29. S. I. Svinolupov and V. V. Sokolov, "On evolution equations with nontrivial conservation laws," Funkts. Anal. Prilozhen., 16, No. 4, 86-87 (1982).
30. I. V. Cherednik, "On conditions of realness in "finite-zone integration," Dokl. Akad. Nauk SSSR, 252, No. 5, 1104-1108 (1980).
31. M. Ablowitz, D. Kaup, A. Newell, and H. Segur, "Method for solving the sine-Gordon equation," Phys. Rev. Lett., 30, 1262-1264 (1973).
32. Y. Akharonov and A. Casher, "Ground state of a spin 1/2 charged particle in a two-dimensional magnetic field," Phys. Rev. A, 19, No. 6, 2461-2464 (1979).
33. M. Atiyah and I. Singer, "The index of elliptic operators on compact manifolds," Bull. Am. Math. Soc., 69, No. 3, 422-433 (1963).
34. J. E. Avron and R. Seiler, "Paramagnetism for nonrelativistic electrons and Euclidean massless Dirac particles," Phys. Rev. Lett., 42, No. 15, 931-934 (1979).
35. H. F. Baker, "Note on the foregoing paper "Commutative ordinary differential operators" by J. L. Burchnall and T. W. Chaundy," Proc. R. Soc., London, A118, 584-593 (1928).
36. H. F. Baker, Abelian Functions, Cambridge Univ. Press (1897).
37. E. Brown, "Bloch electrons in a uniform magnetic field," Phys. Rev., A 133, No. 4, 1038-1045 (1964).
38. J. L. Burchnall and T. W. Chaundy, "Commutative ordinary differential operators," Proc. London Math. Soc., 21, 420-440 (1923).
39. J. L. Burchnall and T. W. Chaundy, "Commutative ordinary differential operators," Proc. R. Soc., London, A 118, 557-583 (1928).
40. P. Dehornoy, "Operateurs differentiels et courbes elliptiques," Compositio Mathematica, 43, No. 1, 71-99 (1981).
41. P. Lax, "Periodic solutions of the KdV equation," Commun. Pure Appl. Math., 38, No. 1, 141-188 (1975).
42. H. McKean and P. Van Moerbeke, "The spectrum of Hill's equation," Invent. Math., 30, No. 3, 217-274 (1975).
43. J. Zak, "Magnetic translation groups," Phys. Rev., A 134, No. 6, 1602-1612 (1964).

MATRIX FINITE-ZONE OPERATORS

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A survey is given of the spectral properties of matrix finite-zone operators. Conditions of the type of J-self-adjointness for such operators and explicit formulas expressing the coefficients of such operators in terms of theta functions are obtained. The simplest examples of such J-self-adjoint, finite-zone operators turn out to be connected with the theory of ovals of plane, real, algebraic curves.

INTRODUCTION

Until relatively recently [the end of (1973)] in the spectral theory of operators with periodic coefficients there were practically no examples where the spectrum and the eigenfunctions of such operators could be explicitly computed (in terms of some special functions). There were also no effective methods of finding the coefficients of operators on the basis of spectral data. For operators with almost-periodic coefficients these question were never even posed.

The situation changed when in 1974 in a cycle of works of Novikov, Dubrovin, Matveev, Its [34, 18, 12, 23] and Lax [48, 49] the class of "finite-zone" periodic and quasiperiodic potentials of the Schrödinger (Sturm-Liouville, Hill) operator was introduced and studied. A program for constructing a broad class of solutions of the Korteweg-de Vries (KdV) equation was formulated and realized on the basis of this class. (Some results of these investigations were also obtained by McKean and van Moerbeke in 1975 [52]. It was later rigorously proved by Marchenko and Ostrovskii [31] that the set of periodic finite-zone potentials is dense in the space of periodic fucntions with given period.) In these works a connection

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was established between the spectral theory of operators with periodic coefficients and algebraic geometry, the theory of finite-dimensional, completely integrable systems, and the theory of nonlinear equations of KdV type. Generalization of this theory to spatially two-dimensional (2 + 1)-systems was realized by Krichever [27, 28]. Krichever's approach also provides a methodologically extremely convenient and transparent exposition of the algebrogeometric procedure of constructing the finite-zone solutions mentioned above of the KdV equations and its many analogues.

The works enumerated constituted the basis for a periodic analogue of the method of the inverse problem in the theory of nonlinear equations [15] (also called the method of finite-zone integration" or "algebrogeometric integration"). The essence of this method (for the case of systems with one spatial variable) consists in the following:

1) finding (by an algebrogeometric method) a broad collection of exact solutions of the nonlinear equation in question which are finite-zone solutions from the point of view of the spectral theory of the corresponding linear operators with periodic or quasiperiodic coefficients;

2) study of the spectral properties of operators with general smooth, periodic coefficients and approximability of an arbitrary periodic solution by smooth, finite-zone solutions.

It should be acknowledged that until very recently this program of finite-zone integration was realized in full scope only for the Korteweg—de Vries equation and the Schrödinger operator connected with it. The fact of the matter is that almost all nonlinear equations integrable by the method of the inverse problem (the nonlinear Schrödinger equation, the sine-Gordon equation, the equations of nonlinear interaction of wave packets, etc.) are associated with the spectral theory of matrix linear differential operators which frequently are not even self-adjoint. Although it is relatively straightforward to construct complex algebrogeometric solutions of these equations (see [14, 21, 22, 24, 11]), attempts to separate out from these real, smooth solutions encountered serious difficulties. The problems of real algebraic geometry arising here turned out to be completely undeveloped (the first serious advances in the solution of these problems in application to the nonlinear Schrödinger equation, the two-dimensional Schrödinger operator, and the sine-Gordon equation were made Cherednik [38-41], although the results obtained in these works are far from being effective). In the same vein almost nothing was known regarding the spectral properties of non-self-adjoint operators with periodic coefficients, i.e., the properties of the Riemann surfaces which arise and the analytic properties of the Bloch eigenfunctions meromorphic on these surfaces.*

The first serious applications of finite-zone solutions and finite-zone operators — in problems of statistical physics, the development of a nonlinear analogue of the WKB method, etc. — have made especially urgent the completion of finite-zone integration of a number of nonlinear equations having important physical applications and the investigation of the spectrum of the corresponding linear operators. This has been done in a number of recent works: for the sine-Gordon equation in [16, 17] and for the nonlinear Schrödinger equation in [17]; the density of finite-zone solutions of the sine-Gordon equation in the space of all periodic solutions was proved in [10]. The spectral properties of the non-self-adjoint matrix operators of second order arising here were first described in these same works. In the present survey we shall present the main ideas of these works. We also present the results obtained by the author on the basis of [11]† regarding the properties of the spectrum of matrix J-self-adjoint linear pencils of operators of higher order. These results have not previously been published.

Several words should be said regarding nonlinear equations associated with matrix linear operators of higher orders. The first examples of such equations — the equations of nonlinear interaction of wave packets (the "three-wave problem") — were found by Zakharov and Manakov (see [35]). Subsequently Manakov noted [30] that the stationary equations of the n-wave problem coincide with the n-dimensional generalization [for the Lie algebra $so(n)$] of the Euler equations of motion of a solid body with a fixed point and were therefore integrable. Mishchenko and Fomenko in [33] gave a direct verification that the integrals of the n-dimensional Euler equations constructed by S. V. Manakov were independent and involutive. In this same work the method of Manakov was used to prove complete integrability in other semisimple

*In particular, the correct formulation of the question of the position of the poles of the Bloch functions was found only very recently in the work of Novikov and Dubrovin [17].
†The main ideas of [11] are presented in Chap. 3, Sec. 2, of [15].

Lie algebras. This method was subsequently generalized to some nonsemisimple Lie algebras [37]. Applications of these results to the construction of integrable flows on symmetric spaces were found in [3]. (We recall [19, 25] that a Hamiltonian system with n degrees of freedom is called completely integrable if it has n independent pairwise commuting first integrals. This concept, which arose in the last century (in the works of Liouville and Bur) on the basis of the method of separation of variables, was found to be applicable to a broader class of systems not admitting, generally speaking, separation of variables (the clearest example is the problem of S. V. Kovalevskaya — see [7]). We note that, in spite of the complete integrability of such systems, it is a very difficult problem to obtain explicit formulas for their solutions; it is still harder to find for them canonical action-angle variables (for example, in the problem of Kovalevskaya these variables had not been computed until very recently). All this restricts the possibilities of applying such integrable systems to solve concrete problems of mechanics and physics. In connection with this it should be noted that for the Hamiltonian systems arising in the theory of finite-zone integration it is possible not only to construct commuting integrals but also to obtain explicit formulas for the solution of these systems in terms of theta functions of Riemann surfaces. The action variables in these systems relative to a large class of Poisson brackets, as became clear after the work of Novikov and Veselov [4], also admit algebrogeometric computation.)

