THE SCHRÖDINGER EQUATION IN A PERIODIC FIELD AND RIEMANN SURFACES

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1. Consider the two-dimensional Schrödinger equation $\hat{H}\psi = E\psi$ or $\hat{H}\psi = i\partial\psi/\partial t$, where $\hat{H} = (i\partial/\partial x - A_1)^2 + (i\partial/\partial y - A_2)^2 + u(x,y)$, the potential $u(x,y)$ and vector potential (A_1, A_2) being periodic in (x,y) with periods T_1, T_2. In the nonstationary case u, A_1, A_2 also depend on the time. The magnetic field is directed along the z-axis and has zero flux: $\int_0^{T_1}\int_0^{T_2} H(x,y)\,dx\,dy = 0$. We wish to find the widest possible class of "integrable" cases of the direct and inverse problems where the eigenfunction ψ and the coefficients of $\hat{H}$ can be exactly determined simultaneously. In the one-dimensional problem, where $\hat{H} = -d^2/dx + u(x)$, the integrable class of "finite-zoned" potentials was discovered and studied in connection with the theory of the Kortweg–de Vries (K.-dV.) equation (see the survey [2]). In the one-dimensional nonstationary problem $\hat{H}\psi - i\partial\psi/\partial t = 0$, the integrable class of potentials $u(x,t)$ was found in [3]. The present work was stimulated, on the one hand, by the method of [3] and, on the other hand, by analogous higher K.-dV. equations, which were discovered by Manakov [4] and preserve the equation $\hat{H}\psi = E_0\psi$ with magnetic field for one level E_0.

2. In the two-dimensional stationary problem $\hat{H}\psi = E\psi$ it is natural to distinguish the Bloch eigenfunctions $\psi(x, y, p_1, p_2)$, where $\psi(x + T_1, y) = e^{ip_1T_1}\psi(x,y)$ and $\psi(x, y + T_2) = e^{ip_2T_2}\psi(x,y)$. Suppose also $\psi(0,0,p_1,p_2) = 1$. The numbers p_1, p_2 are called quasi-momenta. The discrete energy spectrum $\mathcal{E}_n(p_1,p_2)$ is defined for given real p_1, p_2. Clearly $\psi = \psi(x, y, p_1, p_2, n)$.

Definition 1. We say that the Hamiltonian $\hat{H}$ *has good analytic properties* if: a) all of the branches of $\mathcal{E}_n(p_1,p_2)$ extend to all complex values of the quasi-momenta, b) a Bloch function $\psi(x, y, p_1, p_2, n)$ exists for all complex (p_1, p_2) as a meromorphic function of (p_1, p_2) on all n sheets, and c) the complete graph of the multivalent functions $\mathcal{E}_n(p_1,p_2)$ forms a complex manifold $\hat{M}^2$ on which the group $G = Z \times Z$ of translations $G = Z \times Z$, $p_1 \to p_1 + 2\pi n/T_1$, $p_2 \to p_2 + 2\pi m/T_2$ acts. The quotient manifold $M^2 = \hat{M}^2/Z \times Z$ is called the *manifold of quasi-momenta*. A Bloch function $\psi = \psi(x,y,P)$ is defined for the points $P \in M^2$. A function $\mathcal{E}\colon M^2 \to C$ (dispersion law), where $\hat{H}\psi = \mathcal{E}(P)\psi$ and C is the complex energy plane, is also defined.

Definition 2. A Hamiltonian $\hat{H}$ with good analytic properties is said to be *algebraic* if there exists a compact complex manifold W and a meromorphic mapping

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$\mathcal{E}\colon W \to P^1 = (C \cup \infty)$ into the extended energy plane, on which an open (everywhere dense) domain is isomorphic to the manifold of quasi-momenta M^2 with dispersion law $\mathcal{E}\colon M^2 \to C$. The complement $W \setminus M^2 = X_\infty$ is called the *part at infinity.* We require that X_∞ be the union of a finite number of Riemann surfaces (algebraic curves). The fibers of $\mathcal{E}\colon M^2 \to C$, after going over to the completion $W^{\mathcal{E}} \to (C \cup \infty)$, have the form $\mathcal{E}^{-1}(E_0) \subset W$; they are compact Riemann surfaces $X = \mathcal{E}^{-1}(E_0)$. For all $E_0 \neq \infty$ the intersection $X \cap X_\infty$ consists of a finite number of points. The Bloch function $\psi(x, y, P)$ has an essential singularity at the infinite points $P \in X_\infty$.

We enumerate the properties of algebraic Hamiltonians.

