GENERALIZED WITT GROUPS

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In this paper we construct a functor from the category of one-dimensional commutative formal groups to the category of topological Abelian groups. For a multiplicative formal group, this function is the usual Witt functor. We study certain properties of the constructed functor. This functor is then used to describe multiplicative operations in the theory of unitary cobordisms.

1. BASIC DEFINITIONS

1. Let R be a commutative associative ring with unity, R[[X, Y]] the ring of formal power series in two variables, $F(X, Y) \in R[[X, Y]]$, a one-dimensional commutative formal group over R, $I(X) \in R[[X]]$ its inverse element, and

$$\omega(X) = \left(\sum_{i=0}^{\infty} p_i X^i\right) dX \tag{1.1}$$

its canonical invariant differential ($p_0 = 1$). If R is a Q-algebra there then exists a series $l(X) = \sum_{i=0}^{\infty} \frac{p_i}{i+1} X^{i+1}$ such that $\omega(X) = dl(X)$ and

$$F(X, Y) = l^{-1}(l(X) + l(Y)) \tag{1.2}$$

(see [2]). The symbol Λ will denote the topological space of formal power series over R without a free term, with the usual topology of formal power series. We impose on Λ the structure of an Abelian topological group $\Lambda(R, F)$, setting

$$(f +^F g)(X) = F(f(X), g(X)). \tag{1.3}$$

It is trivial to check that the axioms of a topological group hold. Let R not have elements of finite order.

LEMMA 1.4. Let $f(X) = \sum_{i=1}^{\infty} a_i X^i$. Then, the series $f(X)$ is uniquely represented in the form

$$f(X) = \sum_{i=1}^{\infty} {}^F \alpha_i X^i. \tag{1.5}$$

Moreover, if a_m is the first nonzero coefficient in $f(X)$ then $\alpha_i = a_i$ when $i \leq m$.

Proof. Expression (1.5) is meaningful since its right side converges in group $\Lambda(R, F)$. We have an equation over

$$R \otimes Q:\ l\left(\sum_{i=m}^{\infty} a_i X^i\right) = \sum_{i=m}^{\infty} l(\alpha_i X^i).$$

From this we have

$$a_m X^m + o(X^m) = \alpha_m X^m + o(X^m),$$

i.e., $a_m = \alpha_m$, which means that

$$l\left(\sum_{i=m}^{\infty} a_i X^i\right) - l(a_m X^m) = \sum_{i=m+1}^{\infty} l(\alpha_i X^i). \tag{1.6}$$

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Now applying the series for l to both sides of (1.6), we obtain

$$F(f(X), l(a_m X^m)) = \sum_{i=m+1}^{\infty} {}^F\alpha_i X^i. \tag{1.7}$$

But, the left side of (1.7) is a series over R which begins with a power greater than m. The remainder of the proof is an obvious induction.

COROLLARY 1.8. $\Lambda(R, F) \subset \Lambda(R \otimes Q, F)$ is a subgroup.

Example 1.9. Let $F_m(X, Y) = X + Y - XY$ be a multiplicative formal group. Consider the mapping $\lambda : \Lambda \to R[[X]]$, setting $f \xrightarrow{\lambda} 1 - f(X)$. Then, $(f +^F g)(X) = f(X) + g(X) - f(X)g(X) \xrightarrow{\lambda} [1 - f(X)][1 - g(X)]$, i.e., λ is a continuous monomorphism of group $\Lambda(R, F_m)$ into the multiplicative group of invertible elements of ring $R[[X]]$.

2. Definition 2.1. We call the generalized Witt group W(R, F) the set of infinite vectors $x = (x_1, x_2, \ldots)$, where $x_i \in R \forall i$, with the mappings

$$w_n(x) = \sum_{d \mid n} d p_{\frac{n}{d}-1} x_d^{n/d}, \qquad n = 1, 2, \ldots \tag{2.2}$$

defining a collection of homomorphisms in the additive group of ring R.

We shall call $x_1, x_2, \ldots$ the true coordinates of vector x, while $w_1(x), w_2(x), \ldots$ are illusory coordinates. It follows from (2.2) that the transition from true to illusory coordinates is invertible over the ring $R \otimes Q$, i.e., addition in group $W(R \otimes Q, F)$ is univocally defined by (2.1). We now show that W(R, F) is a subgroup in $W(R \otimes Q, F)$.