Recently, a series of works of Adler and van Moerbeke [42-44] were published in which the algebraic structure of the invariant tori for the integrable Euler equations found by S. V. Manakov was studied. (The case of the Lie algebra $so(4)$, where integrability was proved already in [33, 9], is studied in more detail in a recent preprint of Haine (Haine, Geodesic flow on $SO(4)$ and Abelian surfaces).) Explicit formulas for the solutions were not obtained in them. The authors did not pose the question of separating out "real" solutions. More complex nonlinear equations connected with matrix operators of order higher than second were studied in [11, 5] where a Hamiltonian formalism for these equations was also constructed.

As the author discovered, the multidimensional Euler equations also turned out to be very interesting from the point of view of possible algebrogeometric applications. The situation is that the corresponding Riemann surfaces of the spectrum of the associated matrix linear operators turn out to be plane, nonsingular real algebraic curves. For example, for the Euler equations on su(k) the real ovals of these curves are imbedded in one another (i.e., they form a "nest" of k/2 ovals). The possibility of applying the spectral theory of matrix operators to the problem of the classification of plane real curves is now being subject to a thorough analysis.

The survey consists of six sections and an appendix. In the first two sections we present the simplest ideas of the method of finite-zone integration in application to matrix operators of second order and the nonlinear Schrödinger equation connected with them and to the sine-Gordon equation. The material of these sections is based on the works [21, 22, 24, 11, 16, 17]. In the next section we treat the main examples of nonlinear equations connected with matrix operators of higher order. The simplest spectral properties of such operators with periodic coefficients are discussed (following [11]) in Sec. 4. Finite-zone (complex) matrix operators are constructed in Sec. 5. Here we mainly follow the works [11, 26], although the methods of these works had to be considerably improved to obtain explicit formulas in a good form. Finally, in Sec. 6 we treat conditions of J-self-adjointness type for the matrix, finite-zone, linear operators constructed and the solutions of the corresponding nonlinear equations. "Realness" conditions for solutions of the multidimensional Eucler equations in connection with the plane algebraic curves arising here are studied in special detail. In the appendix we have collected for the reader's convenience a list of the basic definitions from the theory of Riemann surfaces and theta functions; it is possible to become acquainted with these concepts in more detail in the survey [13], for example.

The author expresses his gratitude to O. Ya. Viro, S. M. Natanzon, and A. N. Tyurin for a number of useful discussions.

1. Dirac Operator and the Nonlinear Schrödinger Equation

The nonlinear Schrödinger equation (NS±)

$$ir_t = -r_{xx} \pm 2|r|^2 r \qquad (1.1)$$

is obtained from the "complexified" system

$$\left.\begin{aligned} ir_t &= -r_{xx} + 2r^2 q \\ iq_t &= q_{xx} - 2q^2 r \end{aligned}\right\} \tag{1.2}$$

by the reduction $q = \pm\bar{r}$. System (1.2) can be represented as the commutation conditions of the λ-pencils [33]

$$[\partial_x - U,\ \partial_t - V] = 0, \tag{1.3}$$

where

$$U = U(\lambda) = i\lambda\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + i\begin{pmatrix}0 & -q\\ r & 0\end{pmatrix}, \tag{1.4}$$

$$V = V(\lambda) = 2i\lambda^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + 2i\lambda\begin{pmatrix}0 & -q\\ r & 0\end{pmatrix}\begin{pmatrix}0 & -q_x\\ r_x & 0\end{pmatrix} + i\begin{pmatrix}rq & 0\\ 0 & -rq\end{pmatrix}. \tag{1.5}$$

We shall also find useful another normalization of this commutation representation obtained from (1.3) by the obvious transformation

$$\tilde{U} = C^{-1}UC,\ \tilde{V} = C^{-1}VC, \tag{1.6}$$

where

$$C = \begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}, \tag{1.7}$$

$$\tilde{U}(\lambda) = \lambda\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}i(r-q) & r+q\\ r+q & i(q-r)\end{pmatrix}. \tag{1.8}$$

We introduce the linear matrix operator depending on the spectral parameter λ which is connected with the theory of the NS equation

$$L(\lambda) = \partial_x - U(\lambda), \tag{1.9}$$

or in the other representation

$$\tilde{L}(\lambda) = \partial_x - \tilde{U}(\lambda). \tag{1.9'}$$

For $q = \pm\bar{r}$ the operator L possesses the symmetry

$$L^*(\bar{\lambda}) = -J_\pm L(\lambda) J_\pm;\ J_- = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ J_+ = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{1.10}$$

The equation for the "eigenfunctions" has the form

$$L(\lambda)\psi = 0,\ \psi = (\psi_1,\ \psi_2)^T, \tag{1.11}$$

or

$$\tilde{L}(\lambda)\tilde{\psi} = 0,\ \tilde{\psi} = (\tilde{\psi}_1,\ \tilde{\psi}_2)^T. \tag{1.11'}$$

Suppose that the coefficients r, q of the operator L(λ) are periodic with period T, $r(x + T) = r(x)$, $q(x + T) = q(x)$. Then on the space of solutions of Eq. (1.11) there acts the monodromy operator

$$\hat{T}\psi(x) = \psi(x+T). \tag{1.12}$$

If $Y(x, x_0, \lambda) = \begin{pmatrix}y_{11}(x, x_0, \lambda) & y_{12}(x, x_0, \lambda)\\ y_{21}(x, x_0, \lambda) & y_{22}(x, x_0, \lambda)\end{pmatrix}$ is a fundamental matrix of solutions of Eq. (1.11) with initial conditions $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ at the point $x = x_0$, then the matrix $\hat{T}(x_0, \lambda)$ of the monodromy operator (1.12) in the basis $(y_{11}, y_{21})^T$, $(y_{12}, y_{22})^T$ has the form

$$\hat{T}(x_0, \lambda) = \begin{pmatrix}\tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{22}\end{pmatrix} = Y(x_0 + T, x_0, \lambda).$$

Its matrix elements are therefore entire functions of the spectral parameter λ. The matrix $\hat{T}$ is unimodular.

For q = ±$\bar{r}$ the following unitarity conditions holds for the monodromy matrix:

$$q=\bar{r},\quad \hat{T}^*(\bar{\lambda})\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\hat{T}(\lambda)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{1.13}$$

$$q=-\bar{r},\quad \hat{T}^*(\bar{\lambda})\hat{T}(\lambda)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{1.13'}$$

(the asterisk denotes the Hermitian conjugate).

The levels of the spectrum of the periodic and antiperiodic problems are sought from the condition

$$\Delta(\lambda)\equiv\frac{1}{2}\,\mathrm{Sp}\,\hat{T}(x_0,\lambda)=\pm 1. \tag{1.14}$$

We note [21] that for the case $q = \bar{r}$ the eigenvalue problem (1.11') is self-adjoint [for appropriate boundary conditions — periodic or zero boundary conditions $\tilde{\psi}_1(0) = \tilde{\psi}_1(T) = 0$]. The periodic problem (1.11) for $q = \bar{r}$ is also self-adjoint, but the problem $\psi_1(0) = \psi_1(T) = 0$ is not self-adjoint. For the case $q = -\bar{r}$ (NS_-) problem (1.11') is J-self-adjoint, $J = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$. Thus, for the self-adjoint case $q = \bar{r}$ all levels of the spectrum (1.14) of the periodic and antiperiodic problems are real and go to $\pm\infty$ with asymptotics $\lambda_n = \pi n/T$, $-\infty < n < \infty$. For the non-self-adjoint case $q = -\bar{r}$ the situation is more involved, and we shall describe it below.

