{t \to +\infty} \rho(t) > 0 \quad \text{and} \quad \lim_{t \to +\infty} \theta(t) = \frac{\pi}{2}.
$$

Fig. 4. Trajectories in the $(\theta, \rho)$ plane for $\theta_0 = \frac{9\pi}{10}$ and 12 different values of $z_0 < z_s(\theta_0)$. Solid lines: trajectories of $(\theta(t), \rho(t)) = (\theta(t), z(t))$ (slow dynamics). Dashed lines: trajectories of $(\theta^f(s), \rho^f(s))$ (fast dynamics). Dotted lines: trajectories of $(\theta^f(s), z^f(s))$ (fast dynamics).
Finally, the viscosity approximations \((\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t))\) are uniformly close to a rescaled version of \((\rho'(s), \theta'(s), z'(s))\) in a suitable right neighborhood of \(t_0\).

(c) If \(\theta_\varepsilon < \theta_0 \leq \pi\) and \(r_\varepsilon(\theta_0) < z_0 < z_\varepsilon(\theta_0)\), then the viscosity solution \((\rho(t), \theta(t), z(t))\) is discontinuous at a time \(t_1 > t_0\) (see Figs. 4–6). Moreover, the solution \((\rho(t), \theta(t), z(t))\) is the uniform limit of the viscosity approximations \((\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t))\) on every compact subset of \([t_0, t_1) \cup (t_1, +\infty)\). It satisfies

\[
\rho(t) = z(t) \quad \text{for } t \in [t_0, t_1) \cup (t_1, +\infty),
\]

\[
\dot{\rho}(t) < 0 \quad \text{and} \quad \dot{\theta}(t) < 0 \quad \text{for } t \in [t_0, t_1) \cup (t_1, +\infty),
\]

\[
\lim_{t \to +\infty} \rho(t) > 0 \quad \text{and} \quad \lim_{t \to +\infty} \theta(t) = \frac{\pi}{2}.
\]
Finally, the viscosity approximations $\langle \rho_c(t), \theta_c(t), z_c(t) \rangle$ are uniformly close to a rescaled version of $\langle \rho_f(s), \theta_f(s), z_f(s) \rangle$ in a suitable right neighborhood of $t_1$.

Further details on the mechanical interpretation of the behavior of the solutions are given in Sec. 8 using Cartesian coordinates $\langle x(t), y(t) \rangle$, see Figs. 7–9.

Extensions to general $K$ and general loading conditions in the spatially uniform case, and extensions to spatially non-uniform solutions will be considered in other forthcoming papers.

2. Formulation of the Problem and General Results

Let $K$ be a closed convex cone in $M_{\text{sym}}^{n \times n} \times [0, +\infty)$. For every $\zeta \in [0, +\infty)$ we define

$$K(\zeta) := \{\sigma \in M_{\text{sym}}^{n \times n} : (\sigma, \zeta) \in K\}.$$ 

Each set $K(\zeta)$ is closed and convex, and we have

$$K(\zeta) = \langle K(1) \rangle \quad \text{for every } \zeta \in [0, +\infty). \quad (2.1)$$

We assume that $K(1)$ is bounded and that $0 \in K(1)$, hence

$$0 \in K(\zeta) \quad \text{for every } \zeta \in [0, +\infty), \quad (2.2)$$

and

$$|\sigma| \leq M_K \zeta \quad \text{for every } (\sigma, \zeta) \in K, \quad (2.3)$$

for a suitable constant $M_K < +\infty$. 

Fig. 6. Graph of $\theta(t)$ in the plastic regime $t > t_0$ for $a_0 = 2$ and eight different values of $t_0$ and $z_0$. 

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For every closed convex set \( C \subset \mathbb{M}_{\text{sym}}^{n \times n} \) let \( \pi_C : \mathbb{M}_{\text{sym}}^{n \times n} \rightarrow C \) be the minimal distance projection onto \( C \). It follows from (2.1) that

\[
\pi_{K(\zeta)}(\sigma) = \zeta \pi_{K(1)} \left( \frac{1}{\zeta} \sigma \right)
\]  

(2.4)

for every \( \zeta > 0 \) and every \( \sigma \in \mathbb{M}_{\text{sym}}^{n \times n} \).

The following result will be used to prove the existence of a solution to the system (1.2) governing the viscous approximation of the original problem (1.1).

**Lemma 2.1.** The map \( (\sigma, \zeta) \mapsto \pi_{K(\zeta)}(\sigma) \) from \( \mathbb{M}_{\text{sym}}^{n \times n} \times [0, +\infty) \) into \( \mathbb{M}_{\text{sym}}^{n \times n} \) satisfies the Lipschitz estimate

\[
|\pi_{K(\zeta)}(\sigma_2) - \pi_{K(\zeta)}(\sigma_1)| \leq |\sigma_2 - \sigma_1| + 2M_K |\zeta_2 - \zeta_1|
\]

(2.5)

for every \( (\sigma_1, \zeta_1), (\sigma_2, \zeta_2) \in \mathbb{M}_{\text{sym}}^{n \times n} \times [0, +\infty) \).

**Proof.** It is enough to prove the estimate for \( (\sigma_1, \zeta_1), (\sigma_2, \zeta_2) \in \mathbb{M}_{\text{sym}}^{n \times n} \times [0, +\infty) \) with \( 0 < \zeta_1 \leq \zeta_2 \). Since \( \pi_{K(\zeta)} \) has Lipschitz constant 1 on \( \mathbb{M}_{\text{sym}}^{n \times n} \), from (2.3) and (2.4) we obtain

\[
|\pi_{K(\zeta_2)}(\sigma_2) - \pi_{K(\zeta_1)}(\sigma_1)| \\
\leq |\pi_{K(\zeta_2)}(\sigma_2) - \pi_{K(\zeta_2)}(\sigma_1)| + |\pi_{K(\zeta_2)}(\sigma_1) - \pi_{K(\zeta_1)}(\sigma_1)| \\
\leq |\sigma_2 - \sigma_1| + |\zeta_2 \pi_{K(1)} \left( \frac{1}{\zeta_2} \sigma_1 \right) - \zeta_1 \pi_{K(1)} \left( \frac{1}{\zeta_1} \sigma_1 \right)| \\
\leq |\sigma_2 - \sigma_1| + M_K |\zeta_2 - \zeta_1| + \zeta_1 |\pi_{K(1)} \left( \frac{1}{\zeta_2} \sigma_1 \right) - \pi_{K(1)} \left( \frac{1}{\zeta_1} \sigma_1 \right)|.
\]

To prove (2.5) it is enough to show that

\[
\zeta_1 \left| \pi_{K(1)} \left( \frac{1}{\zeta_2} \sigma_1 \right) - \pi_{K(1)} \left( \frac{1}{\zeta_1} \sigma_1 \right) \right| \leq M_K |\zeta_2 - \zeta_1|.
\]

(2.6)

As \( 0 < \zeta_1 \leq \zeta_2 \), we have

\[
\pi_{K(1)} \left( \frac{1}{\zeta_1} \sigma_1 - \frac{\zeta_2 - \zeta_1}{\zeta_1} \pi_{K(1)} \left( \frac{1}{\zeta_2} \sigma_1 \right) \right) \\
= \pi_{K(1)} \left( \frac{1}{\zeta_2} \sigma_1 + \frac{\zeta_2 - \zeta_1}{\zeta_1} \left( \frac{1}{\zeta_2} \sigma_1 - \pi_{K(1)} \left( \frac{1}{\zeta_2} \sigma_1 \right) \right) \right) \\
= \pi_{K(1)} \left( \frac{1}{\zeta_2} \sigma_1 \right).
\]
Since \( \pi_{K(1)} \) has Lipschitz constant 1 on \( \mathcal{M}^{n \times n}_{\text{sym}} \), we obtain
\[
\left| \pi_{K(1)} \left( \frac{1}{\xi_2} \sigma_1 \right) - \pi_{K(1)} \left( \frac{1}{\xi_1} \sigma_1 \right) \right| \leq \frac{\xi_2 - \xi_1}{\xi_1} \left| \pi_{K(1)} \left( \frac{1}{\xi_2} \sigma_1 \right) \right| \leq M_K \frac{\xi_2 - \xi_1}{\xi_1},
\]
which gives (2.6). □

Let us fix \( \xi \in W^{1,1}_{\text{loc}}([0, +\infty); \mathcal{M}^{n \times n}_{\text{sym}}) \). For every \( \varepsilon > 0 \) system (1.2) is equivalent to
\[
\begin{align*}
\varepsilon \dot{e}_\varepsilon(t) &= \varepsilon \dot{\xi}(t) - \mathcal{C} e_\varepsilon(t) + \pi_{K(\xi_\varepsilon(t))}(\mathcal{C} e_\varepsilon(t)), \\
\varepsilon \dot{z}_\varepsilon(t) &= \text{tr}(\pi_{K(\xi_\varepsilon(t))}(\mathcal{C} e_\varepsilon(t))) \text{tr}(\mathcal{C} e_\varepsilon(t) - \pi_{K(\xi_\varepsilon(t))}(\mathcal{C} e_\varepsilon(t))).
\end{align*}
\]
(2.7)

**Lemma 2.2.** For every \( \varepsilon > 0 \) and for every initial condition \( e_\varepsilon(0) = e_0 \) and \( z_\varepsilon(0) = z_0 \geq 0 \) system (2.7) has a unique solution defined for every \( t \in [0, +\infty) \).

**Proof.** As the right-hand sides are locally Lipschitz with respect to \( e \) and \( z \) by Lemma 2.1, it is enough to prove that for every \( T > 0 \) there is a constant \( M_T > 0 \) such that \( |e_\varepsilon(t)| \leq M_T \) and \( |z_\varepsilon(t)| \leq M_T \) for every \( t \in [0, T] \). Since \( 0 \in K(\xi) \) for every \( \xi \in R \) by (2.2), we have \( |\mathcal{C} e_\varepsilon(t) - \pi_{K(\xi_\varepsilon(t))}(\mathcal{C} e_\varepsilon(t))| \leq |\mathcal{C} e_\varepsilon(t)| \leq \beta |e_\varepsilon(t)| \) and \( |\pi_{K(\xi_\varepsilon(t))}(\mathcal{C} e_\varepsilon(t))| \leq |\mathcal{C} e_\varepsilon(t)| \leq \beta |e_\varepsilon(t)| \) for every \( t \in [0, +\infty) \). Therefore, given \( T > 0 \), from the first equation in (2.7) we have
\[
|e_\varepsilon(t)| \leq A_T + \frac{\beta C}{\varepsilon} \int_0^t |e_\varepsilon(s)| \, ds \quad \text{for every } t \in [0, T],
\]
with \( A_T := |e_0| + \int_0^T |\dot{\xi}(s)| \, ds \). It follows from the Gronwall inequality that
\[
|e_\varepsilon(t)| \leq A_T \exp \left( \frac{T \beta_C}{\varepsilon} \right) \quad \text{for every } t \in [0, T].
\]
Then the second equation in (2.7) allows easily to obtain a constant \( M_T > 0 \) such that \( |z_\varepsilon(t)| \leq M_T \) for every \( t \in [0, T] \). □

**Lemma 2.3.** For every \( \varepsilon > 0 \), \( e_\varepsilon \in \mathcal{M}^{n \times n}_{\text{sym}} \), and \( z_0 > 0 \) the solution \((e_\varepsilon, z_\varepsilon)\) of (2.7) with initial condition \( e_\varepsilon(0) = e_0 \) and \( z_\varepsilon(0) = z_0 \) satisfies \( z_\varepsilon(t) > 0 \) for every \( t \in [0, +\infty) \).

**Proof.** Suppose by contradiction that there exists \( t_0 \in (0, +\infty) \) such that \( z_\varepsilon(t_0) = 0 \). Let \( e_\varepsilon^* \) be the solution of the Cauchy problem
\[
\begin{align*}
\varepsilon \dot{e}_\varepsilon^*(t) &= \varepsilon \dot{\xi}(t) - \mathcal{C} e_\varepsilon^*(t), \\
e_\varepsilon^*(t_0) &= e_\varepsilon(t_0),
\end{align*}
\]
(2.8)
and let \( z_\varepsilon^* := 0 \). Then \((e_\varepsilon^*, z_\varepsilon^*)\) would be a solution to (2.7) with \( e_\varepsilon^*(t_0) = e_\varepsilon(t_0) \) and \( z_\varepsilon^*(t_0) = z_\varepsilon(t_0) \). Since the right-hand side of (2.7) is locally Lipschitz with respect to \( e \) and \( z \) by Lemma 2.1, by uniqueness we would have \( z_\varepsilon(t) = z_\varepsilon^*(t) = 0 \) for every \( t \in [0, +\infty) \), which contradicts the assumption \( z_\varepsilon(0) = z_0 > 0 \). □
For the rest of the paper we assume that \( C \xi = \xi \) for every \( \xi \in M_{n \times n} \) and that \( K(\zeta) \) is the closed ball centered at \(-\frac{1}{n} \zeta I\) with radius \( \frac{1}{\sqrt{n}} \zeta \), namely,
\[
K(\zeta) = \left\{ \sigma \in M_{n \times n}^{sym} : \sigma + \frac{1}{n} \zeta I \leq \frac{1}{\sqrt{n}} \zeta \right\},
\]
(2.9)
where \( I \) is the identity matrix in \( M_{n \times n}^{sym} \). In this case \( \sigma(\zeta) = e(\zeta) \) and Eq. (2.7) simplifies to
\[
\begin{align*}
\varepsilon \dot{e}(t) &= \varepsilon \dot{\zeta}(t) - e(\zeta(t)) + \pi_{K(\zeta(t))}(e(\zeta(t))), \\
\varepsilon \dot{z}(t) &= \text{tr}(\pi_{K(\zeta(t))}(e(\zeta(t)))) \text{tr}(e(\zeta(t)) - \pi_{K(\zeta(t))}(e(\zeta(t)))).
\end{align*}
\]
(2.10)
Moreover, the projection onto \( K(\zeta) \) is explicitly given by
\[
\pi_{K(\zeta)}(\sigma) = -\frac{1}{n} \zeta I + \frac{\sigma + \frac{1}{n} \zeta I}{\max(|\sigma + \frac{1}{n} \zeta I|, \frac{1}{\sqrt{n}} \zeta)} \frac{1}{\sqrt{n}} \zeta.
\]
Let us fix \( e_0 \in M_{n \times n}^{sym} \) with \( \text{tr}(e_0) = 0 \) with \( |e_0| = 1 \). In the rest of the paper we consider \( \xi(t) \) of the form
\[
\xi(t) = -\frac{1}{n} a(t) I + \frac{1}{\sqrt{n}} b(t) e_0,
\]
(2.11)
with \( a \) and \( b \) in \( W^{1,1}_{\text{loc}}([0, \infty)) \). In this case \( \sigma(\zeta) \) and \( e(\zeta) \) take the form
\[
\sigma(\zeta) = e(\zeta) = -\frac{1}{n} x(\zeta(t)) I + \frac{1}{\sqrt{n}} y(\zeta(t)) e_0,
\]
(2.12)
where the absolute values of the scalars \( \frac{1}{\sqrt{n}} x(\zeta(t)) \) and \( \frac{1}{\sqrt{n}} y(\zeta(t)) \) represent the norms of the spherical and deviatoric components of the stress, respectively. Moreover (2.10) is equivalent to the system
\[
\begin{align*}
\varepsilon \dot{x}(t) &= \varepsilon \dot{\zeta}(t) - x(\zeta(t)) + z(\zeta(t))(x(\zeta(t)) - z(\zeta(t))) \\
\varepsilon \dot{y}(t) &= \varepsilon \dot{\zeta}(t) - y(\zeta(t)) + z(\zeta(t)) y(\zeta(t)) \\
\varepsilon \dot{z}(t) &= \left( z(\zeta(t)) + \frac{z(\zeta(t))(x(\zeta(t)) - z(\zeta(t)))}{u(\zeta(t))} \right) \left( x(\zeta(t)) - z(\zeta(t)) - \frac{z(\zeta(t))(x(\zeta(t)) - z(\zeta(t)))}{u(\zeta(t))} \right),
\end{align*}
\]
(2.13)
where
\[
u(\zeta(t)) := \max \left\{ z(\zeta(t)), \sqrt{(x(\zeta(t)) - z(\zeta(t)))^2 + y(\zeta(t))^2} \right\}.
\]
The corresponding viscosity solution \((e(t), p(t), \sigma(t), z(t))\) will be given by
\[
\sigma(t) = e(t) = -\frac{1}{n} x(t) I + \frac{1}{\sqrt{n}} y(t) e_0 \quad \text{and}
\]
\[
p(t) = \frac{1}{n} (a(t) - x(t)) I + \frac{1}{\sqrt{n}} (b(t) - y(t)) e_0,
\]
where \( x(t), y(t), \) and \( z(t) \) are left continuous with respect to \( t \) and \( x_\varepsilon(t) \to x(t), \ y_\varepsilon(t) \to y(t), \) and \( z_\varepsilon(t) \to z(t) \) for a.e. \( t \in [0, +\infty) \).

Passing to polar coordinates through (1.6), system (2.13) becomes

\[
\begin{align*}
\varepsilon \dot{p}_\varepsilon(t) & = \varepsilon (\dot{a}(t) \cos \theta_\varepsilon(t) + \dot{b}(t) \sin \theta_\varepsilon(t)) \\
- (p_\varepsilon(t) - z_\varepsilon(t))^{+} & + (z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t))\cos^2 \theta_\varepsilon(t) + 1), \\
\varepsilon \dot{p}_\varepsilon(t) \dot{\theta}_\varepsilon(t) & = -\varepsilon (\dot{a}(t) \sin \theta_\varepsilon(t) - \dot{b}(t) \cos \theta_\varepsilon(t)) \\
+ (p_\varepsilon(t) - z_\varepsilon(t))^{+} & + z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t)) \cos \theta_\varepsilon(t) \sin \theta_\varepsilon(t), \\
\varepsilon \dot{z}_\varepsilon(t) & = (p_\varepsilon(t) - z_\varepsilon(t))^{+} z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t)) \cos \theta_\varepsilon(t),
\end{align*}
\]

where \((\cdot)^{+}\) denotes the positive part. The polar coordinates of a viscosity solution are denoted by \((\rho(t), \theta(t), z(t))\). They are continuous from the left and \((\rho_\varepsilon(t), \theta_\varepsilon(t), z_\varepsilon(t)) \to (\rho(t), \theta(t), z(t))\) for a.e. \( t \in [0, +\infty) \).

Let us fix \( a_0 \) and \( z_0 \), with \( 0 \leq a_0 \leq 2z_0 \) and \( z_0 > 0 \). In the rest of the paper we study the special (strain controlled) loading path

\[
a(t) := a_0 \quad \text{and} \quad b(t) := t,
\]

and the initial conditions

\[
x_\varepsilon(0) = a_0, \quad y_\varepsilon(0) = 0, \quad z_\varepsilon(0) = z_0.
\]

### 2.1. The elastic regime

The solution of (2.13) with loading path (2.15) and initial conditions (2.16) remains in the elastic regime in an interval \([0, t_0]\), where \( t_0 := \sqrt{z_0^2 - (a_0 - z_0)^2} \) is the only positive number such that

\[
\sqrt{(a_0 - z_0)^2 + t_0^2} = z_0.
\]

More precisely we have

\[
x_\varepsilon(t) = a_0, \quad y_\varepsilon(t) = t, \quad z_\varepsilon(t) = z_0,
\]

for every \( t \in [0, t_0] \). Indeed in this interval the functions defined by (2.18) satisfy the inequality \( \sqrt{(x_\varepsilon - z_\varepsilon)^2 + y_\varepsilon^2} \leq z_\varepsilon \), so that the system reduces to

\[
\begin{align*}
\varepsilon \dot{x}_\varepsilon(t) & = \varepsilon \dot{a}(t), \\
\varepsilon \dot{y}_\varepsilon(t) & = \varepsilon \dot{b}(t), \\
\varepsilon \dot{z}_\varepsilon(t) & = 0,
\end{align*}
\]

which is trivially satisfied by (2.15) and (2.18). Therefore the viscosity solution satisfies

\[
x(t) = a_0, \quad y(t) = t, \quad z(t) = z_0,
\]

for every \( t \in [0, t_0] \).
2.2. The inelastic regime

After time $t_0$ the solution exhibits a plastic behavior. To study the solution for $t > t_0$ we use (2.14), which in case (2.15) becomes

$$
\begin{align*}
\varepsilon \rho_{\varepsilon}(t) &= \varepsilon \sin \theta_{\varepsilon}(t) - (\rho_{\varepsilon}(t) - z_{\varepsilon}(t)) + (z_{\varepsilon}(t)(1 + \cos \theta_{\varepsilon}(t)) \cos^2 \theta_{\varepsilon}(t) + 1), \\
\varepsilon \rho_{\varepsilon}(t) \dot{\theta}_{\varepsilon}(t) &= \varepsilon \cos \theta_{\varepsilon}(t) + (\rho_{\varepsilon}(t) - z_{\varepsilon}(t)) + z_{\varepsilon}(t)(1 + \cos \theta_{\varepsilon}(t)) \cos \theta_{\varepsilon}(t) \sin \theta_{\varepsilon}(t), \\
\varepsilon \dot{z}_{\varepsilon}(t) &= (\rho_{\varepsilon}(t) - z_{\varepsilon}(t)) + z_{\varepsilon}(t)(1 + \cos \theta_{\varepsilon}(t)) \cos \theta_{\varepsilon}(t).
\end{align*}
$$

(2.21)

By (2.17), there exists a unique $\theta_0 \in (0, \pi)$ such that

$$
z_0 \cos \theta_0 = a_0 - z_0, \quad z_0 \sin \theta_0 = t_0.
$$

(2.22)

By elementary geometric considerations we have

$$
0 \leq a_0 < z_0 \Rightarrow \frac{\pi}{2} < \theta_0 \leq \pi \quad \text{and} \quad z_0 < a_0 \leq 2z_0 \Rightarrow 0 \leq \theta_0 < \frac{\pi}{2}.
$$

(2.23)

By (2.17), (2.18) and (2.22) we have

$$
\rho_{\varepsilon}(t_0) = z_0, \quad \theta_{\varepsilon}(t_0) = \theta_0, \quad z_{\varepsilon}(t_0) = z_0.
$$

(2.24)

Subtracting the third equation from the first one in (2.21) we obtain the following differential equation for the difference $\rho_{\varepsilon}(t) - z_{\varepsilon}(t)$:

$$
\varepsilon (\dot{\rho}_{\varepsilon}(t) - \dot{z}_{\varepsilon}(t)) = \varepsilon \sin \theta_{\varepsilon}(t) - (\rho_{\varepsilon}(t) - z_{\varepsilon}(t)) + w_{\varepsilon}(t),
$$

(2.25)

where

$$
w_{\varepsilon}(t) := z_{\varepsilon}(t)(1 + \cos \theta_{\varepsilon}(t))^2 \cos \theta_{\varepsilon}(t) + 1.
$$

(2.26)

From (2.21) for every $t \in [t_0, +\infty)$ we obtain

$$
\varepsilon \rho_{\varepsilon}(t) \dot{w}_{\varepsilon}(t) = -\varepsilon z_{\varepsilon}(t)(1 + \cos \theta_{\varepsilon}(t))(1 + 3 \cos \theta_{\varepsilon}(t)) \cos \theta_{\varepsilon}(t) \sin \theta_{\varepsilon}(t)
$$

$$
- (\rho_{\varepsilon}(t) - z_{\varepsilon}(t)) z_{\varepsilon}(t)(1 + \cos \theta_{\varepsilon}(t))^3 \cos \theta_{\varepsilon}(t) v_{\varepsilon}(t),
$$

(2.27)

where

$$
v_{\varepsilon}(t) := z_{\varepsilon}(t)(1 + \cos \theta_{\varepsilon}(t) - 3 \cos^2 \theta_{\varepsilon}(t)) - (\rho_{\varepsilon}(t) - z_{\varepsilon}(t)) \cos \theta_{\varepsilon}(t).
$$

If $a_0 = z_0$, then $\theta_0 = \frac{\pi}{2}$ and, in this case,

$$
\rho_{\varepsilon}(t) := z_0 + \varepsilon \left(1 - \exp \left(-\frac{t - t_0}{\varepsilon}\right)\right), \quad \theta_{\varepsilon}(t) := \frac{\pi}{2}, \quad z_{\varepsilon}(t) := z_0,
$$

(2.28)

is the explicit solution of (2.21) with initial conditions (2.24). Then the viscosity solution obtained by taking the limit as $\varepsilon \to 0$ satisfies

$$
\rho(t) = z_0, \quad \theta(t) = \frac{\pi}{2}, \quad z(t) = z_0 \quad \text{for every} \quad t \in [t_0, +\infty).
$$

(2.29)

Lemma 2.4. If $\theta_0 \neq \frac{\pi}{2}$, then $\theta_{\varepsilon}(t) \neq \frac{\pi}{2}$ for every $t \in [t_0, +\infty)$. 


Lemma 2.6. Suppose \( \theta_0 \neq \frac{\pi}{2} \) and suppose that there exists \( \tau \in [t_0, +\infty) \) such that \( \theta_\varepsilon(\tau) = \frac{\pi}{2} \). Let \( \rho_\varepsilon^\tau \) be the solution of the Cauchy problem
\[
\begin{align*}
\varepsilon \dot{\rho}_\varepsilon^\tau(t) &= \varepsilon - (\rho_\varepsilon^\tau(t) - z_\varepsilon(\tau))^+, \\
\rho_\varepsilon^\tau(\tau) &= \rho_\varepsilon(\tau).
\end{align*}
\]
Then the triple
\[
\rho_\varepsilon^\tau(t), \quad \theta_\varepsilon^\tau(t) := \frac{\pi}{2}, \quad z_\varepsilon^\tau(t) := z_\varepsilon(\tau),
\]
would be a solution of (2.21) which satisfies the Cauchy condition
\[
\rho_\varepsilon^\tau(\tau) := \rho_\varepsilon(\tau), \quad \theta_\varepsilon^\tau(\tau) := \theta_\varepsilon(\tau), \quad z_\varepsilon^\tau(\tau) := z_\varepsilon(\tau).
\]
By uniqueness we must have \( \theta_\varepsilon(t) = \theta_\varepsilon^\tau(t) = \frac{\pi}{2} \) for every \( t \), which contradicts the fact that \( \theta_\varepsilon(t_0) = \theta_0 \neq \frac{\pi}{2} \). This concludes the proof of (2.4).

Lemma 2.5. If \( 0 \leq \theta_0 < \frac{\pi}{2} \), then \( 0 < \theta_\varepsilon(t) < \frac{\pi}{2} \) for every \( t \in (t_0, +\infty) \). If \( \frac{\pi}{2} < \theta_0 \leq \pi \), then \( \frac{\pi}{2} < \theta_\varepsilon(t) < \pi \) for every \( t \in (t_0, +\infty) \).

Proof. Assume \( 0 \leq \theta_0 < \frac{\pi}{2} \). From the second equation in (5.12) it follows that \( \dot{\theta}_\varepsilon(t_0) > 0 \). Therefore the inequalities \( 0 < \theta_\varepsilon(t) < \frac{\pi}{2} \) are satisfied in a right neighborhood of \( t_0 \). If they do not hold for every \( t \in [t_0, +\infty) \), by Lemma 2.4 we can consider the first \( \tau \in (t_0, +\infty) \) such that \( \theta_\varepsilon(\tau) = 0 \). Then \( \theta_\varepsilon(\tau) \leq 0 \). As \( 0 < \theta_\varepsilon(t) < \frac{\pi}{2} \) for every \( t \in [t_0, \tau] \) by Lemma 2.4, from the second equation in (2.21) we obtain \( \rho_\varepsilon(t)\theta_\varepsilon(t) \geq \cos \theta_\varepsilon(t) > 0 \) for every \( t \in [t_0, \tau] \) and \( \rho_\varepsilon(\tau)\theta_\varepsilon(\tau) = 1 \). As \( \rho_\varepsilon(t_0) = z_0 > 0 \), by continuity we have \( \dot{\theta}_\varepsilon(t) > 0 \) for every \( t \in [t_0, \tau] \). This contradicts the inequality \( \dot{\theta}_\varepsilon(\tau) \leq 0 \), and concludes the proof of the first implication. The second one is proved in the same way.

Lemma 2.6. We have \( \rho_\varepsilon(t) > z_\varepsilon(t) \) for every \( t \in (t_0, +\infty) \).

Proof. We deduce from (2.25) that, if \( \rho_\varepsilon(t) = z_\varepsilon(t) \) for some \( t \in [t_0, +\infty) \), then \( \dot{\rho}_\varepsilon(t) - \dot{z}_\varepsilon(t) = \sin \theta_\varepsilon(t) > 0 \), where the inequality follows from (2.28) and Lemma 2.5. Since \( \rho_\varepsilon(t_0) = z_\varepsilon(t_0) \), we conclude that \( \rho_\varepsilon(t) > z_\varepsilon(t) \) for every \( t \in (t_0, +\infty) \).

Lemma 2.7. For every \( t \in (t_0, +\infty) \) the following properties hold:
\[
\rho_\varepsilon(t) > 0, \quad (2.30)
\]
\[
0 \leq \theta_0 < \frac{\pi}{2} \Rightarrow \dot{\theta}_\varepsilon(t) > 0 \quad \text{and} \quad 0 < \theta_0 < \theta_\varepsilon(t) < \frac{\pi}{2}, \quad (2.31)
\]
\[
\frac{\pi}{2} < \theta_0 \leq \pi \Rightarrow \dot{\theta}_\varepsilon(t) < 0 \quad \text{and} \quad \frac{\pi}{2} < \theta_\varepsilon(t) < \theta_0 < \pi, \quad (2.32)
\]
\[
0 \leq \theta_0 < \frac{\pi}{2} \Rightarrow \dot{z}_\varepsilon(t) > 0 \quad \text{and} \quad z_\varepsilon(t) > z_0, \quad (2.33)
\]
\[
\frac{\pi}{2} < \theta_0 \leq \pi \Rightarrow \dot{z}_\varepsilon(t) < 0 \quad \text{and} \quad 0 < z_\varepsilon(t) < z_0. \quad (2.34)
\]
Proof. By Lemma 2.5 from the second equation in (2.21) and from (2.24) we obtain (2.30)–(2.32). Implications (2.33) and (2.34) can be obtained from Lemmas 2.3, 2.5, and 2.6, using the third equation in (2.21).

Lemma 2.8. Assume \( \pi < \theta_0 < \pi \). Then \( \rho_\varepsilon(t) \leq \rho_\varepsilon(s) + \varepsilon \) whenever \( t_0 \leq s \leq t \).

Proof. Let us fix \( s \geq t_0 \) and \( \eta > 0 \). If the inequality
\[
\rho_\varepsilon(t) \leq \rho_\varepsilon(s) + (1 + \eta)\varepsilon \tag{2.35}
\]
is not satisfied for every \( t \geq s \), let \( \tau \) be the first time after \( s \) with \( \rho_\varepsilon(\tau) = \rho_\varepsilon(s) + (1 + \eta)\varepsilon \). Then \( \dot{\rho}_\varepsilon(\tau) \geq 0 \). From the first equation in (2.21) we obtain
\[
\varepsilon \dot{\rho}_\varepsilon(\tau) \leq \varepsilon - (\rho_\varepsilon(\tau) - z_\varepsilon(\tau)).
\]
By (2.34) and by the definition of \( \tau \) we have
\[
\varepsilon \dot{\rho}_\varepsilon(\tau) \leq \varepsilon - (\rho_\varepsilon(s) + (1 + \eta)\varepsilon - z_\varepsilon(s)),
\]
so that Lemma 2.6 gives \( \varepsilon \dot{\rho}_\varepsilon(\tau) \leq -\eta\varepsilon \), which contradicts the inequality \( \dot{\rho}_\varepsilon(\tau) \geq 0 \). This proves that (2.35) holds for every \( t \geq s \). The conclusion can be obtained by taking the limit as \( \eta \to 0 \). □

3. The Slow Dynamics

In this section, we study in detail the behavior of the solutions to the system of the slow dynamics.

3.1. The trajectory of the slow dynamics

In this subsection, we study the equation
\[
r'(\theta) = r(\theta) \frac{r(\theta)(1 + \cos \theta) \sin \theta}{r(\theta)(1 + \cos \theta)^2 + 1}, \tag{3.1}
\]
which describes the trajectories followed along the slow dynamics.

Lemma 3.1. Every solution of (3.1) with \( r(\theta^*) > 0 \) for some \( \theta^* \in [0, \pi] \) is defined for every \( \theta \in [0, \pi] \) and satisfies \( r(\theta) > 0 \) for every \( \theta \in [0, \pi] \) and \( r'(\theta) > 0 \) for every \( \theta \in (0, \pi) \).