Definition 2.3. Mapping $E : W(R, F) \to R[[X]] \otimes Q$, defined by the formula

$$E(X) = l^{-1}\left(\sum_{n=1}^{\infty} \frac{1}{n} w_n(x) X^n\right), \tag{2.4}$$

is called the Artin-Hasse exponent.

THEOREM 2.5. The Artin-Hasse exponent defines an isomorphism of the groups $E : W(R, F) \to \Lambda(R, F)$.

Proof. From (2.4) we have:

$$\begin{aligned} E(x+y) &= l^{-1}\left(\sum_{n=1}^{\infty} \frac{1}{n} w_n(x) X^n + \sum_{n=1}^{\infty} \frac{1}{n} w_n(y) X^n\right) = \\ &= l^{-1}[l(E(x)) + l(E(y))] = E(x) +^F E(y). \end{aligned} \tag{2.6}$$

Furthermore:

$$\begin{aligned} E(x) &= l^{-1}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{d \mid n} d p_{\frac{n}{d}-1} x_d^{n/d}\right) X^n\right) = l^{-1}\left(\sum_{d,k} \frac{1}{k} p_{k-1} (x_d X^d)^k\right) = \\ &= l^{-1}\left(\sum_{d=1}^{\infty} l(x_d X^d)\right) = \sum_{d=1}^{\infty} {}^F x_d X^d. \end{aligned} \tag{2.7}$$

The theorem now follows from (2.6), (2.7), and Lemma 1.4.

COROLLARY 2.8. The structure of the group on W(R, F) is univocally defined by (2.1).

The proof follows directly from (1.8) and (2.5).

Example 2.9. For group F_m of example (1.9) we have: $p_i = 1$ for $i = 1, 2, \ldots$, i.e., formulas (2.2) assume the form $w_n(x) = \sum_{d/n} d x_d^{n/d}$. Consequently, group $W(R, F_m)$ is the additive group of a Witt ring (see [1]). This motivated our choice of name for group W(R, F).

In W(R, F) we introduce the topology induced by the natural valuation: $v(x) = n$, if x_n is the first nonzero true coordinate. Then, E becomes an isomorphism of topological groups.

Let F be the category whose objects are the pairs (R, F), with F being a formal group over ring R. A morphism f of category F is a ring homomorphism f: $R_1 \to R_2$, where $F_2 = F_1^f$, i.e., F_2 is obtained by applying f to the coefficients of F_1. We note that then $p_i^{F_2} = f(p_i^{F_1})$ where p_i is defined by (1.1). We define $W(f) : W(R_1, F_1) \to W(R_2, F_2)$, setting $W(f)(x_1, x_2, \ldots) = (f(x_1), f(x_2), \ldots)$. This, obviously, is a continuous isomorphism of the groups.

There is a universal object (see [3]) in category **F**. We denote it by the symbol (L, U). Since L is a torsion-free ring we have then defined the group W(L, U). If, now, (R, F) is an object of **F** and f: (L, U) $\to$ (R, F) is the canonical homomorphism, then W(R, F) is defined as the image of group W(L, U) under the homomorphism of W(f) into the set of all vectors with coordinates in R. Thus, W becomes a functor from category **F** to the category of topological Abelian groups and their continuous homomorphisms.

2. ENDOMORPHISMS OF FUNCTOR W

1. Definition 1.1. We define the family of mappings by shift V_n: W $\to$ W, setting

$$(V_n(x))_m = \begin{cases} x_{\frac{m}{n}}, & \text{if} \quad n \mid m, \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

LEMMA 1.3. V_n is a monomorphism for n = 1, 2, ...

Proof. We compute the action of V_n on the illusory coordinates. We have:

$$w_m(V_n(x)) = \sum_{n|d|m} dp_{\frac{m}{d}-1} x_{d/n}^{m/d},$$

i.e.,

$$w_m(V_n(x)) = \begin{cases} \sum_{kl=m/n} nkp_{l-1}x_k^l = nw_{m/n}(x), & \text{if} \quad n \mid m, \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

The assertion of the lemma then follows from the obvious injectiveness of mapping (1.2) and the additivity of expression (1.4).