The Bloch eigenfunctions have the form

$$\hat{T}\psi_\pm(x)=\psi_\pm(x+T)=e^{\pm ipT}\psi_\pm(x), \tag{1.15}$$

where $p = p(\lambda)$ is the quasimomentum (the Bloch dispersion law) which is defined up to a period of the reciprocal lattice, $p \to p + 2\pi n/T$. For an arbitrary value of λ there are, generally speaking, two distinct Bloch functions ψ_+ and ψ_- which coalesce at points of the spectrum of the periodic and antiperiodic problems (of odd multiplicity). The permitted zones of the spectrum are determined from the condition that the quasimomentum $p(\lambda)$ be real. In the self-adjoint case $q = \bar{r}$ these zones form, generally speaking, and infinite family of segments of the real axis $-\infty < \lambda < \infty$ with end points at simple points of the spectrum of the periodic and antiperiodic problems (see Fig. 1a). As $|n| \to \infty$ the lengths of the lacunae decrease rapidly; the rate of decrease is determined by the smoothness of the potential r. We now consider the non-self-adjoint case $q = -\bar{r}$. Here the monodromy matrix $\hat{T}(x_0, \lambda)$ is unitary for real λ; therefore, the entire real axis is a permitted zone (the eigenvalues $e^{\pm ipT}$ of a unitary matrix are equal to one in modulus). The entire function $\Delta(\lambda)$ of (1.14) possesses the symmetry

$$\overline{\Delta(\lambda)}=\Delta(\bar{\lambda}), \tag{1.16}$$

and hence the levels (1.14) of the spectrum of the periodic and antiperiodic problems are situated symmetrically with respect to the real axis. Moreover, all real points of the spectrum are degenerate of even multiplicity (otherwise there would be lacunae on the real axis). All simple branch points are situated with respect to the real axis in symmetric pairs λ_n^+, $\lambda_n^- = \overline{\lambda_n^+}$ with asymptotics $\lambda_n^\pm = \pi n/T$ for large n. The imaginary parts of $\lambda_n^\pm$ decrease rapidly as $|n| \to \infty$; the rate of decrease is determined by the smoothness of the potential r. Aside from the real axis, the quasimomentum $p(\lambda)$ assumes real values on certain arcs joining the pairs λ_n^+ and λ_n^- of branch points (on both sheets of the Riemann surface; see Fig. 1b).

We now consider the properties of the Bloch functions. To uniquely determine them it is necessary to prescribe a normalization, setting, for example,

$$\psi_1|_{x=x_0}=1. \tag{1.17}$$

We have the following simple result.

LEMMA 1.1. For any complex periodic functions r(x), q(x) the Bloch eigenfunction extends (in λ) to a single-valued meromorphic function on a two-sheeted Riemann surface of the form

$$w^2-2\Delta(\lambda)\,w+1=0, \tag{1.18}$$

which has branch points at points of the spectrum (of odd multiplicity) of the periodic and antiperiodic problems (1.14). The poles of the function ψ on the surface Γ have the form

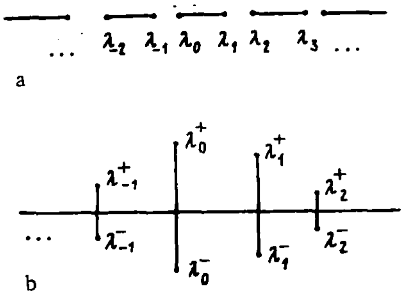

Fig. 1

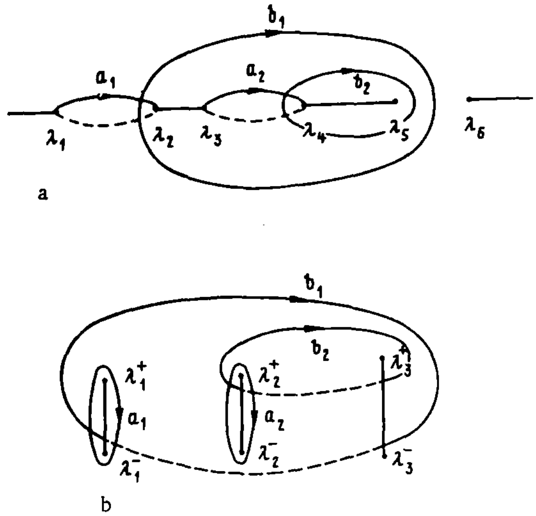

Fig. 2

$$\lambda=\gamma_l(x_0), \quad \tau_{12}(\gamma_l(x_0), x_0)=0, \tag{1.19}$$

$$w=w_l(x_0)=\tau_{22}(\gamma_l(x_0), x_0). \tag{1.19'}$$

The zeros of the first component $\psi_1(x, x_0, \lambda)$ have a form analogous to (1.19), (1.19') with the change $x_0 \to x$. For $\lambda \to \infty$ the function $\psi = \psi_\pm$ has exponential asymptotics $e^{\pm i\lambda(x-x_0)}$.

The proof of the lemma is analogous to the proof of the properties of the Bloch eigenfunction for the Sturm–Liouville operator (see [15] and also [35]).

The Riemann surface Γ on which the Bloch function is meromorphic we call the spectrum of the operator L, and the poles of the Bloch function situated on it we call the complementary spectrum of this operator. It is obvious from (1.19) that the points $\lambda = \gamma_i(x_0)$ of the complementary spectrum are eigenvalues of the operator (1.11) on the segment $[x_0, x_0 + T]$ with zero boundary conditions

$$\psi_1(x_0)=\psi_1(x_0+T)=0. \tag{1.20}$$

From the foregoing considerations it follows that in the self-adjoint case $q = \bar{r}$ all branch points of the Riemann surface Γ are real, while in the non-self-adjoint case $q = -\bar{r}$ all branch points are nonreal and are situated symmetrically with respect to the real axis in pairs (see Fig. 1).

Analogous constructions can be carried out for the transformed equation (1.11'). The spectrum — Riemann surface Γ — is obviously the same as above, but the complementary spectrum $(\tilde{\gamma}_j(x_0), \tilde{w}_j(x_0))$ changes. In particular, in the self-adjoint case $q = \bar{r}$ all points $\lambda = \tilde{\gamma}_j(x_0)$ lie on the real axis one in each lacuna (on one of the sheets of the Riemann surface). This follows from the fact that they are levels of the spectrum of the self-adjoint problem

$$\tilde{\psi}_1(x_0)=\tilde{\psi}_1(x_0+T)=0. \tag{1.20'}$$

The location of the complementary spectrum in the non-self-adjoint case is more involved. It is easy to find only the asymptotics of the quantities $\gamma_n(x_0)$:

$$\gamma_n(x_0) \sim \pi n/T, \quad |n| \to \infty.$$

The unitarity condition (1.13') gives an ineffective relation on the position of the complementary spectrum γ_j: for entire functions of exponential type $1 - \Delta^2(\lambda)$ and $\tau_{12}(\lambda)$, having zeros at the branch points of the Riemann surface and at points of the complementary spectrum, respectively, the expression

$$1-\Delta^2(\lambda)-\tau_{12}(\lambda)\overline{\tau_{12}(\bar{\lambda})} \tag{1.21}$$

must be a complete square of some entire function [namely, the function $\tilde{\Delta} = (\tau_{11} - \tau_{22})/2i$]. This relation can be rewritten in the following form: points of the complementary spectrum must be situated on the Riemann surface Γ at the zeros of a "differential of second kind" of the form

$$\Omega=\left(1+\frac{\tilde{\Delta}(\lambda)}{\sqrt{1-\Delta^2(\lambda)}}\right)d\lambda, \tag{1.21'}$$

which has no singularities on the finite part of Γ and at infinity has asymptotics of the form $d\lambda(1 + O(\lambda^{-1}))$. This notation is more convenient, since it contains only the branch points of the surface Γ (the multiple spectrum cancels). The complex-conjugate points $(\bar{\gamma}_i, \bar{w}_i)$ are also zeros of the differential Ω.

Until now we have spoken only of periodic potentials q, r. It is also possible to introduce almost periodic potentials; in this case the presence for them of an eigenfunction meromorphic on a two-sheeted Riemann surface and possessing on it analytic properties of the type described above is the definition of the class of potentials studied (potentials with "regular analytic properties"; cf. [15]).

Generally speaking, the Riemann surface of the spectrum introduced above has infinite genus. We now introduce the concept of finite-zone operators [for the example of operators of the form (1.9)] which plays a central role below.

Definition 1.1. The operator $L(\lambda)$ is called a finite-zone operator if it possesses an eigenfunction meromorphic (in λ) on a Riemann surface Γ of finite genus.

Definition 1.1'. The operator $L(\lambda)$ is called a finite-zone operator if there is a matrix

$$M=M(\lambda, x)=\begin{pmatrix} m_{11}(\lambda, x) & m_{12}(\lambda, x) \\ m_{21}(\lambda, x) & m_{22}(\lambda, x)\end{pmatrix}$$

which depends on λ in polynomial fashion and commutes with the operator $L(\lambda)$:

$$[L(\lambda), M(\lambda)]=0. \tag{1.22}$$

We have the following result.

THEOREM 1.1. Definitions 1.1 and 1.1' are equivalent.