Proof. Since the null function is a solution of the equation, if \( r(\theta) \) is a solution of (3.1) and \( r(\theta^*) > 0 \) for some \( \theta^* \), then \( r(\theta) > 0 \) for every \( \theta \) by uniqueness. Therefore, the right-hand side of (3.1) is positive for \( \theta \in (0, \pi) \), which implies that \( r'(\theta) > 0 \) on this interval.

To prove the global existence in the whole interval \([0, \pi]\), it is not restrictive to assume \( \theta^* \in (0, \pi) \). The positivity and monotonicity of \( r(\theta) \) imply that \([0, \theta^*]\) is contained in the maximal domain of existence of \( r(\theta) \). To study the problem for \( \theta > \theta^* \) we consider the inequalities
\[
0 < \rho \frac{\rho(1 + \cos \theta) \sin \theta}{\rho(1 + \cos \theta)^2 + 1} < \frac{\rho \sin \theta}{1 + \cos \theta}.
\]
for every $\rho > 0$ and every $\theta \in (0, \pi)$. Using an elementary comparison argument we deduce that the maximal domain of existence of $r(\theta)$ contains $[\theta^*, \pi]$ and

$$r(\theta) \leq r(\theta^*) \frac{1 + \cos \theta^*}{1 + \cos \theta} \quad \text{for every } \theta^* \leq \theta \leq \pi.$$ 

By (3.1) this inequality yields

$$r'(\theta) \leq r(\theta)r(\theta^*)(1 + \cos \theta^*) \quad \text{for every } \theta \in [\theta^*, \pi),$$

and this implies that $\pi$ belongs to the maximal domain of existence of $r(\theta)$. □

Let $\lambda_c$ be the unique negative solution of the equation $1 + \lambda - 3\lambda^2 = 0$, i.e.

$$\lambda_c := -\frac{1}{6} (\sqrt{13} - 1) \simeq -0.43425 \ldots, \quad (3.2)$$

and let

$$\theta_c := \arccos \lambda_c \simeq 2.0200 \ldots \quad (3.3)$$

We consider the function $z_s : [\frac{\pi}{2}, \pi] \rightarrow [\frac{2\pi}{3}, +\infty]$ defined by

$$z_s(\theta) := -\frac{1}{(1 + \cos \theta)^2 \cos \theta} \quad \text{for } \theta \in \left(\frac{\pi}{2}, \pi\right), \quad z_s\left(\frac{\pi}{2}\right) := z_s(\pi) := +\infty, \quad (3.4)$$

and we define

$$z_c := z_s(\theta_c) = \frac{61}{18} + \frac{19}{18} \sqrt{13} \simeq 7.1947 \ldots. \quad (3.5)$$

We shall see that the graph of $z_s$ plays the role of separation line between initial data leading to the slow dynamics and those leading to the fast dynamics.

Finally, let

$$r_c(\theta)$$

be the solution of (3.1) with Cauchy condition $r_c(\theta_c) = z_c$. \quad (3.6)

**Lemma 3.2.** We have $r_c(\theta_c) = z_s(\theta_c)$ and $r_c(\theta) < z_s(\theta)$ for every $\theta \in [\frac{\pi}{2}, \theta_c) \cup (\theta_c, \pi]$.

**Proof.** By direct computation for every $\theta \in (\frac{\pi}{2}, \pi)$ we obtain

$$z_s'(\theta) = -\frac{(1 + 3 \cos \theta) \sin \theta}{(1 + \cos \theta)^3 \cos^2 \theta},$$

$$z_s(\theta) z_s(\theta) (1 + \cos \theta) \sin \theta = -\frac{\sin \theta}{(1 + \cos \theta)^3 \cos \theta (1 - \cos \theta)},$$

so that in the interval $(\frac{\pi}{2}, \pi)$ the inequality

$$z_s'(\theta) > z_s(\theta) \frac{(1 + \cos \theta) \sin \theta}{z_s(\theta) (1 + \cos \theta)^2 + 1} = -\frac{\sin \theta}{(1 + \cos \theta)^3 \cos \theta (1 - \cos \theta)} \quad (3.7)$$

is equivalent to

$$1 + \cos \theta - 3 \cos^2 \theta > 0.$$
Therefore (3.7) is satisfied $\theta < \theta_c$, and the opposite inequality holds for $\theta > \theta_c$. Since $r_c(\theta_c) = z_s(\theta_c)$ by (3.6), the inequality $r_c(\theta) < z_s(\theta)$ for $\theta \neq \theta_c$ follows from a comparison argument.

**Lemma 3.3.** Assume that

$$\theta_c < \theta_0 \leq \pi \quad \text{and} \quad r_c(\theta_0) \leq z_0 < z_s(\theta_0).$$

Let $r_0(\theta)$ be the solution of (3.1) with Cauchy condition $r_0(\theta_0) = z_0$. Then there exists $\theta_1 \in [\theta_c, \theta_0)$ such that

$$r_0(\theta_1) = z_s(\theta_1) \quad \text{and} \quad r_0(\theta) < z_s(\theta) \quad \text{for} \quad \theta \in (\theta_1, \theta_0].$$ (3.9)

If $z_0 > r_c(\theta_0)$, then $\theta_1 > \theta_c$; if $z_0 = r_c(\theta_0)$, then $\theta_1 = \theta_c$.

**Proof.** Since $r_0(\theta_0) = z_0 \geq r_c(\theta_0)$, by comparison we have $r_0(\theta) \geq r_c(\theta)$ for every $\theta \in (0, \pi)$. In particular $r_0(\theta_c) \geq r_c(\theta_c) = z_s(\theta_c)$ and $r_0(\theta_0) = z_0 < z_s(\theta_0)$. Then (3.9) is satisfied by the greatest point $\theta_1$ of $[\theta_c, \theta_0)$ such that $r_0(\theta_1) = z_s(\theta_1)$. If $z_0 > r_c(\theta_0)$, then $r_0(\theta) > r_c(\theta)$ by comparison, and $\theta_1 > \theta_c$ by Lemma 3.2. If $z_0 = r_c(\theta_0)$, then $r_0(\theta) = r_c(\theta)$ by uniqueness, and $\theta_1 = \theta_c$ by Lemma 3.2.

**Lemma 3.4.** Assume one of the following conditions:

$$\frac{\pi}{2} < \theta_2 \leq \theta_c \quad \text{and} \quad z_2 \leq z_s(\theta_2),$$ (3.10)

$$\theta_c < \theta_2 < \pi \quad \text{and} \quad z_2 < r_c(\theta_2).$$ (3.11)

Let $r_2(\theta)$ be the solution of (3.1) with Cauchy condition $r_2(\theta_2) = z_2$. Then

$$r_2(\theta) < z_s(\theta) \quad \text{for} \quad \theta \in \left(\frac{\pi}{2}, \theta_2\right).$$ (3.12)

**Proof.** Assume (3.10). Then (3.7) holds for every $\theta \in \left(\frac{\pi}{2}, \theta_2\right)$ and $r_2(\theta_2) = z_2 \leq z_s(\theta_2)$, so that (3.12) follows from a comparison argument.

Assume (3.11). Since $r_2(\theta_2) = z_2 < r_c(\theta_2)$, by uniqueness we have $r(\theta) < r_c(\theta)$ for every $\theta \in \mathbb{R}$. In particular we have $r_2(\theta) < r_c(\theta) \leq z_s(\theta)$ for every $\theta \in \left(\frac{\pi}{2}, \theta_2\right)$.

**3.2. The system of the slow dynamics**

In this subsection, we study the system

$$\begin{cases} \\
\dot{\rho}^{sl}(t) = \frac{\rho^{sl}(t)(1 + \cos \theta^{sl}(t)) \cos \theta^{sl}(t) \sin \theta^{sl}(t)}{\rho^{sl}(t)(1 + \cos \theta^{sl}(t))^2 \cos \theta^{sl}(t) + 1}, \\
\dot{\theta}^{sl}(t) = \frac{\rho^{sl}(t)(1 + \cos \theta^{sl}(t))^2 \cos \theta^{sl}(t) + \cos \theta^{sl}(t)}{\rho^{sl}(t)(\rho^{sl}(t)(1 + \cos \theta^{sl}(t))^2 \cos \theta^{sl}(t) + 1)}, \\
\end{cases}$$ (3.13)

that will be satisfied during the slow dynamics. Let $\theta_c$ and $z_s$ be the constants defined in (3.3) and (3.6), and let $z_s(\theta)$ and $r_c(\theta)$ be the functions defined in (3.4) and (3.6).
Lemma 3.5. Assume \( 0 \leq \theta_0 < \frac{\pi}{2} \) and let \((\rho^s_0, \theta^s_0)\) be the solution of (3.13) with Cauchy conditions

\[
\rho^s_0(t_0) = z_0 \quad \text{and} \quad \theta^s_0(t_0) = \theta_0.
\]  
(3.14)

Then \((\rho^s_0, \theta^s_0)\) is defined on \([t_0, +\infty)\) and

\[
\dot{\rho}^s(t) > 0 \quad \text{and} \quad \dot{\theta}^s(t) > 0 \quad \text{for} \quad t \in (t_0, +\infty),
\]  
(3.15)

\[
\lim_{t \to +\infty} \rho^s(t) < +\infty \quad \text{and} \quad \lim_{t \to +\infty} \theta^s(t) = \frac{\pi}{2}.
\]  
(3.16)

Proof. Let \(r_0(\theta)\) be the solution of (3.1) with Cauchy condition \(r_0(\theta_0) = z_0\), which is defined for every \(\theta \in [0, \pi)\) by Lemma 3.1. Let us consider the solution \(\theta_s(t)\) of the autonomous equation

\[
\dot{\theta}_s(t) = \frac{r_0(\theta_s(t))(1 + \cos \theta_s(t))^2 \cos \theta_s(t) + \cos \theta_s(t)}{r_0(\theta_s(t))(1 + \cos \theta_s(t))^2 \cos \theta_s(t) + 1}
\]  
(3.17)

with Cauchy condition \(\theta_s(t_0) = \theta_0\). We observe that the right-hand side of (3.17) is positive on \([\theta_0, \frac{\pi}{2}]\) and vanishes for \(\theta = \frac{\pi}{2}\). Then the theory of autonomous equations guarantees that \(\theta_s(t)\) is defined for every \(t \in [t_0, +\infty)\), \(\dot{\theta}_s(t) > 0\) for every \(t \in [t_0, +\infty)\), and \(\theta_s(t) \to \frac{\pi}{2}\) as \(t \to +\infty\).

Let \(\rho_s(t) := r_0(\theta_s(t))\) for every \(t \in [t_0, +\infty)\). Then \((\rho_s(t), \theta_s(t))\) is a solution of (3.13) defined on \([t_0, +\infty)\). Since it satisfies the Cauchy conditions (3.14), by uniqueness we have \((\rho^s_0(t), \theta^s_0(t)) = (\rho_s(t), \theta_s(t))\) for every \(t \in [t_0, +\infty)\). This implies that \(\dot{\rho}^s(t) > 0\) for every \(t \in [t_0, +\infty)\), and that \(\theta^s(t) \to \frac{\pi}{2}\) and \(\rho^s(t) \to r_0(\frac{\pi}{2}) < +\infty\) as \(t \to +\infty\). Since \(r_0(\theta) > 0\) for every \(\theta \in (0, \pi)\) by Lemma 3.1, we obtain \(\dot{\rho}^s(t) = r'_0(\theta_s(t))\dot{\theta}_s(t) > 0\) for every \(t \in (t_0, +\infty)\). \(\square\)

Lemma 3.6. Assume (3.8) and let \((\rho^s_0, \theta^s_0)\) be the solution of (3.13) with Cauchy conditions (3.14). Then there exist \(t_1 \in (t_0, +\infty)\), \(z_1 \in (0, z_0)\) and \(\theta_1 \in [\theta_c, \theta_0)\), such that \((\rho^s_0, \theta^s_0)\) is defined on \([t_0, t_1]\) and

\[
\lim_{t \to t_1} \rho^s_0(t) = z_1, \quad \lim_{t \to t_1} \theta^s_0(t) = \theta_1, \quad z_1 = z_s(\theta_1),
\]  
(3.18)

\[
\lim_{t \to t_1} \dot{\rho}^s_0(t) = -\infty, \quad \lim_{t \to t_1} \dot{\theta}^s_0(t) = -\infty,
\]  
(3.19)

\[
\dot{\rho}^s_0(t) < 0 \quad \text{and} \quad \dot{\theta}^s_0(t) < 0 \quad \text{for} \quad t \in [t_0, t_1),
\]  
(3.20)

\[
\rho^s_0(t) < z_s(\theta^s_0(t)) \quad \text{for every} \quad t \in [t_0, t_1).
\]  
(3.21)

If \(z_0 > r_c(\theta_0)\), then \(\theta_1 > \theta_c\); if \(z_0 = r_c(\theta_0)\), then \(\theta_1 = \theta_c\) and \(z_1 = z_c\).

Proof. Let \(r_0(\theta)\) and \(\theta_1\) be as in Lemma 3.3, and let \(z_1 := z_s(\theta_1)\). Let us consider the solution \(\theta_s(t)\) of the autonomous equation (3.17) with Cauchy condition \(\theta_s(t_0) = \theta_0\). By (3.9) the right-hand side of (3.17) is negative on \((\theta_1, \theta_0)\) and tends to \(-\infty\) for
\[ \theta \rightarrow \theta_1. \] Then the theory of autonomous equations guarantees that there exists \( t_1 > t_0 \) such that \( \theta_1(t) \) is defined for every \( t \in [t_0, t_1) \), \( \dot{\theta}_1(t) < 0 \) for every \( t \in [t_0, t_1) \), and \( \theta_1(t) \rightarrow \theta_1 \) as \( t \rightarrow t_1 \).

Let \( \rho_2(t) := r_0(\theta_2(t)) \) for every \( t \in [t_0, t_1) \). Then \( (\rho_2(t), \theta_2(t)) \) is a solution of (3.13) defined on \([t_0, t_1)\). Since it satisfies the Cauchy conditions (3.14), by uniqueness we have \( (\rho_2(t), \theta_2(t)) = (\rho_0(t), \theta_0(t)) \) for every \( t \in [t_0, t_1) \). This implies that \( \dot{\theta}_0(t) > 0 \) for every \( t \in [t_0, t_1) \), and that \( \theta_0(t) \rightarrow \theta_1 \) and \( \rho_0(t) \rightarrow r_0(\theta_1) = z_1 \) as \( t \rightarrow t_1 \), where the last equality follows from (3.9) and from the definition of \( z_1 \). Since \( r_0'(\theta) > 0 \) for every \( \theta \in (0, \pi) \) by Lemma 3.1, we obtain \( \dot{\rho}_0(t) = r'_0(\theta(t))\dot{\theta}(t) < 0 \) for every \( t \in (t_0, t_1) \). Inequality (3.21) follows from (3.9).

Finally, Lemma 3.3 guarantees that, if \( z_0 > r_c(\theta_0) \), then \( \theta_1 > \theta_c \), and if \( z_0 = r_c(\theta_0) \), then \( \theta_1 = \theta_c \), and hence \( z_1 = z_s(\theta_c) = z_c \).

**Lemma 3.7.** Assume (3.10) or (3.11), let \( t_1 \geq t_0 \), and let \( t^*_1 \rightarrow t_1 \). Then there exists a unique solution \((\rho_2^*, \theta_2^*)\) of (3.13) defined on \((t_1, +\infty)\) such that

\[ \rho_2^*(t^*_1) \rightarrow z_2 \quad \text{and} \quad \theta_2^*(t^*_1) \rightarrow \theta_2. \] (3.22)

Moreover,

\[ \lim_{t \rightarrow t_1} \rho_2^*(t) = z_2 \quad \text{and} \quad \lim_{t \rightarrow t_1} \theta_2^*(t) = \theta_2, \] (3.23)

\[ \dot{\rho}_2^*(t) < 0 \quad \text{and} \quad \dot{\theta}_2^*(t) < 0 \quad \text{for} \quad t \in (t_1, +\infty), \] (3.24)

\[ \lim_{t \rightarrow +\infty} \rho_2^*(t) > 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \theta_2^*(t) = \frac{\pi}{2}, \] (3.25)

\[ \rho_2^*(t) < z_s(\theta_2^*(t)) \quad \text{for every} \quad t \in (t_1, +\infty). \] (3.26)

**Proof.** Let \( r_2(\theta) \) be as in Lemma 3.4. By (3.12) we have

\[ r_2(\theta)(1 + \cos \theta)^2 \cos \theta + 1 > 0 \quad \text{for every} \quad \theta \in \left[ \frac{\pi}{2}, \theta_2 \right). \] (3.27)

Let us consider the autonomous equation

\[ \dot{\theta}_2(t) = \frac{r_2(\theta_2(t))(1 + \cos \theta_2(t))^2 \cos \theta_2(t) + \cos \theta_2(t)}{r_2(\theta_2(t))(r_2(\theta_2(t))(1 + \cos \theta_2(t))^2 \cos \theta_2(t) + 1)}. \] (3.28)

Since the right-hand side of this equation is negative on \( (\frac{\pi}{2}, \pi) \) and vanishes at \( \frac{\pi}{2} \), the theory of autonomous equations guarantees that there exists a unique solution \( \theta_2(t) \) of (3.28) defined for every \( t \in (t_1, +\infty) \) and such that \( \theta_2(t) \rightarrow \theta_2 \) as \( t \rightarrow t_1 \). Moreover, \( \theta_2(t) \) is defined for every \( t \in (t_1, +\infty) \), \( \dot{\theta}_2(t) < 0 \), \( \frac{\pi}{2} < \theta_2(t) < \pi \), and \( \theta_2(t) \rightarrow \frac{\pi}{2} \) as \( t \rightarrow +\infty \). Let \( \rho_2(t) := r_2(\theta_2(t)) \) for every \( t \in (t_1, +\infty) \). From (3.1) and (3.28) it follows that \( (\rho_2(t), \theta_2(t)) \) is a solution of (3.13) defined on \((t_1, +\infty)\) and satisfies (3.23). Moreover, \( \dot{\rho}_2(t) > 0 \) for every \( t \in (t_1, +\infty) \) and \( \rho_2(t) = r_2(\theta_2(t)) \rightarrow r_2(\frac{\pi}{2}) > 0 \) as \( t \rightarrow +\infty \). Since \( \frac{\pi}{2} < \theta_2(t) < \theta_2 \) and \( \rho_2(t) := r_2(\theta_2(t)) \), by (3.27) we have \( \rho_2(t) (1 + \cos \theta_2(t))^2 \cos \theta_2(t) + 1 > 0 \) for every \( t > t_1 \), which proves (3.26).
To prove the uniqueness, let \((\rho^s(t), \theta^s(t))\) be a solution of (3.13) satisfying (3.22). By uniqueness we have \(\theta^s(t) \neq \frac{\pi}{2}\) for every \(t\). As \(\rho^s(t)(1 + \cos \theta^s(t))^2 \cos \theta^s(t) + 1 > 0\) and \(\cos \theta^s(t) < 0\) for \(t\) near \(t_1\), we deduce from the second equation in (3.13) that \(\frac{\pi}{2} < \theta^s(t) < \theta_2\) and \(\dot{\theta}^s(t) < 0\) for every \(t \in (t_1, +\infty)\). It follows that there exists \(r(\theta)\) such that \(\rho^s(t) = r(\theta^s(t))\) for every \(t \in (t_1, +\infty)\) and that \(r(\theta)\) satisfies (3.1). Since \(r(\theta^s(t_1)) \rightarrow z_2\) by (3.22), we conclude that \(r(\theta) = r_2(\theta)\) in a left neighborhood of \(\theta_2\). This implies that \(\theta^s(t)\) satisfies (3.28). By (3.22), \(\theta^s\) and \(\theta_2\) satisfy the same Cauchy condition at \(t_1\), therefore \(\theta^s = \theta_2\) in a right neighborhood of \(t_1\). Since \(\rho^s(t) = r(\theta^s(t))\), \(r(\theta) = r_2(\theta)\), and \(\rho_2(t) := r_2(\theta^s(t))\), we conclude that \(\rho^s(t) = \rho_2(t)\) in a right neighborhood of \(t_1\). The equality is extended to all \(t \in (t_1, +\infty)\) by uniqueness.

\[ \Box \]

4. Behavior Near the Separation Line

In this section, we prove two technical lemmas which describe the behavior of the solutions of the system near the points \((z_s(\theta), \theta, z_s(\theta))\), \(\frac{\pi}{2} < \theta \leq \theta_c\), which correspond to the separation line \(z = z_s(\theta)\) defined by (3.4).

4.1. Behavior near the critical point

In this subsection, we study the behavior of the system (2.21) near the point \((\theta_c, z_c, z_c)\), where \(\theta_c\) and \(z_c\) are the constants defined in (3.3) and (3.5). Let \(w_\epsilon(t)\) be the function defined in (2.26).

Lemma 4.1. Let \(\kappa \geq 1\), let \(t_1 \in [t_0, +\infty)\), and let \(\tau_\epsilon\) be a sequence in \([t_0, +\infty)\). Assume that

\[ |\tau_\epsilon - t_1| \leq \delta, \]

\[ |\rho_\epsilon(\tau_\epsilon) - z_c| + |\theta_\epsilon(\tau_\epsilon) - \theta_c| + |z_\epsilon(\tau_\epsilon) - z_c| \leq \delta, \]

for \(\epsilon\) small enough. Then there exist three constants \(\beta_1 > 0, \beta_2 > 0, \) and \(\delta_0 \in (0, 1)\), a sequence \(\epsilon_\delta\) in \((0, +\infty)\), defined for \(\delta \in (0, \delta_0)\), and a double sequence \(\tau_\epsilon^\delta\) in \([t_0, +\infty)\), defined for \(\delta \in (0, \delta_0)\) and \(\epsilon \in (0, \epsilon_\delta)\), such that

\[ t_1 - \delta \leq \tau_\epsilon^\delta \leq t_1 + \beta_1 \delta, \]

\[ w_\epsilon(\tau_\epsilon^\delta) \geq 0, \]

\[ \theta_\epsilon(\tau_\epsilon^\delta) \leq \theta_c - \kappa \delta, \]

\[ \sup_{\tau_\epsilon \leq t \leq \tau_\epsilon^\delta} (|\rho_\epsilon(t) - z_c| + |\theta_\epsilon(t) - \theta_c| + |z_\epsilon(t) - z_c|) \leq \beta_2 \sqrt{\delta}, \]

for every \(\delta \in (0, \delta_0)\) and every \(\epsilon \in (0, \epsilon_\delta)\).
Proof. We begin by observing that $1 + \cos \theta_c - 3 \cos^2 \theta_c = 0$ and $1 + 3 \cos \theta_c < 0$. Let us fix four constants $a_0, b_0, c_0,$ and $d_0$ such that

$$0 < a_0 < (1 + \cos \theta_c)(1 + 3 \cos \theta_c) \cos \theta_c \sin \theta_c < 1,$$

$$0 < b_0 < -(1 + \cos \theta_c) \cos \theta_c \sin \theta_c < 1,$$

$$0 < c_0 < (1 + \cos \theta_c)^3 \cos^2 \theta_c < 1,$$

$$0 < d_0 < -z_c(1 + \cos \theta_c)^3 \cos \theta_c.$$

By continuity there exists $\eta > 0$ such that

$$\eta < \frac{1}{2} \left( \frac{\theta_c - \pi}{2} \right) < \frac{1}{2} < \frac{1}{2} z_c,$$

$$-\rho < z^2(1 + \cos \theta)^3 \cos \theta(1 + \cos \theta - 3 \cos^2 \theta) < \rho,$$

$$a_0 \rho < z(1 + \cos \theta)(1 + 3 \cos \theta) \cos \theta \sin \theta < \rho,$$

$$b_0 \rho < -z(1 + \cos \theta) \cos \theta \sin \theta < \rho,$$

$$c_0 \rho < z(1 + \cos \theta)^3 \cos^2 \theta < \rho,$$

$$d_0 \rho < -z^2(1 + \cos \theta)^3 \cos \theta,$$

for $|\theta - \theta_c| \leq \eta$, $|\rho - z_c| \leq \eta$, and $|z - z_c| \leq \eta$.

Since the result has to be proved only for sufficiently small $\delta$, we may also assume that

$$\delta < \frac{1}{8}, \quad \delta < \eta, \quad 2\delta < \kappa.$$  \hspace{1cm} (4.8)

We define

$$t^\delta := \inf \{ t \in (\tau_0, +\infty) : \theta_c(t) < \theta_c - \kappa \delta \},$$  \hspace{1cm} (4.9)

$$\alpha^\delta := \inf \{ t \in (\tau_0, +\infty) : |r(t) - z_c| + |\theta(t) - \theta_c| + |z(t) - z_c| > \eta \},$$  \hspace{1cm} (4.10)

$$s^\delta := \min \{ t^\delta, \alpha^\delta \}.$$  \hspace{1cm} (4.11)

From (4.7) we obtain that

$$-\rho(t) < z_c(t)^2(1 + \cos \theta(t))^3 \cos \theta(t)(1 + \cos \theta(t) - 3 \cos^2 \theta(t)) < \rho(t),$$  \hspace{1cm} (4.12)

$$a_0 \rho(t) < z_c(t)(1 + \cos \theta(t))(1 + 3 \cos \theta(t)) \cos \theta(t) \sin \theta(t) < \rho(t),$$  \hspace{1cm} (4.13)

$$b_0 \rho(t) < -z_c(t)(1 + \cos \theta(t)) \cos \theta(t) \sin \theta(t) < \rho(t),$$  \hspace{1cm} (4.14)

$$c_0 \rho(t) < z_c(t)(1 + \cos \theta(t))^3 \cos^2 \theta(t) < \rho(t),$$  \hspace{1cm} (4.15)

$$d_0 \rho(t) < -z_c(t)^2(1 + \cos \theta(t))^3 \cos \theta(t) < \rho(t).$$  \hspace{1cm} (4.16)
for every \( t \in [\tau_\delta, \alpha^\delta_{\varepsilon, \eta}] \). Therefore (2.27) and (4.7) give
\[
\varepsilon \dot{w}_\varepsilon(t) \leq -\varepsilon a_0 + (\rho_\varepsilon(t) - z_\varepsilon(t)) + (\rho_\varepsilon(t) - z_\varepsilon(t))^2 \leq -\varepsilon a_0 + 2(\rho_\varepsilon(t) - z_\varepsilon(t)),
\]
(4.17)
\[
\varepsilon \dot{w}_\varepsilon(t) \geq -\varepsilon - (\rho_\varepsilon(t) - z_\varepsilon(t)) + c_0(\rho_\varepsilon(t) - z_\varepsilon(t))^2 \geq -\varepsilon - (\rho_\varepsilon(t) - z_\varepsilon(t)),
\]
(4.18)
for every \( t \in [\tau_\delta, \alpha^\delta_{\varepsilon, \eta}] \).

Using the second equation in (2.21) we deduce from (4.14) that
\[
\varepsilon \dot{z}_\varepsilon(t) \leq -b_0(\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [\tau_\delta, \alpha^\delta_{\varepsilon, \eta}].
\]
From (4.17) and (4.18) we obtain
\[
-1 + \frac{1}{b_0} \dot{\theta}_\varepsilon(t) \leq \dot{w}_\varepsilon(t) \leq -a_0 - \frac{2}{b_0} \dot{\theta}_\varepsilon(t) \quad \text{for every } t \in [\tau_\delta, \alpha^\delta_{\varepsilon, \eta}].
\]
Integrating we get
\[
w_\varepsilon(t) - w_\varepsilon(\tau_\delta) \geq -(t - \tau_\delta) + \frac{1}{b_0}(\theta_\varepsilon(t) - \theta_\varepsilon(\tau_\delta)),
\]
\[
w_\varepsilon(t) - w_\varepsilon(\tau_\delta) \leq -a_0(t - \tau_\delta) - \frac{2}{b_0}(\theta_\varepsilon(t) - \theta_\varepsilon(\tau_\delta)),
\]
(4.19)
for every \( t \in [\tau_\delta, \alpha^\delta_{\varepsilon, \eta}] \).

Since \( z_\varepsilon(1 + \cos \theta_\varepsilon)^2 \cos \theta_\varepsilon + 1 = 0 \), an elementary estimate of the first derivatives leads to the inequality
\[
|z(1 + \cos \theta)^2 \cos \theta + 1| \leq |z - z_\varepsilon| + 8|\theta - \theta_\varepsilon| \quad \text{for } |z - z_\varepsilon| < \frac{1}{2},
\]
so that (4.2) and (4.8) give
\[
|w_\varepsilon(\tau_\delta)| \leq 8\delta
\]
(4.20)
for \( \varepsilon \) small enough. By (2.32), (4.2), and (4.9) we have
\[
\theta_\varepsilon - \kappa \delta \leq \theta_\varepsilon(t) \leq \theta_\varepsilon(\tau_\delta) \leq \theta_\varepsilon + \delta \leq \theta_\varepsilon + \kappa \delta
\]
(4.21)
for every \( t \in [\tau_\delta, t^\delta_\varepsilon] \), so that (4.19) gives
\[
w_\varepsilon(t) \geq -8\delta - (t - \tau_\delta) - \frac{2\kappa}{b_0} \delta,
\]
(4.22)
\[
w_\varepsilon(t) \leq 8\delta - a_0(t - \tau_\delta) + \frac{4\kappa}{b_0} \delta,
\]
(4.23)
for every \( t \in [\tau_\delta, s^\delta_{\varepsilon, \eta}] \).

Let
\[
\hat{\tau}_\delta := \tau_\delta + \kappa_1 \delta, \quad \text{where } \kappa_1 := \frac{9}{a_0} + \frac{4\kappa}{a_0 b_0}.
\]
(4.24)
Let us show that
\[
s^\delta_{\varepsilon, \eta} \leq \hat{\tau}_\delta + 2\delta.
\]
(4.25)
Suppose, by contradiction, that \( \hat{\tau}_\delta + 2\delta < s^\delta_{\varepsilon, \eta} \). Then by (4.23) we have \( w_\varepsilon(t) \leq -\delta \) for every \( t \in [\hat{\tau}_\delta, s^\delta_{\varepsilon, \eta}] \). Hence, (2.25) and (2.31) imply
\[
\varepsilon(\dot{\rho}_\varepsilon(t) - \dot{z}_\varepsilon(t)) \geq \varepsilon \sin \theta_0 + \delta(\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [\hat{\tau}_\delta, s^\delta_{\varepsilon, \eta}].
\]
By comparison with the solution of the equation we obtain
\[
\rho_\varepsilon(t) - z_\varepsilon(t) \geq \frac{\varepsilon}{\delta} \sin \theta_0 \left( \exp \left( \frac{\delta}{\varepsilon} (t - \hat{\tau}_\delta) \right) - 1 \right) \quad \text{for every } t \in [\hat{\tau}_\delta, s_{\varepsilon,\eta}^\delta]. \tag{4.26}
\]
In particular, we have
\[
\rho_\varepsilon(t) - z_\varepsilon(t) \geq \frac{\varepsilon}{\delta} \sin \theta_0 \left( \exp \left( \frac{\delta^2}{\varepsilon} \right) - 1 \right) \quad \text{for every } t \in [\hat{\tau}_\delta + \delta, s_{\varepsilon,\eta}^\delta],
\]
so that (4.18) gives
\[
\dot{w}_\varepsilon(t) \geq -1 + \frac{1}{\delta} \sin \theta_0 \left( \exp \left( \frac{\delta^2}{\varepsilon} \right) - 1 \right) \left[ -1 + c_0 \frac{\varepsilon}{\delta} \sin \theta_0 \left( \exp \left( \frac{\delta^2}{\varepsilon} \right) - 1 \right) \right]
\]
for every \( t \in [\hat{\tau}_\delta + \delta, s_{\varepsilon,\eta}^\delta] \). For small \( \varepsilon \) we have \(-1 + c_0 \frac{\varepsilon}{\delta} \sin \theta_0 (\exp(\frac{\delta^2}{\varepsilon}) - 1) > 1\), hence
\[
\dot{w}_\varepsilon(t) \geq -1 + \sin \theta_0 \left( \exp \left( \frac{\delta^2}{\varepsilon} \right) - 1 \right) \quad \text{for every } t \in [\hat{\tau}_\delta + \delta, s_{\varepsilon,\eta}^\delta].
\]
Integrating, we obtain
\[
w_\varepsilon(t) \geq w_\varepsilon(\hat{\tau}_\delta + \delta) - (t - \hat{\tau}_\delta - \delta) + \sin \theta_0 \left( \exp \left( \frac{\delta^2}{\varepsilon} \right) - 1 \right) (t - \hat{\tau}_\delta - \delta), \tag{4.27}
\]
for every \( t \in [\hat{\tau}_\delta + \delta, s_{\varepsilon,\eta}^\delta] \). By using (4.22) we get \( w_\varepsilon(\hat{\tau}_\delta + \delta) \geq -\kappa_2 \delta \), with \( \kappa_2 := 9 + \kappa_1 + \frac{4\kappa}{b_0} \), so that (4.27) gives
\[
w_\varepsilon(t) \geq -\kappa_2 \delta + \left[ -1 + \sin \theta_0 \left( \exp \left( \frac{\delta^2}{\varepsilon} \right) - 1 \right) \right] (t - \hat{\tau}_\delta - \delta)
\]
for every \( t \in [\hat{\tau}_\delta + \delta, s_{\varepsilon,\eta}^\delta] \). Using (4.23) for \( t = \hat{\tau}_\delta + 2\delta \), we obtain
\[
\sin \theta_0 \left( \exp \left( \frac{\delta^2}{\varepsilon} \right) - 1 \right) \leq 9 + \kappa_2 + \frac{4\kappa}{b_0},
\]
which leads to a contradiction for \( \varepsilon \) small enough. This concludes the proof of (4.25), which, together with (4.24), gives
\[
s_{\varepsilon,\eta}^\delta \leq \tau_\delta + (\kappa_1 + 2)\delta. \tag{4.28}
\]
From (4.21) we have
\[
|\theta_\varepsilon(t) - \theta_\varepsilon| \leq \kappa \delta \quad \text{for every } t \in [\tau_\delta, s_{\varepsilon,\eta}^\delta]. \tag{4.29}
\]
From (4.22), (4.23) and (4.28) if follows that
\[
|w_\varepsilon(t)| \leq \kappa_3 \delta \quad \text{for every } t \in [\tau_\delta, s_{\varepsilon,\eta}^\delta], \tag{4.30}
\]
where \( \kappa_3 := \kappa_1 + 3 + \frac{4\kappa}{b_0} \). Since the function
\[
(\omega, \theta) \mapsto \frac{\omega - 1}{(1 + \cos \theta)^2 \cos \theta}
\]
is Lipschitz continuous in $\mathbb{R} \times [\theta_c - \eta, \theta_c + \eta]$ and takes the value $z_c$ at $(0, \theta_c)$ by the very definition of $z_c$ (see (3.6)), there exists a constant $L \geq 1$ such that

$$|z_c(t) - z_c| \leq L(|w_c(t)| + |\theta_c(t) - \theta_c|) \quad \text{for every } t \in [\tau_{\delta}, \alpha^{\delta, \eta}].$$

(4.31)

By (4.29)–(4.31), we have

$$|z_c(t) - z_c| \leq \kappa_4 \delta \quad \text{for every } t \in [\tau_{\delta}, s^{\delta, \eta}],$$

(4.32)

where $\kappa_4 := L(\kappa_3 + \kappa)$.

