Supplementary material for the article *Tunable morphing of electroactive dielectric-elastomer balloons*

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1 Derivation of the Stroh formulation

The incremental constitutive relation can be obtained by expanding Eq. (3.1), as

$$
\begin{aligned}
\dot{S}_{\theta r} &= A_{2121}\frac{1}{r}\frac{\partial \dot{u}_r}{\partial \theta} - A_{2121}\frac{\dot{u}_\theta}{r} + (A_{1221}+p)\frac{\partial \dot{u}_\theta}{\partial r} + \Gamma_{122}\dot{D}_\theta,\\
\dot{S}_{\theta\theta} &= A_{1122}\frac{\partial \dot{u}_r}{\partial r} + (A_{2222}+p)\frac{1}{r}\left(\dot{u}_r + \frac{\partial \dot{u}_\theta}{\partial \theta}\right) + A_{2233}\frac{1}{r}\left(\dot{u}_r + \dot{u}_\theta\cot\theta\right) + \Gamma_{221}\dot{D}_r - \dot{p},\\
\dot{S}_{\psi\psi} &= A_{1122}\frac{\partial \dot{u}_r}{\partial r} + A_{2233}\frac{1}{r}\left(\dot{u}_r + \frac{\partial \dot{u}_\theta}{\partial \theta}\right) + (A_{2222}+p)\frac{1}{r}\left(\dot{u}_r + \dot{u}_\theta\cot\theta\right) + \Gamma_{221}\dot{D}_r - \dot{p},\\
\dot{S}_{r\theta} &= A_{1212}\frac{\partial \dot{u}_\theta}{\partial r} + (A_{1221}+p)\frac{1}{r}\frac{\partial \dot{u}_r}{\partial \theta} - (A_{1221}+p)\frac{\dot{u}_\theta}{r} + \Gamma_{122}\dot{D}_\theta,\\
\dot{S}_{\theta r} &= A_{2121}\frac{1}{r}\frac{\partial \dot{u}_r}{\partial \theta} - A_{2121}\frac{\dot{u}_\theta}{r} + (A_{1221}+p)\frac{\partial \dot{u}_\theta}{\partial r} + \Gamma_{122}\dot{D}_\theta,\\
\dot{E}_r &= -\frac{\partial \dot{\phi}}{\partial r} = \Gamma_{111}\frac{\partial \dot{u}_r}{\partial r} + \Gamma_{221}\left[\frac{1}{r}\left(\dot{u}_r + \frac{\partial \dot{u}_\theta}{\partial \theta}\right) + \frac{1}{r}\left(\dot{u}_r + \dot{u}_\theta\cot\theta\right)\right] + K_{011}\dot{D}_r,\\
\dot{E}_\theta &= -\frac{1}{r}\frac{\partial \dot{\phi}}{\partial \theta} = \Gamma_{122}\left[\frac{\partial \dot{u}_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial \dot{u}_r}{\partial \theta} - \dot{u}_\theta\right)\right] + K_{22}\dot{D}_\theta.
\end{aligned}
\tag{S1}
$$

The components of the incremental equilibrium equation (3.3) and the incremental Maxwell equation (3.5) read

$$
\begin{aligned}
&\frac{\partial \dot{S}_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \dot{S}_{\theta r}}{\partial \theta} + \frac{1}{r}\left(2\dot{S}_{rr} - \dot{S}_{\theta\theta} - \dot{S}_{\psi\psi} + \dot{S}_{\theta r}\cot\theta\right) = 0,\\
&\frac{\partial \dot{S}_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \dot{S}_{\theta\theta}}{\partial \theta} + \frac{1}{r}\left[\left(\dot{S}_{\theta\theta} - \dot{S}_{\psi\psi}\right)\cot\theta + 2\dot{S}_{r\theta} + \dot{S}_{\theta r}\right] = 0,
\end{aligned}
\tag{S2}
$$

and

$$
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\dot{D}_r\right) + \frac{1}{r\sin\theta}\frac{\partial\left(\dot{D}_\theta\sin\theta\right)}{\partial \theta} = 0,
\tag{S3}
$$

respectively.