This assertion is important and well known in the theory of finite-zone integration; it first appeared and was used in the work of Novikov [34] for the case of the Schrödinger operator (see also [15]). We shall give here a sketch of the proof of the fact that Definition 1.1' implies Definition 1.1. We introduce the characteristic polynomial

$$R(\lambda, \nu)=\det(i\nu-M(\lambda, x))=-\nu^2+\det M(\lambda, x) \tag{1.23}$$

of the matrix $M(\lambda, x)$ which does not depend on x by (1.22). The eigenvectors of the matrix $M = M(\lambda, x)$ have the form

$$M\xi=i\nu\xi, \quad \xi=\begin{pmatrix} 1 \\ \frac{i\nu-m_{11}(\lambda)}{m_{12}(\lambda)}\end{pmatrix}, \quad \xi=\xi(\lambda, x) \tag{1.24}$$

and are therefore meromorphic on a Riemann surface Γ of the form

$$R(\lambda, \nu)=-\nu^2+\det M(\lambda, x)=0. \tag{1.25}$$

It is obvious that the two-sheeted (over the λ plane) surface Γ is algebraic, i.e., has finite genus. We consider the function ψ which is a common eigenfunction of the commuting operators $L(\lambda)$ and $M(\lambda)$:

$$L(\lambda)\psi=0, \quad M(\lambda, x)\psi=i\nu\psi, \quad \psi=(\psi_1, \psi_2)^T \tag{1.26}$$

with the normalization condition

$$\psi_1|_{x=x_0}=1. \tag{1.26'}$$

This function has the form

$$\psi(x, x_0, \lambda)=Y(x, x_0, \lambda)\xi(\lambda, x_0). \tag{1.27}$$

It is therefore meromorphic on the Riemann surface Γ. Its analytic properties can be investigated without difficulty.

It is more difficult to prove the opposite implication (the derivation of Definition 1.1' from Definition 1.1) (see [15]). Actually, it follows from the fact that finite-zone potentials (in the sense of Definition 1.1) are solutions of certain completely determined nonlinear differential equations.

We shall now treat a very simple but important example. Suppose that the matrix $M(\lambda)$ commuting with $L(\lambda)$ has the form

$$M(\lambda)=(\lambda-\alpha)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\begin{pmatrix}0 & -q\\ r & 0\end{pmatrix}. \tag{1.28}$$

The commutation condition $[L(\lambda), M(\lambda)] = 0$ leads to the following formulas for r, q:

$$r=r_0e^{-2i\alpha x},\quad q=q_0e^{2i\alpha x}, \tag{1.29}$$

where r_0, q_0 are constants. We shall consider the non-self-adjoint case $q_0 = -\bar{r}_0$. We set $r_0=\rho_0e^{-i\varphi_0}$. The Riemann surface Γ – the spectrum of the operator $L(\lambda)$ with coefficients of the form (1.29) – has the form

$$\det|\nu-M(\lambda)|=\nu^2-(\lambda-\alpha)^2-\rho_0^2=0. \tag{1.30}$$

This is a two-sheeted surface of genus 0. Its branch points have the form

$$\lambda_\pm=\alpha\pm i\rho_0. \tag{1.31}$$

There is one point of the complementary spectrum (γ, ν) [for the normalization (1.9')] of the form

$$\gamma(x_0)=\alpha+i\rho_0\sin(2\alpha x_0+\varphi_0),\quad \nu(x_0)=-i\rho_0\cos(2\alpha x_0+\varphi_0).$$

As x_0 varies, the point $(\gamma(x_0), \nu(x_0))$ traverses the cycle formed by gluing together two copies of the segment $[\lambda_-, \lambda_+]$ at their end points (branch points). We orient this cycle so that on intersecting with the real axis λ on the upper sheet where $\nu > 0$ decrease of the imaginary part of λ correspond to the positive direction of tranversal. We denote the cycle so oriented by a.

We shall consider in more detail the periodic case where $\alpha = \pi N/T$, T is the period, and N is an integer. It may then be assumed that the Riemann surface (1.30) with branch points (1.31) is obtained from the Riemann surface $\nu^2 = \lambda^2$ of the zero potential with $\rho_0 = 0$ by opening up the N-th twofold eigenlevel $\lambda_N = \pi N/T$ of the periodic (or antiperiodic) problem. We point out that in passing through a period $[0, T]$ the point $(\gamma(x_0), \nu(x_0))$ turns on the cycle a with degree 2N. In other words, the pole of the Bloch function appearing on perturbation of the zero potential remembers "its birthplace" on the λ axis in terms of the degree of winding the corresponding a-cycle. It is clear that the same holds for any periodic potential $r(x)$, because of the connectedness of the collection of operators (1.9) with $q = -\bar{r}$. More precisely, in transversing a period $[0, T]$ the poles $(\gamma_j(x_0), \nu_j(x_0))$ of the Bloch function turn on some cycles a_j on the Riemann surface Γ; the position of each of these cycles is analogous to the position of the cycle a described above. In the homology classes of these cycles it is possible to select representers (we also denote them by a_j) which under projection onto the λ plane go over into pairwise nonintersecting arcs with end points at complex-conjugate pairs of branch points $[\lambda_j^+, \lambda_j^- = \overline{\lambda_j^+}]$. We order these cycles according to their intersection with the real axis λ and orient them in a manner similar to the orientation of the cycle a. Then the degree of winding the poles of the Bloch function on the a-cycles corresponding to them will increase monotonically.*

An analogous assertion is also valid in the almost-periodic case if the winding number is correctly defined. This can be used for the unique determination of the homology classes of the a-cycles along which the poles of the Bloch functions on the Riemann surface move (for more details see [17]).

The most characteristic properties of finite-zone potentials, treated above for the simplest example, remain in force for a general finite-zone potential as well. To conclude this section we shall present explicit formulas for finite-zone potentials of an operator $L(\lambda)$ of the form (1.9).

1) NS_+. In this case the Riemann surface Γ of genus g has the form

$$\nu^2=P_{2g+2}(\lambda)=\lambda^{2g+2}+\ldots, \tag{1.32}$$

where all the zeros $\lambda_1,\ldots,\lambda_{2g+2}$ of the polynomial $P_{2g+2}(\lambda)$ are real and distinct. Let $\lambda_1 < \ldots < \lambda_{2g+2}$. The lacunae in the spectrum have the form $\lambda_{2j-1} < \lambda < \lambda_{2j}$, $1 \leqslant j \leqslant g + 1$. Over each lacuna there is precisely one pole $(\tilde{\gamma}_j, \tilde{\nu}_j)$ of the Bloch function of the Bloch function of the operator (1.9'). We denote by P_+, P_- the infinitely distant points of the

*For the sine-Gordon equation an analogous observation was made in [51].

surface (1.35): $P_\pm=\{\lambda=\infty,\ \nu\lambda^{-g-1}=\pm 1\}$. Suppose the canonical basis of cycles $a_1,\dots,a_g$, $b_1,\dots,b_g$ has the form shown in Fig. 2a. Under the action of the antiinvolution $\tau:(\lambda,\ \nu)\to(\bar\lambda,\ \bar\nu)$ this basis transforms as follows:

$$\tau(a_i)=-a_i,\quad \tau(b_i)=b_i,\quad i=1,\dots,g. \tag{1.33}$$

The matrix of periods $B=(B_{jk})$ is real. The potential $r=\bar q$ has the form

$$r(x)=r_0\exp\left\{ix\int_{P_-}^{P_+}\Omega\right\}\frac{\theta(ixU+z_0+\Delta)}{\theta(ixU+z_0)\,\varepsilon(P_+,P_-)}, \tag{1.34}$$

where $\Omega=\Omega^{(1)}_{P_+}-\Omega^{(1)}_{P_-}$ is the normalized Abelian differential of second kind with double poles at P_+, P_-; U is its vector of b-periods, and $\Delta=A(P_+)-A(P_-)$ (where A is the Abel mapping); $\varepsilon(P_+,P_-)$ is a constant depending only on the Riemann surface (cf. Sec. 5 below), $|r_0|=1$, and z_0 is a purely imaginary vector.

2) NS$_-$. In this case the Riemann surface Γ of genus g has the form (1.32) where all the zeros of the polynomial $P_{2g+2}(\lambda)$ are nonreal, $\lambda_j^-=\overline{\lambda_j^+}$, $j=1,\dots,g+1$. We choose a basis of cycles $a_1,\dots,a_g$, $b_1,\dots,b_g$ as shown in Fig. 2b. Under the action of the antiinvolution τ: $(\lambda,\ \nu)\to(\bar\lambda,\ \bar\nu)$ these cycles transform as follows:

$$\tau(a_i)=-a_i,\quad \tau(b_i)=b_i+\sum_{k\neq i}a_k. \tag{1.35}$$

The matrix of periods possesses a symmetry of the form

$$\bar B=2\pi i\begin{pmatrix}0&1\dots&1\\1&0\dots&1\\ \vdots&&\\1&1\dots1&0\end{pmatrix}+B. \tag{1.36}$$

The potential $r=-\bar q$ has the form (1.34) where the vector z_0 can assume the values

$$z_0=ir_0,\quad r_0\in\mathbf{R}^g. \tag{1.37}$$

Smooth solutions of the NS$_+$ and NS$_-$ equations are obtained from these formulas by the substitution

$$z_0\mapsto z_0+itV,\quad \exp ix\int_{P_-}^{P_+}\Omega\mapsto\exp\left\{ix\int_{P_-}^{P_+}\Omega+it\int_{P_-}^{P_+}\Omega^{(2)}\right\},$$

where $\Omega^{(2)}=\Omega^{(2)}_{P_+}-\Omega^{(2)}_{P_-}$ is the normalized differential of second kind with poles of third order at the points P_+, P_-; V is its vector of b-periods.