1. For an algebraic Hamiltonian $\hat{H}$ on a fiber $X = \mathcal{E}^{-1}(E_0)$, $E_0 \neq \infty$, there are precisely two infinite points $P_1 \cup P_2 = X \cup X_\infty$; if w_1, w_2 are local parameters in the vicinity of the points P_1, P_2 on X, then the Bloch function has the asymptotic behavior

$$\psi \sim e^{k_1(x+iy)}[c_1(x,y) + c_1(x,y)^{\mu(x,y)}/k_1 + O(1/k_1^2)],$$
$$\psi \sim e^{k_2(x-iy)}[c_1(x,y) + c_1(x,y)^{\nu(x,y)}/k_2 + O(1/k_2^2)],$$

where $k_1 = 1/w_1$, $k_2 = 1/w_2$, $k_1 \to \infty$, $k_2 \to \infty$.

2. The divisor D of the poles of ψ on $X = \mathcal{E}^{-1}(E_0)$ has degree $n(D) = g$, $D = D_1 + \cdots + D_g$, where g is the genus of the curve X if X is the general fiber of E. The divisor D does not depend on x or y.

We usually introduce a more general class of complex quasi-periodic weakly algebraic Hamiltonians $\hat{H}$. It is required that there exist a "Bloch" eigenfunction $\psi(x, y, P)$ such that: a) the differential $d\psi/\psi = (\psi_x\, dx + \psi_y\, dy)/\psi$ is quasi-periodic with the same group of periods as $\hat{H}$, b) ψ is meromorphic on a complex manifold M^2 and, as in Definition 1, $\hat{H}\psi = \mathcal{E}(P)\psi$, $\mathcal{E}\colon M^2 \to C$, c) an energy level $\mathcal{E} = E_0$ in M^2 can be completed to a complex algebraic curve X, and d) properties 1 and 2 (see above) are valid.

3. We now turn to the solution of the inverse problem.

Lemma 1. *If a pair of points P_1, P_2 and a divisor $D = D_1 + \cdots + D_g$ are given on an arbitrary Riemann surface X of genus g, then there exists a function $\psi(x, y, P)$ with pole divisor D and the asymptotic behavior indicated in property* 1*. This function is uniquely determined to within a common factor $c_1(x, y) \to c_1(x, y)f(x, y)$, $c_2 \to c_2 f$.*

The construction of this function is carried out according to the scheme of [1], which has already been repeatedly used in the works of the authors, Matveev and Its (see the survey [2] and the paper [3]).

Lemma 2. *The function $\psi(x, y, P)$ constructed in Lemma* 1 *satisfies the equation $\hat{H}\psi = 0$, where*

$$\hat{H} = -\partial^2/\partial z\, \partial\bar{z} + A_{\bar{z}}\partial/\partial\bar{z} + v(x,y) = (i\partial/\partial x - A_1)^2 + (i\partial/\partial y - A_2)^2 + u(x,y),$$
$$c_1 = 1, \quad c_2 = c(x,y),$$
$$A_1 + iA_2 = A_{\bar{z}} = \partial \ln c(x,y)/\partial z, \quad A_z = A_1 - iA_2 = 0,$$
$$v(x,y) = -2\partial\mu/\partial\bar{z}.$$

The functions $A_1, A_2, u(x,y)$ and the differential $d\ln\psi = (\psi_x\,dx + \psi_y\,dy)/\psi$ are almost periodic with a common group of periods depending only on the curve X and the pair of points $P_1, P_2 \in X$.

For any anti-involution $T\colon X \to X$ the group $H_1(X)$ has a basis of cycles $a_i, b_i \in H_1(X)$ such that $a_i \cdot b_i = \delta_{ij}$, $T_* a_i = a_i$, $T_* b_j = -b_j$. We choose a basis $(\omega_1, \dots, \omega_g)$ of differentials of the first kind such that $\oint_{a_i} \omega_j = 2\pi i \delta_{ij}$. The matrix $B_{kj} = \oint_{b_k} \omega_j$ is real and $T^*(\omega_k) = -\bar{\omega}_k$. We choose two differentials Ω_α of the second kind having the form $\Omega_\alpha \sim (dw_\alpha / w_\alpha^2 +$ a regular differential) near the points P_α and such that $\oint_{a_k} \Omega_\alpha = 0$. Let $U_{j\alpha} = \oint_{b_j} \Omega_\alpha$. If $T(P_1) = P_2$, then one of the relations $U_{j1} = \pm U_{j2}$ is valid if $T^*_{w_1} = \pm \bar{w}_2$. Let $D(x,y) = \sum_{j=1}^{g} D_j(x,y)$ be the divisor of zeros of $\psi(x,y,P)$. By the scheme of [1], in every case we get

$$(zU_{k1} + \bar{z}U_{k2}) = \sum_{j=1}^{g} \int_{D_j}^{D_(x,y)} \omega_k, \quad z = x + iy$$

