INVERSE PROBLEM FOR PERIODIC FINITE-ZONED POTENTIALS IN THE THEORY OF SCATTERING

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Let u(x) be a smooth periodic real function, and let $L = -(d^2/dx^2) + u(x)$ be the Sturm–Liouville operator. The spectrum of L on the real line consists of a collection of intervals called Bloch admissible zones or Lyapunov stability zones. It is the purpose of this paper to describe the class of potentials having a finite number of zones. The fact that this class is nontrivial can be deduced from Ince [5]: the potentials of the Lamé equation $u(x) = n(n+1)\wp(x)$ are n + 1-zoned (here $\wp(x)$ is the Weierstrass function). For n = 1, this result was given again by Akhiezer; starting from the Gel'fand–Levitan and Marchenko results (see [1] and [2]), he began to solve this problem for certain special spectral densities; the idea of the method used in this paper is essentially that proposed by Akhiezer. Most recently, starting from the nonlinear Kurteweg–de Vries equation (KV-equation), S. P. Novikov proved the following theorem: if u(x−ct) is a solution of the N-th analog of the KV-equation (see Theorem 2 below), then the potential u(x) is N + 1-zoned (see [4]). Novikov conjectured that this theorem gives all finite-zoned potentials. The proof of this conjecture follows from Theorems 1 and 2 below.

To state our result, we introduce the following notation. Let $E_1, \ldots, E_{2N+1}$ be the boundaries of the spectral zones, let Γ_N be the hyperelliptic Riemann surface $W^2 = \prod_{i=1}^{2N+1}(E - E_i)$, let $\pi\colon \Gamma_N \to C$ be its canonical projection onto the E-plane, let S^N be the N-th symmetric power, and let $J(\Gamma_N)$ be the Jacobian variety. There exists a birational isomorphism $\alpha\colon S^N\Gamma_N \to J(\Gamma_N)$, and therefore it is natural to define the algebraic function $\sigma_1\colon J(\Gamma_N) \xrightarrow{\alpha^{-1}} S^N\Gamma_N \xrightarrow{S^N\pi} S^N C \to C$, where the last mapping is a summation. Let ω be an Abel differential of the second kind with second-order poles at infinity and zero periods with respect to cycles around the cuts E_{2j-1}, E_{2j}; let iU_j be the conjugate periods of ω. The vector (U_j) gives a constant vector field on the torus $J(\Gamma_N)$.

THEOREM 1. Any N + 1-zoned potential u(x) with zone boundaries $E_1, \ldots, E_{2N+1}$ is determined by assigning a point Q on $J(\Gamma_N)$ and is the restriction of the function $-2\sigma_1 + \Sigma E_i$ to the rectilinear winding along the field (U_j) which passes through Q.

THEOREM 2. There exist constants c and c_i which are symmetric functions of $E_1, \ldots, E_{2N+1}$, such that, for any potential u(x) constructed in Theorem 1, the function u(x−ct) is a solution of the equation (the definition of the integrals I_k of the KV-equation is given below)

$$\frac{\partial}{\partial t}u = \frac{\partial}{\partial x}\sum_{i=1}^{N} c_i \frac{\delta}{\delta u} I_{2i+3}.$$

We introduce in the space of solutions of the equation $L\psi = E\psi$ a basis of Bloch functions $\psi(x, E; x_0)$, $\bar{\psi}(x, E; x_0)$ defined by the conditions: $\hat{T}\psi = \lambda\psi$, $\psi(x, E; x_0) = 1$, where $\hat{T}$ is the matrix of the monodromy $(a/b, b/a)$ (see [4]). Let $\chi = i\psi'/\psi$.