2. Non-Self-Adjoint Operators Connected with the Sine-Gordon Equation

As was discovered in [20], the sine-Gordon equation (sG)

$$u_{tt}-u_{xx}+\sin u=0 \tag{2.1}$$

admits the Lax commutation relation

$$\dot{\mathscr{L}}=[\mathscr{A},\ \mathscr{L}] \tag{2.2}$$

with matrix operators of fourth order

$$\mathscr{L}=-\begin{pmatrix}i\sigma_2&0\\0&0\end{pmatrix}\partial_x+\begin{pmatrix}\frac{iu}{4}\sigma_1&\exp\frac{iu}{2}\sigma_3\\ \exp\frac{iu}{2}\sigma_3&0\end{pmatrix}, \tag{2.3}$$

$$\mathscr{A}=\begin{pmatrix}-1&0\\0&0\end{pmatrix}\partial_x+2\begin{pmatrix}0&i\sigma_2\exp\frac{iu}{2}\sigma_3\\ i\left(\exp\frac{iu}{2}\sigma_3\right)\sigma_2&0\end{pmatrix}; \tag{2.4}$$

here $v=u_t+u_x$; σ_1, σ_2, σ_3 are the Pauli matrices,

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{2.5}$$

The operator $\mathscr{L}$ is J-self-adjoint, $\mathscr{L}^*=J\mathscr{L}J$, where $J=\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix}$.

The spectral problem $\mathscr{L}f=\lambda f$ for operator (2.3) reduces to the problem

$$\tilde{L}(\lambda)\psi=0,\quad \tilde{L}(\lambda)=\partial_x-\tilde{U}(\lambda), \tag{2.6}$$

$$\tilde{U}(\lambda)=-i\lambda\sigma_2-\frac{iv}{4}\sigma_3-\frac{1}{16i\lambda}(\cos u\sigma_2-\sin u\sigma_1). \tag{2.7}$$

The Lax representation (2.2) can be rewritten in the form of a commutation condition for λ-pencils

$$[\partial_x-\tilde{U}(\lambda),\ \partial_t-\tilde{V}(\lambda)]=0, \tag{2.8}$$

$$\tilde{V}(\lambda)=-i\lambda\sigma_2-\frac{iv}{4}\sigma_3+\frac{1}{16i\lambda}(\cos u\sigma_2-\sin u\sigma_1). \tag{2.9}$$

In conical variables $\xi=\frac{1}{2}(x+t)$, $\eta=\frac{1}{2}(x-t)$, where the sG equation can be rewritten in the form

$$u_{\xi\eta}=\sin u, \tag{2.10}$$

it is more convenient to use another normalization of the commutation representation

$$[\partial_\xi-U(\lambda),\ \partial_\eta-V(\lambda)]=0, \tag{2.11}$$

which is connected with $\tilde{U}$, $\tilde{V}$ by a transformation of the type (1.6): $U(\lambda)$ has the form (1.4) with $r=-q=u_\xi/2$,

$$V(\lambda)=\frac{1}{4i\lambda}\begin{pmatrix}\cos u & -i\sin u\\ i\sin u & -\cos u\end{pmatrix}. \tag{2.12}$$

The non-self-adjointness of the operator $\tilde{L}(\lambda)$ arises here due to the nontrivial entry of the spectral parameter λ (actually, due to the fact that the original operator $\mathscr{L}$ was non-self-adjoint). The operator $\tilde{L}(\lambda)$ possesses (for real u, v) the symmetries (2.13), (2.14)

$$\tilde{L}^*(\bar{\lambda})=-\tilde{L}(\lambda), \tag{2.13}$$

$$\tilde{L}^T(-\lambda)=-\sigma_1\tilde{L}(\lambda)\sigma_1. \tag{2.14}$$

The presence of the second symmetry (2.14) complicates the spectral properties of the operator $\tilde{L}(\lambda)$ as compared with the spectral properties of the operator $\tilde{L}(\lambda)$ of Sec. 1.

The periodicity conditions have the form

$$u(x+T)=u(x)+2\pi Q,\quad v(x+T)=v(x), \tag{2.15}$$

where Q is an integer called the topological charge (i.e., $\exp iu$ is a periodic function). The monodromy matrix $\hat{T}$ of the operator $\tilde{L}(\lambda)$ is unimodular, and in the standard basis of solutions with initial conditions $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ at the point x_0 it possesses a unitarity property of the form (1.13'). Considering the symmetry (2.14), we obtain the form of the monodromy matrix

$$\hat{T}=\hat{T}(x_0,\lambda)=\begin{pmatrix}a(x_0,\lambda^2) & \lambda b(x_0,\lambda^2)\\ -\lambda\bar{b}(x_0,\bar{\lambda}^2) & \bar{a}(x_0,\lambda^2)\end{pmatrix}, \tag{2.16}$$

where $a(x_0, z)$, $b(x_0, z)$ are analytic in $z=\lambda^2$ everywhere except 0 and ∞ where they have essential singularities of exponential type.

Let

$$a_R(z)=\frac{1}{2}(a(x_0,z)+\bar{a}(x_0,\bar{z})),\quad z=\lambda^2, \tag{2.17}$$

be half the trace of the monodromy matrix, and let

$$a_I(x_0,z)=\frac{1}{2i}(a(x_0,z)-\bar{a}(x_0,\bar{z})). \tag{2.18}$$

The spectrum of the periodic and antiperiodic problems is situated symmetrically with respect to the real and imaginary axes of λ and has the form.

$$\lambda_n = \pm\sqrt{z_n},\ 1-a_R^2(z_n)=0, \tag{2.19}$$

where the point z_n problem are situated symmetrically with respect to the real z axis. All real points z_n with $z_n > 0$ are degenerate of even multiplicity. The remaining points z_n are distributed in pairs or complex conjugates z_n^+, $z_n^- = \overline{z_n^+}$ or both numbers z_n^+, z_n^- are real and negative. For $n \to \infty$ the points $z_n^\pm$ have the asymptotics [10, 51]

$$\begin{aligned} z_n^\pm &= \frac{n^2\pi^2}{T^2} + \Delta_\infty + O\left(\frac{1}{n^2}\right), \\ z_{-n}^\pm &= 2^{-8}\left[\frac{T^2}{n^2\pi^2} + \frac{\Delta_0}{n^3} + O\left(\frac{1}{n^6}\right)\right], \end{aligned} \tag{2.20}$$

where Δ_0, Δ_∞ are real constants. The imaginary parts of the quantities $z_n^\pm$ are small for $|n| \to \infty$, and their rate of decay is determined by the smoothness of u.

Because of these asymptotics, on the negative real semiaxis there are only a finite number of points of the spectrum of the periodic and antiperiodic problems. Moreover, this implies that for large $|n|$ all nonreal points of the spectrum are simple.

The Bloch eigenfunctions $\psi_\pm$ are determined by the conditions

$$\tilde{L}(\lambda)\psi_\pm = 0,\ \psi_\pm(x+T) = \exp[\pm ip(\lambda)T]\psi_\pm(x), \tag{2.20'}$$

where $p = T^{-1}\arccos a_R$ is the quasimomentum. The permitted zones of the spectrum are obtained for real $p(\lambda)$. As in Sec. 1 (for the case $q = -\bar{r}$), from the unitarity of the matrix $\hat{T}$ we obtain: the entire real axis λ is a permitted zone. On the z plane this means that the real z semiaxis from 0 to ∞ is a permitted zone.

The Riemann surface $\tilde{\Gamma}$ of the Bloch function $\psi_\pm(\lambda)$ normalized by the condition $\psi_\pm^1|_{x=x_0} = 1$ has the form

$$\tilde{\mu}(\lambda^2) = \pm\sqrt{1-a_R^2(\lambda^2)} \tag{2.21}$$

[the spectrum of the operator $\tilde{L}(\lambda)$]. It is convenient to introduce another Riemann surface Γ of the form

$$\mu(z) = \pm\sqrt{z(1-a_R^2(z))}, \tag{2.22}$$

which is a two-sheeted covering of the z plane (the role of the spectral parameter will henceforth be played by z).

A typical form of the Riemann surface Γ is shown in Fig. 3. Typical here means that complex-conjugate pairs of branch points (z_n^+, $z_n^- = \overline{z_n^+}$) and (z_m^+, $z_m^- = \overline{z_m^+}$) do not coalesce. We note that coalescence of such pairs in the space of Riemann surfaces of given genus occurs on a subset of codimension 2.