(to within a lattice in C^n). Suppose the anti-involution T has at least g real (fixed) ovals that are independent in $H_1(X)$. These ovals provide a semibasis of cycles $a_1, \dots, a_g$. We take a divisor of poles of the form $D = \sum_{j=1}^{g} D_j$, where the point D_j lies on the oval a_j. Suppose $T(P_1) = P_2$ and $T^*(w_1) = \bar{w}_2$.

Lemma 3. *If the points P_1, P_2 lie outside the ovals $(a_1, \dots, a_g)$ and the poles D_j lie on the different ovals a_j, then the functions iA_1, iA_2, u are smooth and real.*

This yields a sufficient (but not necessary) condition for the boundedness of the coefficients $iA_1, iA_2, u(x,y)$.

Let $\theta(\eta_1, \dots, \eta_n)$ be the Riemann θ-function constructed from the matrix $B_{kj} = \oint_{b_j} \omega_k$. The above lemmas imply

Theorem 1. *Suppose $\psi(x,y,P)$ has the asymptotic behavior*

$$\psi \sim e^{k_1 z}(1 + \mu(x,y)/k_1 + \cdots) \quad \text{and} \quad \psi \sim c(x,y)^{k_2 \bar{z}}(1 + \nu(x,y)/k_2 + \cdots),$$

near arbitrary points P_1 and P_2 on X, where $w_1 = 1/k_1$ and $w_2 = 1/k_2$ are local parameters on X in the vicinity of P_1 and P_2.

Then the coefficients of $\hat{H}$ have the form

$$u(x,y) = -2\frac{\partial^2}{\partial z\,\partial \bar{z}} \ln \theta(\vec{U}_1 z + \vec{U}_2 \bar{z} + \vec{W}(D)),$$

$$A_{\bar{z}} = A_1 + iA_2 = \frac{\partial}{\partial z} \ln \left[\frac{\theta(\vec{U}_1 z + \vec{U}_2 \bar{z} + \vec{V}_1 + \vec{W}(D))}{\theta(\vec{U}_1 z + \vec{U}_2 \bar{z} + \vec{V}_2 + \vec{W}(D))} \right],$$

$A_z = A_1 - iA_2 = 0$, $D = D_1 + \cdots + D_g$ is a divisor of poles,

$$W_j(D) = \sum_k \int_Q^{D_k} \omega_j + \frac{1}{2} - \frac{1}{2} B_{jj} + \sum_{s \neq j} \oint_{a_j} \left(\int_Q^t \omega_s \right) \omega_j(t), \quad t \in a_j, \quad V_{\alpha j} = \int_Q^{P_\alpha} \omega_j;$$

Q is a fixed point. The function $\psi(x,y,P)$ satisfies the equation given by the formula $\hat{H}\psi = E_0\psi$ and is given by the formula

$$\psi(x,y,P) = \exp\left\{ z \int_Q^P \Omega_1 + \bar{z} \int_Q^P \Omega_2 \right\} \frac{\theta(\vec{U}_1 z + \vec{U}_2 \bar{z} + \vec{W}(D) + \vec{f}(P))\,\theta(\vec{W}(D))}{\theta(\vec{f}(P) + \vec{W}(D))\,\theta(\vec{U}_1 z + \vec{U}_2 \bar{z} + \vec{W}(D))},$$

where $\vec{f}(P) = (f_j(P))$, $f_j(P) = \int_Q^P \omega_j$.

The magnetic field is directed along the third axis and has the form

$$H(x,y) = \frac{\partial^2}{\partial x\,\partial y} \ln \frac{\theta(\vec{U}_1 z + \vec{U}_2 \bar{z} + \vec{W}(D) + \vec{V}_1)}{\theta(\vec{U}_1 z + \vec{U}_2 \bar{z} + \vec{W}(D) + \vec{V}_2)}$$