By Lemmas 2.6 and 2.8 we have

$$z_c(t) \leq \rho_c(t) \leq \rho_c(\tau_{\delta}) + \varepsilon \quad \text{for every } t \in [\tau_{\delta}, +\infty),$$

so that for $\varepsilon$ small enough (4.2) and (4.32) give

$$z_c - \kappa_4 \delta \leq \rho_c(t) \leq z_c + \delta + \varepsilon \leq z_c + 2\delta \quad \text{for every } t \in [\tau_{\delta}, s^{\delta, \eta}],$$

which implies

$$|\rho_c(t) - z_c| \leq \kappa_4 \delta \quad \text{for every } t \in [\tau_{\delta}, s^{\delta, \eta}].$$

(4.33)

Taking into account (4.10) and (4.11), if

$$\kappa_4 \delta < \eta,$$

(4.34)

from (4.29), (4.32) and (4.33) we obtain $s^{\delta, \eta}_c < \alpha^{\delta, \eta}_c$, hence

$$s^{\delta, \eta}_c = t^\delta_c.$$

(4.35)

Therefore (4.28) yields

$$t^\delta_c \leq \tau_{\delta} + (\kappa_1 + 2)\delta,$$

(4.36)

which implies

$$\theta_c(t^\delta_c) \leq \theta_c - \kappa \delta.$$

(4.37)

By (2.32), we have

$$\frac{\pi}{2} < \theta_c(t) \leq \theta_c - \kappa \delta \quad \text{for every } t \in [t^\delta_c, +\infty).$$

(4.38)

Since the function $\theta \mapsto 1 + \cos \theta - 3 \cos^2 \theta$ is concave on $[\pi/2, \theta_c]$, vanishes at $\theta_c$, and takes the value 1 at $\pi/2$, using the inequality $\theta_c - \pi/2 < \frac{\pi}{2}$ we obtain

$$1 + \cos \theta - 3 \cos^2 \theta \geq 2(\theta_c - \theta) \quad \text{for every } \theta \in \left[\frac{\pi}{2}, \theta_c\right].$$

(4.39)

It follows from (4.38) that

$$1 + \cos \theta_c(t) - 3 \cos^2 \theta_c(t) \geq 2(\theta_c - \theta_c(t)) \geq 2\kappa \delta \quad \text{for every } t \in [t^\delta_c, +\infty).$$

(4.40)

Let us define

$$\tau^\delta_c := \inf\{t \in (t^\delta_c, +\infty) : w_c(t) > 0\},$$

(4.41)

$$s^{\delta, \eta}_c := \min\{\tau^\delta_c, \alpha^{\delta, \eta}_c\}.$$
From (2.27), (4.13), and (4.16) we obtain
\[
\varepsilon \dot{w}_\varepsilon(t) \geq -\varepsilon + 2d_0 \kappa \delta (\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta, \eta}_\varepsilon]. \tag{4.43}
\]
Since \( w_\varepsilon(t) \leq 0 \) for every \( t \in [t^\delta_\varepsilon, \tau^\delta_\varepsilon] \), using (2.25) and (2.32) we get
\[
\dot{\rho}_\varepsilon(t) - \dot{z}_\varepsilon(t) \geq \sin \theta_0 \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta, \eta}_\varepsilon],
\]
therefore
\[
\rho_\varepsilon(t) - z_\varepsilon(t) \geq \rho_\varepsilon(t^\delta_\varepsilon) - z_\varepsilon(t^\delta_\varepsilon) + \sin \theta_0 (t - t^\delta_\varepsilon)
\geq \sin \theta_0 (t - t^\delta_\varepsilon) \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta, \eta}_\varepsilon],
\]
where the last inequality follows from Lemma 2.6. Using (4.43) we obtain
\[
\varepsilon \dot{w}_\varepsilon(t) \geq -\varepsilon + 2d_0 \kappa \sin \theta_0 \delta (t - t^\delta_\varepsilon) \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta, \eta}_\varepsilon],
\]
which gives
\[
w_\varepsilon(t) \geq w_\varepsilon(t^\delta_\varepsilon) - (t - t^\delta_\varepsilon) + \kappa_5 \frac{\delta}{\varepsilon} (t - t^\delta_\varepsilon)^2 \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta, \eta}_\varepsilon],
\]
with \( \kappa_5 := d_0 \kappa \sin \theta_0 \). Using (4.30) we obtain
\[
w_\varepsilon(t) \geq -\kappa_3 \delta - (t - t^\delta_\varepsilon) + \kappa_5 \frac{\delta}{\varepsilon} (t - t^\delta_\varepsilon)^2 \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta, \eta}_\varepsilon],
\]
hence
\[
w_\varepsilon(t) \geq -\kappa_3 \delta - \frac{2}{\kappa_5} \frac{\varepsilon}{\delta} + \frac{2}{\kappa_5} \frac{\delta}{\varepsilon} (t - t^\delta_\varepsilon)^2 \quad \text{for every } t \in [t^\delta_\varepsilon, \sigma^{\delta, \eta}_\varepsilon]. \tag{4.44}
\]
Since \( w_\varepsilon(t) \leq 0 \) for every \( t \in [t^\delta_\varepsilon, \sigma^{\delta, \eta}_\varepsilon] \), this implies
\[
(\sigma^{\delta, \eta}_\varepsilon - t^\delta_\varepsilon)^2 \leq \frac{2\kappa_3}{\kappa_5} \frac{\varepsilon}{\delta} + \frac{4}{\kappa_5^2 \delta^2} \varepsilon^2,
\]
so that
\[
\sigma^{\delta, \eta}_\varepsilon - t^\delta_\varepsilon \leq \delta \tag{4.45}
\]
for \( \varepsilon \) small enough.

Using the second equation in (2.21) we deduce from (4.7) and (4.14) that
\[
\varepsilon \dot{\theta}_\varepsilon(t) \geq -\frac{2}{z_c} \varepsilon - (\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [\tau^\delta_\varepsilon, \alpha^{\delta, \eta}_\varepsilon].
\]
From (2.27), (4.13), (4.16), and (4.40) we obtain
\[
\varepsilon \dot{w}_\varepsilon(t) \geq -\varepsilon + 2d_0 (\theta_\varepsilon - \theta_\varepsilon(t))(\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [t^\delta_\varepsilon, \alpha^{\delta, \eta}_\varepsilon].
\]
As \( |\theta_\varepsilon - \theta_\varepsilon(t)| < \eta \) for every \( t \in [\tau^\delta_\varepsilon, \alpha^{\delta, \eta}_\varepsilon] \), from the last two inequalities we obtain
\[
\dot{w}_\varepsilon(t) \geq -1 - 2d_0 (\theta_\varepsilon - \theta_\varepsilon(t)) \dot{\theta}_\varepsilon(t) - \frac{2d_0}{z_c} \eta \quad \text{for every } t \in [t^\delta_\varepsilon, \alpha^{\delta, \eta}_\varepsilon].
\]
Let $\varphi_\varepsilon(t) := (\theta_\varepsilon - \theta_\varepsilon(t))^2$. The previous inequality gives
\[ w_\varepsilon(t) \geq -a_1 + d_0 \varphi_\varepsilon(t) - \frac{2d_0}{z_\varepsilon} \eta \] for every $t \in [t_\varepsilon^\delta, \alpha_\varepsilon^{\delta,\eta}]$.

so that
\[ w_\varepsilon(t) - w_\varepsilon(t_\varepsilon^\delta) \geq -a_1(t - t_\varepsilon^\delta) + d_0(\varphi_\varepsilon(t) - \varphi_\varepsilon(t_\varepsilon^\delta)) - \frac{2d_0}{z_\varepsilon} \eta(t - t_\varepsilon^\delta) \]
for every $t \in [t_\varepsilon^\delta, \alpha_\varepsilon^{\delta,\eta}]$.

Since $w_\varepsilon(t) \leq 0$ for every $t \in [t_\varepsilon^\delta, \tau_\varepsilon^\delta]$, the previous inequality, together with (4.30), (4.35), (4.45) and (4.37) gives
\[ \varphi_\varepsilon(t) \leq \kappa_0 \delta \]
with $\kappa_0^2 := \kappa^2 + \frac{1}{d_0}(\kappa_3 + a_1 + \frac{2d_0}{z_\varepsilon} \eta)$. It follows that
\[ |\theta_\varepsilon(t) - \theta_\varepsilon| \leq \kappa_0 \sqrt{\delta} \]
for every $t \in [t_\varepsilon^\delta, \sigma_\varepsilon^{\delta,\eta}]$. (4.46)

These inequalities, together with (4.8) and (4.31), imply that
\[ |z_\varepsilon(t) - z_\varepsilon| \leq \kappa_\tau \sqrt{\delta} \]
for every $t \in [t_\varepsilon^\delta, \sigma_\varepsilon^{\delta,\eta}]$, (4.48)

where $\kappa_\tau := L(\kappa_3 + \kappa_6 + 1)$.

By Lemmas 2.6 and 2.8 we have
\[ z_\varepsilon(t) \leq \rho_\varepsilon(t) \leq \rho_\varepsilon(\tau_\varepsilon^\delta) + \varepsilon \]
for every $t \in [\tau_\varepsilon^\delta, +\infty)$,

so that for $\varepsilon$ small enough (4.54) and (4.48) give
\[ z_\varepsilon - \kappa_\tau \sqrt{\delta} \leq \rho_\varepsilon(t) \leq z_\varepsilon + \delta + \varepsilon \leq z_\varepsilon + 2\delta \]
for every $t \in [t_\varepsilon^\delta, \sigma_\varepsilon^{\delta,\eta}]$,

which implies
\[ |\rho_\varepsilon(t) - z_\varepsilon| \leq \kappa_\tau \sqrt{\delta} \]
for every $t \in [t_\varepsilon^\delta, \sigma_\varepsilon^{\delta,\eta}]$. (4.49)

There exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ inequalities (4.8) and (4.34) are satisfied and
\[ \kappa_\tau \sqrt{\delta} < \eta. \]

It follows from (4.46), (4.48) and (4.49) that $\sigma_\varepsilon^{\delta,\eta} < \alpha_\varepsilon^{\delta,\eta}$ for $\varepsilon$ small enough, hence
\[ \sigma_\varepsilon^{\delta,\eta} = \tau_\varepsilon^\delta, \]
which implies $w_\varepsilon(\tau_\varepsilon^\delta) \geq 0$. This proves (4.4) for $\varepsilon$ small enough.

Inequality (4.5) follows from (4.38). If $\delta \in (0, \delta_0)$ we have (4.35) and (4.50) for $\varepsilon$ small enough, so that (4.3) follows from (4.28) and (4.45), with $\beta_1 := \kappa_1 + 3$, while
(4.6) follows from (4.29), (4.32), (4.33), (4.46), (4.48) and (4.49), with \( \beta_2 := 3\kappa_4 + 3\kappa_7 \).

\[ \text{Lemma 4.2.} \]

4.2. **Behavior near the left branch of the separation line**

The following lemma will be used to study the behavior of the system when \( \theta_0 < \frac{\pi}{2} \) and \( z_0 = z_s(\theta_0) \), where \( z_s(\theta) \) is the function defined in (3.4). Note that (4.53) is always satisfied when \( \theta_1 < \theta_c \) and \( \delta \) is small.

**Lemma 4.2.** Let \( t_1 \geq t_0, \frac{\pi}{2} < \theta_1 \leq \theta_c \), \( z_1 = z_s(\theta_1) \), \( \kappa_1 > 0 \) and \( \delta_0 \in (0, 1) \). For every \( \delta \in (0, \delta_0) \) let \( \varepsilon_\delta \in (0, +\infty) \) and for every \( \varepsilon \in (0, \varepsilon_\delta) \) let \( \tau_\varepsilon^\delta \in [t_0, +\infty) \). Assume that for every \( \delta \in (0, \delta_0) \) and every \( \varepsilon \in (0, \varepsilon_\delta) \)

\[
|\tau_\varepsilon^\delta - t_1| \leq \delta, \tag{4.51}
\]

\[
w_\varepsilon(\tau_\varepsilon^\delta) \geq 0, \tag{4.52}
\]

\[
\theta_\varepsilon(\tau_\varepsilon^\delta) \leq \theta_c - \kappa_1 \delta, \tag{4.53}
\]

\[
|\rho_\varepsilon(\tau_\varepsilon^\delta) - z_1| + |\theta_\varepsilon(\tau_\varepsilon^\delta) - \theta_1| + |z_\varepsilon(\tau_\varepsilon^\delta) - z_1| \leq \sqrt{\delta}. \tag{4.54}
\]

Then there exist \( \delta_1 > 0 \), a sequence \( \varepsilon_\delta \in (0, +\infty) \), defined for \( \delta \in (0, \delta_1) \), a double sequence \( t_\varepsilon^\delta \in [t_0, +\infty) \), defined for \( \delta \in (0, \delta_1) \) and \( \varepsilon \in (0, \varepsilon_\delta) \), and two constants \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \), such that

\[
t_1 - \delta \leq \tau_\varepsilon^\delta \leq t_\varepsilon^\delta \leq t_1 + 2\delta, \tag{4.55}
\]

\[
w_\varepsilon(t_\varepsilon^\delta) \geq \delta^2, \tag{4.56}
\]

\[
|\rho_\varepsilon(t_\varepsilon^\delta) - z_\varepsilon(\tau_\varepsilon^\delta)| \leq \gamma_1 \frac{1}{\delta^2} \varepsilon, \tag{4.57}
\]

\[
\sup_{\tau_\varepsilon^\delta \leq t \leq t_\varepsilon^\delta} (|\rho_\varepsilon(t) - z_1| + |\theta_\varepsilon(t) - \theta_1| + |z_\varepsilon(t) - z_1|) \leq \gamma_2 \sqrt{\delta}, \tag{4.58}
\]

for every \( \delta \in (0, \delta_1) \) and every \( \varepsilon \in (0, \varepsilon_\delta) \).

**Proof.** Since \( z_1 = z_s(\theta_1) \), we have \( z_1(1 + \cos \theta_1)^2 \cos \theta_1 + 1 = 0 \). An elementary estimate of the first derivatives leads to the inequality \( |z(1 + \cos \theta)^2 \cos \theta + 1| \leq |z - z_1| + 8|\theta - \theta_1| \) for \( |z - z_1| < \frac{1}{2} \), so that (4.54) gives

\[
|w_\varepsilon(\tau_\varepsilon^\delta)| \leq 8\sqrt{\delta} \tag{4.59}
\]

for \( \varepsilon \) small enough. By (2.32) and (4.53) we have

\[
\frac{\pi}{2} < \theta_\varepsilon(t) \leq \theta_c - \kappa_1 \delta \quad \text{for every } t \in [\tau_\varepsilon^\delta, +\infty). \tag{4.60}
\]

It follows from (4.46) that

\[
1 + \cos \theta_\varepsilon(t) - 3 \cos^2 \theta_\varepsilon(t) \geq 2(\theta_c - \theta_\varepsilon(t)) \geq 2\kappa_1 \delta \quad \text{for every } t \in [\tau_\varepsilon^\delta, +\infty). \tag{4.61}
\]
Let us define
\[ d_0 := -\frac{1}{2} z_1 (1 + \cos \theta_1)^3 \cos \theta_1. \] (4.62)

Since
\[ z_1 (1 + \cos \theta_1)^3 \cos \theta_1 (1 + \cos \theta_1 - 3 \cos^2 \theta_1) < z_1, \]
\[ -(1 + \cos \theta_1) \cos \theta_1 \sin \theta_1 < 1, \]
\[ 0 < d_0 < -z_1 (1 + \cos \theta_1)^3 \cos \theta_1, \]
by continuity there exists \( \eta > 0 \) such that
\[ 2\eta < z_1 \quad \text{and} \quad 2\eta < \theta_1 - \frac{\pi}{2}, \] (4.63)
\[ z(1 + \cos \theta)^3 \cos \theta [z(1 + \cos \theta - 3 \cos^2 \theta) - (\rho - z) \cos \theta] < z_1 \rho, \] (4.64)
\[ -z(1 + \cos \theta) \cos \theta \sin \theta < \rho, \] (4.65)
\[ d_0 \rho < -z^2 (1 + \cos \theta)^3 \cos \theta, \] (4.66)
for \( |\theta - \theta_1| \leq \eta, |\rho - z_1| \leq \eta, \) and \( |z - z_1| \leq \eta. \) Moreover,
\[ -2\rho \leq z(1 + \cos \theta)(1 + 3 \cos \theta) \cos \theta \sin \theta \leq 2\rho \] (4.67)
for every \( 0 < z \leq \rho \) and every \( \theta \in [\frac{\pi}{2}, \pi]. \)

We set
\[ \lambda := 3 + 4z_1 \quad \text{and} \quad \gamma_1 := \frac{6}{d_0 \kappa_1}. \] (4.68)

Since the result has to be proved only for sufficiently small \( \delta, \) we may also assume that
\[ \delta < \frac{1}{\lambda} < \frac{1}{3}, \quad \delta < \eta < \frac{1}{2} z_1, \quad \delta < \gamma_1, \quad \delta < \frac{d_0 \kappa_1 \sin \theta_1}{2}. \] (4.69)

For every \( \varepsilon > 0 \) and \( \delta > 0 \) we define
\[ \bar{\tau}_\varepsilon^\delta := \inf \{ t \in (\tau_\varepsilon^\delta, +\infty) : w_\varepsilon(t) > \delta^2 \}, \] (4.70)
\[ \bar{t}_\varepsilon^\delta := \inf \{ t \in (\bar{\tau}_\varepsilon^\delta, +\infty) : w_\varepsilon(t) > w_\varepsilon(\tau_\varepsilon^\delta) + \lambda \delta^2 \}, \] (4.71)
\[ t_\varepsilon^\delta := \inf \left\{ t \in (\bar{\tau}_\varepsilon^\delta, +\infty) : \rho_\varepsilon(t) - z_\varepsilon(t) < \gamma_1 \frac{1}{\delta^2} \varepsilon \right\}, \] (4.72)
\[ \alpha_\varepsilon^\delta \eta := \inf \{ t \in (\tau_\varepsilon^\delta, +\infty) : |\rho_\varepsilon(t) - z_1| + |\theta_\varepsilon(t) - \theta_1| + |z_\varepsilon(t) - z_1| \eta \}, \] (4.73)
\[ \hat{\tau}_\varepsilon^\delta \eta := \min \{ \bar{\tau}_\varepsilon^\delta, \alpha_\varepsilon^\delta \eta \}, \quad \bar{s}_\varepsilon^\delta := \min \{ t_\varepsilon^\delta, \tau_\varepsilon^\delta, \bar{\tau}_\varepsilon^\delta + \delta^2 \}, \quad \hat{s}_\varepsilon^\delta \eta := \min \{ \hat{\tau}_\varepsilon^\delta, \alpha_\varepsilon^\delta \eta \}. \] (4.74)

Since \( z_\varepsilon(t) \leq \rho_\varepsilon(t) \) for every \( t \in [t_0, +\infty) \) by Lemma 2.6, from (4.67) we obtain that
\[ -2\rho_\varepsilon(t) \leq z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t)(1 + 3 \cos \theta_\varepsilon(t)) \cos \theta_\varepsilon(t) \sin \theta_\varepsilon(t) \leq 2\rho_\varepsilon(t) \] (4.75)
for every $t \in [t_0, +\infty)$. By (4.65) and (4.66) we have
\[ -z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t)) \cos \theta_\varepsilon(t) \sin \theta_\varepsilon(t) < \rho_\varepsilon(t), \]  
(4.76)
and
\[ d_0 \rho_\varepsilon(t) < -z_\varepsilon(t)^2(1 + \cos \theta_\varepsilon(t))^3 \cos \theta_\varepsilon(t), \]  
(4.77)
for every $t \in [\tau_\varepsilon^\delta, \alpha_\varepsilon^{\delta, \eta}]$. From (2.27), (4.61), (4.75) and (4.77) we obtain
\[ \varepsilon \dot{w}_\varepsilon(t) \geq -2\varepsilon + 2d_0(\theta_\varepsilon - \theta(t))(\rho_\varepsilon(t) - z_\varepsilon(t)) \geq -2\varepsilon + 2d_0 \kappa_1 \delta (\rho_\varepsilon(t) - z_\varepsilon(t)) \]  
(4.78)
for every $t \in [\tau_\varepsilon^\delta, \alpha_\varepsilon^{\delta, \eta}]$.

Since $w_\varepsilon(t) \leq \delta^2$ for every $t \in [\tau_\varepsilon^\delta, \bar{\tau}_\varepsilon^\delta]$ by (4.70), from (2.25) and (2.32) we obtain that
\[ \varepsilon (\dot{\rho}_\varepsilon(t) - \dot{z}_\varepsilon(t)) \geq \varepsilon \sin \theta_0 - \delta^2 (\rho_\varepsilon(t) - z_\varepsilon(t)) \]  
for every $t \in [\tau_\varepsilon^\delta, \bar{\tau}_\varepsilon^\delta]$.

By comparison, we have
\[ \rho_\varepsilon(t) - z_\varepsilon(t) \geq \varepsilon \frac{\sin \theta_0}{\delta^2} \left[ 1 - \exp \left( -\frac{\delta^2}{\varepsilon} (t - \tau_\varepsilon^\delta) \right) \right] \]  
for every $t \in [\tau_\varepsilon^\delta, \bar{\tau}_\varepsilon^\delta]$,
so that (4.78) gives
\[ \dot{w}_\varepsilon(t) \geq -2 + \frac{2d_0 \kappa_1 \sin \theta_0}{\delta} \left[ 1 - \exp \left( -\frac{\delta^2}{\varepsilon} (t - \tau_\varepsilon^\delta) \right) \right] \]  
for every $t \in [\tau_\varepsilon^\delta, \bar{\tau}_\varepsilon^\delta]$.

By (4.69) we have $-2 + \frac{2d_0 \kappa_1 \sin \theta_0}{\delta} > 2$. Integrating (4.79) and using the definition of $\bar{\tau}_\varepsilon^\delta$ and (4.52) we obtain
\[ \delta^2 \geq w_\varepsilon(t) \geq 2(t - \tau_\varepsilon^\delta) - \frac{2d_0 \kappa_1 \sin \theta_0}{\delta^3} \varepsilon \]  
for every $t \in [\tau_\varepsilon^\delta, \bar{\tau}_\varepsilon^\delta]$.

This inequality implies
\[ \bar{\tau}_\varepsilon^{\delta, \eta} - \tau_\varepsilon^\delta \leq \frac{1}{2} \delta^2 + \frac{d_0 \kappa_1 \sin \theta_0}{\delta^3} \varepsilon \leq \frac{2}{3} \delta^2, \]  
(4.80)
for $\varepsilon$ small enough.

Using the second equation in (2.21), we deduce from (4.69) and (4.76) that
\[ \varepsilon \dot{\theta}_\varepsilon(t) \geq -\frac{2}{z_1} \varepsilon - (\rho_\varepsilon(t) - z_\varepsilon(t)) \]  
for every $t \in [\tau_\varepsilon^\delta, \alpha_\varepsilon^{\delta, \eta}]$.

As $|\theta_\varepsilon - \theta(t)| < \frac{\varepsilon}{2}$ for every $t \in [t_0, +\infty)$ by (2.32), from (4.78) we obtain
\[ \dot{w}_\varepsilon(t) \geq -2 - 2d_0(\theta_\varepsilon - \theta_\varepsilon(t)) \dot{\theta}_\varepsilon(t) - \frac{2d_0 \pi}{z_1} \geq -2 - 2d_0(\theta_1 - \theta_\varepsilon(t)) \dot{\theta}_\varepsilon(t) - \frac{2d_0 \pi}{z_1} \]  
for every $t \in [\tau_\varepsilon^\delta, \alpha_\varepsilon^{\delta, \eta}]$, where the last inequality follows from the inequalities $\dot{\theta}_\varepsilon(t) < 0$ and $\theta_\varepsilon \geq \theta_1$. Let $\varphi_\varepsilon(t) := (\theta_1 - \theta_\varepsilon(t))^2$. The previous inequality gives
\[ \dot{w}_\varepsilon(t) \geq -2 + d_0 \varphi_\varepsilon(t) - \frac{2d_0 \pi}{z_1} \]  
for every $t \in [\tau_\varepsilon^\delta, \alpha_\varepsilon^{\delta, \eta}]$. 
so that 

$$w_\varepsilon(t) - w_\varepsilon(\tau^\delta_\varepsilon) \geq -2(t - \tau^\delta_\varepsilon) + d_0(\varphi_\varepsilon(t) - \varphi_\varepsilon(\tau^\delta_\varepsilon)) - \frac{2d_0 \pi}{z_1} (t - \tau^\delta_\varepsilon)$$

for every $t \in [\tau^\delta_\varepsilon, \alpha^\delta_\varepsilon]$. By (4.71) and (4.74), we have $w_\varepsilon(t) - w_\varepsilon(\tau^\delta_\varepsilon) \leq \lambda \delta^2$ for every $t \in [\tau^\delta_\varepsilon, \tilde{s}^\delta_\varepsilon]$ and $\tilde{s}^\delta_\varepsilon - \tau^\delta_\varepsilon \leq \delta^2$. Therefore the previous inequality, together with (4.54), gives 

$$\varphi_\varepsilon(t) \leq \kappa_2^2 \delta^2 \quad \text{for every } t \in [\tau^\delta_\varepsilon, \tilde{s}^\delta_\varepsilon],$$

with $\kappa_2^2 := 1 + (\lambda^2 + \frac{2\pi}{z_1})$. It follows that 

$$|\theta_\varepsilon(t) - \theta_1| \leq \kappa_2 \delta \quad \text{for every } t \in [\tau^\delta_\varepsilon, \tilde{s}^\delta_\varepsilon].$$

(4.82)

Since the function 

$$(\omega, \theta) \mapsto \frac{\omega - 1}{(1 + \cos \theta)^2 \cos \theta}$$

is Lipschitz continuous in $\mathbb{R} \times [\theta_1 - \eta, \theta_1 + \eta]$ and takes the value $z_1$ at $(0, \theta_1)$ by the hypothesis $z_1 = z_\varepsilon(\theta_1)$, there exists a constant $L \geq 1$ such that 

$$|z_\varepsilon(t) - z_1| \leq L(|w_\varepsilon(t)| + |\theta_\varepsilon(t) - \theta_1|) \quad \text{for every } t \in [\tau^\delta_\varepsilon, \alpha^\delta_\varepsilon].$$

(4.83)

Since for $\varepsilon$ small enough 

$$|w_\varepsilon(t)| \leq 9\sqrt{\delta} \quad \text{for every } t \in [\tau^\delta_\varepsilon, \tilde{s}^\delta_\varepsilon]$$

(4.84)

by (4.59), (4.69), (4.73) and (4.74), from (4.82) we obtain 

$$|z_\varepsilon(t) - z_1| \leq L(9 + \kappa_2)\sqrt{\delta} \quad \text{for every } t \in [\tau^\delta_\varepsilon, \tilde{s}^\delta_\varepsilon].$$

(4.85)

By Lemmas 2.6 and 2.8 we have 

$$z_\varepsilon(t) \leq \rho_\varepsilon(t) \leq \rho_\varepsilon(\tau^\delta_\varepsilon) + \varepsilon \quad \text{for every } t \in [\tau^\delta_\varepsilon, +\infty),$$

so that for $\varepsilon$ small enough (4.54) and (4.85) give 

$$z_1 - L(9 + \kappa_2)\sqrt{\delta} \leq \rho_\varepsilon(t) \leq z_1 + \delta + \varepsilon \leq z_1 + 2\delta \quad \text{for every } t \in [\tau^\delta_\varepsilon, \tilde{s}^\delta_\varepsilon],$$

which implies 

$$|\rho_\varepsilon(t) - z_1| \leq L(9 + \kappa_2)\sqrt{\delta} \quad \text{for every } t \in [\tau^\delta_\varepsilon, \tilde{s}^\delta_\varepsilon].$$

(4.86)

Let $\gamma_2 := \kappa_2 + 18L + 2L\kappa_2$. We choose $\delta_1 \in (0, \delta_0)$ such that for every $\delta \in (0, \delta_1)$ inequalities (4.69) are satisfied and $\gamma_2 \sqrt{\delta} < \eta$. Taking into account (4.73), from (4.83), (4.85), and (4.86) for every $\delta \in (0, \delta_1)$ we obtain $\tilde{s}^\delta_\varepsilon < \alpha^\delta_\varepsilon$ for $\varepsilon$ small enough, hence 

$$\tilde{s}^\delta_\varepsilon = \tilde{s}^\delta_\varepsilon < \alpha^\delta_\varepsilon.$$

(4.87)
By (4.74) and (4.81) we have \( \tilde{\sigma}^{\delta_\eta}_\varepsilon \leq \tilde{s}^{\delta}_\varepsilon < \alpha^{\delta_\eta}_\varepsilon \), hence \( \sigma^{\delta_\eta}_\varepsilon = \tilde{\tau}^{\delta}_\varepsilon \) and

\[
\tilde{\tau}^{\delta}_\varepsilon \leq \tilde{s}^{\delta}_\varepsilon \leq \tau^{\delta}_\varepsilon + \frac{2}{3} \delta^2.
\] (4.88)

By the definition of \( \tilde{\tau}^{\delta}_\varepsilon \) this implies that

\[
w_\varepsilon(\tilde{\tau}^{\delta}_\varepsilon) \geq \delta^2 \quad \text{and} \quad w_\varepsilon(\tau^{\delta}_\varepsilon) \geq w_\varepsilon(\tilde{\tau}^{\delta}_\varepsilon).
\] (4.89)

By (4.69), (4.73) and (4.78) we have \( \dot{w}_\varepsilon(t) \geq \frac{9}{8} \) for every \( t \in [\tilde{\tau}^{\delta}_\varepsilon, \tilde{s}^{\delta}_\varepsilon] \), so that by (4.89)

\[
\lambda \delta^2 \geq w_\varepsilon(t) - w_\varepsilon(\tilde{\tau}^{\delta}_\varepsilon) \geq w_\varepsilon(t) - w_\varepsilon(\tau^{\delta}_\varepsilon) \geq \frac{9}{\delta} (t - \tilde{\tau}^{\delta}_\varepsilon) \quad \text{for every} \quad t \in [\tilde{\tau}^{\delta}_\varepsilon, \tilde{s}^{\delta}_\varepsilon].
\] (4.90)

By (4.69) this implies \( \tilde{s}^{\delta}_\varepsilon - \tilde{\tau}^{\delta}_\varepsilon \leq \frac{1}{\delta} \delta^2 \), which, together with (4.88), gives

\[
\tilde{s}^{\delta}_\varepsilon \leq \tau^{\delta}_\varepsilon + \delta^2.
\] (4.91)