The incremental incompressibility constraint can be derived from Eq. (3.7) as

$$\frac{\partial \dot{u}_r}{\partial r} + \frac{1}{r}\left(\dot{u}_r + \frac{\partial \dot{u}_\theta}{\partial \theta}\right) + \frac{1}{r}\left(\dot{u}_r + \dot{u}_\theta \cot\theta\right) = 0. \tag{S4}$$

We combine Eqs. (3.9) and (S4) to obtain

$$U_r' = -\frac{1}{r}\left(U_r + MU_\theta\right), \tag{S5}$$

where the prime denotes the derivative with respect to r. From Eqs. $(S1)_{4,7}$ and (S4) we get

$$U_\theta' = -\frac{1}{\alpha}\left[(S_{rr}-\alpha)\frac{M}{r}U_r + \frac{\alpha - S_{rr}}{r}U_\theta + \frac{\Gamma_{122}M}{K_{22}r}\Phi + \frac{1}{r}(r\Sigma_{r\theta})\right]. \tag{S6}$$

Substitution of Eq. (3.9) into Eq. $(S1)_6$ gives

$$\Phi' = \frac{2(\Gamma_{111}-\Gamma_{221})}{r}U_r - \frac{\Gamma_{111}-\Gamma_{221}}{r}MU_\theta - \frac{K_{11}}{r}(r\Delta_r). \tag{S7}$$

Rewriting Eqs. $(S1)_7$, (S3) and (3.9) results in

$$(r\Delta_r)' = \frac{1}{r}\left[\frac{S_{rr}\Gamma_{22}M^2}{\alpha K_{22}}U_r + \frac{S_{rr}\Gamma_{22}M}{\alpha K_{22}}U_\theta - \left(\frac{1}{K_{22}} + \frac{\Gamma_{122}^2}{\alpha K_{22}^2}\right)M^2\Phi - r\Delta_r - \frac{\Gamma_{122}M}{\alpha K_{22}}(r\Sigma_{r\theta})\right]. \tag{S8}$$

Then, we substitute Eq. (3.9) into Eqs. $(S1)_{1,2,3,5,11}$ and $(S2)_1$ to get

$$(r\Sigma_{rr})' = \frac{1}{r}\left[\kappa_{11}U_r + \kappa_{12}U_\theta + \kappa_{13}(r\Delta_r) + r\Sigma_{rr} + \frac{(\alpha - S_{rr})M}{\alpha}(r\Sigma_\theta) + \frac{S_{rr}\Gamma_{122}M^2}{\alpha K_{22}}\Phi\right]. \tag{S9}$$

where

$$\kappa_{11} = 2(2A_{1111} - 4A_{1122} + A_{2222} + A_{2233} + 3p) + \frac{(\alpha - S_{rr})^2}{\alpha}M^2, \quad \kappa_{12} = \frac{(\alpha - S_{rr})^2}{\alpha}. \tag{S10}$$

Similarly, from Eqs. (3.9), $(S1)_{1,2,3,5}$ and $(S2)_2$, we have

$$(r\Sigma_{r\theta})' = \frac{1}{r}\left[\kappa_{21}U_r + \kappa_{22}U_\theta + \kappa_{13}U_z - M(r\Sigma_{rr}) - \frac{2\alpha - S_{rr}}{\alpha}(r\Sigma_{r\theta}) + \frac{S_{rr}K_{122}M}{\alpha K_{22}}\Phi\right]. \tag{S11}$$

Finally we can write Eqs. (S5)-(S11) in the form (3.11).