Definition 1.5. We define the family of Frobenius mappings F_n: W $\to$ W by their action on isomorphic group Λ. Let $\xi_1, \ldots, \xi_n$ be the formal n-th roots of X. We set

$$F_n(f(X)) = \sum_{i=1}^{n} {}^F f(\xi_i). \tag{1.6}$$

The right side of (1.6), being symmetric in the ξ_i by virtue of the commutativity of group F, is the kernel of R[[X]]. The additivity of mapping F_n is obvious.

LEMMA 1.7.

$$w_m(F_n(x)) = w_{mn}(x). \tag{1.8}$$

Proof. Due to the simplicity of the computations we verify our assertion on element E((1, 0, . . .)) = X. We have:

$$F_n(X) = l^{-1}(l(\xi_1) + \ldots + l(\xi_n)) = l^{-1}((\xi_1 + \ldots + \xi_n) + \\ + \frac{p_1}{2}(\xi_1^2 + \ldots + \xi_n^2) + \ldots\Big) = l^{-1}(p_{n-1}X + p_{\frac{2n-1}{2}}X^2 + \ldots).$$

But, $w_m(1, 0, 0, \ldots) = p_{m-1}$. Now, (1.8) follows from the previous computations and (1.2.4).

THEOREM 1.9. V_n and F_n have the following properties:

(1) $V_m \circ V_n = V_{mn}$,

(2) $F_m \circ F_n = F_{mn}$,

(3) $F_n \circ V_m = V_m \circ F_n$ when (m, n) = 1,

(4) $F_n \circ V_n$ is multiplication by n in Z-module W,

(5) (1/n) $V_n \circ F_n$ is the projector of W(R $\otimes$ **Q**, F) on vector x such that $w_m(x) = 0$ when $n \nmid m$,

(6) $\pi_p = \sum_{(n,p)=1} \frac{\mu(n)}{n} V_n \circ F_n$ is the projector of W(R $\otimes$ Z_p, F) on vector x such that $w_m(x) = 0$ when $m \neq p^h$ (μ is the Möbius function).

Proof. By virtue of (1.4) and (1.8), the action of F_n and V_n on illusory components in generalized Witt groups is identical to the action of the shift and Frobenius homomorphisms in ordinary Witt groups.

With this in mind, we reduce assertions (1)-(6) of the theorem to the corresponding assertions about ordinary Witt groups, proven by computations on the illusory components (see, for example, [1]).

Now, let ring R be an algebra over the ring of p-adic integer Z_p.

Definition 1.10. Formal group F is p-typical if, in group $W(R, F)$, $w_m(x) = 0$ when $m \neq p^h$.

The equivalence of this definition with that given by Cartier (see [4]) follows immediately from (1.8).

THEOREM 1.11. Let $f_p(X) = E(\pi_p(1))$. Then, formal group $G_p(X, Y) = f_p^{-1} F(f_p(X), f_p(Y))$ is p-typical.

Proof. Follows directly from assertion (6) of Theorem (1.9).

3. CONNECTION WITH TOPOLOGY

1. Let F(u, v) be the formal group of geometric cobordisms, and

$$g(u) = \sum_{i=0}^{\infty} \frac{[CP^i]}{i+1} u^{i+1} \in U^*(CP^\infty) \otimes Q \tag{1.1}$$

its logarithm (Mishchenko series) (see [6]). Here, $u \in U^*(CP^\infty) = \Omega_U[[u]]$ is a universal element, i.e., in this case the coefficients of the invariant differential (1.1.1) have the form $p_i = [CP^i]$, and formulas (1.2.2) take the form

$$w_n(x) = \sum_{d|n} d\,[CP^{n/d-1}]\, x_d^{n/d}. \tag{1.2}$$

LEMMA 1.3. Let $\varphi \in A^U \otimes Q$ be the multiplicative operation in the theory of unitary cobordisms; then, $\varphi(g(u)) = g(u)$.

Proof. By definition, g(u) is a primitive element under the mapping $U^*(CP^\infty) \otimes Q \to U^*(CP^\infty) \hat{\otimes} U^*(CP^\infty) \otimes Q$, induced by the H-structure on CP^∞. Any operation of A^U commutes with the diagonal, i.e., the multiplicative operation takes primitive elements into primitive ones. But, the Ω_U-module of primitive elements is one-dimensional and, since $\varphi(g(u)) = u + o(u)$, then $\varphi(g(u)) = g(u)$.