Let us prove that

\[ t^{\delta}_\varepsilon < \tilde{s}^{\delta}_\varepsilon \] (4.92)

for \( \varepsilon \) small enough. We argue by contradiction. If \( t^{\delta}_\varepsilon \geq \tilde{s}^{\delta}_\varepsilon \), then (4.91) and the definition of \( \tilde{s}^{\delta}_\varepsilon \) imply that \( \tilde{s}^{\delta}_\varepsilon = \tilde{t}^{\delta}_\varepsilon \) and \( \tilde{t}^{\delta}_\varepsilon \leq \tilde{\tau}^{\delta}_\varepsilon + \delta^2 \). Recalling (4.53) we obtain

\[
w_\varepsilon(\tilde{t}^{\delta}_\varepsilon) = w_\varepsilon(\tau^{\delta}_\varepsilon) + \lambda \delta^2 \geq \lambda \delta^2.
\]

Let \( \sigma^{\delta}_\varepsilon \) be the last time in \( [\tau^{\delta}_\varepsilon, \tilde{t}^{\delta}_\varepsilon] \) such that \( w_\varepsilon(\sigma^{\delta}_\varepsilon) = \omega^{\delta}_\varepsilon := w_\varepsilon(\tau^{\delta}_\varepsilon) + \delta^2 \) and let \( \dot{\sigma}^{\delta}_\varepsilon \) be the first time in \( [\sigma^{\delta}_\varepsilon, t^{\delta}_\varepsilon] \) such that \( w_\varepsilon(\dot{\sigma}^{\delta}_\varepsilon) = \dot{\omega}^{\delta}_\varepsilon := w_\varepsilon(\tau^{\delta}_\varepsilon) + 2\delta^2 \). Let us prove that for \( \varepsilon \) small enough there exists \( t^{\delta}_\varepsilon \in [\sigma^{\delta}_\varepsilon, \dot{\sigma}^{\delta}_\varepsilon] \) such that

\[
\rho_\varepsilon(t^{\delta}_\varepsilon) - z_\varepsilon(t^{\delta}_\varepsilon) < \sqrt{\varepsilon}.
\] (4.93)

We argue by contradiction. If \( \rho_\varepsilon(t) - z_\varepsilon(t) \geq \sqrt{\varepsilon} \) for every \( t \in [\sigma^{\delta}_\varepsilon, \dot{\sigma}^{\delta}_\varepsilon] \), for \( \varepsilon \) small enough from (4.78) we obtain

\[
\dot{w}_\varepsilon(t) \geq -2 + 2d_0 \kappa_1 \delta \frac{1}{\sqrt{\varepsilon}} \geq d_0 \kappa_1 \delta \frac{1}{\sqrt{\varepsilon}} \quad \text{for every} \quad t \in [\sigma^{\delta}_\varepsilon, \dot{\sigma}^{\delta}_\varepsilon],
\] (4.94)

so that \( w_\varepsilon \) is increasing on \( [\sigma^{\delta}_\varepsilon, \dot{\sigma}^{\delta}_\varepsilon] \). Therefore there exists a function \( u_\varepsilon : [\omega^{\delta}_\varepsilon, \dot{\omega}^{\delta}_\varepsilon] \rightarrow [\sigma^{\delta}_\varepsilon, \dot{\sigma}^{\delta}_\varepsilon] \) such that

\[
\rho_\varepsilon(t) - z_\varepsilon(t) = u_\varepsilon(w_\varepsilon(t)) \quad \text{for every} \quad t \in [\sigma^{\delta}_\varepsilon, \dot{\sigma}^{\delta}_\varepsilon].
\] (4.95)

By (2.25) we have

\[
\varepsilon(\dot{\rho}_\varepsilon(t) - \dot{z}_\varepsilon(t)) \leq \varepsilon - (\rho_\varepsilon(t) - z_\varepsilon(t))w_\varepsilon(t) \quad \text{for every} \quad t \in [\sigma^{\delta}_\varepsilon, \dot{\sigma}^{\delta}_\varepsilon].
\] (4.96)

From (4.94) and (4.96) we obtain

\[
u^{\prime}_\varepsilon(\omega) \leq \frac{1}{d_0 \kappa_1 \delta \sqrt{\varepsilon}} - \frac{1}{d_0 \kappa_1 \sqrt{\varepsilon}} u_\varepsilon(\omega) \omega \quad \text{for every} \quad \omega \in [\omega^{\delta}_\varepsilon, \dot{\omega}^{\delta}_\varepsilon].
\]
By comparison with the solution of the equation we obtain

\[ u_\varepsilon(\omega) \leq (\rho_\varepsilon(\sigma^\varepsilon) - z_\varepsilon(\sigma^\varepsilon)) \exp\left(-\frac{1}{2d_0 \kappa_1} \frac{1}{\sqrt{\varepsilon}} (\omega - \omega^\varepsilon)^2\right) \]

\[ + \frac{1}{d_0 \kappa_1} \sqrt{\varepsilon} \int_0^\omega \exp\left(-\frac{1}{2d_0 \kappa_1} \frac{1}{\sqrt{\varepsilon}} ((\omega - \omega^\varepsilon)^2 - s^2)\right) ds \]

for every \( \omega \in [\omega^\varepsilon, \tilde{\omega}^\varepsilon] \). For \( \omega = \tilde{\omega}^\varepsilon \) we obtain from (4.58) and (4.95)

\[ \rho_\varepsilon(\sigma^\varepsilon) - z_\varepsilon(\sigma^\varepsilon) = u_\varepsilon(\tilde{\omega}^\varepsilon) \leq \gamma_\varepsilon \delta \exp\left(-\frac{\delta^2}{2d_0 \kappa_1} \frac{1}{\sqrt{\varepsilon}}\right) + \frac{1}{d_0 \kappa_1} \sqrt{\varepsilon} \mu_\varepsilon, \]

with \( \mu_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). Since the right-hand side of this inequality is less than \( \sqrt{\varepsilon} \) for \( \varepsilon \) small enough, we have contradicted the assumption \( \rho_\varepsilon(t) - z_\varepsilon(t) \geq \sqrt{\varepsilon} \) for every \( t \in [\sigma^\varepsilon, \tilde{\sigma}^\varepsilon] \). This concludes the proof of (4.93).

As \( w_\varepsilon(t) \geq \delta^2 \) for every \( t \in [\sigma^\varepsilon, \tilde{\sigma}^\varepsilon] \), from (2.25) we obtain

\[ \varepsilon(\dot{\rho}_\varepsilon(t) - \dot{z}_\varepsilon(t)) \leq \varepsilon - \delta^2 (\rho_\varepsilon(t) - z_\varepsilon(t)) \]

for every \( t \in [\sigma^\varepsilon, \tilde{\sigma}^\varepsilon] \).

By comparison with the solution of the equation we get

\[ \rho_\varepsilon(t) - z_\varepsilon(t) \leq \frac{\varepsilon}{\delta^2} + (\rho_\varepsilon(\tilde{t}^\varepsilon) - z_\varepsilon(\tilde{t}^\varepsilon)) \exp\left(-\frac{\delta^2}{\varepsilon} (t - \tilde{t}^\varepsilon)\right) \]

for every \( t \in [\tilde{t}^\varepsilon, \tilde{t}^\varepsilon] \).

By (4.93) we have \( \rho_\varepsilon(\tilde{t}^\varepsilon) - z_\varepsilon(\tilde{t}^\varepsilon) < \sqrt{\varepsilon} \), so that

\[ \rho_\varepsilon(t) - z_\varepsilon(t) < \frac{\varepsilon}{\delta^2} + \sqrt{\varepsilon} \exp\left(-\frac{\delta^2}{\varepsilon} (t - \tilde{t}^\varepsilon)\right) \]

for every \( t \in [\tilde{t}^\varepsilon, \tilde{t}^\varepsilon] \). (4.97)

By (4.64) we have

\[ z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t))^3 \cos \theta_\varepsilon(t)[z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t) - 3 \cos^2 \theta_\varepsilon(t)) - (\rho_\varepsilon(t) - z_\varepsilon(t))] \]

\[ \times \cos \theta_\varepsilon(t) \]

for every \( t \in [\tau^\varepsilon, \alpha^\varepsilon] \), so that (2.27) and (4.75) imply

\[ \varepsilon \dot{\omega}_\varepsilon(t) \leq 2\varepsilon + z_1(\rho_\varepsilon(t) - z_\varepsilon(t)) \]

for every \( t \in [\tau^\varepsilon, \alpha^\varepsilon] \).

By (4.97) this yields

\[ \varepsilon \dot{\omega}_\varepsilon(t) \leq 2\varepsilon + z_1 \frac{\varepsilon}{\delta^2} + z_1 \sqrt{\varepsilon} \exp\left(-\frac{\delta^2}{\varepsilon} (t - \tilde{t}^\varepsilon)\right) \]

for every \( t \in [\tilde{t}^\varepsilon, \tilde{t}^\varepsilon] \).

Since \( w_\varepsilon(\tilde{t}^\varepsilon) \leq w_\varepsilon(\tau^\varepsilon) + 2\delta^2 \), integrating we obtain

\[ w_\varepsilon(t) - w_\varepsilon(\tau^\varepsilon) - 2\delta^2 \leq \left(2 + \frac{z_1}{\delta^2}\right)(t - \tilde{t}^\varepsilon) + \frac{z_1}{\delta^2} \sqrt{\varepsilon} \left[1 - \exp\left(-\frac{\delta^2}{\varepsilon} (t - \tilde{t}^\varepsilon)\right)\right] \]
for every $t \in [\tilde{t}_e^b, \tilde{t}_0^b]$. Taking $t = \tilde{t}_e^b$ we find that
\[(\lambda - 2)\delta^2 \leq \left( 2 + \frac{z_1}{\delta^2} \right)(\tilde{t}_e^b - \tilde{t}_e^b) + \frac{z_1}{\delta^2} \sqrt{\varepsilon} \leq \left( 2 + \frac{z_1}{\delta^2} \right)(\tilde{t}_e^b - \tilde{t}_e^b) + \delta^2\]
for $\varepsilon$ small enough. Since $2\delta^2 < z_1$ by (4.69), the previous inequality gives $(\lambda - 3)\delta^2 \leq \frac{2z_1}{\delta^2}(\tilde{t}_e^b - \tilde{t}_e^b)$ for $\varepsilon$ small enough, hence $\tilde{t}_e^b - \tilde{t}_e^b \geq \frac{\lambda - 3}{2z_1}\delta^4$. If we apply (4.97) with $t = \tilde{t}_e^b$ we obtain
\[\rho_e(\tilde{t}_e^b) - z_e(\tilde{t}_e^b) < \frac{\varepsilon}{\delta} + \sqrt{\varepsilon} \exp\left(-\frac{2\delta^6}{\varepsilon}\right),\]
which gives $\rho_e(\tilde{t}_e^b) - z_e(\tilde{t}_e^b) < \frac{\varepsilon}{\delta}$ for $\varepsilon$ small enough. By the definition of $t_e^b$ this implies $t_e^b < \tilde{t}_e^b$, which violates our hypothesis $t_e^b \geq \tilde{t}_e^b$ and concludes the proof of (4.92).

From (4.89), (4.90) and (4.92), we obtain $w_e(t_e^b) \geq \delta^2$, which proves (4.56). Inequalities (4.55) follow from (4.51), (4.69), (4.91) and (4.92). Inequality (4.57) follows from the definition of $t_e^b$, and (4.58) follows from (4.83), (4.85) and (4.86).

\[\square\]

5. Continuous Evolution

In this section, we consider two cases where the viscosity solution $(\rho, \theta, z)$ is continuous. In the first case $0 \leq \theta_0 < \frac{\pi}{2}$ and the system exhibits a hardening behavior by (2.33). In the second case $\frac{\pi}{2} < \theta_0 \leq \pi$, so that we have a softening behavior by (2.34), and we consider an additional condition on $z_0$ which implies that the viscosity solution $(\rho, \theta, z)$ is continuous. We begin by stating the result in the case of hardening, that will be proved in the next subsection.

**Theorem 5.1.** Assume that $0 \leq \theta_0 < \frac{\pi}{2}$ and let $(\rho_0^{sl}, \theta_0^{sl})$ be defined as in Lemma 3.5. Then
\[\rho(t) = z(t) = \rho_0^{sl}(t) \quad and \quad \theta(t) = \theta_0^{sl}(t) \quad for \ every \ t \in [t_0, +\infty). \quad (5.1)\]
Moreover,
\[\sup_{t_0 \leq t \leq \tau} \left( |\rho_e(t) - \rho(t)| + |\theta_e(t) - \theta(t)| + |z_e(t) - z(t)| \right) \to 0 \quad (5.2)\]
for every $\tau \in (t_0, +\infty)$.

We now state the result in the case of softening with continuous evolution that will be proved in Sec. 5.3. Let $\theta_c$ be the constant defined in (3.3), and let $z_s(\theta)$ and $r_c(\theta)$ be the functions defined in (3.4) and (3.6).

**Theorem 5.2.** Assume one of the following conditions:
\[\frac{\pi}{2} < \theta_0 < \theta_c \quad and \quad z_0 < z_s(\theta_0), \quad (5.3)\]
\[\theta_c < \theta_0 < \pi \quad and \quad z_0 < r_c(\theta_0). \quad (5.4)\]
Let \((\rho^s_2, \theta^s_2)\) be defined as in Lemma 3.7 with \(t_1 = t_0, \theta_2 = \theta_0,\) and \(z_2 = z_0.\) Then
\[
\rho(t) = z(t) = \rho^s_2(t) \quad \text{and} \quad \theta(t) = \theta^s_2(t) \quad \text{for every} \quad t \in [t_0, +\infty).
\] (5.5)

Assume that
\[
\theta_c < \theta_0 < \pi \quad \text{and} \quad z_0 = r_c(\theta_0).
\] (5.6)

Let \((\rho^s_0, \theta^s_0)\) and \(t_1\) be defined as in Lemma 3.6, and let \((\rho^s_2, \theta^s_2)\) be defined as in Lemma 3.7 with \(\theta_2 = \theta_c\) and \(z_2 = z_c.\) Then
\[
\rho(t) = z(t) = \begin{cases} 
\rho^s_0(t) & \text{if} \ t \in [t_0, t_1), \\
z_c & \text{if} \ t = t_1, \\
\rho^s_2(t) & \text{if} \ t \in (t_1, +\infty),
\end{cases} \quad \theta(t) = \begin{cases} 
\theta^s_0(t) & \text{if} \ t \in [t_0, t_1), \\
\theta_c & \text{if} \ t = t_1, \\
\theta^s_2(t) & \text{if} \ t \in (t_1, +\infty).
\end{cases}
\] (5.7)

In both cases we have
\[
\sup_{t_0 \leq t \leq \tau} \left( |\rho_c(t) - \rho(t)| + |\theta_c(t) - \theta(t)| + |z_c(t) - z(t)| \right) \to 0 \quad (5.8)
\]
for every \(\tau \in (t_0, +\infty).\)

In the proof we shall use the following general result on continuous dependence on a parameter, whose proof can be found in Refs. 5 and 6 (see also Ref. 1).

**Theorem 5.3.** Let \(f_\varepsilon\) and \(f_0\) be Carathéodory functions defined on \([a, b] \times \mathbb{R}^m\) with values in \(\mathbb{R}^m,\) let \(t_\varepsilon, t_0 \in [a, b],\) and let \(x_\varepsilon, x_0 \in \mathbb{R}^m.\) Assume that there exist two constants \(L > 0\) and \(M > 0\) such that
\[
|f_\varepsilon(t, x_2) - f_\varepsilon(t, x_1)| \leq L|x_2 - x_1|, \\
|f_\varepsilon(t, x)| \leq M,
\]
for every \(\varepsilon > 0,\) every \(t \in [a, b],\) and every \(x, x_1, x_2 \in \mathbb{R}^m.\) Let \(y_\varepsilon(t)\) and \(y_0(t)\) be the solutions of the Cauchy problems
\[
\begin{cases} 
\dot{y}_\varepsilon(t) = f_\varepsilon(t, y(t)), \\
y_\varepsilon(t_\varepsilon) = x_\varepsilon,
\end{cases} \quad \begin{cases} 
\dot{y}_0(t) = f_\varepsilon(t, y(t)), \\
y_0(t_0) = x_0.
\end{cases}
\]
If \(t_\varepsilon \to t_0, x_\varepsilon \to x_0,\) and for every \(x \in \mathbb{R}^m\)
\[
\int_a^t f_\varepsilon(s, x)ds \to \int_a^t f(s, x)ds \quad \text{uniformly for} \ t \in [a, b],
\]
then \(y_\varepsilon(t) \to y_0(t)\) uniformly for \(t \in [a, b].\)

**5.1. Hardening**

In this subsection we prove Theorem 5.1 about the hardening regime.

**Proof.** By Lemma 2.5 we deduce from (2.25) that
\[
\varepsilon(\dot{\rho}_c(t) - \dot{z}_c(t)) \leq \varepsilon - (\rho_c(t) - z_c(t))^+ \quad \text{for every} \ t \in [t_0, +\infty).
\]
As \( \rho_\varepsilon(t_0) - z_\varepsilon(t_0) = 0 \), by comparison we obtain that
\[
\rho_\varepsilon(t) - z_\varepsilon(t) \leq \varepsilon (1 - e^{-\frac{1}{\varepsilon}(|t-t_0|)}) \leq \varepsilon \quad \text{for every } t \in [t_0, +\infty).
\]  

(5.9)

Let us define
\[
\psi_\varepsilon(t) := \frac{1}{\varepsilon} (\rho_\varepsilon(t) - z_\varepsilon(t)).
\]  

(5.10)

By Lemma 2.6 and (5.9) we have
\[
0 \leq \psi_\varepsilon(t) \leq 1 \quad \text{for every } t \in [t_0, +\infty).
\]  

(5.11)

Passing to a subsequence, we may assume that \( \psi_\varepsilon \rightharpoonup \psi \) weakly in \( L^\infty([t_0, +\infty)) \), with \( 0 \leq \psi \leq 1 \) a.e. on \([t_0, +\infty)\).

From (2.21) we obtain
\[
\begin{aligned}
\hat{\rho}_\varepsilon(t) &= \sin \theta_\varepsilon(t) - \psi_\varepsilon(t)(z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t))\cos^2 \theta_\varepsilon(t) + 1), \\
\hat{\rho}_\varepsilon(t) \hat{\theta}_\varepsilon(t) &= \cos \theta_\varepsilon(t) + \psi_\varepsilon(t) z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t)) \cos \theta_\varepsilon(t) \sin \theta_\varepsilon(t), \\
\dot{z}_\varepsilon(t) &= \psi_\varepsilon(t) z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t)) \cos \theta_\varepsilon(t).
\end{aligned}
\]  

(5.12)

By Lemma 2.6 and (2.33) we have \( \rho_\varepsilon(t) \geq z_\varepsilon(t) \geq z_0 \) for every \([t_0, +\infty)\). Therefore we can apply Theorem 5.3 and we obtain that \( \rho_\varepsilon \to \rho, \theta_\varepsilon \to \theta, \) and \( z_\varepsilon \to z \) uniformly on compact subsets of \([t_0, +\infty)\), where \((\rho, \theta, z)\) is the solution of the Cauchy problem
\[
\begin{aligned}
\dot{\rho}(t) &= \sin \theta(t) - \psi(t)(z(t)(1 + \cos \theta(t)) \cos^2 \theta(t) + 1), \\
\rho(t) \dot{\theta}(t) &= \cos \theta(t) + \psi(t) z(t)(1 + \cos \theta(t)) \cos \theta(t) \sin \theta(t), \\
\dot{z}(t) &= \psi(t) z(t)(1 + \cos \theta(t)) \cos \theta(t), \\
\rho(t_0) &= z_0, \quad \theta(t_0) = \theta_0, \quad z(t_0) = z_0.
\end{aligned}
\]  

(5.13)

By (2.31), passing to the limit we obtain we have hence
\[
0 < \theta_0 \leq \theta(t) \leq \frac{\pi}{2} \quad \text{for every } t \in [t_0, +\infty).
\]  

(5.14)

By Lemma 2.6 and (5.9) \( \rho_\varepsilon - z_\varepsilon \to 0 \) strongly in \( L^\infty([t_0, +\infty)) \), hence \( \rho(t) = z(t) \) for every \( t \in [t_0, +\infty) \). From the first and third equations in (5.13) we obtain
\[
\sin \theta(t) = \psi(t)(z(t)(1 + \cos \theta(t))^2 \cos \theta(t) + 1) \quad \text{for a.e. } t \in [t_0, +\infty).
\]

By (5.14) we have \( \sin \theta(t) > 0 \) for every \( t \in [t_0, +\infty) \), hence
\[
\psi(t) = \frac{\sin \theta(t)}{\rho(t)(1 + \cos \theta(t))^2 \cos \theta(t) + 1} \quad \text{for a.e. } t \in [t_0, +\infty).
\]  

(5.15)

It follows that \((\rho, \theta)\) satisfies the system of the slow dynamics (3.13) in \([t_0, +\infty)\) with initial conditions (3.14), therefore
\[
(\rho(t), \theta(t)) = (\rho_0(t), \theta_0(t)) \quad \text{for every } t \in [t_0, +\infty).
\]

Since the limit does not depend on the subsequence, we obtain (5.2).
5.2. Convergence to the slow dynamics

In this subsection, we prove a general result on the convergence of the solutions of (2.21) to the solutions of the system of the slow dynamics. Let \( z_s(\theta) \) be the function defined in (3.4).

Lemma 5.1. Assume that

\[
t_0 \leq t_s < \tau < +\infty, \quad \frac{\pi}{2} < \theta_s \leq \pi, \quad 0 < z_s < z_s(\theta_s). \tag{5.16}
\]

Let \((\rho^s_t, \theta^s_t)\) be the solution of (3.13) with Cauchy conditions

\[
\rho^s(t_s) = z_s \quad \text{and} \quad \theta^s(t_s) = \theta_s, \tag{5.17}
\]

and let \( t^* \) be a sequence in \([t_0, +\infty)\). Assume that

\[
\rho^s(t) < z_s(\theta^s(t)) \quad \text{for every } t \in [t_s, \tau], \tag{5.18}
\]

\[
t^* \rightarrow t_s, \quad \rho^s(t^*_\epsilon) \rightarrow z_s, \quad \theta^s(t^*_\epsilon) \rightarrow \theta_s, \quad z^s(t^*_\epsilon) \rightarrow z_s, \tag{5.19}
\]

\[
0 \leq \rho^s(t^*_\epsilon) - z_s(t^*_\epsilon) \leq \kappa \epsilon, \tag{5.20}
\]

for some \( \kappa \geq 0 \) independent of \( \epsilon \). Then

\[
\sup_{t^*_\epsilon \leq t \leq \tau} (|\rho^s(t) - \rho^s(t)| + |\theta^s(t) - \theta^s(t)| + |z^s(t) - \rho^s(t)|) \rightarrow 0. \tag{5.21}
\]

Proof. For every \( \alpha \in \mathbb{R} \) and \( \eta > 0 \) we define \( q^\eta_\alpha : \mathbb{R} \rightarrow \mathbb{R} \) as the minimum distance projection into the interval \([\alpha - \eta, \alpha + \eta]\), i.e.

\[
q^\eta_\alpha(\beta) := \begin{cases} 
\alpha - \eta, & \text{if } \beta < \alpha - \eta, \\
\beta, & \text{if } \alpha - \eta \leq \beta \leq \alpha + \eta, \\
\alpha + \eta, & \text{if } \beta > \alpha + \eta. 
\end{cases} \tag{5.22}
\]

Since the inequality in (5.18) is strict, from (3.4) we obtain

\[
\rho^s(t)(1 + \cos \theta^s(t))^2 \cos \theta^s(t) + 1 > 0 \quad \text{for every } t \in [t_s, \tau].
\]

By continuity there exists \( \eta > 0 \) such that

\[
q^\eta_{\rho^s(t)}(\rho)(1 + \cos q^\eta_{\theta^s(t)}(\theta))^2 \cos q^\eta_{\theta^s(t)}(\theta) + 1 \geq \eta \tag{5.23}
\]

for every \( t \in [t_s, \tau], \theta \in \mathbb{R}, \rho \in \mathbb{R} \). Since \((z_s, \frac{\pi}{2})\) is a constant solution of (3.13), we have \( \frac{\pi}{2} < \theta^s(t) < \frac{3}{2} \pi \) for every \( t \in [t_s, \tau] \). Therefore the second equation in (3.13) implies that \( \theta^s(t) < 0 \), hence \( \frac{\pi}{2} < \theta^s(t) \leq \theta_s < \pi \) for every \( t \in [t_s, \tau] \). We deduce that, if \( \eta \) is small enough, we have

\[
\sin q^\eta_{\theta^s(t)}(\theta) \geq \eta \quad \text{for every } t \in [t_s, \tau] \text{ and every } \theta \in \mathbb{R}. \tag{5.24}
\]

Since \( \rho^s(t) > 0 \) for every \( t \in [t_s, \tau] \), we may assume that

\[
\rho^s(t) \geq 2\eta \quad \text{for every } t \in [t_s, \tau]. \tag{5.25}
\]
Finally, we may also assume that
\[ \kappa \eta < 1, \tag{5.26} \]
where \( \kappa \) is the constant in (5.20).

Let us fix \( \eta \) satisfying (5.23)–(5.26), and let \(( \rho^\eta_v(t), \theta^\eta_v(t), z^\eta_v(t) ), t \in [t^*_v, \tau] \), be the solutions of the systems
\[
\begin{align*}
\varepsilon \dot{\rho}^\eta_v(t) &= \varepsilon \sin \tilde{\theta}^\eta_v(t) - (\rho^\eta_v(t) \\
&\quad - z^\eta_v(t))^+(z^\eta_v(t)(1 + \cos \tilde{\theta}^\eta_v(t)) \cos^2 \tilde{\theta}^\eta_v(t) + 1), \\
\varepsilon \max\{\rho^\eta_v(t), \eta\} \dot{\theta}^\eta_v(t) &= \varepsilon \cos \tilde{\theta}^\eta_v(t) + (\rho^\eta_v(t) - z^\eta_v(t)) + \tilde{z}^\eta_v(t) \\
&\quad \times (1 + \cos \tilde{\theta}^\eta_v(t)) \cos \tilde{\theta}^\eta_v(t), \\
\varepsilon \dot{z}^\eta_v(t) &= (\rho^\eta_v(t) - z^\eta_v(t)) + \tilde{z}^\eta_v(t)(1 + \cos \tilde{\theta}^\eta_v(t)) \cos \tilde{\theta}^\eta_v(t),
\end{align*}
\tag{5.27}\
\]
with Cauchy conditions
\[ \rho^\eta_v(t^*_v) = \rho_v(t^*_v), \quad \theta^\eta_v(t^*_v) = \theta_v(t^*_v), \quad z^\eta_v(t^*_v) = z_v(t^*_v), \tag{5.28} \]
where \( \tilde{\theta}^\eta_v(t) := q^{\eta}_{\rho_v(t)}(\theta^\eta_v(t)) \) and \( \tilde{z}^\eta_v(t) := q^{\eta}_{\rho_v(t)}(z^\eta_v(t)) \). By subtracting the third equation from the first one in (5.27) we get
\[
\varepsilon (\dot{\rho}^\eta_v(t) - \dot{z}^\eta_v(t)) = \varepsilon \sin \tilde{\theta}^\eta_v(t) - (\rho^\eta_v(t) - z^\eta_v(t))^+(\tilde{z}^\eta_v(t) \\
&\quad \times (1 + \cos \tilde{\theta}^\eta_v(t)) \cos \tilde{\theta}^\eta_v(t) + 1). \tag{5.29}\
\]

Therefore we deduce from (5.23) that
\[
\varepsilon (\dot{\rho}^\eta_v(t) - \dot{z}^\eta_v(t)) \leq \varepsilon - \eta (\rho^\eta_v(t) - z^\eta_v(t))^+ \quad \text{for every } t \in [t_v, \tau].
\]

As \( 0 \leq \rho^\eta_v(t^*_v) - z^\eta_v(t^*_v) \leq \kappa \varepsilon \) by (5.20) and (5.28), by comparison we obtain that
\[
\rho^\eta_v(t) - z^\eta_v(t) \leq \left( \frac{\kappa \varepsilon - \varepsilon}{\eta} \right) \exp \left( -\frac{\eta}{\varepsilon} (t - t^*_v) \right) + \frac{\varepsilon}{\eta} \leq \frac{\varepsilon}{\eta} \quad \text{for every } t \in [t^*_v, \tau], \tag{5.30}\
\]
where the last inequality follows from (5.26). Let us prove that
\[
\rho^\eta_v(t) - z^\eta_v(t) > 0 \quad \text{for every } t \in [t_v, \tau]. \tag{5.31}\
\]

If not, let \( \tau \) be the first time in \((t_v, \tau)\] such that \( \rho^\eta_v(\tau) - z^\eta_v(\tau) = 0 \). Clearly we have \( \dot{\rho}^\eta_v(\tau) - \dot{z}^\eta_v(\tau) \leq 0 \). By (5.24) and (5.29) we have \( \dot{\rho}^\eta_v(\tau) - \dot{z}^\eta_v(\tau) = \sin \tilde{\theta}^\eta_v(\tau) > 0 \), which contradicts the inequality \( \dot{\rho}^\eta_v(\tau) - \dot{z}^\eta_v(\tau) \leq 0 \) and concludes the proof of (5.31).