The poles $(\gamma_n(x_0), \mu_n(x_0))$ of the Bloch function on Γ have the form

$$b(x_0,\ \gamma_n(x_0)) = 0,\ \mu_n(x_0) = -a_I(x_0,\ \gamma_n(x_0)) \tag{2.23}$$

(points of the complementary spectrum). For $|n| \to \infty$ we have

$$\gamma_n(x_0) \approx z_n^\pm. \tag{2.24}$$

The points of the complementary spectrum are in one-to-one correspondence with the pairs z_n^+, z_n^-.

Suppose that on the negative real semiaxis z there are k real zones of the spectrum $[z_1^+, z_1^-], \ldots, [z_k^+, z_k^-]$. Then the collection of operators $\tilde{L}(\lambda)$ with given spectrum Γ consists of 2^k connected components [40, 10]. These components can be numbered by collections $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ where all $\sigma_i = \pm 1$. Then there exist k pairwise distinct integers $q_1, \ldots, q_k$ such that the topological charge Q of the potential u corresponding to the component with index σ is equal to ([10])

$$Q = Q(\sigma) = \sum_{l=1}^{k} \sigma_l q_l. \tag{2.25}$$

A complete list of real finite-zone solutions of the sG equation together with explicit theta-function formulas can be found in [16] (similar results were obtained independently in [2]).

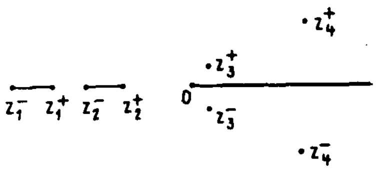

Fig. 3

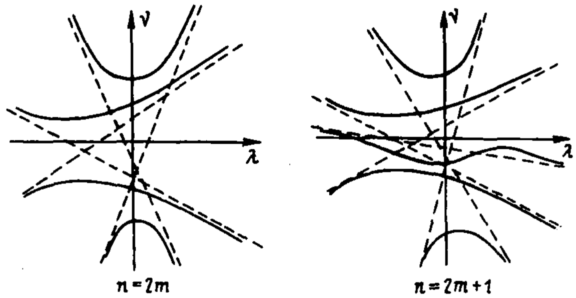

Fig. 4

3. Examples of Nonlinear Equations Connected with Matrix Operators of Higher Orders

We consider the matrix n × n linear operator

$$L_A(\lambda)=i\partial_x+A\lambda-U_A. \tag{3.1}$$

Here

$$A=\mathrm{diag}\,(a_1,\dots,a_n) \tag{3.2}$$

is a constant, real, diagonal matrix of n-th order. We shall assume that the quantities a_i are pairwise distinct. The matrix U_A has zero diagonal elements.

We consider another such operator

$$\begin{gathered}L_B(\lambda)=i\partial_t+B\lambda-U_B,\\ B=\mathrm{diag}\,(b_1,\dots,b_n),\end{gathered} \tag{3.3}$$

where the matrices B, U_B satisfy conditions analogous to those formulated above. The condition of commutation of the λ-pencils

$$[L_A(\lambda),\ L_B(\lambda)]=0 \tag{3.4}$$

is equivalent to the system

$$[A,\ U_B]=[B,\ U_A], \tag{3.5}$$

$$U_{A_t}-U_{B_x}=i\,[U_A,\ U_B]. \tag{3.6}$$

Equation (3.5) can be solved explicitly in the form

$$U_A=[A,\ V],\quad U_B=[B,\ V], \tag{3.7}$$

where the diagonal elements of the matrix V = V(x, t) can also be assumed to be zero. After this we obtain a single matrix nonlinear equation for the function V:

$$[A,\ V_t]-[B,\ V_x]=i\,[[A,\ V],\ [B,\ V]]. \tag{3.8}$$

For applications solutions of the system (3.8) satisfying the "realness conditions" are of interest:

$$V^*=-JVJ, \tag{3.9}$$

where the asterisk denotes the Hermitian conjugate, and J is a diagonal matrix with ones or minus ones on the diagonal,

$$J=\mathrm{diag}\,(\pm 1,\dots,\pm 1). \tag{3.10}$$

For the operators $L_A(\lambda)$, $L_B(\lambda)$ in this case there is the symmetry

$$L_A^*(\bar{\lambda})=JL_A(\lambda)\,J,\quad L_B^*(\bar{\lambda})=JL_B(\lambda)\,J. \tag{3.11}$$

We call the symmetry (3.11) the J-Hermitian property of the operator pencils $L_A(\lambda)$, $L_B(\lambda)$. For example, for n = 3 (suppose $a_1 > a_2 > a_3$), J = 1 or J = diag (−1, 1, 1) or J = diag (1, −1, 1) the system (3.8) describes various types of interaction of three-wave packets in a medium with a quadratic nonlinearity [35]. The case of a purely imaginary matrix $V = i\tilde{V}$, where Eqs. (3.8) reduce to equations for a purely real symmetric matrix V, is also physically interesting. System (3.8) with conditions of realness of the type (3.9) admits an analogous interpretation also for n > 3 [6, 35].

Another important application of these equations found by Manakov [30] is the integration of the Euler equations for the motion of a multidimensional solid body [1]. These equations have the form

$$\dot{M}=[M, \Omega], \quad M=I\Omega+\Omega I, \tag{3.12}$$

where I is the inertia operator of the solid body,

$$I=\operatorname{diag}(I_1,\ldots,I_n). \tag{3.13}$$

They are obtained from (3.8) if we set

$$A=I^2, \quad B=I, \quad [I, V]=i\Omega, \tag{3.14}$$

and there is no dependence on x. For applications general solutions of the system (3.8) with realness condition (3.9) which do not depend on x are of interest [35]:

$$[A, V_t]=i[[A, V], [B, V]]. \tag{3.15}$$

4. Spectral Properties of Matrix Operators with Periodic Coefficients

We consider a J-Hermitian operator $L(\lambda) \equiv L_A(\lambda)$ of the form (3.1) with a periodic potential $U \equiv U_A$, $U(x + T) = U(x)$. Suppose that $Y = Y(x, x_0, \lambda)$ is a fundamental matrix of solutions of the equation

$$L(\lambda)Y=0, \quad Y|_{x=x_0}=1. \tag{4.1}$$

We consider the monodromy matrix $\hat{T} = \hat{T}(x_0, \lambda) = Y(x_0 + T, x_0, \lambda)$. For the monodromy matrix the relation of unitarity is satisfied:

$$\hat{T}^*(\bar{\lambda})J\hat{T}(\lambda)=J. \tag{4.2}$$

Let $J = \operatorname{diag}(\sigma_1,\ldots,\sigma_n)$, $\sigma_k = \pm1$. We introduce a diagonal matrix $\varepsilon = \operatorname{diag}(\varepsilon_1,\ldots,\varepsilon_n)$ such that

$$\varepsilon^2=J. \tag{4.3}$$

As in Secs. 1, 2, it can be proved that the Bloch eigenfunction $\psi = (\psi^1,\ldots,\psi^n)^T$ with the normalization condition

$$(\varepsilon_1\psi^1+\ldots+\varepsilon_n\psi^n)_{x=x_0}=1 \tag{4.4}$$

is meromorphic on a Riemann surface Γ of the form

$$F(\lambda, \mu)=\det(\mu-\hat{T}(\lambda))=0. \tag{4.5}$$

This n-sheeted Riemann surface Γ is called the spectrum of the operator L(λ) with periodic coefficients. We note that the branch points of this surface for n > 2 are, generally speaking, not connected with the spectrum of the period or antiperiodic problems. "Realness" of this Riemann surface Γ follows from the symmetry (3.11) of the operator L(λ).

LEMMA 4.1. A Riemann surface Γ of form (4.5) admits an antiholomorphic involution τ of the form

$$\tau(\lambda, \mu)=(\bar{\lambda}, \bar{\mu}^{-1}). \tag{4.6}$$

Proof. By (4.3) we have

$$0=\overline{F(\lambda, \mu)}=\det|\bar{\mu}-\hat{T}^*(\lambda)|=\det|\bar{\mu}-J\hat{T}^{-1}(\bar{\lambda})J|=(-1)^n\bar{\mu}^n\det\hat{T}^{-1}\det|\bar{\mu}^{-1}-\hat{T}(\bar{\lambda})|=\mathrm{const}\cdot F(\bar{\lambda}, \bar{\mu}^{-1}).$$

The proof of the lemma is complete.