Let $\varphi(u)$ be a formal power series in $f(u)$. As is known (see [6]), from the series $f(u) \in \Omega_U[[u]] \otimes Q$ one reconstitutes, univocally, the multiplicative operation $\varphi \in A^U \otimes Q$. We denote by $g^\varphi(u)$ the series obtained from g(u) by the action of φ on its coefficients. We have:

$$g(u) = \varphi(g(u)) = g^\varphi(f(u)),$$

consequently,

$$g(f^{-1}(u)) = g^\varphi(u),$$

i.e.,

$$f^{-1}(u) = g^{-1}(g^\varphi(u)). \tag{1.4}$$

Let $x \in W(\Omega_U \otimes Q, F)$ be a vector such that $w_n(x) = \varphi[CP^{n-1}]$. We then obtain from (1.1), (1.4), and (1.2.3) that $f^{-1}(u) = E(x)$, i.e., the Artin-Hasse exponent established the connection between the action of the multiplicative operation on the coefficient ring and its action on the cobordisms of infinite-dimensional projective space.

COROLLARY 1.5. Let $w_1, w_2, \ldots, \in \Omega_U$. There exists multiplicative operation $\varphi \in A^U$, with $\varphi[CP^{n-1}] = w_n$, if and only if there exists a collection $x_1, x_2, \ldots \in \Omega_U$ such that

$$w_n = \sum_{d|n} d\,[CP^{\frac{n}{d}-1}]\, x_d^{n/d}, \qquad x_1 = 1.$$

Proof. This follows directly from the previous discussion and from (1.2).

This latest corollary makes it possible to obtain diverse information about the multiplicative operations of A^U. For example:

COROLLARY 1.6. For any multiplicative operation $\varphi \in A^U$ we have:

(1) $Td(\varphi[CP^{p^h-1}]) \equiv 1 \pmod p$,

(2) $L(\varphi[CP^{p^h-1}]) \equiv 1 \pmod p$ for odd p,

(3) $L(\varphi[CP^{2k-1}]) \equiv 0 \pmod 2$,

(4) $\chi(\varphi[CP^{n-1}]) \equiv 0 \pmod n$.

Here, Td is the Todd genus, L is the Hirzebruch L-genus, and χ is the Euler characteristic.

Proof. All these assertions are of the same type, so we shall prove only the first. Let $x \in W(\Omega_U, F)$ be a vector such that $w_n(x) = \varphi[CP^{n-1}]$. It exists, by virtue of (1.5). Let $y_i = \mathrm{Td}\, x_i$. Then, by virtue of (1.2) we have: $Tdw_n(x) = \Sigma_{d/n} dy_d^{n/d}$. In particular $Tdw_{p^h}(x) = \sum_{i=0}^{n} p^i y_{p^i}^{p^{n-i}}$. But, $y_1 = w_1(x) = \varphi(1) = 1$, whence follows the required assertion.

COROLLARY 1.7. Operation $\varphi_p \in A^U \otimes Z_p$, corresponding to vector $\pi_p(1)$, acts as follows:

$$\varphi_p[CP^{p^h-1}] = [CP^{p^h-1}],$$

$$\varphi_p[CP^n] = 0 \text{ when } n \neq p^h - 1.$$

Proof. This follows directly from assertion (6) of Theorem (2.1.9) and from (1.1) (see [6]).

LITERATURE CITED

1. E. Witt, "Zyklische Körper und Algebren der Charakteristik P von Grad p^n," J. Reine und Angew. Math., 176, No. 3, 126-140 (1936).
2. Honda, "Formal groups and zeta functions," in the collection of translations: Matematika, 13, No. 6, 3-21 (1969).
3. M. Lazard, "Sur les groupes de Lie formels a un parametre," Bull. Soc. Math. France, 83, 251-274 (1955).
4. D. Mumford, Lectures on Curves on an Algebraic Surface, Princeton Univ. Press (1966).
5. P. Cartier, "Sur les groupes formels abelienne," Comptes rendus, 265, No. 1, 49-129 (1967).
6. S. P. Novikov, "Methods of algebraic topology from the point of view of cobordism theory," Izv. Akad. Nauk SSSR, Ser. Matem., 31, No. 4, 855-951 (1967).
7. D. Quillen, "On the formal group laws," Bull. Amer. Math. Soc., 75, No. 6, 1293-1298 (1969).