Let us define
\[
\psi^\eta_v(t) := \frac{1}{\varepsilon} (\rho^\eta_v(t) - z^\eta_v(t)). \tag{5.32}\
\]

By (5.30) and (5.31) we have
\[
0 \leq \psi^\eta_v(t) \leq \frac{1}{\eta} \quad \text{for every } t \in [t_v, \tau]. \tag{5.33}\
\]
Passing to a subsequence, we may assume that \( \psi^n_\varepsilon \to \psi^n \) weakly* in \( L^\infty([t_*, \tau]) \) as \( \varepsilon \to 0 \), with \( 0 \leq \psi^n \leq \frac{1}{\eta} \) a.e. on \([t_*, \tau] \). From (5.27) we obtain

\[
\begin{align*}
\dot{\rho}^n_\varepsilon(t) &= \sin \tilde{\theta}^n_\varepsilon(t) - \psi^n_\varepsilon(t) \tilde{z}^n_\varepsilon(t) (1 + \cos \tilde{\theta}^n_\varepsilon(t)) \cos^2 \tilde{\theta}^n_\varepsilon(t) + 1), \\
\max\{\rho^n_\varepsilon(t), \eta\} \dot{\theta}^n_\varepsilon(t) &= \cos \tilde{\theta}^n_\varepsilon(t) + \psi^n_\varepsilon(t) \tilde{z}^n_\varepsilon(t) (1 + \cos \tilde{\theta}^n_\varepsilon(t)) \cos \tilde{\theta}^n_\varepsilon(t) \sin \tilde{\theta}^n_\varepsilon(t), \\
\dot{z}^n_\varepsilon(t) &= \psi^n_\varepsilon(t) \tilde{z}^n_\varepsilon(t) (1 + \cos \tilde{\theta}^n_\varepsilon(t)) \cos \tilde{\theta}^n_\varepsilon(t).
\end{align*}
\]

We can regard (5.34) as a sequence of systems whose right-hand sides are given by

\[
\begin{align*}
F^n_\varepsilon(t, \rho, \theta, z) := \sin q^n_{\theta^e(t)}(\theta) - \psi^n_\varepsilon(t) (q^n_{\theta^e(t)}(z)(1 + \cos q^n_{\theta^e(t)}(\theta)) \cos^2 q^n_{\theta^e(t)}(\theta) + 1), \\
G^n_\varepsilon(t, \rho, \theta, z) := \cos q^n_{\theta^e(t)}(\theta) + \psi^n_\varepsilon(t) q^n_{\theta^e(t)}(z) \\
&\quad \times (1 + \cos q^n_{\theta^e(t)}(\theta)) \cos q^n_{\theta^e(t)}(\theta) \sin q^n_{\theta^e(t)}(\theta), \\
H^n_\varepsilon(t, \rho, \theta, z) := \psi^n_\varepsilon(t) q^n_{\theta^e(t)}(z) (1 + \cos q^n_{\theta^e(t)}(\theta)) \cos q^n_{\theta^e(t)}(\theta).
\end{align*}
\]

By Theorem 5.3 we have \( \rho^n_\varepsilon \to \rho^n, \theta^n_\varepsilon \to \theta^n, \) and \( z^n_\varepsilon \to z^n \) uniformly on \([t_*, \tau] \), where \( (\rho^n, \theta^n, z^n) \) is the solution of the system

\[
\begin{align*}
\dot{\rho}^n(t) &= \sin \tilde{\theta}^n(t) - \psi^n(t) \tilde{z}^n(t) (1 + \cos \tilde{\theta}^n(t)) \cos^2 \tilde{\theta}^n(t) + 1), \\
\max\{\rho^n(t), \eta\} \dot{\theta}^n(t) &= \cos \tilde{\theta}^n(t) + \psi^n(t) \tilde{z}^n(t) (1 + \cos \tilde{\theta}^n(t)) \cos \tilde{\theta}^n(t) \sin \tilde{\theta}^n(t), \\
\dot{z}^n(t) &= \psi^n(t) \tilde{z}^n(t) (1 + \cos \tilde{\theta}^n(t)) \cos \tilde{\theta}^n(t),
\end{align*}
\]

with \( \tilde{\theta}^n(t) := q^n_{\theta^e(t)}(\theta^n(t)) \) and \( \tilde{z}^n(t) := q^n_{\theta^e(t)}(z^n(t)) \). Moreover

\[
\rho^n(t_*) = z_*, \quad \theta^n(t_*) = \theta_*, \quad z^n(t_*) = z_*.
\]

By (5.30) and (5.31) \( \rho^n_\varepsilon \to \rho^n \) and \( z^n_\varepsilon \to z^n \) strongly in \( L^\infty([t_*, \tau]) \) as \( \varepsilon \to 0 \), hence \( \rho^n(t) = z^n(t) \) for every \( t \in [t_*, \tau] \). From the first and third equations in (5.35) we obtain

\[
\sin \tilde{\theta}^n(t) = \psi^n(t) (\tilde{\rho}^n(t) (1 + \cos \tilde{\theta}^n(t))^2 \cos \tilde{\theta}^n(t) + 1) \quad \text{for a.e. } t \in [t_*, \tau],
\]

hence

\[
\psi^n(t) = \frac{\sin \tilde{\theta}^n(t)}{\rho^n(t) (1 + \cos \tilde{\theta}^n(t))^2 \cos \tilde{\theta}^n(t) + 1} \quad \text{for a.e. } t \in [t_*, \tau].
\]

It follows that \((\rho^n, \theta^n)\) satisfies the system

\[
\begin{align*}
\dot{\rho}^n(t) &= \tilde{\rho}^n(t) (1 + \cos \tilde{\theta}^n(t)) \cos \tilde{\theta}^n(t) \sin \tilde{\theta}^n(t), \\
\max\{\rho^n(t), \eta\} \dot{\theta}^n(t) &= \tilde{\rho}^n(t) (1 + \cos \tilde{\theta}^n(t))^2 \cos \tilde{\theta}^n(t) + 1, \\
\end{align*}
\]

\[
\begin{align*}
\dot{\rho}^n(t) &= \tilde{\rho}^n(t) (1 + \cos \tilde{\theta}^n(t)) \cos \tilde{\theta}^n(t) \sin \tilde{\theta}^n(t), \\
\max\{\rho^n(t), \eta\} \dot{\theta}^n(t) &= \tilde{\rho}^n(t) (1 + \cos \tilde{\theta}^n(t))^2 \cos \tilde{\theta}^n(t) + 1, \\
\end{align*}
\]

\[
\begin{align*}
\dot{\rho}^n(t) &= \tilde{\rho}^n(t) (1 + \cos \tilde{\theta}^n(t)) \cos \tilde{\theta}^n(t) \sin \tilde{\theta}^n(t), \\
\max\{\rho^n(t), \eta\} \dot{\theta}^n(t) &= \tilde{\rho}^n(t) (1 + \cos \tilde{\theta}^n(t))^2 \cos \tilde{\theta}^n(t) + 1,
\end{align*}
\]
with Cauchy conditions
\[ \rho^0(t_\ast) = z_\ast \quad \text{and} \quad \theta^0(t_\ast) = \theta_\ast. \] (5.37)

By (5.25) we have \( \max\{\rho_{\ast s}^0(t), \eta\} = \rho_{\ast s}^0(t) \) in a neighborhood of \([t_\ast, \tau]\). Moreover, by (5.22) we have \( q_{\ast s}^0(t, \rho_{\ast s}^0(t)) = \rho_{\ast s}^0(t) \) and \( q_{\ast s}^0(t, \theta_{\ast s}^0(t)) = \theta_{\ast s}^0(t) \) in a neighborhood of \([t_\ast, \tau]\). Since \( \rho_{\ast s}^0(t), \theta_{\ast s}^0(t) \) is the solution of (3.13) with Cauchy conditions (5.17), it satisfies also (5.36) with Cauchy conditions (5.37). By uniqueness we have \( \rho^0(t) = \rho_{\ast s}^0(t) \) and \( \theta^0(t) = \theta_{\ast s}^0(t) \) in a neighborhood of \([t_\ast, \tau]\).

Since the limit does not depend on the subsequence, we conclude that \( \rho_{\ast s}^\eta \to \rho_{\ast s}^0 \), \( \theta_{\ast s}^\eta \to \theta_{\ast s}^0 \), and \( z_{\ast s}^\eta \to z_{\ast s}^0 \) uniformly in a neighborhood of \([t_\ast, \tau]\) as \( \varepsilon \to 0 \). Then for \( \varepsilon \) small enough we have \( \tilde{\theta}_{\ast s}^\eta(t) := q_{\rho_{\ast s}^\eta(t)}(\theta_{\ast s}^\eta(t)) = \theta_{\ast s}^\eta(t) \) and \( \tilde{z}_{\ast s}^\eta(t) := q_{\rho_{\ast s}^\eta(t)}(z_{\ast s}^\eta(t)) = z_{\ast s}^\eta(t) \), and, recalling (5.26), \( \max\{\rho_{\ast s}^\eta(t), \eta\} = \rho_{\ast s}^\eta(t) \) in a neighborhood of \([t_\ast, \tau]\). From (5.27) we deduce that \( \rho_{\ast s}^\eta(t), \theta_{\ast s}^\eta(t), z_{\ast s}^\eta(t) \) satisfies (2.21) in a neighborhood of \([t_\ast, \tau]\) for \( \varepsilon \) small enough. Since, by (5.28), \( \rho_{\ast s}^\eta(t), \theta_{\ast s}^\eta(t), z_{\ast s}^\eta(t) \) and \( \rho_\varepsilon, \theta_\varepsilon, z_\varepsilon \) satisfy the same Cauchy condition at \( t_\varepsilon^\ast \), by uniqueness we have that \( \rho_{\ast s}^\eta(t), \theta_{\ast s}^\eta(t), z_{\ast s}^\eta(t) = (\rho_\varepsilon, \theta_\varepsilon, z_\varepsilon) \) on \([t_\ast, \tau]\) for \( \varepsilon \) small enough. It follows that \( \rho_\varepsilon \to \rho_{\ast s}^0 \), \( \theta_\varepsilon \to \theta_{\ast s}^0 \), and \( z_\varepsilon \to z_{\ast s}^0 \) uniformly in a neighborhood of \([t_\ast, \tau]\) as \( \varepsilon \to 0 \). As \( t_\varepsilon^\ast \to t_\ast \), this concludes the proof of (5.21). \( \square \)

### 5.3. Softening with continuous evolution

In this subsection, we prove Theorem 5.2 describing the softening regime with a continuous evolution.

**Proof.** Let us fix \( \tau \in (t_0, +\infty) \). Assume either \( \frac{\pi}{2} < \theta_0 \leq \theta_c \) and \( z_0 < z_s(\theta_0) \), or \( \theta_c < \theta_0 < \pi \) and \( z_0 < r_s(\theta_0) \). Then we can apply Lemma 5.1 with \( t_\ast = t_0, \theta_\ast = \theta_0, z_\ast = z_0, t_\varepsilon^\ast = t_\varepsilon, t_\varepsilon^0 = \tau \ast, \theta_\varepsilon = \theta_\varepsilon, z_\varepsilon = z_\varepsilon, \) and \( \kappa = 0 \), since (5.18) is a consequence of (3.26). Therefore (5.5) and (5.8) follow from (5.21).

Assume \( \frac{\pi}{2} < \theta_0 < \theta_c \) and \( z_0 = z_s(\theta_0) \). To deal with the behavior of the solutions near \( t_0 \) we apply Lemma 4.2 with \( t_1 = t_0, \theta_1 = \theta_0, z_1 = z_0 = z_s(\theta_0), \kappa_1 = 1, \tau_\varepsilon^\delta = t_0 \), and \( 0 < \delta_0 < \theta_c - \theta_0 \). Let \( \delta_1, \gamma_1, \gamma_2 \), and \( t_\varepsilon^\delta \) be the constants and the double sequence given by Lemma 4.2, and let \( \delta_1 \) be a decreasing sequence in \((0, \delta_1)\) converging to 0. For every \( k \) we have

\[ |t_\varepsilon^\delta(t) - t_0| \leq 2\delta_k, \] (5.38)

\[ w_\varepsilon(t_\varepsilon^\delta) \geq \delta_k^2, \] (5.39)

\[ |\rho_\varepsilon(t_\varepsilon^\delta) - z_\varepsilon(t_\varepsilon^\delta)| \leq \gamma_1 \frac{1}{\delta_k^\varepsilon}, \] (5.40)

\[ \sup_{t_0 \leq t \leq t_\varepsilon^\delta} |\rho_\varepsilon(t) - z_0| + |\theta_\varepsilon(t) - \theta_0| \leq \gamma_2 \sqrt{\delta_k}, \] (5.41)
for $\varepsilon$ small enough. Using a diagonal argument and \eqref{5.40}, we may assume that for every $k$ there exist three constants $t^\varepsilon_0$, $\theta^0_0$, and $z^0_0$ such that

$$
t^\varepsilon_k \to t^\varepsilon_0, \quad \rho^\varepsilon(t^\varepsilon_k) \to z^0_0, \quad \theta^\varepsilon(t^\varepsilon_k) \to \theta^0_0, \quad z^\varepsilon(t^\varepsilon_k) \to z^0_0, $$

as $\varepsilon \to 0$ along a suitable sequence independent of $k$. By \eqref{5.38}, \eqref{5.39}, and \eqref{5.41} for every $k$ we have

$$
|t^\varepsilon_k - t_0| \leq 2\delta_k, \tag{5.43}
$$

$$
z^0_k (1 + \cos \theta^0_0)^2 \cos \theta^0_0 + 1 \geq \delta_k^2, \tag{5.44}
$$

$$
|\theta^0_k - \theta_0| + |z^0_k - z_0| \leq \gamma_2 \sqrt{\delta_k}. \tag{5.45}
$$

Inequality \eqref{5.44} implies that $z^0_k < z^k_0$. Let $(\rho^s_{t^\varepsilon_k}, \theta^s_{t^\varepsilon_k})$ be the solution of \eqref{3.13} with Cauchy conditions

$$
\rho^s_{t^\varepsilon_k}(t^\varepsilon_k) = z^0_0 \quad \text{and} \quad \theta^s_{t^\varepsilon_k}(t^\varepsilon_k) = \theta^0_0. \tag{5.46}
$$

By \eqref{3.26} and \eqref{5.44} we have

$$
\rho^s_{t^\varepsilon_k}(t) < z_\varepsilon(\theta^s_{t^\varepsilon_k}(t)) \quad \text{for every } t \in [t^\varepsilon_k, \tau]. \tag{5.47}
$$

We can apply Lemma \ref{lem:5.1} with $t_\varepsilon = t^\varepsilon_k$, $z_\varepsilon = z^0_0$, $\theta_\varepsilon = \theta^0_0$, $(\rho^s_{t^\varepsilon_k}, \theta^s_{t^\varepsilon_k}) = (\rho^s_{t^\varepsilon_k}, \theta^s_{t^\varepsilon_k})$, $t^\varepsilon_0 = t^\varepsilon_k$, and $\kappa = \gamma_1 \frac{1}{\delta_k}$. Indeed, \eqref{5.17} follows from \eqref{5.46}, \eqref{5.18} from \eqref{5.47}, \eqref{5.19} from \eqref{5.42}, and \eqref{5.20} from \eqref{5.40}. We conclude that for every $k$

$$
\sup_{t^\varepsilon_k \leq t \leq \tau} \left( |\rho^\varepsilon(t) - \rho^s_{t^\varepsilon_k}(t)| + |\theta^\varepsilon(t) - \theta^s_{t^\varepsilon_k}(t)| + |z^\varepsilon(t) - \rho^s_{t^\varepsilon_k}(t)| \right) \to 0, \tag{5.48}
$$

as $\varepsilon \to 0$ along a sequence satisfying \eqref{5.42}.

We deduce from \eqref{5.48} that $\rho^s_{t^\varepsilon_k}(t) = \rho^s_{t^\varepsilon_k}(t)$ and $\theta^s_{t^\varepsilon_k}(t) = \theta^s_{t^\varepsilon_k}(t)$ for every $t \in [t^\varepsilon_k, \tau] \cap [t^\varepsilon_k, \tau]$. Let $\tau_0 := \inf_k t^\varepsilon_k$. Then there exists a solution $(\rho^s, \theta^s)$ of \eqref{3.13} in $(\tau_0, \tau)$ such that $\rho^s(t) = \rho^s_{t^\varepsilon_k}(t)$ and $\theta^s(t) = \theta^s_{t^\varepsilon_k}(t)$ for every $t \in [t^\varepsilon_k, \tau]$. Since $t^\varepsilon_k \to t_0$ as $k \to \infty$ by \eqref{5.43}, while $\rho^s(t^\varepsilon_k) \to z_0$ and $\theta^s(t^\varepsilon_k) \to \theta_0$ by \eqref{5.45} and \eqref{5.46}, the uniqueness result proved in Lemma \ref{lem:3.7} implies that $(\rho^s, \theta^s) = (\rho^s_{t^\varepsilon_k}, \theta^s_{t^\varepsilon_k})$ on $(t_0, \tau]$, hence $\rho^s(t) = \rho^s_{t^\varepsilon_k}(t)$ and $\theta^s(t) = \theta^s_{t^\varepsilon_k}(t)$ for every $t \in [t^\varepsilon_k, \tau]$. As the limit does not depend on the sequence satisfying \eqref{5.42}, the limit in \eqref{5.48} holds as $\varepsilon \to 0$.

Since

$$
|\rho^\varepsilon(t) - \rho^s_{t^\varepsilon_k}(t)| + |\theta^\varepsilon(t) - \theta^s_{t^\varepsilon_k}(t)| + |z^\varepsilon(t) - \rho^s_{t^\varepsilon_k}(t)|
\leq |\rho^\varepsilon(t) - z_0| + |\theta^\varepsilon(t) - \theta_0| + |z^\varepsilon(t) - z_0| + 2|z_0 - \rho^s_{t^\varepsilon_k}(t)| + |\theta_0 - \theta^s_{t^\varepsilon_k}(t)|,
$$

it follows from \eqref{3.23} and \eqref{5.41} that there exists a sequence $\omega_k \to 0$ such that

$$
\sup_{t_0 < t \leq t^\varepsilon_k} \left( |\rho^\varepsilon(t) - \rho^s_{t^\varepsilon_k}(t)| + |\theta^\varepsilon(t) - \theta^s_{t^\varepsilon_k}(t)| + |z^\varepsilon(t) - \rho^s_{t^\varepsilon_k}(t)| \right) \leq \omega_k.
$$
By (5.48) we have
\[
\limsup_{\varepsilon \to 0} \sup_{t_0 < t \leq \tau} (|\rho_c(t) - \rho_0^s(t)| + |\theta_c(t) - \theta_0^s(t)| + |z_c(t) - \rho_0^s(t)|) \leq \omega_k,
\]
which gives (5.8) as \( k \to \infty \).

Assume that \( \theta_c < \theta_0 < \pi \) and \( z_0 = r_c(\theta_0) \), and let \((\rho_0^s, \theta_0^s)\) and \( t_1 \) be defined as in Lemma 3.6. Let us fix a decreasing sequence \( \delta_k \to 0 \). Since \( \rho_0^s(t) \to z_c \) and \( \theta_0^s(t) \to \theta_c \) as \( t \to t_1 \) by Lemma 3.6, there exists a sequence \( \tau_{\delta_k} \) such that
\[
t_1 - \delta_k < \tau_{\delta_k} < t_1, \quad |\rho_0^s(\tau_{\delta_k}) - z_c| < \frac{1}{6} \delta_k, \quad |\theta_0^s(\tau_{\delta_k}) - \theta_c| < \frac{1}{6} \delta_k. \tag{5.49}
\]
We can apply Lemma 5.1 with \( t_\varepsilon = t_0, \theta_\varepsilon = \theta_0, z_\varepsilon = z_0, \tau = \tau_{\delta_k}, t_\varepsilon^* = t_0 \), and \( \kappa = 0 \). Indeed, \( z_0 = r_c(\theta_0) < z_s(\theta_0) \) by Lemma 3.2, and (5.18) follows from (3.20). By (5.21) we have
\[
\sup_{t_0 < t \leq \tau_{\delta_k}} (|\rho_c(t) - \rho_0^s(t)| + |\theta_c(t) - \theta_0^s(t)| + |z_c(t) - \rho_0^s(t)|) \leq \frac{1}{2} \delta_k \tag{5.50}
\]
for \( \varepsilon \) small enough. By (5.49) and (5.50) we have also
\[
|\rho_c(\tau_{\delta_k}) - z_c| + |\theta_c(\tau_{\delta_k}) - \theta_c| + |z_c(\tau_{\delta_k}) - z_c| \leq \delta_k
\]
for \( \varepsilon \) small enough. Then we can apply Lemma 4.1 with \( \kappa = 1 \), and we obtain a constant \( \beta \geq 1 \) and, for every \( k \), a sequence \( \tau_{\delta_k} \) in \([t_0, +\infty)\), such that
\[
t_1 - \delta_k \leq \tau_{\delta_k} \leq \tau_{\delta_k}^* \leq t_1 + \beta \delta_k, \tag{5.51}
\]
\[
w_\varepsilon(\tau_{\delta_k}) \geq 0, \tag{5.52}
\]
\[
\theta_\varepsilon(\tau_{\delta_k}) \leq \theta_c - \delta_k, \tag{5.53}
\]
\[
\sup_{\tau_{\delta_k} \leq t \leq \tau_{\delta_k}^*} (|\rho_c(t) - z_c| + |\theta_c(t) - \theta_c| + |z_c(t) - z_c|) \leq \sqrt{\beta} \sqrt{\delta_k}, \tag{5.54}
\]
for \( \varepsilon \) small enough.

We now apply Lemma 4.2 with \( \kappa_1 = \frac{1}{3} \) and obtain two constants \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \), and, for every \( k \), a new sequence \( t_{\delta_k}^* \) in \([t_0, +\infty)\), such that
\[
t_1 - \delta_k \leq \tau_{\delta_k} \leq t_{\delta_k}^* \leq t_1 + 2\beta \delta_k, \tag{5.55}
\]
\[
w_\varepsilon(t_{\delta_k}^*) \geq \frac{1}{\beta^2} \delta_k^2, \tag{5.56}
\]
\[
|\rho_\varepsilon(t_{\delta_k}^*) - z_\varepsilon(t_{\delta_k}^*)| \leq \frac{\gamma_1}{\beta^2} \frac{1}{\delta_k} \varepsilon, \tag{5.57}
\]
\[
\sup_{\tau_{\delta_k} \leq t \leq t_{\delta_k}^*} (|\rho_c(t) - z_c| + |\theta_c(t) - \theta_c| + |z_c(t) - z_c|) \leq \gamma_2 \sqrt{\beta} \sqrt{\delta_k}, \tag{5.58}
\]
for \( \varepsilon \) small enough.
Using a diagonal argument and (5.40), we may assume that for every $k$ there exist three constants $t_1^k$, $\theta_1^k$, and $z_1^k$ such that

$$t_\varepsilon^k \to t_1^k, \quad \rho_\varepsilon(t_\varepsilon^k) \to z_1^k, \quad \theta_\varepsilon(t_\varepsilon^k) \to \theta_1^k, \quad z_\varepsilon(t_\varepsilon^k) \to z_1^k,$$  

(5.59)

as $\varepsilon \to 0$ along a suitable sequence independent of $k$. By (5.55), (5.57), and (5.58) for every $k$ we have

$$|t_1^k - t_1| \leq 2\beta \delta_k,$$  

(5.60)

$$z_1^k(1 + \cos \theta_1^k) \cos \theta_1^k + 1 \geq \frac{1}{\beta^2} \delta_k^2,$$  

(5.61)

$$|\theta_1^k - \theta_c| + |z_1^k - z_c| \leq \gamma_2 \sqrt{\beta} \delta_k.$$  

(5.62)

Inequality (5.44) implies that $z_0^k < z_s(t_0^k)$. Let $(\rho_{s, 0}^k, \theta_{s, 0}^k)$ be the solution of (3.13) with Cauchy conditions

$$\rho_{s, 0}^k(t_1^k) = z_1^k \quad \text{and} \quad \theta_{s, 0}^k(t_1^k) = \theta_1^k.$$  

(5.63)

By (3.26) and (5.44) we have

$$\rho_{s, 0}^k(t) < z_s(\theta_{s, 0}^k(t)) \quad \text{for every} \quad t \in [t_1^k, \tau].$$  

(5.64)

We can apply Lemma 5.1 with $t_k = t_1^k$, $z_k = z_1^k$, $\theta = \theta_1^k$, $t_\varepsilon^k = t_1^k$, $(\rho_{s, s}^k, \theta_{s, s}^k) = (\rho_{s, 0}^k, \theta_{s, 0}^k)$, and $\kappa = \frac{\gamma_1}{2} \frac{1}{\beta^2}$. Indeed, (5.17) follows from (5.63), (5.18) from (5.64), (5.19) from (5.59), and (5.20) from (5.57). We conclude that for every $k$

$$\sup_{t_k^k \leq t \leq \tau} (|\rho_\varepsilon(t) - \rho_{s, 0}^k(t)| + |\theta_\varepsilon(t) - \theta_{s, 0}^k(t)| + |z_\varepsilon(t) - \rho_{s, 0}^k(t)|) \to 0,$$  

(5.65)

as $\varepsilon \to 0$ along a sequence satisfying (5.59)

We deduce from (5.65) that $\rho_{s, 0}^k(t) = \rho_{s, 0}^k(t)$ and $\theta_{s, 0}^k(t) = \theta_{s, 0}^k(t)$ for every $t \in [t_1^k, \tau] \cap [t_1^k, \tau]$. Let $\tau := \inf_k t_1^k$. Then there exists a solution $(\rho_{s, s}^k, \theta_{s, s}^k)$ of (3.13) in $(\tau, \tau]$ such that $\rho_{s, 0}^k(t) = \rho_{s, 0}^k(t)$ and $\theta_{s, 0}^k(t) = \theta_{s, 0}^k(t)$ for every $t \in [t_1^k, \tau]$. Let $(\rho_{s, s}^k, \theta_{s, s}^k)$ be defined as in Lemma 3.7 with $\theta_2 = \theta_c$ and $z_2 = z_c$. Since $t_1^k \to t_1$ as $k \to \infty$ by (5.55), while $\rho_{s, 0}^k(t_1^k) \to z_c$ and $\theta_{s, 0}^k(t_1^k) \to \theta_c$ by (5.62) and (5.63), the uniqueness result proved in Lemma 3.7 implies that $(\rho_{s, 0}^k, \theta_{s, 0}^k) = (\rho_{s, 0}^k, \theta_{s, 0}^k)$ on $(t_1, \tau]$, hence $\rho_{s, 0}^k(t) = \rho_{s, 0}^k(t)$ and $\theta_{s, 0}^k(t) = \theta_{s, 0}^k(t)$ for every $t \in [t_1^k, \tau]$. As the limit does not depend on the sequence satisfying (5.59), the limit in (5.65) holds as $\varepsilon \to 0$.

From (5.50), (5.51), and (5.65) we obtain (5.7), except for $t \in (t_1 - \delta_k, t_1 + 2\beta \delta_k)$.

As $k \to \infty$ we obtain (5.7) on $[t_0, +\infty)$.

Since

$$|\rho_\varepsilon(t) - \rho(t)| + |\theta_\varepsilon(t) - \theta(t)| + |z_\varepsilon(t) - z(t)|$$

$$\leq |\rho_\varepsilon(t) - z_c| + |\theta_\varepsilon(t) - \theta_c| + |z_\varepsilon(t) - z_c| + 2|z_c - \rho(t)| + |\theta_c - \theta(t)|,$$
it follows from (3.18), (3.23), (5.54), and (5.58) that there exists a sequence $\omega_k \to 0$ such that
\[
\sup_{t_k \leq t \leq t_k^*} \left( |\rho_\varepsilon(t) - \rho(t)| + |\theta_\varepsilon(t) - \theta(t)| + |z_\varepsilon(t) - z(t)| \right) \leq \omega_k.
\]
By (5.50) and (5.65) we have
\[
\lim_{\varepsilon \to 0} \sup_{t_0 \leq t \leq t_0^*} \sup_{t_k \leq t \leq t_k^*} \left( |\rho_\varepsilon(t) - \rho(t)| + |\theta_\varepsilon(t) - \theta(t)| + |z_\varepsilon(t) - z(t)| \right) \leq \omega_k,
\]
which gives (5.8) as $k \to \infty$.

Assume that $\theta = \theta_\varepsilon$ and $z_0 = z_\varepsilon$, and let $(\rho_{\varepsilon k}^*, \theta_{\varepsilon k}^*)$ be defined as in Lemma 3.7 with $t_1 = t_0$, $\theta_2 = \theta_\varepsilon$, and $z_2 = z_\varepsilon$. Then we can apply Lemma (4.1) with $\kappa = 1$, $t_1 = \tau^\varepsilon = t_0$. Given a decreasing sequence $\delta_k \to 0$, we obtain a constant $\beta \geq 1$ and, for every $k$, a sequence $\tau_{\varepsilon k}^*$ in $[t_0, +\infty]$ which satisfies (5.51)–(5.54) for $\varepsilon$ small enough. Then the proof can be concluded as in the previous case, replacing $t_1$ by $t_0$.

\[\square\]

6. The Fast Dynamics

In this section, we study in detail the behavior of the solutions to the system of the fast dynamics.

6.1. The trajectory of the fast dynamics

In this subsection, we study the system
\[
\begin{cases}
\vartheta'(z) = -\cos \vartheta(z) - \frac{1}{z(1 + \cos \vartheta(z)) \cos \vartheta(z)}, \\
\vartheta'(z) = \frac{\sin \vartheta(z)}{\vartheta(z)},
\end{cases}
\]
that describes the trajectories followed along the fast dynamics. Using Cartesian coordinates, we consider the functions
\[
x(z) := z + \varrho(z) \cos \vartheta(z) \quad \text{and} \quad y(z) := \varrho(z) \sin \vartheta(z),
\]
and (6.1) is equivalent to
\[
\begin{cases}
x'(z) = -\frac{1}{z(1 + \cos \vartheta(z))}, \\
y'(z) = -\frac{\tan \vartheta(z)}{z(1 + \cos \vartheta(z))},
\end{cases}
\]
where
\[
\cos \vartheta(z) = \frac{x(z) - z}{\sqrt{(x(z) - z)^2 + y(z)^2}} \quad \text{and} \quad \tan \vartheta(z) = \frac{y(z)}{x(z) - z}.
\]
Let us fix $z_1 > 0$ and $\frac{\pi}{2} < \theta_1 < \pi$ and consider the Cauchy conditions
\[
\varrho(z_1) = z_1 \quad \text{and} \quad \vartheta(z_1) = \theta_1,
\]
that in Cartesian coordinates become
\[ x(z_1) = z_1(1 + \cos \theta_1) \quad \text{and} \quad y(z_1) = z_1 \sin \theta_1. \]  
(6.5)

Let \( \theta_c \) be the constant defined in (3.3), and let \( z_s(\theta) \) and \( r_c(\theta) \) be the functions defined in (3.4) and (3.6).

**Lemma 6.1.** Let \( z_1 > 0 \) and \( \frac{\pi}{2} < \theta_1 < \pi \). Assume one of the following two conditions:

\[ z_1 > z_s(\theta_1), \]  
(6.6)

\[ z_1 = z_s(\theta_1) \quad \text{and} \quad \theta_1 > \theta_c. \]  
(6.7)

Then there exists \( z_2 \in (0, z_1) \) such that (6.1) with Cauchy condition (6.4) has a solution \((\varrho, \vartheta)\) defined in \([z_2, z_1]\) such that

\[ \varrho(z_2) = z_2 \quad \text{and} \quad \varrho(z) > z \quad \text{for} \quad z \in (z_2, z_1). \]  
(6.8)

Let \( \theta_2 := \vartheta(z_2) \). Then we have

\[ \frac{\pi}{2} < \theta_2 < \vartheta(z) < \theta_1 < \pi \quad \text{for} \quad z \in (z_2, z_1), \]  
(6.9)

\[ \varrho'(z) > 0 \quad \text{and} \quad \vartheta'(z) > 0 \quad \text{for} \quad z \in (z_2, z_1), \]  
(6.10)

\[ z_2 \geq \frac{27}{4} \Rightarrow \theta_2 < \theta_c, \]  
(6.11)

\[ \varrho'(z_2) > 1 \quad \text{and} \quad z_2 < z_s(\theta_2). \]  
(6.12)

**Proof.** Let us consider the solution \((\varrho, \vartheta)\) of (6.1) with Cauchy condition (6.4) on its maximal left interval of existence \([z_c, z_1]\). By the singularities of the right-hand side we have that \( z, \varrho(z), \cos \vartheta(z), \) and \( 1 + \cos \vartheta(z) \) cannot vanish for \( z \in (z_c, z_1] \) so that \( z_c \geq 0 \) and \( \frac{\pi}{2} < \vartheta(z) < \pi \) for every \( z \in (z_c, z_1] \). Then (6.1) implies (6.10), which gives (6.9).

Let us define

\[ \theta_e := \lim_{z \to z_c} \vartheta(z) = \inf_{z > z_c} \vartheta(z) \geq \frac{\pi}{2}, \]

\[ \rho_e := \lim_{z \to z_c} \varrho(z) = \inf_{z > z_c} \varrho(z) \geq 0. \]

If \( z_c = 0 \), from the first equation in (6.1) we would have \( \varrho'(z) \geq \frac{1}{2z} \) for every \( z \in (0, z_1] \), and this contradicts the fact that the limit \( \rho_e \) is finite. Therefore \( z_c > 0 \).

We now show that \( \rho_e < z_c \). If not, we would have \( \rho_e \geq z_c > 0 \), and hence \( \theta_e > \frac{\pi}{2} \), otherwise the solution could be continued by solving a new Cauchy problem at \( z_c \). By the second equation in (6.1) we have \( \vartheta'(z) \to \frac{1}{\rho_e} \) as \( z \to z_c \). Thus the first equation in (6.1) gives

\[ \varrho'(z) \geq \frac{1}{2z_1|\cos \vartheta(z)|} \geq \frac{\rho_e}{4z_1|z - z_c|}, \]
for \( z \) near \( z_e \), which contradicts again the finiteness of \( \rho_e \). This proves that \( \rho_e < z_e \).

It is convenient to introduce the function \( \omega : [z_e, z_1) \to \mathbb{R} \) defined by

\[
\omega(z) := z(1 + \cos \vartheta(z))^2 \cos \vartheta(z) + 1. \tag{6.13}
\]

It follows from (6.1) that

\[
\varrho'(z) - 1 = -\frac{\omega(z)}{z(1 + \cos \vartheta(z)) \cos \vartheta(z)}. \tag{6.14}
\]

Using (6.1) we obtain

\[
\omega'(z)\varrho(z)(1 + \cos \vartheta(z))^{-2} = \varrho(z) \cos \vartheta(z) - z(1 - \cos \vartheta(z))(1 + 3 \cos \vartheta(z))
\]

\[
= \varrho(z) \cos \vartheta(z) - z(1 + 2 \cos \vartheta(z) - 3 \cos^2 \vartheta(z)). \tag{6.15}
\]

If (6.6) holds, then \( \varrho'(z_1) < 1 \), so that \( \varrho(z) > z \) for all \( z < z_1 \) close to \( z_1 \). If, instead, (6.7) holds, then \( \omega'(z_1) \) has the same sign as \(-1 - \cos \theta_1 + 3 \cos^2 \theta_1\), which is positive by (6.6). Therefore \( \omega'(z_1) > 0 \) and \( \omega(z_1) = 0 \), hence \( \omega(z) < 0 \) for all \( z < z_1 \) close to \( z_1 \).

From (6.14) we deduce that \( \varrho'(z) < 1 \), and hence \( \varrho(z) > z \) for all \( z < z_1 \) close to \( z_1 \).