$$\mathsf{G}_1 = \begin{bmatrix} -2 & M & 0 \\ -\dfrac{(\alpha-\sigma_{rr})M}{\alpha} & \dfrac{\alpha-\sigma_{rr}}{\alpha} & 0 \\ -\dfrac{\sigma_{rr}\beta M^2}{\alpha} & \dfrac{\sigma_{rr}\beta M}{\alpha} & -1 \end{bmatrix}, \quad \mathsf{G}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{1}{\alpha} & \dfrac{\beta M}{\alpha} \\ 0 & -\dfrac{\beta M}{\alpha} & -\left(\dfrac{1}{K_{022}} + \dfrac{\beta^2}{\alpha}\right)M^2 \end{bmatrix},$$
$$\mathsf{G}_3 = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{bmatrix}, \quad \mathsf{G}_4 = \begin{bmatrix} 1 & \dfrac{(\alpha-\sigma_{rr})M}{\alpha} & -\dfrac{\sigma_{rr}\beta M^2}{\alpha} \\ -M & -\dfrac{2\alpha-\sigma_{rr}}{\alpha} & \dfrac{\sigma_{rr}\beta M}{\alpha} \\ 0 & 0 & 0 \end{bmatrix}, \tag{S12}$$

and

$$
\begin{aligned}
&A_{01212}-\tfrac{\Gamma_{0122}^2}{K_{022}}=\alpha, \quad \frac{\Gamma_{0122}}{K_{022}}=\beta,\\
&A_{01221}+p-\tfrac{\Gamma_{0122}^2}{K_{022}}=A_{01221}+A_{01212}-\sigma_{rr}-A_{01221}-\tfrac{\Gamma_{0122}^2}{K_{022}}=\alpha-\sigma_{rr}\\
&\gamma_{21}=A_{02121}-\tfrac{\Gamma_{0122}^2}{K_{022}},\\
&\kappa_{11}=2\left(2A_{01111}-4A_{01122}+A_{02222}+A_{02233}+3p\right)+M^2\left(\gamma_{21}-\tfrac{(\alpha-\sigma_{rr})^2}{\alpha}\right),\\
&\kappa_{12}=\kappa_{21}=\left[\tfrac{(\alpha-\sigma_{rr})^2}{\alpha}-\gamma_{21}-\left(2A_{01111}-4A_{01122}+A_{02222}+A_{02233}+3p\right)\right]M,\\
&\kappa_{22}=-\left[M^2\left(2A_{01122}-A_{01111}-A_{02222}-2p\right)+\left(A_{02222}+p-A_{02233}-\gamma_{21}\right)+\tfrac{(\alpha-\sigma_{rr})^2}{\alpha}\right],\\
&\kappa_{13}=-\kappa_{31}=-2\left(\Gamma_{0111}-\Gamma_{0221}\right),\\
&\kappa_{23}=-\kappa_{32}=-\left(\Gamma_{0221}-\Gamma_{0111}\right)M,\\
&\kappa_{33}=-K_{011}.
\end{aligned}
\tag{S13}
$$

2 Incremental coefficients for a Gent material

From Eqs. (2.35) and (4.1), the radial stresses in the DE and elastic layers can be obtained as

$$
\begin{aligned}
T_{rr}^d=&-\frac{\mu^d G^d}{2}\sum_{j=1}^{6}\left.\frac{2\ln\left(\lambda-\eta_j^d\right)\left(\eta_j^d\right)^3-\ln\left(\lambda-\eta_j^d\right)-3\ln\left(\lambda-\eta_j^d\right)\eta_j^d-G^d\ln\left(\lambda-\eta_j^d\right)\eta_j^d}{3\left[\left(\eta_j^d\right)^3-\eta_j^d\right]-G^d\eta_j^d}\right|_{\lambda_i}^{\lambda}\\
&+2\mu^d G^d\ln\left(\frac{\lambda}{\lambda_i}\right)+\frac{\varepsilon}{2}\left(\frac{Vr_mr_i}{r_i-r_m}\right)^2\left(\frac{1}{r^4}-\frac{1}{r_i^4}\right)-P,
\end{aligned}
\tag{S14}
$$