PROPOSITION 1. The function $\chi = \chi(x, E)$ does not depend on the choice of the point x_0 and is periodic in x; if $\chi_I = 1/2(\chi_R'/\chi_R)$, $\chi_R = \dfrac{k\sqrt{1-a_R^2}}{a_I + b_I}$; and for $k \to \infty$ we have the asymptotic expansion $\chi_R(x, E) \sim k + \sum_{n=0}^{\infty}\chi_{2n+1}(x)/(2k)^{2n+1}$ $(k^2 = E)$, then $I_{2n+1} = \int_T \chi_{2n+1}(x)\,dx$ are polynomial integrals of the KV-equation (see [4]).

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PROPOSITION 2. For a finite-zoned potential, $\chi_R(x, E)$ has the form

$$\chi_R(x, E) = \sqrt{\prod_{i=1}^{2N+1} (E - E_i)} \Big/ \prod_{i=1}^{N} (E - \gamma_i(x)),$$

where the poles $\gamma_i(x)$ are real and lie one-by-one in prohibited zones for any x.

PROPOSITION 3. The function $\psi(x, E; x_0)$ can be extended to a meromorphic function on $\Gamma_N \setminus \infty$, which function has, for $x \neq x_0$, N poles $E = \gamma_i(x_0)$, N zeros $E = \gamma_i(x)$, and an essential singularity at infinity, where we have asymptotically, $\psi \sim e^{ik(x-x_0)}$.

As Akhiezer has noted, analytic properties of a function on a surface Γ_N similar to those described above allow one to reconstruct its zeros from its poles, thus solving the inversion problem of Jacobi (see [3]). More precisely: we choose a basis of holomorphic differentials $\omega_1, \ldots, \omega_N$ on Γ_N, normed by the condition $\oint\limits_{E_{2j-1}E_{2j}} \omega_k = 2\pi i \delta_{jk}$.

PROPOSITION 4. The zeros $\gamma_1(x), \ldots, \gamma_N(x)$ of the function $\psi(x, E; x_0)$ are determined by the equation on the Jacobian variety

$$\int\limits_{\gamma_1(x_0)}^{\gamma_1(x)} \omega_j + \ldots + \int\limits_{\gamma_N(x_0)}^{\gamma_N(x)} \omega_j = U_j(x - x_0) \qquad (j = 1, \ldots, N).$$

Now the expression in Theorem 1 can be obtained from Proposition 4 and the asymptotic expression for χ_R.

PROPOSITION 5. We have $\frac{\delta}{\delta u} \int\limits_T \chi_R dx = \frac{1}{2\chi_R}$.

Hence, it follows in the finite-zoned case that there are linear recurrence relations for the coefficient in the series on the left-hand side, which proves Theorem 2.

Remark 1. From Proposition 3 it follows that the poles γ_j of the function χ are eigenvalues in the discrete spectrum of the operator L for one of two problems: on the half-line $(-\infty, x_0]$ or $[x_0, \infty)$ with zero boundary conditions, i.e., they are conditional eigenvalues in the terminology of Shabat [6]. It is not difficult to write a system of differential equations with conditional eigenvalues γ_j, generalizing the equations of [6]:

$$\gamma_j' = 2i \sqrt{\prod_k (\gamma_j - E_k)} \Big/ \prod_{k \neq j} (\gamma_j - \gamma_k), \qquad j = 1, \ldots, N.$$

2. It turns out that the dependence on time of the potential u(x), by virtue of the above KV-equations, is also given by various rectilinear windings on the torus $J(\Gamma_N)$. From this it follows immediately that the tori $J(\Gamma_N)$ are identical with the tori constructed in [4] as the level surfaces of the commuting collection of integrals of the stationary problem for the above KV-equations. For the original KV-equation $\dot{u} = 6uu' - u'''$, the derivatives with respect to time of the γ_j have the form

$$\dot{\gamma}_j = 8i\Big(\sum_{k \neq j} \gamma_k - \frac{1}{2}\sum E_k\Big) \sqrt{\prod_k (\gamma_j - E_k)} \Big/ \prod_{k \neq j} (\gamma_j - \gamma_k).$$

We will give further applications to the theory of the KV-equation in a subsequent work.

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