On the other hand the inequality \( \rho_e < z_e \) gives \( \varrho(z) < z \) for all \( z > z_e \) close to \( z_e \). Therefore there exists the greatest point \( z_2 \) in \((z_e, z_1)\) such that \( \varrho(z_2) = z_2 \). Condition (6.8) is clearly satisfied, and implies

\[
\varrho'(z_2) \geq 1. \tag{6.16}
\]

By (6.6), (6.7), (6.16), and (6.14) we have

\[
\omega(z_1) \leq 0 \quad \text{and} \quad \omega(z_2) \geq 0. \tag{6.17}
\]

Since \( \cos \theta_2 > \cos \theta_1 \) by (6.9) and (6.10), if \( \cos \theta_1 \geq \lambda_c \) we have also \( \cos \theta_2 > \lambda_c \), where \( \lambda_c \) is the constant defined in (3.2). Therefore to prove (6.11) we may assume

\[
\cos \theta_1 \leq \lambda_c \quad \text{and} \quad z_2 \geq \frac{27}{4}, \tag{6.18}
\]

and we want to prove that \( \cos \theta_2 > \lambda_c \). We argue by contradiction, assuming (6.18) and

\[
\cos \theta_2 \leq \lambda_c < -\frac{1}{3}, \tag{6.19}
\]

since \( \omega'(z_2)(1 + \cos \theta_2)^{-2} = -1 - \cos \theta_2 + 3 \cos^2 \theta_2 \) by (6.15), inequality (6.19) gives

\[
\omega'(z_2) \geq 0. \tag{6.20}
\]

As \( \omega'(z_1)(1 + \cos \theta_1)^{-2} = -1 - \cos \theta_1 + 3 \cos^2 \theta_1 \) by (6.15), we have also

\[
\omega'(z_1) \geq 0. \tag{6.21}
\]

By (6.17) there exists a minimum point \( z_m \) of \( \omega \) in \((z_2, z_1)\) and a maximum point \( z_M \) in \([z_2, z_m)\). By (6.21) we have

\[
\omega'(z_m) = 0, \tag{6.22}
\]
and by (6.20) we have
\[ \omega'(z_M) = 0 \quad \text{and} \quad \omega''(z_M) \leq 0. \tag{6.23} \]

We want to prove that
\[ \cos \vartheta(z_m) > -\frac{9}{10}. \tag{6.24} \]

As \( \vartheta \) is increasing by (6.10), this inequality is trivial if \( \cos \theta_1 > -\frac{9}{10} \), so we may assume that
\[ -1 < \cos \theta_1 \leq -\frac{9}{10}. \tag{6.25} \]

To prove (6.24) we argue by contradiction and assume that \( \cos \vartheta(z_m) \leq -\frac{9}{10} \). Let
\[ \eta := 1 + \cos \theta_1, \quad \text{so that} \quad 0 < \eta \leq \frac{19}{10} \quad \text{and} \quad \sin \theta_1 = \sqrt{2 - \eta \sqrt{\eta}}. \]
By (6.3) and (6.9) we have
\[ x'(z) \leq 0 \quad \text{and} \quad y'(z) \geq 0, \]
so that, by (6.2),
\[
\varrho(z) \cos \vartheta(z) \geq z_1 \eta - z \quad \text{and} \quad \varrho(z) \sin \vartheta(z) \leq z_1 \sqrt{2 - \eta \sqrt{\eta}} \\
\text{for every} \ z \in [z_2, z_1].
\tag{6.26}
\]

This implies
\[ z_2 > z_1 \eta \quad \text{and} \quad \varrho(z)^2 \leq (z - z_1 \eta)^2 + z_1^2 (2 - \eta) \eta \leq z^2 + 2 z_1^2 \eta, \]

hence
\[ \frac{\varrho(z)^2}{z^2} \leq 1 + 2 \frac{z_1^2}{z^2} \eta \quad \text{for every} \ z \in [z_2, z_1]. \tag{6.27} \]

Since \( \cos \vartheta(z_m) \leq -\frac{9}{10} \), by (6.15) and (6.22) the polynomial
\[ P_m(\lambda) := \varrho(z_m) \lambda - z_m (1 + 2 \lambda - 3 \lambda^2) \]
has a zero in the interval \((-1, -\frac{9}{10})\). As \( P_m(0) = -z_m < 0 \), this implies \( P_m(-\frac{9}{10}) \leq 0 \), hence \( \varrho(z_m) > \frac{323}{100} z_m > 3 z_m \). By (6.27) we obtain
\[ z_m \leq \frac{4}{7} z_1 \sqrt{\eta}. \]
As \( -1 + \eta = \cos \theta_1 \leq \cos \vartheta(z) \leq \cos \vartheta(z_m) \leq -\frac{9}{10} \)
for every \( z \in [z_m, z_1] \), we have
\[ 0 < \sin \vartheta(z) \leq \frac{1}{2} \quad \text{for every} \ z \in [z_m, z_1]. \]

Since the function \( \lambda \mapsto -\lambda (1 + \lambda) \) is increasing in \([-1, -\frac{1}{2}]\), from (6.6) we obtain
\[ 1 \leq z_1 \eta^2 (1 - \eta) \leq z_1 \eta (1 + \cos \vartheta(z)) \quad \text{and} \quad 1 \leq z_1 \eta^2 (1 - \eta) \leq -z_1 \eta (1 + \cos \vartheta(z)) \]
for every \( z \in [z_m, z_1] \). By (6.3) we have
\[ x'(z) \geq -\frac{z_1}{z} \eta \quad \text{and} \quad y'(z) \leq \frac{1}{2} \frac{z_1}{z} \eta \quad \text{for every} \ z \in [z_m, z_1]. \]

Integrating we obtain
\[ x(z_1) - x(z_1 \sqrt{\eta}) \geq z_1 \eta \log \sqrt{\eta} \quad \text{and} \quad y(z_1) - y(z_1 \sqrt{\eta}) \leq -\frac{1}{2} z_1 \eta \log \sqrt{\eta}. \]

As \( x(z_1) = z_1 \eta \) and \( y(z_1) = z_1 \sqrt{2 - \eta \sqrt{\eta}} \), we deduce from (6.2) that
\[
\varrho(z_1 \sqrt{\eta}) \cos \vartheta(z_1 \sqrt{\eta}) \leq -z_1 (\sqrt{\eta} - \eta + \eta \log \sqrt{\eta}), \\
\varrho(z_1 \sqrt{\eta}) \sin \vartheta(z_1 \sqrt{\eta}) \geq z_1 \left( \sqrt{2 - \eta \sqrt{\eta}} + \frac{1}{2} \eta \log \sqrt{\eta} \right).
\]
As \( \sqrt{\eta} \log \sqrt{\eta} \geq \frac{1}{\sqrt{10}} \log \frac{1}{\sqrt{10}} \geq -\frac{5}{12} \), we have \( \sqrt{\eta} - \eta + \eta \log \sqrt{\eta} > 0 \) and \( \sqrt{2 - \eta} \times \sqrt{\eta} + \frac{1}{2} \eta \log \sqrt{\eta} > 0 \) for every \( \eta \in (0, \frac{1}{10}] \), so that
\[
\vartheta(z_1 \sqrt{\eta})^2 \geq z_1^2 ((\sqrt{\eta} - \eta + \eta \log \sqrt{\eta})^2 + \left( \sqrt{2 - \eta} \sqrt{\sqrt{\eta} + \frac{1}{2} \eta \log \sqrt{\eta}} \right)^2)
\]
\[
\geq z_1^2 \eta ((1 - \sqrt{\eta})^2 + 2 \sqrt{\eta} \log \sqrt{\eta} + 2 - \eta + \sqrt{2 \sqrt{\eta} \log \sqrt{\eta}})
\]
\[
\geq z_1^2 \eta (3 - 2 \sqrt{\eta} + \frac{7}{2} \sqrt{\eta} \log \sqrt{\eta}) \geq z_1^2 \eta \left( \frac{37}{24} - 2 \sqrt{\eta} \right)
\]
\[
\geq z_1^2 \eta \left( \frac{110}{81} - \frac{200}{81} \sqrt{\eta} \right) \geq \frac{100}{81} z_1^2 \eta (1 - \sqrt{\eta})^2 = \frac{100}{81} \frac{2}{9} (\sqrt{\eta} - \eta)^2
\]
for every \( \eta \in (0, \frac{1}{10}] \). Therefore (6.26) implies that \( \cos^2 \vartheta(z_1 \sqrt{\eta}) \leq \frac{81}{100} \). As \( z_m < z_1 \sqrt{\eta} \), we have \( \cos \vartheta(z_m) > \cos \vartheta(z_1 \sqrt{\eta}) \geq -\frac{9}{10} \), which contradicts our assumption \( \cos \vartheta(z_m) \leq -\frac{9}{10} \), and concludes the proof of (6.24).

Let \( \lambda_M := \cos \vartheta(x_M) \). As \( z_2 \leq z_M < z_m \) we have
\[
-\frac{9}{10} < \lambda_M < -\frac{1}{3}. \tag{6.28}
\]
Since \( \omega'(z_M) = 0 \) by (6.23), using (6.1) and the equality
\[
\frac{z_M}{\rho(z_M)} = \frac{\lambda_M}{(1 - \lambda_M)(1 + 3 \lambda_M)}
\]
that can be deduced from (6.15), we obtain
\[
\omega''(z_M) \vartheta(z_M) \frac{1 + 3 \lambda_M}{1 - \lambda_M} = -\frac{1}{z_M} (1 + 3 \lambda_M) - (1 + \lambda_M)(2 + 6 \lambda_M + 7 \lambda^2_M - 3 \lambda^3_M).
\]
By (6.18), (6.23) and (6.28) we have
\[
(1 + \lambda_M)(2 + 6 \lambda_M + 7 \lambda^2_M - 3 \lambda^3_M) + \frac{4}{27} (1 + 3 \lambda_M) \leq 0. \tag{6.29}
\]
Let us consider the polynomial \( P(\lambda) := (1 + \lambda)(2 + 6 \lambda + 7 \lambda^2 - 3 \lambda^3) + \frac{4}{27}(1 + 3 \lambda) = \frac{35}{27} + \frac{70}{9} \lambda + 13 \lambda^2 + 4 \lambda^3 - 3 \lambda^4 \) and its derivative \( P'(\lambda) = \frac{70}{9} + 26 \lambda + 12 \lambda^2 - 12 \lambda^3 = \frac{5}{9}(2 + 3 \lambda)(19 + 30 \lambda - 18 \lambda^2) \). Since \( P'(\lambda) \) vanishes at \( -\frac{2}{3}, -\frac{1}{6}(3 \sqrt{7} - 5) \), and \( \frac{1}{6}(3 \sqrt{7} + 5) \), we deduce that \( P(\lambda) \) has two local minima on \( \left[-\frac{1}{10}, -\frac{1}{3}\right] \) at the points \( -\frac{1}{10} \) and \( -\frac{1}{6}(3 \sqrt{7} - 5) \). By direct computation we see that \( P(-\frac{1}{6}(3 \sqrt{7} - 5)) = \frac{3053}{108} - \frac{21}{2} \sqrt{7} > \frac{52339}{720000} = P(-\frac{1}{10}) \), so that \( P(\lambda) > 0 \) for every \( \lambda \in \left[-\frac{1}{10}, -\frac{1}{3}\right] \). This contradicts (6.29) and concludes the proof of the implication (6.11).

Let us prove (6.12). By (6.14) it is enough to prove that \( \vartheta'(z_2) > 1 \). We argue by contradiction, taking (6.16) into account. If \( \vartheta'(z_2) = 1 \), by (6.14) we have \( \omega(z_2) = 0 \), hence
\[
z_2 = -\frac{1}{(1 + \cos \theta_2)^2 \cos \theta_2} \geq \frac{27}{4}.
\]
Since, by (6.15), \( \omega'(z_2)(1 + \cos \theta_2)^{-2} = -1 - \cos \theta_2 + 3 \cos^2 \theta_2 \), by (6.11) we have \( \omega'(z_2) < 0 \). As \( \omega(z) = 0 \), this implies \( \omega(z) < 0 \) for every \( z > z_2 \) close to \( z_2 \), so that by (6.14) \( \varrho'(z) < 1 \), hence \( \varrho(z) < z \) for every \( z > z_2 \) close to \( z_2 \), which contradicts (6.8).

### 6.2. The system of the fast dynamics

We study now the solutions \((\rho^f(s), \theta^f(s), z^f(s))\) of the system of the fast dynamics

\[
\begin{align*}
\dot{\rho}^f(s) &= -(\rho^f(s) - z^f(s))(z^f(s)(1 + \cos \theta^f(s))\cos^2 \theta^f(s) + 1), \\
\dot{\theta}^f(s) &= (\rho^f(s) - z^f(s))z^f(s)(1 + \cos \theta^f(s)) \cos \theta^f(s) \sin \theta^f(s), \\
\dot{z}^f(s) &= (\rho^f(s) - z^f(s))z^f(s)(1 + \cos \theta^f(s)) \cos \theta^f(s),
\end{align*}
\]

(6.30)

under the additional condition \( \rho^f(s) > z^f(s) > 0 \). In Cartesian coordinates this system is written as

\[
\begin{align*}
\dot{x}^f(s) &= -(x^f(s) - z^f(s)) \left( 1 - \frac{z^f(s)}{\rho^f(s)} \right), \\
\dot{y}^f(s) &= -y^f(s) \left( 1 - \frac{z^f(s)}{\rho^f(s)} \right), \\
\dot{z}^f(s) &= \left( z^f(s) + \frac{(x^f(s) - z^f(s))y^f(s)}{\rho^f(s)} \right)(x^f(s) - z^f(s)) \left( 1 - \frac{z^f(s)}{\rho^f(s)} \right),
\end{align*}
\]

(6.31)

where \( \rho^f(s) := \sqrt{(x^f(s) - z^f(s))^2 + y^f(s)^2} \). Let \( \theta_c \) be the constant defined in (3.3), and let \( z_4(\theta) \) be the function defined in (3.4).

**Lemma 6.2.** Let \( z_1 > 0 \) and \( \frac{\pi}{2} < \theta_1 < \pi \). Assume (6.6) or (6.7). Then there exists a solution of (6.30) such that

\[
\lim_{s \to -\infty} \rho^f(s) = z_1, \quad \lim_{s \to -\infty} \theta^f(s) = \theta_1, \quad \lim_{s \to -\infty} z^f(s) = z_1,
\]

(6.32)

\[
\rho^f(s) > z^f(s) \quad \text{for every } s \in \mathbb{R}.
\]

(6.33)

The solution satisfying (6.32) and (6.33) is unique up to time translations, i.e. all such solutions have the form \((\rho^f(s - s_0), \theta^f(s - s_0), z^f(s - s_0))\) for some \( s_0 \in \mathbb{R} \). Moreover \( \rho^f(s) = \varrho(z^f(s)) \) and \( \theta^f(s) = \vartheta(z^f(s)) \) for every \( s \in \mathbb{R} \), where \((\varrho, \vartheta)\) is the solution of (6.1) with Cauchy conditions (6.4). Finally,

\[
\lim_{s \to -\infty} \rho^f(s) = z_2, \quad \lim_{s \to -\infty} \theta^f(s) = \theta_2, \quad \lim_{s \to -\infty} z^f(s) = z_2,
\]

(6.34)

where \( z_2 \) and \( \theta_2 \) are defined as in Lemma 6.1.

**Proof.** Let \((\varrho, \vartheta)\) be the solution of (6.1) with Cauchy conditions (6.4), and let \( z^f(s) \) be a solution of the autonomous equation

\[
\dot{z}^f(s) = (\varrho(z^f(s)) - z^f(s))z^f(s)(1 + \cos \vartheta(z^f(s))) \cos \vartheta(z^f(s)),
\]

(6.36)
with \( z_2 < z^f(s) < z_1 \) for some \( s \). By Lemma 6.1 we have \((\rho(z) - z)z(1 + \cos \vartheta(z)) \leq 0\) for every \( z \in [z_2, z_1] \), with equality only at \( z = z_2 \) and \( z = z_1 \). Then the theory of autonomous equations implies that \( z^f(s) \) is defined for every \( s \in \mathbb{R} \) and satisfies

\[
\lim_{s\to -\infty} z^f(s) = z_1, \quad \lim_{s\to +\infty} z^f(s) = z_2, \quad \dot{z}^f(s) < 0 \quad \text{for every } s \in \mathbb{R}.
\] (6.37)

Let us define

\[
\rho^f(s) := \rho(z^f(s)) \quad \text{and} \quad \vartheta^f(s) := \vartheta(z^f(s)).
\] (6.38)

By (6.1) and (6.36) \((\rho^f(s), \vartheta^f(s), z^f(s))\) is a solution of (6.30). Since \( \rho(z_1) = z_1 \) and \( \vartheta(z_1) = \theta_1 \) by (6.4), condition (6.32) follows from (6.37). Since \( z_2 < z^f(s) < z_1 \) for every \( s \in \mathbb{R} \), inequality (6.33) follows from (6.8). Finally, (6.34) and (6.35) follow from (6.10), (6.37), and (6.38).

Suppose that \((\rho_*(s), \theta_*(s), z_*(s))\) is another solution of (6.30) satisfying (6.32) and (6.33). By uniqueness it is easy to see that \( \theta_*(s) \neq \frac{\pi}{2} \) and \( \theta_*(s) \neq \pi \) for every \( s \in \mathbb{R} \). Recalling (6.32), we deduce that \( \frac{\pi}{2} < \theta_*(s) < \pi \), so that (6.33) and the third equation in (6.30) imply that \( \dot{\theta}_*(s) < 0 \) for every \( s \in \mathbb{R} \). Then \( z_*(s) \to z_\infty^* < z_1 \) as \( s \to +\infty \).

Since \( \theta_*(s) \) is decreasing, there exist two functions \( \rho_* \) and \( \vartheta_* \), defined on \( (z_\infty^*, z_1) \), such that

\[
\rho_*(s) := \rho_*(z_*(s)) \quad \text{and} \quad \theta_*(s) := \vartheta_*(z_*(s)).
\] (6.39)

It follows from (6.30) that \((\rho_*, \vartheta_*)\) satisfy (6.1) on \((z_\infty^*, z_1)\), and we deduce from (6.32) that \( \rho_*(z) \to z_1 \) and \( \vartheta_*(z) \to \theta_1 \) as \( z \to z_1 \). By (6.4) \((\rho, \vartheta)\) satisfies the same Cauchy conditions at \( z_0 \). By uniqueness we have \((\rho_*, \vartheta_*) = (\rho, \vartheta)\) on \((\max\{z_2, z_\infty^*\}, z_1)\).

Therefore (6.30) and (6.39) imply that \( z_*(s) \) is a solution of (6.36) and \( z_2 < z_*(s) < z_1 \) for \( s \) large enough (recall (6.32) and the monotonicity of \( z_*(s) \)). Then the theory of autonomous equations ensures that there exists \( s_0 \in \mathbb{R} \) such that \( z_*(s) = z^f(s - s_0) \) for \( s \) large enough. Since \((\rho_*, \vartheta_*) = (\rho, \vartheta)\) near \( z_1 \), by (6.39) we have \( \rho_*(s) = \rho^f(s - s_0) \) and \( \vartheta_*(s) = \theta^f(s - s_0) \) for \( s \) large enough. These equalities are extended to every \( s \in \mathbb{R} \) by the uniqueness of the solutions of a Cauchy problem for (6.30).

7. Discontinuous Evolution

In this subsection, we consider the case where \( \frac{\pi}{2} < \theta_0 < \pi \) and the viscosity solution \((\rho, \theta, z)\) has a discontinuity at a time \( t_1 \geq t_0 \) determined by the initial conditions. This solution follows the slow dynamics in \([t_0, t_1]\), has a jump at time \( t_1 \), governed by the system of the fast dynamics, and finally follows again the slow dynamics \((t_1, +\infty)\) with initial conditions at \( t_1 \) determined by the end point of the trajectory of the fast dynamics.

Let \( \theta_e \) be the constant defined in (3.3), and let \( z_e(\theta) \) and \( r_e(\theta) \) be the functions defined in (3.4) and (3.6). The first theorem deals with the case \( t_1 > t_0 \).

**Theorem 7.1.** Assume

\[
\theta_e < \theta_0 \leq \pi \quad \text{and} \quad r_e(\theta_0) < z_0 < z_e(\theta_0),
\] (7.1)
and let \( (\rho_0^{sl}, \theta_0^{sl}) \) and \( t_1, z_1, \) and \( \theta_1 \) be defined as in Lemma 3.6. Let \( (\rho^f, \theta^f, z^f) \), \( z_2, \) and \( \theta_2 \) be defined as in Lemma 6.2, and let \( (\rho_2^{sl}, \theta_2^{sl}) \) be defined as in Lemma 3.7. Then

\[
\rho(t) = z(t) = \begin{cases} 
\rho_0^{sl}(t) & \text{if } t \in [t_0, t_1], \\
\rho_2^{sl}(t) & \text{if } t \in (t_1, +\infty),
\end{cases} \\
\theta(t) = \begin{cases} 
\theta_0^{sl}(t) & \text{if } t \in [t_0, t_1], \\
\theta_2^{sl}(t) & \text{if } t \in (t_1, +\infty).
\end{cases}
\]  

(7.2)

Moreover there exist three sequences of real numbers \( t_\varepsilon^1, \tau_\varepsilon^1, \) and \( s_\varepsilon \) such that for every \( \tau > t_1 \) we have

\[
t_0 < t_\varepsilon^1 < \tau_\varepsilon^1 \quad \text{and} \quad \lim_{\varepsilon \to 0} t_\varepsilon^1 = \lim_{\varepsilon \to 0} \tau_\varepsilon^1 = t_1, 
\]

(7.3)

\[
\sup_{t_0 \leq t \leq t_\varepsilon^1} \left( |\rho_\varepsilon(t) - \rho_0^{sl}(t)| + |\theta_\varepsilon(t) - \theta_0^{sl}(t)| + |z_\varepsilon(t) - \rho_0^{sl}(t)| \right) \to 0, 
\]

(7.4)

\[
\sup_{t_\varepsilon^1 \leq t \leq \tau_\varepsilon^1} \left( |\rho_\varepsilon(t) - \rho_\varepsilon^f(t)| + |\theta_\varepsilon(t) - \theta_\varepsilon^f(t)| + |z_\varepsilon(t) - z_\varepsilon^f(t)| \right) \to 0, 
\]

(7.5)

\[
\sup_{\tau_\varepsilon^1 \leq t \leq \tau} \left( |\rho_\varepsilon(t) - \rho_2^{sl}(t)| + |\theta_\varepsilon(t) - \theta_2^{sl}(t)| + |z_\varepsilon(t) - \rho_2^{sl}(t)| \right) \to 0, 
\]

(7.6)

where

\[ \rho_\varepsilon^f(t) := \rho^f \left( \frac{1}{\varepsilon} t - s_\varepsilon \right), \quad \theta_\varepsilon^f(t) := \theta^f \left( \frac{1}{\varepsilon} t - s_\varepsilon \right), \quad z_\varepsilon^f(t) := z^f \left( \frac{1}{\varepsilon} t - s_\varepsilon \right). \]  

(7.7)

We now consider the case in which the discontinuity time is \( t_0. \)

**Theorem 7.2.** Let \( z_0 > 0 \) and \( \frac{\pi}{2} < \theta_0 < \pi. \) Assume one of the following two conditions:

\[ z_0 > z_s(\theta_0), \]  

(7.8)

\[ z_0 = z_s(\theta_0) \quad \text{and} \quad \theta_0 > \theta_s. \]  

(7.9)

Let \( (\rho^f, \theta^f, z^f), z_2, \theta_2 \) be defined as in Lemma 6.2 with \( z_1 = z_0 \) and \( \theta_1 = \theta_0, \) and let \( (\rho_2^{sl}, \theta_2^{sl}) \) be defined as in Lemma 3.7 with \( t_1 = t_0. \) Then

\[ \rho(t) = z(t) = \rho_2^{sl}(t) \quad \text{and} \quad \theta(t) = \theta_2^{sl}(t) \quad \text{for every } t \in (t_0, +\infty). \]  

(7.10)

Moreover, there exist two sequences of real numbers \( \tau_\varepsilon^1 \) and \( s_\varepsilon \) such that for every \( \tau > t_0 \) we have

\[ t_0 < \tau_\varepsilon^1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \tau_\varepsilon^1 = t_0, \]  

(7.11)

\[
\sup_{t_0 \leq t \leq \tau_\varepsilon^1} \left( |\rho_\varepsilon(t) - \rho_\varepsilon^f(t)| + |\theta_\varepsilon(t) - \theta_\varepsilon^f(t)| + |z_\varepsilon(t) - z_\varepsilon^f(t)| \right) \to 0, 
\]

(7.12)

\[
\sup_{\tau_\varepsilon^1 \leq t \leq \tau} \left( |\rho_\varepsilon(t) - \rho_2^{sl}(t)| + |\theta_\varepsilon(t) - \theta_2^{sl}(t)| + |z_\varepsilon(t) - \rho_2^{sl}(t)| \right) \to 0, 
\]

(7.13)

where \( \rho_\varepsilon^f, \theta_\varepsilon^f, \) and \( z_\varepsilon^f \) are defined in (7.7).

The proof of both theorems will be given in Sec. 7.4.
7.1. Transition to the fast dynamics

We now describe the behavior of the system in a small time interval \([\tau_\varepsilon, t^1_\varepsilon]\), where \(\rho_\varepsilon(t) - z_\varepsilon(t)\) passes from a size of order \(\varepsilon\) to a size of order \(\varepsilon^{1-\alpha}\) with \(\alpha \in (0, \frac{1}{2})\). After \(t^1_\varepsilon\) the system will be governed by the fast dynamics. Let \(\theta_\varepsilon\) be the constant defined in (3.3), and let \(z_\varepsilon(\theta)\) and \(w_\varepsilon(t)\) be the functions defined in (3.4) and (2.26).

Lemma 7.1. Let \(z_1 > 0\) and \(\frac{\pi}{2} < \theta_1 < \pi\). Assume (6.6) or (6.7). Let \(t_1 \in [t_0, +\infty)\), let \(\alpha \in (0, \frac{1}{2})\), and let \(\tau_\varepsilon\) be a sequence in \([t_0, +\infty)\) such that

\[\tau_\varepsilon \to t_1, \quad \rho_\varepsilon(\tau_\varepsilon) \to z_1, \quad \theta_\varepsilon(\tau_\varepsilon) \to \theta_1, \quad z_\varepsilon(\tau_\varepsilon) \to z_1. \tag{7.14}\]

Then there exists a sequence \(t^1_\varepsilon\) in \([t_0, +\infty)\) such that

\[\tau_\varepsilon < t^1_\varepsilon \quad \text{and} \quad t^1_\varepsilon \to t_1, \tag{7.15}\]

\[w_\varepsilon(t^1_\varepsilon) \leq -\varepsilon, \tag{7.16}\]

\[\rho_\varepsilon(t^1_\varepsilon) - z_\varepsilon(t^1_\varepsilon) \geq \varepsilon^{1-\alpha}, \tag{7.17}\]

\[\sup_{\tau_\varepsilon \leq t \leq t^1_\varepsilon} (|\rho_\varepsilon(t) - z_\varepsilon| + |\theta_\varepsilon(t) - \theta_1| + |z_\varepsilon(t) - z_1|) \to 0. \tag{7.18}\]

Proof. As \(z_1 \geq z_\varepsilon(\theta_1)\), we have \(z_1(1 + \cos \theta_1)^2 \cos \theta_1 + 1 \leq 0\), so that (7.14) gives

\[\lim_{\varepsilon \to 0} w_\varepsilon(\tau_\varepsilon) \leq 0. \tag{7.19}\]

Under the assumption (6.6) we have \(z_1(1 + \cos \theta_1)^2 \cos \theta_1 + 1 < 0\). Therefore there exists \(\eta > 0\) such that

\[z(1 + \cos \theta)^2 \cos \theta + 1 \leq -\eta \quad \text{for} \quad |\theta - \theta_1| \leq 2\eta \quad \text{and} \quad |z - z_1| \leq 2\eta. \tag{7.20}\]

If, instead, (6.7) holds, then we have \(\cos \theta_1 < \cos \theta_\varepsilon = \lambda_\varepsilon < -\frac{1}{2}\) by (3.2) and (3.3). This implies that \((1 + \cos \theta_1)(1 + 3 \cos \theta_1) \cos \theta_1 \sin \theta_1 > 0\) and \((1 + \cos \theta_1)^3 \cos \theta_1 (1 + \cos \theta_1 - 3 \cos^2 \theta_1) > 0\). Therefore there exists \(0 < \eta < \frac{1}{2} z_1\) such that

\[z(1 + \cos \theta)(1 + 3 \cos \theta) \cos \theta \sin \theta > \eta \rho, \tag{7.21}\]

\[z(1 + \cos \theta)^3 \cos \theta[z(1 + \cos \theta - 3 \cos^2 \theta) - (\rho - z) \cos \theta] > \eta \rho, \tag{7.21}\]

for \(|\rho - z_1| \leq 2\eta, |\theta - \theta_1| \leq 2\eta, \) and \(|z - z_1| \leq 2\eta\).

In both cases (6.6) and (6.7), we define

\[\tilde{t}^1_\varepsilon := \inf\{t \in (\tau_\varepsilon, +\infty) : w_\varepsilon(t) < -\varepsilon\}, \tag{7.22}\]

\[t^1_\varepsilon := \inf\{t \in (\tilde{t}^1_\varepsilon, +\infty) : \rho_\varepsilon(t) - z_\varepsilon(t) > \varepsilon^{1-\alpha}\}, \tag{7.23}\]

\[\alpha^\eta_\varepsilon := \inf\{t \in (\tau_\varepsilon, +\infty) : |\rho_\varepsilon(t) - z_1| + |\theta_\varepsilon(t) - \theta_1| + |z_\varepsilon(t) - z_1| < 2\eta\}, \tag{7.24}\]

\[s^\eta_\varepsilon := \min\{t^1_\varepsilon, \alpha^\eta_\varepsilon\}, \quad \tilde{s}^\eta_\varepsilon := \min\{\tilde{t}^1_\varepsilon, \alpha^\eta_\varepsilon\}. \tag{7.25}\]
By (7.14) for \( \varepsilon \) small enough we have
\[
|\rho_\varepsilon(t_\varepsilon) - z_1| < \eta, \quad |\theta_\varepsilon(t_\varepsilon) - \theta_1| < \eta, \quad |z_\varepsilon(t_\varepsilon) - z_1| < \eta. \tag{7.26}
\]

If (6.7) holds, by (7.21) for \( \varepsilon \) small enough we have
\[
z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t)(1 + 3 \cos \theta_\varepsilon(t)) \cos \theta_\varepsilon(t) \sin \theta_\varepsilon(t) \eta \rho_\varepsilon(t),

z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t))^3 \cos \theta_\varepsilon(t)[z_\varepsilon(t)(1 + \cos \theta_\varepsilon(t) - 3 \cos^2 \theta_\varepsilon(t))

- (\rho_\varepsilon(t) - z_\varepsilon(t)) \cos \theta_\varepsilon(t)] \eta \rho_\varepsilon(t)
\]
for every \( t \in [\tau_\varepsilon, \alpha_\varepsilon^n] \). Therefore, using Lemma 2.6 and (2.27), for \( \varepsilon \) small enough we obtain
\[
\varepsilon \dot{w}_\varepsilon(t) < -\varepsilon \eta - \eta (\rho_\varepsilon(t) - z_\varepsilon(t)) \leq -\varepsilon \eta \quad \text{for every } t \in [\tau_\varepsilon, \alpha_\varepsilon^n]. \tag{7.27}
\]
This implies
\[
w_\varepsilon(t) \leq w_\varepsilon(\tau_\varepsilon) - \eta (t - \tau_\varepsilon) \quad \text{for every } t \in [\tau_\varepsilon, \alpha_\varepsilon^n], \tag{7.28}
\]
which gives
\[
0 \leq \tilde{s}_\varepsilon^n - \tau_\varepsilon \leq \frac{1}{\eta} \max\{w_\varepsilon(\tau_\varepsilon), 0\} \tag{7.29}
\]
for \( \varepsilon \) small enough. Recalling (7.27), we have
\[
w_\varepsilon(t) < -\varepsilon \quad \text{for every } t \in (\tilde{s}_\varepsilon^n, \alpha_\varepsilon^n]. \tag{7.30}
\]

If (6.6) holds, for \( \varepsilon \) small enough we have \( w_\varepsilon(\tau_\varepsilon) < -\varepsilon \) by (7.20) and (7.26), so that \( \tilde{s}_\varepsilon^n = \tilde{\tau}_\varepsilon^n = \tau_\varepsilon \). In this case (7.30) follows directly from (7.20) and (7.24) for \( \varepsilon \leq \eta \), while (7.29) is trivial.