and

$$
\begin{aligned}
T_{rr}^e=&\frac{\mu^e G^e}{2}\sum_{j=1}^{6}\left.\frac{2\ln\left(\lambda-\eta_j^e\right)\left(\eta_j^e\right)^3-\ln\left(\lambda-\eta_j^e\right)-3\ln\left(\lambda-\eta_j^e\right)\eta_j^e-G^e\ln\left(\lambda-\eta_j^e\right)\eta_j^e}{3\left[\left(\eta_j^e\right)^3-\eta_j^e\right]-G^e\eta_j^e}\right|_{\lambda}^{\lambda_o}\\
&-2\mu^e G^e\ln\left(\frac{\lambda_o}{\lambda}\right),
\end{aligned}
\tag{S15}
$$

respectively, where η_j^s $(s=d,e;\ j=1-6)$ are the six roots of the polynomial equation $1-\left(3+G^s\right)\left(\eta^s\right)^4+2\left(\eta^s\right)^6=0$.

Further, the circumferential stresses in the balloon can be obtained from Eq. (2.37) as

$$
T_{\theta\theta}^d=T_{rr}^d-\frac{G^d\mu^d}{G^d-\left(\lambda^{-4}+2\lambda^2-3\right)}\left(\lambda^{-4}-\lambda^2\right)-\varepsilon\left[\frac{Vr_ir_m}{r^2(r_m-r_i)}\right]^2 \tag{S16}
$$

at the DE layer, and

$$
T_{\theta\theta}^e=T_{rr}^e-\frac{G^e\mu^e}{G^e-\left(\lambda^{-4}+2\lambda^2-3\right)}\left(\lambda^{-4}-\lambda^2\right) \tag{S17}
$$

at the elastic layer, respectively.

As a result, the deformation of the solid can be solved using the interfacial condition

$$
T_{rr}^e\left(r_m\right)=T_{rr}^d\left(r_m\right). \tag{S18}
$$

We note that the displacements and normal stresses at the interfacial face of the layers are continuous, but there will be a jump for the circumferential stresses at the interface [1].

The non-zero components of the tensors containing the electro-elastic moduli for the considered

Gent elastomer (with the superscript s omitted) read

$$
\begin{gathered}
A_{1111} = \frac{\mu G \left[1 + \lambda^4 (G+3) - 2\lambda^6\right]}{\left[1 - \lambda^4 (G+3) + 2\lambda^6\right]^2} + \varepsilon D_r^2, \qquad A_{1122} = A_{1133} = \frac{2\mu G \lambda^6}{\left[1 - \lambda^4 (G+3) + 2\lambda^6\right]^2}, \\
A_{1212} = A_{1313} = -\frac{\mu G}{1 - \lambda^4 (G+3) + 2\lambda^6} + \varepsilon D_r^2, \\
A_{2121} = A_{3131} = A_{2323} = A_{3232} = -\frac{\mu G \lambda^6}{1 - \lambda^4 (G+3) + 2\lambda^6}, \\
A_{2222} = A_{3333} = \frac{\mu G \lambda^6 \left[\lambda^4 (G+3) - 1\right]}{\left[1 - \lambda^4 (G+3) + 2\lambda^6\right]^2}, \qquad A_{2233} = \frac{2\mu G \lambda^{12}}{\left[1 - \lambda^4 (G+3) + 2\lambda^6\right]^2}, \\
\Gamma_{111} = 2\Gamma_{122} = 2\Gamma_{212} = 2\Gamma_{133} = 2\Gamma_{313} = 2\varepsilon D_r, \qquad K_{11} = K_{22} = K_{33} = \varepsilon.
\end{gathered}
\tag{S19}
$$

References

[1] S. Roccabianca, M. Gei, and D. Bigoni. Plane strain bifurcations of elastic layered structures subject to finite bending: theory versus experiments. *IMA Journal of Applied Mathematics*, 75(4):525–548, 2010.