The permitted zones (Lyapunov stability zones) on the Riemann surface Γ are defined by the condition

$$|\mu(\lambda)|=1\Leftrightarrow \operatorname{Im} p(\lambda)=0 \tag{4.7}$$

[$p(\lambda) = (iT)^{-1}\ln\mu(\lambda)$ is the quasimomentum]. The Bloch function will here be bounded on the entire line. From Lemma 4.1 there immediately follows the

COROLLARY. Let J = 1. Then the complete preimage of the real line λ on the surface Γ is a permitted zone on which there are no branch points.

Proof. Because of (4.2), for J = 1 and real λ the matrix $\hat{T}(\lambda)$ is unitary. Its eigenvalues are analysis unimodular. We shall show that for real λ there are no branch points. Indeed, branch points of the Riemann surface Γ can arise only under coalescence of pairs of eigenvalues μ_i, μ_j of the matrix. Even if such coalescence occurs the corresponding eigenvectors remain independent, i.e., branching does not occur (a singularity under imbedding in $\mathbf{C}^2$ occurs). The proof of the corollary is complete.

Thus, for J = 1 all branch points of the Riemann surface Γ are nonreal and situated in pairs symmetric relative to the antiinvolution (4.6).

For $n \to \infty$ the Riemann surface Γ has n "infinitely distant points" $P_1,\ldots,P_n$ where as $P \to P_k$ $(\lambda \to \infty)$ the quasimomentum $p(\lambda) \sim ia_k\lambda$ (we recall that all the numbers a_k are distinct). We note that these points are fixed points relative to the antiinvolution τ of (4.6). The next result is proved in analogy with Sec. 1.

LEMMA 4.2. The poles of the surface Γ of the Bloch function $\psi = (\psi^1,\ldots,\psi^n)^T$ normalized by condition (4.4) are in one-to-one correspondence with pairs of branch points of the surface Γ. As $P \to P_k$ $(\lambda \to \infty)$ the function ψ has asymptotics of the form

$$\psi^j=\varepsilon_k^{-1}\left(\delta_k{}^j+\frac{v_k{}^j(x)}{\lambda}+O(\lambda^{-2})\right)e^{i\lambda a_k(x-x_0)},\quad j=1,\ldots,n. \tag{4.8}$$

Here $V = (v_k^j)$ is defined by the condition $[A, V] = U$ (see Sec. 3).

We further consider in more detail the finite-zone case where, by definition, there exists a matrix $M = M(\lambda, x)$, depending on λ in polynomial fashion, such that

$$[L(\lambda), M(\lambda, x)]=0. \tag{4.9}$$

It may be assumed that the matrix $M(\lambda)$ has the form

$$M(\lambda)=C\lambda^N+\text{lower terms}\ ,\ C=\operatorname{diag}(c_1,\ldots,c_n). \tag{4.10}$$

$$M^*(\bar{\lambda})=JM(\lambda)J. \tag{4.11}$$

We consider only the case where all the numbers $c_1,\ldots,c_n$ are pairwise distinct. The coefficients of the characteristic polynomial of the matrix $M(\lambda, x)$ are "integrals," i.e., they do not depend on x. They determine the spectrum of the operator $L(\lambda)$ [see formula (4.12) below], i.e., the Riemann surface Γ.

Definition. A smooth potential $V = V(x)$ of the operator $L(\lambda)$ is called strongly bounded if for any real diagonal matrix B Eq. (3.8) with the initial condition $V|_{t=0} = V(x)$ has smooth bounded solutions $V = V(x, t)$.

For strong boundedness of a finite-zone potential $V(x)$ with given spectrum Γ it suffices taht the "integrals" described above — the coefficients of the equation of the surface Γ — define a compact manifold in the space of matrices V. For example, for J = 1 any smooth potential is strongly bounded. We shall henceforth consider only strongly bounded, finite-zone potentials.

We have the following result.

LEMMA 4.3. The matrix $M(\lambda)$ commutes with the monodromy matrix $\hat{T}(\lambda)$.

For the proof see [15]. This implies

THEOREM 4.1. a) A Riemann surface Γ of the form (4.5) is given by the algebraic equation

$$R(\lambda, \nu)=\nu^n+r_1(\lambda)\nu^{n-1}+\ldots+r_n(\lambda)=\det|\nu-M(\lambda, x)|=0 \tag{4.12}$$

and, in particular, it has finite genus [equal to $N\frac{n(n-1)}{2}-(n-1)$ in the case of general position].

b) The antiinvolution (4.6) is given in the coordinates (λ, ν) by the equality

$$\tau(\lambda, \nu)=(\bar{\lambda}, \bar{\nu}). \tag{4.13}$$

c) The Riemann surface Γ with antiinvolution τ belongs to separating type.

We shall sketch the proof of this theorem. Part a) follows from Lemma 4.3, since the eigenvectors of the matrix M will be the Bloch functions [here we used the fact that for large λ the eigenvectors of the matrix $M(\lambda)$ are pairwise independent (4.10)]. The surface (4.12) is invariant relative to the antiinvolution (4.13) by (4.11). This antiinvolution coincides with the antiinvolution (4.6), since $p = p(\lambda, \mu)$ is real for real λ, μ (this can be shown; see [11], formulas (26), (32), (36)). This proves part b). To prove c) we consider the operator $L(\lambda)$ commuting with $\tilde{L}(\lambda)$ of the form

$$\tilde{L}(\lambda) = i\partial_y + J\lambda - [J, V],\ [A, V] = U,\ [L(\lambda), \tilde{L}(\lambda)] = 0. \tag{4.14}$$

Its Bloch functions are meromorphic on the same surface Γ. For the operator $\tilde{L}(\lambda)$ (in the periodic case) the problem

$$\psi(y+T) = \exp i\tilde{p}T\psi(y) \tag{4.15}$$

is self-adjoint. Therefore, for real $\tilde{p}$ (i.e., in the permitted zones) the eigenvalue λ is necessarily real. Thus, in this case the permitted zones coincide with the real ovals $\{(\lambda, \nu) = (\bar{\lambda}, \bar{\nu})\}$ of the surface Γ. Now these permitted zones are given by the equation

$$\operatorname{Im} \tilde{p} = 0, \tag{4.16}$$

where $\tilde{p}$ is the quasimomentum for the operator $\tilde{L}(\lambda)$ ($\operatorname{Im}\tilde{p}$ in a single-valued function on Γ). Thus, the entire Riemann surface Γ, after removal of the real part, decomposes into two disconnected components $\Gamma^+ = \{\operatorname{Im}\tilde{p} > 0\}$ and $\Gamma^- = \{\operatorname{Im}\tilde{p} < 0\}$. The proof of the theorem is complete.

Remark. For $J = 1$ the operator $\tilde{L}(\lambda)$ has the form $\tilde{L}(\lambda) = i\partial_y + \lambda$, $\tilde{p} = \lambda$. The equation $\operatorname{Im}\lambda = 0$ exactly distinguishes the real ovals of the surface Γ (see the corollary of Lemma 4.1).

It is not hard to show that the differential of quasimomentum $d\tilde{p}$ is an Abelian differential on Γ (cf. [15]). It preserves sign on the real part of the surface Γ and is even everywhere nonnegative for the natural orientation on this real part as the boundary $\Gamma^+ = \{\operatorname{Im}\tilde{p} > 0\}$. In neighborhoods of the points $P_1, \ldots, P_n$ it has the form

$$d\tilde{p} = \sigma_k d\lambda + \ldots, \quad P \to P_k. \tag{4.17}$$

Conclusion. The sign of the differential $d\lambda$ in a neighborhood of the point P_k is equal to σ_k, $k = 1, \ldots, n$.

This assertion determines the condition on the choice on the surface Γ of a spectral parameter λ leading to the construction of J-Hermitian operator pencils, $J = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$.

We shall now consider in more detail the properties of the Bloch function ψ. As already mentioned, the poles of the function are in one-to-one correspondence with pairs of branch points. From this it is possible to conclude that the number of poles of ψ is equal to $g + n - 1$, where g is the genus. We shall show that it is possible to obtain necessary conditions on the distribution of these poles.