In both cases (6.6) and (6.7), from (2.25), (2.32), and (7.30) we obtain
\[
\dot{\rho}_\varepsilon(t) - \dot{z}_\varepsilon(t) \geq \sin \theta_0 \quad \text{for every } t \in (\tilde{s}_\varepsilon^n, \alpha_\varepsilon^n]. \tag{7.31}
\]
Integrating this inequality we obtain \( \rho_\varepsilon(t) - z_\varepsilon(t) \geq (t - \tilde{s}_\varepsilon^n) \sin \theta_0 \) for every \( t \in (\tilde{s}_\varepsilon^n, \alpha_\varepsilon^n] \). As \( \rho_\varepsilon(t) - z_\varepsilon(t) \leq \varepsilon^{1\alpha} \) for every \( t \in (\tilde{t}_\varepsilon, \tilde{t}_\varepsilon') \) by (7.23), from (7.25) we obtain
\[
\tilde{s}_\varepsilon^n - \tilde{s}_\varepsilon^n \leq \frac{1}{\sin \theta_0} \varepsilon^{1\alpha} \tag{7.32}
\]
for \( \varepsilon \) small enough. From (7.14), (7.19), (7.29), and (7.32) it follows that
\[
s_\varepsilon^n \to t_1 \quad \text{as } \varepsilon \to 0. \tag{7.33}
\]

As \( 0 < z_\varepsilon(t) \leq z_0 \) for every \( t \in [t_0, +\infty) \) by (2.34), using the third equation in (2.21) we obtain
\[
\varepsilon \ddot{z}_\varepsilon(t) \geq -2z_0 (\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [\tau_\varepsilon, \alpha_\varepsilon^n], \tag{7.34}
\]
so that (7.27) gives for \( \varepsilon \) small enough
\[
\dot{w}_\varepsilon(t) \leq \frac{\eta}{2z_0} \dot{z}_\varepsilon(t) \quad \text{for every } t \in [\tau_\varepsilon, \alpha_\varepsilon^\eta].
\]

Since \( w_\varepsilon(t) \geq -\varepsilon \) for every \( t \in (\tau_\varepsilon, \bar{s}_\varepsilon^\eta) \) by (7.22) and (7.25), we deduce that
\[
-\varepsilon - w_\varepsilon(t) \leq w_\varepsilon(t) - w_\varepsilon(\tau_\varepsilon) \leq \frac{\eta}{2z_0} (z_\varepsilon(t) - z_\varepsilon(\tau_\varepsilon)) \quad \text{for every } t \in (\tau_\varepsilon, \bar{s}_\varepsilon^\eta),
\]
so that (7.27) gives
\[
\bar{s}_\varepsilon(t) \geq \frac{2z_0}{\eta} \max\{w_\varepsilon(\tau_\varepsilon), 0\} - \varepsilon \quad \text{for every } t \in [\tau_\varepsilon, \bar{s}_\varepsilon^\eta]. \tag{7.35}
\]

Since \( \rho_\varepsilon(t) - z_\varepsilon(t) \leq \varepsilon^{-\alpha} \) for every \( t \in (\bar{s}_\varepsilon^\eta, s_\varepsilon^\eta) \) by (7.23) and (7.25), from (7.34) we get
\[
\dot{z}_\varepsilon(t) \geq -2z_0 \varepsilon^{-\alpha} \quad \text{for every } t \in (\bar{s}_\varepsilon^\eta, s_\varepsilon^\eta).
\]

Integrating and using (7.32) we obtain
\[
z_\varepsilon(t) - z_\varepsilon(\bar{s}_\varepsilon^\eta) \geq -2z_0 \varepsilon^{-\alpha} (t - \bar{s}_\varepsilon^\eta) \geq -\frac{2z_0}{\sin \theta_0} \varepsilon^{1-2\alpha} \quad \text{for every } t \in (\bar{s}_\varepsilon^\eta, s_\varepsilon^\eta),
\]
which, together with (2.34) and (7.35), gives for \( \varepsilon \) small enough
\[
z_\varepsilon(\tau_\varepsilon) - \frac{2z_0}{\eta} \max\{w_\varepsilon(\tau_\varepsilon), 0\} - \varepsilon - \frac{2z_0}{\sin \theta_0} \varepsilon^{1-2\alpha} \leq z_\varepsilon(t) \leq z_\varepsilon(\tau_\varepsilon)
\quad \text{for every } t \in [\tau_\varepsilon, s_\varepsilon^\eta].
\]

By (7.14) and (7.19) this implies
\[
\sup_{\tau_\varepsilon \leq t \leq s_\varepsilon^\eta} |z_\varepsilon(t) - z_1| \to 0 \quad \text{as } \varepsilon \to 0. \tag{7.36}
\]

By Lemmas 2.6 and 2.8 we have \( z_\varepsilon(t) \leq \rho_\varepsilon(t) \leq \rho_\varepsilon(\tau_\varepsilon) + \varepsilon \) for every \( t \in [\tau_\varepsilon, +\infty) \). Therefore (7.14) and (7.36) give
\[
\sup_{\tau_\varepsilon \leq t \leq s_\varepsilon^\eta} |\rho_\varepsilon(t) - z_1| \to 0 \quad \text{as } \varepsilon \to 0, \tag{7.37}
\]
so that for \( \varepsilon \) small enough we obtain \( \rho_\varepsilon(t) \geq \frac{1}{2} z_1 \) for every \( t \in [\tau_\varepsilon, s_\varepsilon^\eta] \).

Since, by (7.36), \( z_\varepsilon(t) \leq 2z_1 \) for \( \varepsilon \) small enough, from the second equation in (2.21) we obtain
\[
\varepsilon \dot{z}_\varepsilon(t) \geq -\frac{2}{z_1} \varepsilon - 4(\rho_\varepsilon(t) - z_\varepsilon(t)) \quad \text{for every } t \in [\tau_\varepsilon, s_\varepsilon^\eta], \tag{7.38}
\]
so that (7.27) gives
\[
\dot{w}_\varepsilon(t) \leq \frac{\eta}{4} \dot{\theta}_\varepsilon(t) + \frac{\eta}{2z_1} \quad \text{for every } t \in [\tau_\varepsilon, s_\varepsilon^\eta].
\]
Assume (7.2). Convergence to the fast dynamics follows from (7.36), (7.37) and (7.40). Since

\[ w_1 \leq 1696 G. \ Dal \ Maso \ & \ A. \ DeSimone \]

Integrating and using (7.32) we obtain

\[ \theta_\varepsilon(t) \geq \theta_\varepsilon(t) - \left( \frac{4}{\eta} + \frac{2}{\eta z_1} \right) \max\{w_\varepsilon(\varepsilon), 0\} - \varepsilon \quad \text{for every } t \in [\varepsilon, s_\varepsilon^0], \]

so that by (7.29) for \( \varepsilon \) small enough we have

\[ \theta_\varepsilon(t) \geq \theta_\varepsilon(t) - \left( \frac{4}{\eta} + \frac{2}{\eta z_1} \right) \max\{w_\varepsilon(\varepsilon), 0\} - \varepsilon \quad \text{for every } t \in [\varepsilon, s_\varepsilon^0]. \] (7.39)

Since \( \rho_\varepsilon(t) - z_\varepsilon(t) \leq \varepsilon^{1-\alpha} \) for every \( t \in (s_\varepsilon^0, s_\varepsilon^1] \) by (7.23) and (7.25), from (7.38) we get

\[ \dot{\theta}_\varepsilon(t) \geq -\frac{2}{z_1} - 4\varepsilon^{-\alpha} \quad \text{for every } t \in (s_\varepsilon^0, s_\varepsilon^1]. \]

Integrating and using (7.32) obtain

\[ \theta_\varepsilon(t) - \theta_\varepsilon(s_\varepsilon^0) \geq -\left( \frac{2}{z_1} + 4\varepsilon^{-\alpha} \right) (t - s_\varepsilon^0) \geq -\frac{2\varepsilon^\alpha + 4z_1}{z_1 \sin \theta_0} \varepsilon^{1-2\alpha} \quad \text{for every } t \in (s_\varepsilon^0, s_\varepsilon^1], \]

which, together with (2.32) and (7.39), gives for \( \varepsilon \) small enough

\[ \theta_\varepsilon(t) - \theta_\varepsilon(s_\varepsilon^0) \geq -\left( \frac{4}{\eta} + \frac{2}{\eta z_1} \right) \max\{w_\varepsilon(\varepsilon), 0\} - \varepsilon - \frac{2\varepsilon^\alpha + 4z_1}{z_1 \sin \theta_0} \varepsilon^{1-2\alpha} \leq \theta_\varepsilon(t) \leq \theta_\varepsilon(t) \]

for every \( t \in [\varepsilon, s_\varepsilon^0] \). By (7.14) and (7.19) this implies

\[ \sup_{t_\varepsilon^0 \leq t \leq s_\varepsilon^0} |\theta_\varepsilon(t) - \theta_\varepsilon| \rightarrow 0. \] (7.40)

From (7.36), (7.37) and (7.40) we deduce that \( s_\varepsilon^0 < \alpha_\varepsilon^0 \) for \( \varepsilon \) small enough. By (7.25) this implies \( s_\varepsilon^0 = t_\varepsilon^1 \) and \( s_\varepsilon^1 = t_\varepsilon^1 \). Therefore (7.33) gives (7.15), and (7.18) follows from (7.36), (7.37) and (7.40). Since \( t_\varepsilon^1 < +\infty \), by (7.22) we have \( w_\varepsilon(t_\varepsilon^1) \leq -\varepsilon \), so that (7.16) follows from (7.27). Since \( t_\varepsilon^1 < +\infty \), inequality (7.17) follows from the definition of \( t_\varepsilon^1 \) given in (7.23). \( \square \)

7.2. Convergence to the fast dynamics

Assume \( \frac{\pi}{2} < \theta_0 \leq \pi \), and let

\[ z_\varepsilon^\infty := \lim_{t \rightarrow +\infty} z_\varepsilon(t) \geq 0. \]

By (2.34) there exist functions \( \varrho_\varepsilon \) and \( \vartheta_\varepsilon \) defined on \( [z_\varepsilon^\infty, z_0] \) such that

\[ \varrho_\varepsilon(t) = \varrho_\varepsilon(z_\varepsilon(t)) \quad \text{and} \quad \vartheta_\varepsilon(t) = \vartheta_\varepsilon(z_\varepsilon(t)) \quad \text{for every } t \in [t_0, +\infty). \] (7.41)

From Lemma 2.6 it follows that

\[ \varrho_\varepsilon(z) > z \quad \text{for every } z \in (z_\varepsilon^\infty, z_0), \] (7.42)
and from (2.32) it follows that
\[ \frac{\pi}{2} < \vartheta_{\varepsilon}(z) < \theta_0 < \pi \quad \text{for every } z \in (z_{\varepsilon}^\infty, z_0). \] (7.43)

By (2.32) and (2.34) we have
\[ \vartheta'_{\varepsilon}(z) > 0 \quad \text{for every } z \in (z_{\varepsilon}^\infty, z_0). \]

By (2.21) on the intervals \((z_{\varepsilon}^\infty, z_{\varepsilon}^0)\) the functions \((\varrho_{\varepsilon}, \vartheta_{\varepsilon})\) are solutions to the system
\[
\begin{aligned}
\vartheta'_{\varepsilon}(z) & = -\cos \vartheta_{\varepsilon}(z) - \frac{1}{\varrho_{\varepsilon}(z)(1 + \cos \vartheta_{\varepsilon}(z))} \cos \vartheta_{\varepsilon}(z) + \varepsilon \frac{\sin \vartheta_{\varepsilon}(z)}{F(z, \varrho_{\varepsilon}(z), \vartheta_{\varepsilon}(z))}, \\
\varrho'_{\varepsilon}(z) & = \sin \vartheta_{\varepsilon}(z) + \varepsilon \frac{\cos \vartheta_{\varepsilon}(z)}{\varrho_{\varepsilon}(z)F(z, \varrho_{\varepsilon}(z), \vartheta_{\varepsilon}(z))},
\end{aligned}
\]
(7.44)

where
\[ F(z, \rho, \theta) := (\rho - z)(1 + \cos \theta) \cos \theta. \]

Let \(z_1 > 0\) and \(\frac{\pi}{2} < \theta_1 < \pi\). Assume (6.6) or (6.7), and let \((\varrho, \vartheta), z_2, \theta_2\) be defined as in Lemma 6.1. Let us fix \(\eta > 0\) such that \(z_2(1 + \cos \theta_1)|\cos \theta_2| > \eta\). This implies that
\[ z(1 + \cos \vartheta(z))|\cos \vartheta(z)| > \eta \quad \text{for every } z \in [z_2, z_1]. \] (7.45)

Given \(\alpha \in (0, \frac{1}{2})\), we consider the auxiliary systems
\[
\begin{aligned}
(\varrho_{\varepsilon}^\gamma)'(z) & = -\cos \vartheta_{\varepsilon}^\gamma(z) + \frac{1}{G^\gamma(z, \varrho_{\varepsilon}^\gamma(z))} - \varepsilon \frac{|\sin \vartheta_{\varepsilon}^\gamma(z)|}{F_{\varepsilon}^\gamma(z, \varrho_{\varepsilon}^\gamma(z), \vartheta_{\varepsilon}^\gamma(z))}, \\
(\vartheta_{\varepsilon}^\gamma)'(z) & = \sin \vartheta_{\varepsilon}^\gamma(z) \max\{\varrho_{\varepsilon}^\gamma(z), \eta\} + \varepsilon \frac{|\cos \vartheta_{\varepsilon}^\gamma(z)|}{F_{\varepsilon}^\gamma(z, \varrho_{\varepsilon}^\gamma(z), \vartheta_{\varepsilon}^\gamma(z))},
\end{aligned}
\]
(7.46)

where
\[ G^\gamma(z, \theta) := \max\{z(1 + \cos \theta)|\cos \theta|, \eta\}, \]
\[ F_{\varepsilon}^\gamma(z, \rho, \theta) := G^\gamma(z, \theta) \max\{\rho - z, \varepsilon^{1-\alpha}\}. \]

Note that all solutions of (7.46) are defined for every \(z \in \mathbb{R}\).

**Lemma 7.2.** Let \(z_1 > 0\) and \(\frac{\pi}{2} < \theta_1 < \pi\). Assume (6.6) or (6.7). Let \(\alpha \in (0, \frac{1}{2})\), let \(\rho_{\varepsilon}^1, \theta_{\varepsilon}^1,\) and \(z_{\varepsilon}^1\) be three sequences such that
\[
\begin{aligned}
\rho_{\varepsilon}^1 & \to z_1, \quad \theta_{\varepsilon}^1 \to \theta_1, \quad z_{\varepsilon}^1 \to z_1, \\
z_{\varepsilon}^1(1 + \cos \theta_{\varepsilon}^1)^2 \cos \theta_{\varepsilon}^1 + 1 & < 0, \quad \rho_{\varepsilon}^1 \geq z_{\varepsilon}^1 + \varepsilon^{1-\alpha},
\end{aligned}
\]
(7.47)

and let \((\varrho_{\varepsilon}^1, \vartheta_{\varepsilon}^1)\) be the solution of (7.46) with Cauchy conditions
\[
\begin{aligned}
\varrho_{\varepsilon}^1(\varepsilon_{\varepsilon}) & = \rho_{\varepsilon}^1 \quad \text{and} \quad \vartheta_{\varepsilon}^1(\varepsilon_{\varepsilon}) = \theta_{\varepsilon}^1,
\end{aligned}
\]
(7.49)
Then there exists $z_1^* \in (0, z_1)$, depending on $\eta$ in (7.45), but not on $\varepsilon$, such that for $\varepsilon$ small enough we have
\begin{equation}
\vartheta_\varepsilon^\eta(z) > z + \varepsilon^{1-\alpha} \quad \text{for every } z \in [z_1^*, z_1].
\end{equation}

**Proof.** Let us fix $\delta > 0$. From the second equation in (7.46) we have $-\frac{1}{\eta} \leq (\vartheta_\varepsilon^\eta)'(z) \leq \frac{1}{\eta} + \frac{c}{\eta}$ for every $z \in \mathbb{R}$, so that for $\varepsilon$ small enough we have $|(|\vartheta_\varepsilon^\eta)'(z)| \leq 1 + \frac{1}{\eta}$. Recalling (7.49), by integrating we get $|\vartheta_\varepsilon^\eta(z) - \vartheta_\varepsilon^\eta(1)| < (1 + \frac{1}{\eta})(z_1^*-z_1+\delta)$ for every $z \in [z_1^*, z_1^*]$. Using (7.47) for $\varepsilon$ small enough we obtain
\begin{equation}
|\vartheta_\varepsilon^\eta(z) - \vartheta_\varepsilon^\eta(1)| < \left(1 + \frac{1}{\eta}\right)2\delta \quad \text{for every } z \in [z_1^*, z_1^*].
\end{equation}

Suppose that $z_1 > z_s(\theta_1)$, so that
\[-\cos \theta_1 - \frac{1}{z_1(1 + \cos \theta_1) \cos \theta_1} < 1.
\]
By continuity there exist $\delta_1 > 0$ such that
\[-\cos \theta + \frac{1}{G(\theta_1, \theta_1)} = -\cos \theta + \frac{1}{\max\{z(1 + \cos \theta) | \cos \theta\}, \eta} < 1 \quad \text{for } |\theta - \theta_1| \leq \delta_1 \text{ and } |z - z_1| \leq \delta_1.\]
Let us fix $\delta > 0$ with
\[\left(1 + \frac{1}{\eta}\right)2\delta < \delta_1.\]

Since $z_1^* < z_1 + \delta_1$ for $\varepsilon$ small enough by (7.47), using (7.51), (7.52), (7.53), and the first equation in (7.46) we deduce that $(\vartheta_\varepsilon^\eta)'(z) < 1$ for every $z \in [z_1^*, z_1^*]$. As $\vartheta_\varepsilon^\eta(z_1^*) \geq z_1^* + \varepsilon^{1-\alpha}$ by (7.48), after integration we obtain (7.50).

Suppose now that $\theta_1 < \theta_1 < \pi$ and $z_1 = z_s(\theta_1)$. Let us consider the function
\[\omega_\varepsilon^\eta(z) := z(1 + \cos \vartheta_\varepsilon^\eta(z))^2 \cos \vartheta_\varepsilon^\eta(z) + 1.\]
From (7.46) we obtain
\[(\omega_\varepsilon^\eta)'(z) = \alpha_\varepsilon^\eta(z)((\max\{\vartheta_\varepsilon^\eta(z), \eta\} - z) \cos \vartheta_\varepsilon^\eta(z) - z(1 + \cos \vartheta_\varepsilon^\eta(z) - 3\cos^2 \vartheta_\varepsilon^\eta(z)))
- \beta_\varepsilon^\eta(z)(z(1 + 3 \cos \vartheta_\varepsilon^\eta(z))) \sin \vartheta_\varepsilon^\eta(z),\]
where $\alpha_\varepsilon^\eta(z) \geq 0$ and $\beta_\varepsilon^\eta(z) \geq 0$ for every $z \in \mathbb{R}$. Since $\theta_1 < \theta_1 < \pi$, by (3.2) and (3.3) we have $\cos \theta_1 < \cos \theta_1 = \lambda_1 < -\frac{1}{3}$. This implies that $1 + \cos \theta_1 - 3\cos^2 \theta_1 < 0$ and $z_1(1 + 3 \cos \theta_1) \sin \theta_1 < 0$. As $z_1 > \eta$ by (7.45), by continuity there exists $\delta_1 > 0$ such that
\[(\max\{\rho, \eta\} - z) \cos \theta - z(1 + \cos \theta - 3\cos^2 \theta) > 0,
\]
for $|\rho - z_1| \leq \delta_1$, $|\theta - \theta_1| \leq \delta_1$, and $|z - z_1| \leq \delta_1$. Let us fix $\delta > 0$ satisfying (7.53).
From the first equation in (7.46) we have \(-1 - \frac{\alpha}{\delta} \leq (\phi_\epsilon^\gamma)'(z) \leq 1 + \frac{1}{\eta}\) for every \(z \in \mathbb{R}\), so that for \(\varepsilon\) small enough we have \(|(\phi_\epsilon^\gamma)'(z)| \leq 1 + \frac{1}{\eta}\). Recalling (7.49), by integrating we get \(|\phi_\epsilon^\gamma(z) - \rho_\epsilon^1| < (1 + \frac{1}{\eta})(z^0_\epsilon - z_1 + \delta)\) for every \(z \in [z^1_\epsilon, z^2_\epsilon]\). Using (7.47) for \(\varepsilon\) small enough we obtain \(|\phi_\epsilon^\gamma(z) - z_1| < \delta_1\) for every \(z \in [z^1_\epsilon, z^2_\epsilon]\).

Since \(z^2_\epsilon < z_1 + \delta_1\) for \(\varepsilon\) small enough by (7.47), using (7.51), (7.53), (7.55), and (7.56) we obtain \((\omega_\epsilon^\gamma)'(z) \geq 0\) for every \(z \in [z^1_\epsilon, z^2_\epsilon]\). By (7.48), (7.49), and (7.54) we have \(\omega_\epsilon^\gamma(z_1^1) < 0\). It follows that \(\omega_\epsilon^\gamma(z) < 0\) for every \(z \in [z^1_\epsilon, z^2_\epsilon]\). This implies

\[-\cos \vartheta_\epsilon^\gamma(z) - \frac{1}{1 + \cos \vartheta_\epsilon^\gamma(z)} \vartheta_\epsilon^\gamma(z) < 1\]

for every \(z \in [z^1_\epsilon, z^2_\epsilon]\), and hence

\[-\cos \vartheta_\epsilon^\gamma(z) + \frac{1}{G^\gamma(z, \cos \vartheta_\epsilon^\gamma(z))} < 1\]

for every \(z \in [z^1_\epsilon, z^2_\epsilon]\).

Therefore the first equation in (7.46) gives \((\phi_\epsilon^\gamma)'(z) < 1\) for every \(z \in [z^1_\epsilon, z^2_\epsilon]\). As \(\phi_\epsilon^\gamma(z^1_\epsilon) \geq z^1_\epsilon + \varepsilon^{1-\alpha}\) by (7.48), after integration we obtain (7.50).

\[\text{Lemma 7.3.}\]

Let \(z_1 > 0\) and \(\frac{\alpha}{2} < \theta_1 < \pi\). Assume (6.6) or (6.7). Let \(t_1 \in [t_0, +\infty)\), let \(\alpha \in (0, \frac{1}{2})\), and let \(t_\epsilon^1\) be a sequence in \([t_0, +\infty)\) such that

\[t_\epsilon^1 \to t_1, \quad \rho_\epsilon(t_\epsilon^1) \to z_1, \quad \vartheta_\epsilon(t_\epsilon^1) \to \theta_1, \quad z_\epsilon(t_\epsilon^1) \to z_1, \quad \text{for } \varepsilon \to 0.\]

(7.57)

Let \(w_\epsilon(t_\epsilon^1) < 0\), \(\rho_\epsilon(t_\epsilon^1) - z_\epsilon(t_\epsilon^1) \geq \varepsilon^{1-\alpha}\). (7.58)

Let \(z_\epsilon^1 := z_\epsilon(t_\epsilon^1), \) let \((\vartheta_\epsilon, \varrho_\epsilon)\) be the functions defined in (7.41), and let \((\varrho, \vartheta)\) be the solution of (6.1) with Cauchy condition (6.4). Then for \(\varepsilon\) small enough there exists \(z_\epsilon^2 \in (z^\infty_\epsilon, z^1_\epsilon)\) such that

\[\varrho_\epsilon(z_\epsilon^1) = z_\epsilon^2 + \varepsilon^{1-\alpha} \quad \text{and} \quad \varrho_\epsilon(z) > z + \varepsilon^{1-\alpha} \quad \text{for } z \in (z^2_\epsilon, z^1_\epsilon).\]

(7.59)

Let \(\vartheta_\epsilon^2 := \vartheta_\epsilon(z_\epsilon^2).\) Then

\[z_\epsilon^2 \to z_2 \quad \text{and} \quad \vartheta_\epsilon^2 \to \theta_2,\]

where \(z_2\) and \(\theta_2\) are defined as in Lemma 6.1. Moreover,

\[\sup_{z^2_\epsilon \leq z \leq z^1_\epsilon} (|\varrho_\epsilon(z) - \varrho(z)| + |\vartheta_\epsilon(z) - \vartheta(z)|) \to 0 \quad \text{as } \varepsilon \to 0.\]

(7.60)

\[\text{Proof.}\]

Let us define

\[\rho_\epsilon^1 := \rho_\epsilon(t_\epsilon^1), \quad \theta_\epsilon^1 := \theta_\epsilon(t_\epsilon^1), \quad z_\epsilon^1 := z_\epsilon(t_\epsilon^1).\]

(7.61)

We consider the auxiliary system

\[
\begin{align*}
(\varrho_\epsilon^\gamma)'(z) &= -\cos \vartheta_\epsilon^\gamma(z) + \frac{1}{\max\{z(1 + \cos \vartheta_\epsilon^\gamma(z))|\cos \vartheta_\epsilon^\gamma(z)|, \eta, \eta\}}, \\
(\vartheta_\epsilon^\gamma)'(z) &= \frac{\sin \vartheta_\epsilon^\gamma(z)}{\max\{\varrho_\epsilon^\gamma(z), \eta\}},
\end{align*}
\]

(7.62)
whose solutions are defined for every \( z \in \mathbb{R} \). Since \( 1/F^\eta_\varepsilon(z, \rho, \theta) \leq \varepsilon^\alpha/\eta \) for every \( (z, \rho, \theta) \in \mathbb{R}^3 \), by (7.58) and (7.61) the solutions \((\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)\) considered in Lemma 7.2 converge uniformly on compact subsets of \( \mathbb{R} \) to the solution \((\varrho^\eta, \vartheta^\eta)\) of (6.62) with Cauchy conditions

\[
\varrho^\eta(z_1) = z_1 \quad \text{and} \quad \vartheta^\eta(z_1) = \theta_1.
\] (7.63)

As \( \varrho'(z_2) > 0 \) by (6.12), there exists \( z^*_2 \in (0, z_2) \) such that \((\varrho, \vartheta)\) is defined on \([z^*_2, z_1]\) and

\[
\varrho(z) < z \quad \text{for every} \quad z \in [z^*_2, z_2].
\] (7.64)

By (7.45) we may also suppose that

\[
z(1 + \cos \vartheta(z))|\cos \vartheta(z)| > \eta \quad \text{for every} \quad z \in [z^*_2, z_1].
\] (7.65)

Since \((\varrho, \vartheta)\) is a solution of (6.1), inequality (7.65) implies that \((\varrho, \vartheta)\) is a solution of (6.62). Since \((\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)\) and \((\varrho, \vartheta)\) satisfy the same Cauchy conditions at \( z_1 \) by (6.4) and (7.63), we conclude that \((\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon) = (\varrho, \vartheta)\) on \([z^*_2, z_1]\). Therefore \((\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)\) converges to \((\varrho, \vartheta)\) uniformly on \([z^*_2, z_1]\). By (7.64) and (7.65) for \( \varepsilon \) small enough we have \( \varrho^\eta_\varepsilon(z^*_2) < z^*_2 \) and

\[
z(1 + \cos \vartheta^\eta_\varepsilon(z))|\cos \vartheta^\eta_\varepsilon(z)| > \eta \quad \text{for every} \quad z \in [z^*_2, z_1].
\] (7.66)

Let \( z^*_1 \) be the constant introduced in Lemma 7.2. Since \( \varrho^\eta_\varepsilon(z) > z + \varepsilon^{1-\alpha} \) for every \( z \in [z^*_1, z^*_2]\), we can consider the greatest point \( z^*_\varepsilon \) of \([z^*_2, z^*_1]\) such that \( \varrho^\eta_\varepsilon(z^*_\varepsilon) = z^*_\varepsilon + \varepsilon^{1-\alpha} \), and we have

\[
\varrho^\eta_\varepsilon(z^*_\varepsilon) = z^*_\varepsilon + \varepsilon^{1-\alpha} \quad \text{and} \quad \varrho^\eta_\varepsilon(z) > z + \varepsilon^{1-\alpha} \quad \text{for} \quad z \in (z^*_\varepsilon, z^*_1). \] (7.67)

The uniform convergence of \((\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)\) to \((\varrho, \vartheta)\) on \([z^*_2, z^*_1]\) implies that \( z^*_\varepsilon \to z_2 \) and \( \vartheta^\eta_\varepsilon(z^*_\varepsilon) \to \vartheta(z_2) = \theta_2 \).

From (7.66) and (7.67) we deduce that \((\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)\) satisfies (7.44) in the interval \([z^*_2, z^*_1]\). Since \((\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon)\) and \((\varrho_\varepsilon, \vartheta_\varepsilon)\) satisfy the same Cauchy conditions at \( z^*_1 \) by (7.41), (7.49) and (7.61), we conclude that \((\varrho^\eta_\varepsilon, \vartheta^\eta_\varepsilon) = (\varrho_\varepsilon, \vartheta_\varepsilon)\) on \([z^*_\varepsilon, z^*_1]\). This implies

\[
\sup_{z^*_\varepsilon \leq z \leq z_1} (|\varrho_\varepsilon(z) - \varrho(z)| + |\vartheta_\varepsilon(z) - \vartheta(z)|) \to 0
\]

and the convergence \( \vartheta^2_\varepsilon := \vartheta_\varepsilon(z^*_\varepsilon) = \vartheta^\eta_\varepsilon(z^*_\varepsilon) \to \theta_2 \).

\[\square\]

**Lemma 7.4.** Under the assumptions of Lemma 7.3, let \( \tau^1_\varepsilon \) be the time such that

\[
z_\varepsilon(\tau^1_\varepsilon) = z^*_\varepsilon,
\] (7.68)

and let \( \eta \) be the constant in (7.45). Then

\[
0 < \tau^1_\varepsilon - t^1_\varepsilon < \frac{1}{\eta} \varepsilon^\alpha
\]

for \( \varepsilon \) small enough.
Lemma 7.5. Under the assumptions of Lemma 7.3, let \((\rho^f, \theta^f, z^f)\) be defined as in Lemma 6.2, and let \(\tau^1_\varepsilon\) be the time introduced in Lemma 7.4. Then for every \(\varepsilon > 0\) there exists \(s_\varepsilon \in \mathbb{R}\) such that

\[
\sup_{t^1_\varepsilon \leq t \leq t^1_\varepsilon} (|\rho_\varepsilon(t) - \rho^f_\varepsilon(t)| + |\theta_\varepsilon(t) - \theta^f_\varepsilon(t)| + |z_\varepsilon(t) - z^f_\varepsilon(t)|) \to 0, \tag{7.70}
\]

where \(\rho^f_\varepsilon, \theta^f_\varepsilon,\) and \(z^f_\varepsilon\) are defined in (7.7).

Proof. Let \(\rho^1_\varepsilon, \theta^1_\varepsilon,\) and \(z^1_\varepsilon\) be defined as in (7.61). By (7.41) we have \(\rho_\varepsilon(t) = \rho_\varepsilon(z_\varepsilon(t))\) and \(\theta_\varepsilon(t) = \theta_\varepsilon(z_\varepsilon(t))\) for every \(t \in [0_\varepsilon, +\infty)\), where \((\rho_\varepsilon, \theta_\varepsilon)\) satisfies (7.44) and the Cauchy condition

\[
\rho_\varepsilon(z^1_\varepsilon) = \rho^1_\varepsilon \quad \text{and} \quad \theta_\varepsilon(z^1_\varepsilon) = \theta^1_\varepsilon. \tag{7.71}
\]

Moreover, \(z_\varepsilon\) satisfies (7.69), so that the function \(\zeta_\varepsilon(s) := z_\varepsilon(\varepsilon s)\) satisfies the equation

\[
\dot{\zeta}_\varepsilon(s) = (\rho_\varepsilon(\zeta_\varepsilon(s))) - \zeta_\varepsilon(s))(1 + \cos \theta_\varepsilon(\zeta_\varepsilon(s))) \cos \theta_\varepsilon(\zeta_\varepsilon(s)) \tag{7.72}
\]

for \(s \in \left[\frac{1}{2} t_0, +\infty\right)\). As explained at the end of the proof of Lemma 7.3 we have \((\rho^0_\varepsilon, \theta^0_\varepsilon) = (\rho_\varepsilon, \theta_\varepsilon)\) on \([z^2_\varepsilon, z^1_\varepsilon]\). By (7.68) the function \(\zeta_\varepsilon\) satisfies also the equation

\[
\dot{\zeta}_\varepsilon(s) = (\rho^0_\varepsilon(\zeta_\varepsilon(s)) - \zeta_\varepsilon(s))(1 + \cos \theta^0_\varepsilon(\zeta_\varepsilon(s))) \cos \theta^0_\varepsilon(\zeta_\varepsilon(s)) \tag{7.73}
\]

for \(s \in \left[\frac{1}{2} t^1_\varepsilon, \frac{1}{2} \tau^1_\varepsilon\right]\).

By (7.57) and (7.61) we have \(z^1_\varepsilon \to z_1,\) while \(z^2_\varepsilon \to z_2\) by Lemma 7.3. Since \(z_2 < z^f(0) < z_1\) by the monotonicity of \(z^f\) and by (6.32) and (6.35), we have that \(z^2_\varepsilon < z^f(0) < z^1_\varepsilon\) for \(\varepsilon\) small enough. By (7.61) and (7.68) we have \(\zeta_\varepsilon(\frac{1}{2} t^1_\varepsilon) = z^1_\varepsilon\) and \(\zeta_\varepsilon(\frac{1}{2} \tau^1_\varepsilon) = z^2_\varepsilon\). Since \(\zeta_\varepsilon\) is decreasing by (2.34), there exists a unique \(s_\varepsilon \in \left(\frac{1}{2} t^1_\varepsilon, \frac{1}{2} \tau^1_\varepsilon\right)\) such that \(\zeta_\varepsilon(s_\varepsilon) = z^f(0)\).