4. The coefficients of the linear operators

$$\hat{H} = \frac{\partial^2}{\partial z\,\partial \bar{z}} - \frac{\partial \ln c}{\partial z}\frac{\partial}{\partial \bar{z}} - u,$$

found in this note from a curve X, a pair of points P_1, P_2 and a divisor D, satisfy certain nonlinear equations. Any algebraic function f on X with poles of orders m_1, m_2 only at the points P_1, P_2 induces, by the scheme of [3], an operator $\hat{H}_f$ such that $\hat{H}_f \psi = f\psi$, where

$$\hat{H}_f = \left(\frac{\partial}{\partial z}\right)^{m_1} + \left(\frac{\partial}{\partial \bar{z}}\right)^{m_2} + \sum_{i=1}^{m_1} a_i(x,y)\left(\frac{\partial}{\partial z}\right)^{m_1-i} + \sum_{j=1}^{m_2} b_j(x,y)\left(\frac{\partial}{\partial \bar{z}}\right)^{m_2-j}.$$

The following relations hold:

$$[\hat{H}_f, \hat{H}] = D_{(f)}\hat{H}, \quad [\hat{H}_f, \hat{H}_g] = D_{(f,g)}\hat{H},$$

where $D_f, D_{(f,g)}$ are differential operators, and f and g are functions on X with poles only at the points P_1, P_2. These relations are equivalent to equations on the coefficients $c(x,y), u(x,y)$.

We consider an example that arose in the course of a discussion between the authors and A. R. Its on the relationship of the results of the present note with the Sin-Gordon equation. Suppose two functions f and g on X have a pole of second order only at the points P_1, P_2 respectively, with $k^2 \sim f$, ${k'}^2 \sim g$. Then

$$\hat{H}_f = \frac{\partial^2}{\partial z^2} - 2\frac{\partial \mu}{\partial z}, \quad \hat{H}_g = \frac{\partial^2}{\partial \bar{z}^2} - 2\frac{\partial \ln c}{\partial \bar{z}}\frac{\partial}{\partial \bar{z}}.$$

From the relations $[\hat{H}_f, \hat{H}] = D_{(f)}\hat{H}$, $[\hat{H}_g, \hat{H}] = D_{(g)}\hat{H}$, $[\hat{H}_f, \hat{H}_g] = D_{(f,g)}\hat{H}$ we obtain the collection of nonlinear equations

$$v_{zz} - v(c_{zz}/c) = 0, \quad 2u_{zz} = (c_{zz}/c)_{\bar{z}z},$$
$$v_{\bar{z}\bar{z}} - v(c_{\bar{z}\bar{z}}/c) = 0, \quad 2u_{\bar{z}\bar{z}} = (c_{\bar{z}\bar{z}}/c)_{z\bar{z}}, \quad v = u/c.$$

Nontrivial solutions of these equations are obtained from the compatibility conditions $\alpha_{z\bar{z}} = \phi(\alpha)$, where $\alpha = \ln c$, $\phi(\alpha) = ae^{2\alpha} + be^{-2\alpha}$, $u = \kappa c^2 = \kappa e^{2\alpha}$, $\kappa = \text{const} = 2a$, $b = \text{const}$.

For Liouville's equation $\Delta\alpha = e^{-2\alpha}$ we have $u \equiv 0$. The relation $u = \kappa c^2$ is not obvious from the formulas of Theorem 1. After making the change of variables $z \to x' + t = \xi$, $\bar{z} \to x' - t = \eta$, we get

$$\hat{H} \to \hat{H} = \frac{\partial^2}{\partial \eta\,\partial \xi} - \frac{\partial \alpha}{\partial \xi}\frac{\partial}{\partial \eta} - u.$$

The equation $\hat{H}\psi = 0$ takes the form

$$i\frac{\partial \psi_1}{\partial t} = i\frac{\partial \psi_1}{\partial x'} + c_1\psi_2, \quad \psi = \psi_1,$$
$$i\frac{\partial \psi_2}{\partial t} = -i\frac{\partial \psi_2}{\partial x'} + c_2\psi_1, \quad \psi_2 = \frac{i}{c_2}\frac{\partial \psi_1}{\partial \xi}, \quad u = -c_1c_2, \quad \alpha = -\ln c_1.$$

When $u = \kappa c^2$ we will have $c_1 = -c$, $c_2 = \kappa c$ (the inverse problem for this equation, when c_1 and c_2 decreases as $|x| \to \infty$, was first considered in [5]).

Finally, we note that solutions of the equation

$$\left(i\frac{\partial}{\partial t} - \Delta - a(x,y,t)\frac{\partial}{\partial y} - u\right)\psi = 0$$

can be obtained in an analogous manner under the assumption that ψ has the asymptotic behavior ($\psi \sim e^{kx+ik^2t}$, $\psi \sim ce^{k'y+ik^2t}$).

The methods of this note can be generalized to dimensions $n > 2$, it being always necessary that the spectral data uniquely determining the operator $\hat{H}$ be given on a Riemann surface X.

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