Let \(\zeta^\triangledown_\varepsilon\) be the maximal solution of (7.73) with Cauchy condition \(\zeta^\triangledown_\varepsilon(0) = z^f(0)\) and let \(\zeta^\triangledown_\varepsilon\) be the solution of (7.72) on \((0, 0)\) with Cauchy condition \(\zeta^\triangledown_\varepsilon(0) = z^f(0)\). The theory of autonomous systems implies that \(\zeta^\triangledown_\varepsilon\) is defined in a neighborhood of the interval \([0, +\infty)\), is decreasing, and satisfies \(\zeta^\triangledown_\varepsilon(s) \to z_1\) as \(s \to +\infty\). Taking into account (7.42) and (7.43), the theory of autonomous systems guarantees that \(\zeta^\triangledown_\varepsilon\) is defined on the whole interval \((0, 0]\), is decreasing, and satisfies \(\zeta^\triangledown_\varepsilon(s) \to z_1\) as
$s \to -\infty$. By uniqueness we have

$$\zeta_\varepsilon(s) = \zeta^\varepsilon(s - s_\varepsilon) \quad \text{for every } s \in \left[\frac{1}{\varepsilon} t_\varepsilon^1, \frac{1}{\varepsilon} t_\varepsilon^1 \right],$$

(7.74)

$$\zeta_\varepsilon(s) = \zeta^{\varepsilon}(s - s_\varepsilon) \quad \text{for every } s \in \left[\frac{1}{\varepsilon} t_\varepsilon^0, \frac{1}{\varepsilon} t_\varepsilon^0 \right].$$

(7.75)

In the proof of Lemma 7.3 we have seen that $(q^\varepsilon_q, \vartheta^\varepsilon_q) \to (q, \vartheta)$ as $\varepsilon \to 0$ uniformly on $[z_\varepsilon^2, z_0]$, where $(q, \vartheta)$ is the solution of (6.1) with Cauchy conditions (6.4), and $\delta_2 > 0$ satisfies (7.65). Continuing as in the same proof we can construct $z^\varepsilon_q$, with $z^\varepsilon_q \to z_2$ as $\varepsilon \to 0$, such that

$${q^\varepsilon_q(z^\varepsilon_q) = z^\varepsilon_q \quad \text{and} \quad q^\varepsilon_q(z) > z \quad \text{for } z \in (z^\varepsilon_q, z^1_\varepsilon).}$$

Let us prove that $\zeta^\varepsilon_q$ converges to $z^f$ uniformly on $[s_0, +\infty)$ for every $s_0 < 0$. Let us fix $\lambda > 0$. By (6.35) we can find $s_0 \in (0, +\infty)$ such that $|z^f(s) - z_2| < \lambda$ for any $s \in [s_2, +\infty)$. Since $(q^\varepsilon_q, \vartheta^\varepsilon_q) \to (q, \vartheta)$ as $\varepsilon \to 0$ uniformly on $[z_\varepsilon^2, z_0]$, and $z^f$ satisfies (6.36), we have $\zeta^\varepsilon_q \to z^f$ uniformly in $[s_0, s_2]$ for every $s_0 < 0$. For $s \geq s_2$ the monotonicity of $\zeta^\varepsilon_q$ gives

$$|\zeta^\varepsilon_q(s) - z^f(s)| \leq |\zeta^\varepsilon_q(s) - z^\varepsilon_q| + |z^\varepsilon_q - z_2| + |z_2 - z^f(s)|$$

$$\leq \zeta^\varepsilon_q(s_2) - z^\varepsilon_q + |z^\varepsilon_q - z_2| + \lambda$$

$$\leq |\zeta^\varepsilon_q(s_2) - z^f(s_2)| + 2|z^\varepsilon_q - z_2| + 2\lambda,$$

so that

$$\sup_{s_0 \leq s \leq s_2} |\zeta^\varepsilon_q(s) - z^f(s)| \leq \sup_{s_0 \leq s \leq s_2} |\zeta^\varepsilon_q(s) - z^\varepsilon_q| + 2|z^\varepsilon_q - z_2| + 2\lambda.$$

Since $z^\varepsilon_q$ tends to $z_2$ and $\lambda$ is arbitrary, the uniform convergence of $\zeta^\varepsilon_q$ to $z^f$ on compact subsets of $\mathbb{R}$ implies the uniform convergence on all of $[s_0, +\infty)$. Let us prove now that $\zeta^{\varepsilon}_{\varphi}$ converges to $z^f$ uniformly on $(\infty, 0]$. We first observe that for every $s_0 < 0$ there exists $\varepsilon_0 > 0$ such that $\zeta^{\varepsilon}_{\varphi}(s) = \zeta^{\varepsilon}(s)$ for every $s \in [s_0, 0]$ and every $\varepsilon \in (0, \varepsilon_0)$. Indeed, by the uniform convergence of $\zeta^\varepsilon_q$ to $z^f$ and the properties of $z^f$ listed Lemma 6.2 there exists $\varepsilon_0$ such that $z^\varepsilon_q < \zeta^\varepsilon_q(s) < z^1_\varepsilon$ for every $s \in [s_0, 0]$ and every $\varepsilon \in (0, \varepsilon_0)$. As observed in the proof of Lemma 7.3 we have $(q^\varepsilon_q, \vartheta^\varepsilon_q) = (q_\varepsilon, \vartheta_\varepsilon)$ on $[z^\varepsilon_q, z^1_\varepsilon]$. This implies that on the interval $[s_0, 0]$ the function $\zeta^{\varepsilon}_{\varphi}$ is in fact solution of (7.72). Since it satisfies the same Cauchy conditions as $\zeta^{\varepsilon}_{\varphi}$, by uniqueness we have $\zeta^{\varepsilon}_{\varphi}(s) = \zeta^{\varepsilon}_{\varphi}(s)$ for every $s \in [s_0, 0]$. Let us fix $\lambda > 0$. By (6.32) we can find $s_0 \in (0, 0)$ such that $|p^\varepsilon_q(s) - z_1| < \lambda$ for any $s \in (0, s_0)$. Since $\zeta^{\varepsilon}_{\varphi} = \zeta^{\varepsilon}_{\varphi}$ on $[s_0, 0]$ for $\varepsilon$ small enough, we have that $\zeta^{\varepsilon}_{\varphi} \to z^f$ uniformly on $[s_0, 0]$. For $s \leq s_0$ the monotonicity of $\zeta^{\varepsilon}_{\varphi}$ gives

$$|\zeta^{\varepsilon}_{\varphi}(s) - z^f(s)| \leq |\zeta^{\varepsilon}_{\varphi}(s) - z_1| + |z_1 - z^f(s)|$$

$$\leq z_1 - \zeta^{\varepsilon}_{\varphi}(s_0) + \lambda \leq |\zeta^{\varepsilon}_{\varphi}(s_0) - z^f(s_0)| + 2\lambda,$$
We now describe the behavior of the system in a small time interval $\tau$. Transition to the slow dynamics so that

$$
\sup_{s \leq 0} |\zeta_{\varepsilon}^{\ominus}(s) - z^f(s)| \leq \sup_{s_0 \leq s \leq 0} |\zeta_{\varepsilon}^{\ominus}(s) - z^f(s)| + 2\lambda.
$$

Since $\lambda$ is arbitrary, the uniform convergence of $\zeta_{\varepsilon}^{\ominus}$ to $z^f$ on compact subsets of $(-\infty, 0]$ implies the uniform convergence on all of $(-\infty, 0]$.

By (7.74) and (7.75), the uniform convergence of $\zeta_{\varepsilon}^{\ominus}$ and $\zeta_{\varepsilon}^{\oplus}$ to $z^f$ gives

$$
\sup_{\frac{1}{2} t \leq s \leq \frac{1}{2} t_1} |\zeta_{\varepsilon}(s) - z^f(s - s_\varepsilon)| \to 0.
$$

Since $z_{\varepsilon}(t) = \zeta_{\varepsilon}(\frac{\varepsilon}{t})$, this implies

$$
\sup_{t_1 \leq t \leq \tau_1} |z_{\varepsilon}(t) - z_{\varepsilon}^f(t)| \to 0. \tag{7.76}
$$

By Lemma 6.2 we have $\rho^f(s) = \rho(z^f(s))$ and $\theta^f(s) = \theta(z^f(s))$ for every $s \in \mathbb{R}$. By (2.34) and (7.68) we have $z_{\varepsilon}^2 \leq z_{\varepsilon}(t) \leq z_{\varepsilon}^1$ for every $t \in [t_1, \tau_1]$. It follows from (7.60) that

$$
\sup_{t_1 \leq t \leq \tau_1} (|\rho_{\varepsilon}(t) - \rho(z_{\varepsilon}(t))| + |\theta_{\varepsilon}(t) - \theta(z_{\varepsilon}(t))|) \to 0.
$$

Since $\rho^f(t) = \rho(z_{\varepsilon}^f(t))$ and $\theta^f(t) = \theta(z_{\varepsilon}^f(t))$, using (7.76) and the uniform continuity of $\rho$ and $\theta$ we obtain

$$
\sup_{t_1 \leq t \leq \tau_1} (|\rho_{\varepsilon}(t) - \rho(z_{\varepsilon}(t))| + |\theta_{\varepsilon}(t) - \theta(z_{\varepsilon}(t))|) \leq \sup_{t_1 \leq t \leq \tau_1} (|\rho_{\varepsilon}(t) - \rho(z_{\varepsilon}(t))| + |\theta_{\varepsilon}(t) - \theta(z_{\varepsilon}(t))|) + \sup_{t_1 \leq t \leq \tau_1} (|\rho(z_{\varepsilon}(t)) - \rho(z_{\varepsilon}^f(t))| + |\theta(z_{\varepsilon}(t)) - \theta(z_{\varepsilon}^f(t))|) \to 0,
$$

which, together with (7.76), gives (7.70).

\[ \square \]

7.3. Transition to the slow dynamics

We now describe the behavior of the system in a small time interval $[\tau_1, t_\varepsilon]$ after which the system is governed by the slow dynamics. During this transition $\rho_{\varepsilon}(t) - z_{\varepsilon}(t)$ decreases from the value $\varepsilon^{1-\alpha}$, attained at $t = \tau_1$, to a value of order $\varepsilon$, attained at $t = t_\varepsilon$.

**Lemma 7.6.** Let $\frac{\alpha}{2} < \theta_2 \leq \pi$, let $0 < z_2 < z_\varepsilon(\theta_2)$, let $t_1 \in [t_0, +\infty)$, let $\alpha \in (0, \frac{1}{2})$, and let $\tau_1$ be a sequence in $[t_0, +\infty)$ such that

$$
\tau_1 \to t_1, \quad \rho_{\varepsilon}(\tau_1) \to z_2, \quad \theta_{\varepsilon}(\tau_1) \to \theta_2, \quad z_{\varepsilon}(\tau_1) \to z_2, \tag{7.77}
$$

$$
\rho_{\varepsilon}(\tau_1) - z_{\varepsilon}(\tau_1) = \varepsilon^{1-\alpha}. \tag{7.78}
$$

Then there exist a sequence $t_\varepsilon$ in $[t_0, +\infty)$ and a constant $\beta_1 > 0$ such that

$$
\tau_1 < t_\varepsilon \quad \text{and} \quad t_\varepsilon \to t_1 \quad \text{as} \quad \varepsilon \to 0, \tag{7.79}
$$

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Moreover, by Lemma 2.6 we have
\[ \sup_{\tau^\varepsilon_1 \leq \tau^\varepsilon_1} \left( |\rho^\varepsilon(t) - z^\varepsilon| + |\theta^\varepsilon(t) - \theta^\varepsilon| + |z^\varepsilon(t) - z^\varepsilon| \right) \to 0 \quad \text{as } \varepsilon \to 0. \] (7.81)

**Proof.** As \( z^\varepsilon < z^\varepsilon(\theta^\varepsilon) \), we have \( z^\varepsilon(1 + \cos \theta^\varepsilon)^2 \cos \theta^\varepsilon + 1 > 0 \). Let \( \kappa > 0 \) be such that \( z^\varepsilon(1 + \cos \theta^\varepsilon)^2 \cos \theta^\varepsilon + 1 > \frac{2}{\kappa} \). Under our hypotheses, by continuity there exists \( \eta > 0 \), with \( \eta < \frac{z^\varepsilon}{2} \), such that
\[ z(1 + \cos \theta)^2 \cos \theta + 1 \geq \frac{2}{\kappa} \quad \text{for } |\theta - \theta^\varepsilon| < \eta \quad \text{and} \quad |z - z^\varepsilon| < \eta. \] (7.82)

We define
\[ t^\varepsilon := \inf \{ t \in (\tau^\varepsilon_1, +\infty) : \rho^\varepsilon(t) - z^\varepsilon(t) \geq \kappa \varepsilon \}, \] (7.83)
\[ \alpha^\varepsilon := \inf \{ t \in (\tau^\varepsilon_1, +\infty) : |\theta^\varepsilon(t) - \theta^\varepsilon| + |z^\varepsilon(t) - z^\varepsilon| > \eta \}, \] (7.84)
\[ s^\varepsilon := \min \{ t^\varepsilon, \alpha^\varepsilon \}. \] (7.85)

Since \( \rho^\varepsilon(t) - z^\varepsilon(t) \geq \kappa \varepsilon \) for every \( t \in [\tau^\varepsilon_1, t^\varepsilon] \), from (7.82) we obtain
\[ (z^\varepsilon(t)(1 + \cos \theta^\varepsilon(t))^2 \cos \theta^\varepsilon(t) + 1)(\rho^\varepsilon(t) - z^\varepsilon(t)) \geq 2 \varepsilon \quad \text{for every } t \in [\tau^\varepsilon_1, s^\varepsilon]. \]

Therefore (2.25) gives
\[ \dot{\rho}^\varepsilon(t) - \dot{z}^\varepsilon(t) \leq -\varepsilon \quad \text{for every } t \in [\tau^\varepsilon_1, s^\varepsilon], \] (7.86)
which, after integration, yields
\[ s^\varepsilon - \tau^\varepsilon_1 \leq \varepsilon^{1-\alpha}. \] (7.87)

By Lemma 2.6, (7.78), and (7.86) we have \( 0 < \rho^\varepsilon(t) - z^\varepsilon(t) \leq \rho^\varepsilon(\tau^\varepsilon_1) - z^\varepsilon(\tau^\varepsilon_1) = \varepsilon^{1-\alpha} \) for every \( t \in [\tau^\varepsilon_1, s^\varepsilon] \). Since \( 0 < z^\varepsilon(t) \leq z_0 \) by (2.34), from the third equation in (2.21) we have
\[ \ddot{z}^\varepsilon(t) \geq -2z^\varepsilon(t)\varepsilon^{-\alpha} \geq -2z_0 \varepsilon^{-\alpha} \quad \text{for every } t \in [\tau^\varepsilon_1, s^\varepsilon], \]
which, together with (2.34) and (7.87), implies
\[ z^\varepsilon(\tau^\varepsilon_1) \geq z^\varepsilon(t) \geq z^\varepsilon(\tau^\varepsilon_1) - 2z_0 \varepsilon^{-\alpha}(t - \tau^\varepsilon_1) \geq z^\varepsilon(\tau^\varepsilon_1) - 2z_0 \varepsilon^{1-2\alpha} \]
for every \( t \in [\tau^\varepsilon_1, s^\varepsilon] \),
which gives
\[ |z^\varepsilon(t) - z^\varepsilon| \leq |z^\varepsilon(\tau^\varepsilon_1) - z^\varepsilon| + 2z_0 \varepsilon^{1-2\alpha} \quad \text{for every } t \in [\tau^\varepsilon_1, s^\varepsilon]. \] (7.88)

From the second equation in (2.21) we have
\[ \rho^\varepsilon(t) \dot{\rho}^\varepsilon(t) \leq -1 - 2z_0 \varepsilon^{-\alpha} \quad \text{for every } t \in [\tau^\varepsilon_1, s^\varepsilon]. \]
Moreover, by Lemma 2.6 we have
\[ \rho^\varepsilon(t) > z^\varepsilon(t) \geq z_2 - \eta > \frac{z_2}{2} \quad \text{for every } t \in [\tau^\varepsilon_1, s^\varepsilon]. \] (7.89)
Thus, recalling (2.32),
\begin{equation}
0 \geq \dot{\theta}_\varepsilon(t) \geq -\frac{2}{z_2} - 4\varepsilon^{-\alpha} \quad \text{for every } t \in [\tau^1_\varepsilon, s^\eta_\varepsilon],
\end{equation}
and, integrating and using (7.87), we obtain
\begin{equation}
\theta_\varepsilon(\tau^1_\varepsilon) \geq \theta_\varepsilon(t) \geq \theta_\varepsilon(t^1_\varepsilon) - \left(\frac{2}{z_2} + 4\varepsilon^{-\alpha}\right)(t - \tau^1_\varepsilon) \geq \theta_\varepsilon(t^1_\varepsilon) - \left(\frac{2}{z_2} + 4\right)\varepsilon^{1-2\alpha}
\end{equation}
for every \( t \in [\tau^1_\varepsilon, s^\eta_\varepsilon] \), which gives
\begin{equation}
|\theta_\varepsilon(t) - \theta_2| \leq |\theta_\varepsilon(t^1_\varepsilon) - \theta_2| + \left(\frac{2}{z_2} + 4\right)\varepsilon^{1-2\alpha} \quad \text{for every } t \in [\tau^1_\varepsilon, s^\eta_\varepsilon].
\end{equation}

By (7.77) we have \( |z_\varepsilon(t^1_\varepsilon) - z_2| + \varepsilon^{1-2\alpha} + |\theta_\varepsilon(t^1_\varepsilon) - \theta_2| + \left(\frac{2}{z_2} + 4\right)\varepsilon^{1-2\alpha} < \eta \) for \( \varepsilon \) small enough. Therefore (7.88) and (7.91) give \( s^\eta_\varepsilon < \alpha^\eta_\varepsilon \) for \( \varepsilon \) small enough. By (7.85) this implies \( s^\eta_\varepsilon = t_\varepsilon \), so that (7.87) gives \( t_\varepsilon - \tau^1_\varepsilon \leq \varepsilon^{1-\alpha} \), which concludes the proof of (7.79). Since \( t_\varepsilon < +\infty \), inequality (7.80) follows from the definition of \( t_\varepsilon \) given in (7.83), while (7.81) follows from (7.88) and (7.91).

**7.4. Softening with discontinuity**

In this subsection, we prove Theorems 7.1 and 7.2 describing the softening regime with a discontinuity.

**Proof.** (Proof of Theorem 7.1) Let us prove that there exists a sequence \( \tau_\varepsilon \) in \([t_0, +\infty)\) such that
\begin{equation}
\tau_\varepsilon \to t_1, \quad \rho_\varepsilon(\tau_\varepsilon) \to z_1, \quad \theta_\varepsilon(\tau_\varepsilon) \to \theta_1, \quad z_\varepsilon(\tau_\varepsilon) \to z_1,
\end{equation}
\begin{equation}
\sup_{t_0 \leq t \leq \tau_\varepsilon} (|\rho_\varepsilon(t) - \rho^s_0(t)| + |\theta_\varepsilon(t) - \theta^s_\varepsilon(t)| + |z_\varepsilon(t) - \rho^s_0(t)|) \to 0.
\end{equation}

Let us fix an integer \( k > 0 \). We can apply Lemma 5.1 with \( t_\varepsilon = t_0, \tau = t_1 - \frac{1}{k} \), \( \theta_\varepsilon = \theta_0, z_\varepsilon = z_0 \), and \( t^*_\varepsilon = t_0 \). Indeed, (5.18) follows from (3.21). By (5.21) we have
\begin{equation}
\sup_{t_0 \leq t \leq t_1 - \frac{1}{k}} (|\rho_\varepsilon(t) - \rho^s_0(t)| + |\theta_\varepsilon(t) - \theta^s_\varepsilon(t)| + |z_\varepsilon(t) - \rho^s_0(t)|) \to 0.
\end{equation}

Let us fix a decreasing sequence \( \varepsilon_k \to 0 \). There exists a decreasing sequence \( \varepsilon_k \to 0 \) such that for every \( \varepsilon \in (0, \varepsilon_k] \) we have
\begin{equation}
\sup_{t_0 \leq t \leq t_1 - \frac{1}{k}} (|\rho_\varepsilon(t) - \rho^s_0(t)| + |\theta_\varepsilon(t) - \theta^s_\varepsilon(t)| + |z_\varepsilon(t) - \rho^s_0(t)|) \leq a_k.
\end{equation}

We now define \( \tau_\varepsilon := t_1 - \frac{1}{k} \) for every \( \varepsilon \in (\varepsilon_{k+1}, \varepsilon_k] \). Then \( \tau_\varepsilon \to t_1 \) as \( \varepsilon \to 0 \), and (7.93) follows from (7.94). From (7.93) we obtain, in particular,
\begin{equation}
|\rho_\varepsilon(\tau_\varepsilon) - \rho^s_0(\tau_\varepsilon)| + |\theta_\varepsilon(\tau_\varepsilon) - \theta^s_\varepsilon(\tau_\varepsilon)| + |z_\varepsilon(\tau_\varepsilon) - \rho^s_0(\tau_\varepsilon)| \to 0.
\end{equation}

Since \( \tau_\varepsilon \to t^1_\varepsilon \), this implies (7.92) thanks to (3.18).
Let us fix \( \alpha \in (0, \frac{1}{2}) \). By Lemma 7.1 there exists a sequence \( t^*_\varepsilon \) in \([t_0, +\infty)\) which satisfies (7.15)-(7.18). By (3.18) we have

\[
\sup_{\tau_\varepsilon \leq t \leq t^*_\varepsilon} (|\rho_{\varepsilon}^{sl}(t) - z_1| + |\theta_{\varepsilon}^{sl}(t) - \theta_1|) \to 0.
\]

Together with (7.18) and (7.93), this proves (7.4).

By Lemmas 7.4 and 7.5 there exists \( \tau_1^\varepsilon > t^*_\varepsilon \) such that (7.70) holds and \( \tau_1^\varepsilon \to t_1 \) as \( \varepsilon \to 0 \). This proves (7.5) and concludes the proof of (7.3).

By Lemma 7.6 there exists a sequence \( t_\varepsilon \) in \([t_0, +\infty)\) which satisfies (7.79)-(7.81).

By (3.23) we have

\[
\sup_{\tau_1^\varepsilon \leq t \leq t_\varepsilon} (|\rho_{\varepsilon}^{sl}(t) - z_2| + |\theta_{\varepsilon}^{sl}(t) - \theta_2|) \to 0,
\]

which, together with (7.81), gives

\[
\sup_{\tau_1^\varepsilon \leq t \leq t_\varepsilon} (|\rho_{\varepsilon}(t) - \rho_{\varepsilon}^{sl}(t)| + |\theta_{\varepsilon}(t) - \theta_{\varepsilon}^{sl}(t)| + |z_{\varepsilon}(t) - \rho_{\varepsilon}^{sl}(t)|) \to 0. \tag{7.95}
\]

We can apply now Lemma 5.1 with \( t_\varepsilon = t_1, \theta_\varepsilon = \theta_2, z_\varepsilon = z_2 \), and \( t^*_\varepsilon = \tau_1^\varepsilon \). Indeed, hypothesis (5.18) follows from (3.26), while (5.19) and (5.20) are satisfied thanks to (7.79)-(7.81). We deduce that (5.21) holds with \( \rho_{\varepsilon}^{sl} = \rho_{\varepsilon}^{sl} \). Together with (7.95), this proves (7.6). Equalities (7.2) follow from (7.3), (7.4) and (7.6).}

**Proof.** (Proof of Theorem 7.2) Let us fix \( \alpha \in (0, \frac{1}{2}) \). We apply Lemma 7.1 with \( z_1 = z_0, \theta_1 = \theta_0, t_1 = \tau_\varepsilon = t_0 \), and we find a sequence \( t^*_\varepsilon \) in \([t_0, +\infty)\) which satisfies (7.15)-(7.18) with \( \tau_\varepsilon = t_0 \). In particular, we have

\[
|t^*_\varepsilon - t_1| + |\rho_{\varepsilon}(t^*_\varepsilon) - z_1| + |\theta_{\varepsilon}(t^*_\varepsilon) - \theta_1| + |z_{\varepsilon}(t^*_\varepsilon) - z_1| \to 0. \tag{7.96}
\]

By Lemmas 7.4 and 7.5 there exists \( t^*_\varepsilon > t^*_\varepsilon \) which satisfies (7.11) and (7.70). In particular, we have

\[
|\rho_{\varepsilon}(t^*_\varepsilon) - \rho_{\varepsilon}(t^*_\varepsilon)| + |\theta_{\varepsilon}(t^*_\varepsilon) - \theta_{\varepsilon}(t^*_\varepsilon)| + |z_{\varepsilon}(t^*_\varepsilon) - z^*_\varepsilon| \to 0,
\]

which, together with (7.96), gives

\[
|\rho_{\varepsilon}(t^*_\varepsilon) - z_1| + |\theta_{\varepsilon}(t^*_\varepsilon) - \theta_1| + |z_{\varepsilon}(t^*_\varepsilon) - z_1| \to 0. \tag{7.97}
\]

By (6.32), (6.34), and (7.7) we have

\[
|\rho_{\varepsilon}(t) - z_1| + |\theta_{\varepsilon}(t) - \theta_1| + |\rho_{\varepsilon}(t) - z_1| \\
\leq |\rho_{\varepsilon}(t^*_\varepsilon) - z_1| + |\theta_{\varepsilon}(t^*_\varepsilon) - \theta_1| + |\rho_{\varepsilon}(t^*_\varepsilon) - z_1|
\]

for every \( t \in (-\infty, t^*_\varepsilon] \), so that (7.97) gives

\[
\sup_{t^*_\varepsilon \leq t \leq t^*_\varepsilon} (|\rho_{\varepsilon}(t) - z_1| + |\theta_{\varepsilon}(t) - \theta_1| + |\rho_{\varepsilon}(t) - z_1|) \to 0.
\]

Together with (7.18) and (7.70) this proves (7.12).
We apply Lemma 7.6 with $t_1 = t_0$ and find a sequence $t_\epsilon$ converging to $t_0$ which satisfies (7.79)–(7.81). As in the proof of Theorem 7.1 we obtain (7.95). We can apply now Lemma 5.1 with $t_s = t_0$, $\theta_s = \theta_2$, $z_s = z_2$, and $t_\epsilon = t_\epsilon$. Indeed, hypothesis (5.18) follows from (3.26), while (5.19) and (5.20) are satisfied thanks to (7.79)–(7.81). We deduce that (5.21) holds with $\rho_s = \rho_2$. Together with (7.95), this proves (7.13). Equalities (7.10) follow from (7.11)–(7.13). \qed

8. Mechanical Interpretation of the Results

We conclude the paper with some comments on the mechanical interpretation of our results. We first recall that the scalar variables $x$ and $y$ are related to the stress by the formula

$$\sigma(t) = e(t) = -\frac{1}{n} x(t) I + \frac{1}{\sqrt{n}} y(t) e_0,$$

where $e_0 \in M_{n \times n}^{\text{sym}}$ is a fixed traceless matrix with unit norm. It follows that $-x(t)$ is the trace of the stress, so that, with the usual sign conventions, $\frac{x(t)}{n}$ is the pressure. The scalar $\frac{1}{\sqrt{n}} |y(t)|$ is the norm of the deviatoric part of the stress, usually denoted by $q$ in soil mechanics. For simplicity, in what follows we will call $x$ and $y$ the pressure coefficient and the deviatoric stress coefficient.

By (1.6) we have

$$x(t) = z(t) + \rho(t) \cos \theta(t) \quad \text{and} \quad y(t) = \rho(t) \sin \theta(t).$$

Since $\rho(t) = z(t)$ for every viscosity solution (Theorems 5.1, 5.2, 7.1, and 7.2), we conclude that

$$x(t) = \rho(t)(1 + \cos \theta(t)) \quad \text{and} \quad y(t) = \rho(t) \sin \theta(t) \quad \text{for every } t \in [t_0, +\infty).$$

From the above-mentioned theorems and from Lemmas 3.5–3.7 it follows that $\rho(t) > 0$ and $\frac{\pi}{2} \leq \theta(t) \leq \pi$ for every $t \in [t_0, +\infty)$. Moreover, $\theta(t) = \pi$ only at the initial time $t = t_0$ for the special loading program corresponding to $a_0 = 0$ (i.e. in the absence of a preconsolidation pressure, see (1.4)) so that $t_0 = 0$. Using also (2.20) we deduce that

$$x(t) \geq 0 \quad \text{and} \quad y(t) \geq 0 \quad \text{for every } t \in [0, +\infty),$$

and that $x(t) = 0$ if and only if $t = t_0 = 0$ and $a_0 = 0$, while $y(t) = 0$ if and only if $t = 0$.

Plastic behavior starts at $t = t_0$. The initial data for the plastic regime are given by

$$x_0 := x(t_0) = z_0(1 + \cos \theta_0) = a_0 \quad \text{and} \quad y_0 := y(t_0) = z_0 \sin \theta_0 = t_0,$$

respectively. In Cartesian coordinates the separation line $\rho = z_s(\theta)$ and the critical line $\rho = r_c(\theta)$ of the $(\rho, \theta)$ plane introduced in (3.4) and (3.6) become the parametric
curves defined by

\[ x_s(\theta) := z_s(\theta)(1 + \cos \theta) \quad \text{and} \quad y_s(\theta) := z_s(\theta) \sin \theta \quad \text{for} \ \theta \in \left(\frac{\pi}{2}, \pi\right), \]

\[ x_c(\theta) := r_c(\theta)(1 + \cos \theta) \quad \text{and} \quad y_c(\theta) := r_c(\theta) \sin \theta \quad \text{for} \ \theta \in \left[\frac{\pi}{2}, \pi\right]. \]

The critical point \((\theta_c, z_c)\) becomes

\[ x_c := z_c(1 + \cos \theta_c) \quad \text{and} \quad y_c := z_c \sin \theta_c. \]

Fig. 7. Phase diagram in the \((x, y)\) plane. Dark gray region (including the thick line): initial data \((x_0, y_0)\) of the plastic regime producing a continuous evolution. Light gray region: initial data producing a discontinuity at time \(t_1 > t_0\). White region: initial data producing a discontinuity at time \(t_1 = t_0\). The dotted line is composed of fixed points and separates softening behavior (above the line) from hardening behavior (below the line).
The phase diagram in the \((x, y)\) plane is obtained from Fig. 3 by a change of variables and is shown in Fig. 7.

The trajectories of \((x(t), y(t))\) are shown in Fig. 8, while Fig. 9 illustrates the behavior of \(x(t)\) and \(y(t)\) as functions of \(t\). Note that, by our choice of the loading program (1.4), \(t\) is proportional to the norm of the imposed deviatoric strain. Moreover, we note that the straight line \(x = y\) (critical state line) is composed of fixed points. Each trajectory \(x(t), y(t)\) tends to a fixed point as \(t \to \infty\). The region below the critical state line is invariant, and all solutions therein display a hardening.

Fig. 8. Trajectories of \((x(t), y(t))\) in the plastic regime for several values of the initial data \((x_0, y_0)\). The evolution for \(t > t_0\) is obtained following the trajectory through \((x_0, y_0)\) in the sense of the arrow. Solid lines: slow dynamics. Dashed lines: fast dynamics. Dotted line: fixed points.
behavior, namely, $z(t)$ is increasing. Moreover, $y(t)$ is increasing. Both of these properties follow from (3.15) and Theorem 5.1.

In the region above the critical state line the trajectories exhibit softening, namely, $z(t)$ is decreasing. Some trajectories are continuous and follow the system of the slow dynamics. Other trajectories in this region exhibit a discontinuity, which may occur either at $t = t_0$ or at $t > t_0$. They follow the system of the slow dynamics in the intervals of continuity, and their trajectories follow instantaneously the system of the fast dynamics at the jump time. These different behaviors are described in Fig. 7.

The monotonicity of $z(t)$ in the intervals of slow dynamics follows from (3.20), (3.24), and Theorem 5.2. For what concerns the jump governed by the fast dynamics, we observe that under the assumptions of Lemma 6.1, the solution $(x, y)$ of (6.3) with Cauchy condition (6.5) satisfies $x'(z) < 0$ and $y'(z) > 0$ for every $z \in (z_2, z_1)$ in view of (6.9). Finally in the intervals of fast dynamics described by the solution $(x^f, y^f, z^f)$ of (6.31), $\dot{x}^f(s) > 0$, $\dot{y}^f(s) < 0$, and $\dot{z}^f(s) < 0$ for every $s \in \mathbb{R}$ in view of (6.34).
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References