Let $(\lambda, 1), \ldots, (\lambda, n)$ be points of the surface Γ lying over the point λ. To them there correspond n Bloch eigenfunctions $\psi_1(x, \lambda), \ldots, \psi_n(x, \lambda)$ normalized, for example, by condition (4.4). We suppose that λ is not a branch point, so that these functions are linearly independent. We arrange their coordinates in the matrix $(\psi_i^j(x, \lambda))$. Let $(\varphi_i^j(x, \lambda))$ be the inverse matrix. We set

$$\Psi_i^j(x, y, P) = \psi_k^j(x, \lambda)\varphi_i^k(y, \lambda), \quad P = (\lambda, k). \tag{4.18}$$

This definition does not depend on the original ordering of the points $(\lambda, 1), \ldots, (\lambda, n)$ nor on the normalization of the eigenfunctions. The matrix-valued function $\Psi(x, y, P) = (\Psi_i^j(x, y, P))$ is thus a single-valued function on the Riemann surface Γ. We note that if

$$G(x, y, \lambda) = \begin{cases} \sum_{k=1}^{n} \Psi(x, y, (\lambda, k)), & x \leqslant y \\ 0, & x > y, \end{cases} \tag{4.19}$$

then the function $G(x, y, \lambda)$ is the Green matrix of the operator $L(\lambda)$. We note that the columns of the matrix $\Psi(x, y, P)$ are eigenfunctions of the operator $L(\lambda)$ which differ only in normalization. The rows are eigenfunctions for the formally adjoint operator

$$L^+(\lambda) = -i\partial_y + \lambda A - U^T. \tag{4.20}$$

The rank of the matrix Ψ is equal to one.

We set

$$g(x, P) = \Psi(x, x, P). \tag{4.21}$$

LEMMA 4.4. The matrix-valued function $g(x, P) = (g_i^j(x, P))$ on Γ possesses the following properties:

a) The values of the function $g(x, P)$ for fixed λ on different sheets of the surface Γ are a system of projectors for the matrix $M(\lambda)$, i.e., $g^2 = g$, $g(x, (\lambda, k))g(x, (\lambda, l)) = 0$ for $k \neq l$ (k, l are the numbers of the sheets),

$$\sum_{k=1}^{n} g(x, (\lambda, k)) = 1, \quad \sum_{k=1}^{n} \nu(\lambda, k) g(x, (\lambda, k)) = M(\lambda). \tag{4.22}$$

b) The differentials

$$\Omega_i^j(x, P) = g_i^j(x, P)\, d\lambda \tag{4.23}$$

are meromorphic on Γ and have poles only at the infinitely distant points $P_1, \ldots, P_n$ of the Riemann surface Γ: $\Omega_i^i(x, P)$ has a double pole at P_i, while for $i \neq j$, $\Omega_i^j(x, P)$ has simple poles at the points P_i, P_j, whereby

$$\Omega_i^i(x, P) = d\lambda (1 + O(\lambda^{-2})), \quad P \to P_i \tag{4.24}$$

$$\Omega_i^j(x, P) = \begin{cases} v_i^j(x)(\lambda^{-1} + O(\lambda^{-2}))\, d\lambda, & P \to P_j \\ -v_i^j(x)(\lambda^{-1} + O(\lambda^{-2}))\, d\lambda, & P \to P_i. \end{cases} \tag{4.25}$$

c) Let

$$S_k(\lambda) = r_k(\lambda) + r_{k-1}(\lambda) M(\lambda) + \ldots + M^k(\lambda), \quad k = 1, \ldots, n-1, \tag{4.26}$$

where the polynomials $r_k(\lambda)$ are defined in formula (4.12). Then the matrix $g(x, P)$ has the form

$$g(x, P) = \frac{\nu^{n-1} + S_1(\lambda)\nu^{n-2} + \ldots S_{n-1}(\lambda)}{R_\nu(\lambda, \nu)}. \tag{4.27}$$

For the proof see [11].

The Bloch functions $\psi^j(x, P)$ normalized by condition (4.4) have the form

$$\psi^j(x, P) = \frac{\Psi_j^j(x, x_0, P)}{g_j(x_0, P)}, \tag{4.28}$$

where

$$g_j(x, P) = \sum_{l=0}^{n} \varepsilon_l g_j^l(x, P). \tag{4.29}$$

Similarly, the Bloch functions $\psi_1^+(y, P), \ldots, \psi_n^+(y, P)$ of the formally adjoint operator (4.20) normalized by the condition

$$(\varepsilon_1 \psi_1^+ + \ldots + \varepsilon_n \psi_n^+)_{y = x_0} = 1 \tag{4.30}$$

have the form

$$\psi_j^+(y, P) = \frac{\Psi_j^j(x_0, y, P)}{g^j(x_0, P)}, \tag{4.31}$$

where

$$g^j(x, P) = \sum_{l=1}^{n} \varepsilon_l g_l^j(x, P). \tag{4.32}$$

We introduce the meromorphic differential $\Omega = \Omega(x, P)$ by setting

$$\Omega(x, P) = \frac{g_j(x, P)\, g^j(x, P)}{g_j^j(x, P)}\, d\lambda = \sum_{k, l=1}^{n} \varepsilon_k \varepsilon_l g_k^l\, d\lambda. \tag{4.33}$$

Its properties are as follows:

a) The differential Ω has double poles at the points $P_1, \ldots, P_n$ of the form

$$\Omega = \sigma_k d\lambda (1 + O(\lambda^{-1})), \quad P \to P_k, \quad k = 1, \ldots, n. \tag{4.34}$$

b) The divisor of zeros of the differential $\Omega(x_0, P)$ has the form $D + D^+$, where D are the poles of the functions ψ^j and D^+ are the poles of the functions ψ_i^+; the degrees of the divisors D and D^+ are equal to $g + n - 1$.

We note that the differential $\Omega = \Omega(x_0, P)$ is connected with the differentials Ω_i^j of the form (4.23) by the relations

$$\psi^j(x, P)\,\psi_i^+(x, P)\,\Omega = \Omega_i^j(x, P). \tag{4.35}$$

We return to obtaining necessary conditions that the finite-zone operator pencil $L(\lambda)$ be J-Hermitian. In the J-Hermitian case the formally dual functions $\psi_j^+(x, P)$ normalized by condition (4.30) have the form

$$\psi_j^+(x, P) = \overline{\psi^j(x, \tau(P))}\sigma_j, \tag{4.36}$$

where the bar denotes the complex conjugate, and τ is the antiinvolution (4.13). Therefore, the differential $\Omega(x, P)$ is symmetric,

$$\overline{\Omega(x, \tau(P))} = \Omega(x, P), \tag{4.37}$$

and its divisor of zeros has the form

$$D + \tau(D). \tag{4.38}$$

We have thus proved the following result.

THEOREM 4.2. The necessary conditions on the position of the divisor D of poles of the Bloch function ψ normalized by condition (4.4) have the form: $D + \tau(D)$ is the divisor of zeros of a meromorphic differential with twofold poles at the points $P_1, \ldots, P_n$ and principal parts of the form (4.34).

We note also that the differentials $\Omega_i^j = g_i^j d\lambda$ possess a symmetry of the form

$$\overline{\Omega_i^j(\tau(P))} = \sigma_i\sigma_j\Omega_j^i(P). \tag{4.39}$$

This obviously follows from formula (4.35).

It will be shown below that this condition is also sufficient.

We now consider the skew-symmetric case

$$U^T = -U, \quad V^T = V, \quad J = 1, \tag{4.40}$$

where the matrices U, V are pure imaginary. The operator $L(\lambda)$ possesses in this case the additional symmetry

$$L^+(\lambda) = -L(-\lambda) \tag{4.41}$$

(we recall that the cross is used to denote the formally adjoint). For the monodromy matrix $\hat{T}(\lambda)$ there is the "orthogonality relation"

$$\hat{T}(\lambda)\hat{T}^T(-\lambda) = 1. \tag{4.42}$$

On the Riemann surface of the Bloch function there acts the holomorphic involution σ, where

$$\sigma(\lambda, \mu) = (-\lambda, \mu^{-1}). \tag{4.43}$$

The involution σ commutes with the antiinvolution τ. The points $P_1, \ldots, P_n$ are fixed points relative to the involution $\sigma: \sigma(P_j) = P_j$. For odd n there is also the fixed point $P_0 = (O, 1) \in \Gamma$.

In the finite-zone case it is possible to take the matrix $M(\lambda)$ in the form

$$M(\lambda) = C\lambda^{2k+1} + \ldots \quad M^T(-\lambda) = -M(\lambda). \tag{4.44}$$

In the coordinates (λ, ν) the involution σ acts as follows:

$$\sigma(\lambda, \nu) = (-\lambda, -\nu). \tag{4.45}$$

Then the Riemann surface Γ of the form (4.12) and of finite genus $g = (2k+1)\frac{n(n-1)}{2} - (n-1)$ covers the Riemann surface Γ_0 of genus g_0 in two-sheeted fashion, $\Gamma \to \Gamma_0 = \Gamma/\sigma$, where

$$g_0 = km(2m-1) + m^2 - 2m + 1, \quad n = 2m, \tag{4.46}$$

$$g_0 = m(2km + k + m - 1), \quad n = 2m+1 \tag{4.47}$$

(all formulas are given for the case of general position). We shall obtain necessary conditions on the position of the divisor D of poles of the Bloch functions. We have the following result.

THEOREM 4.3. The differential Ω of the form (4.33) constructed above is antisymmetric relative to the involution σ:

$$\Omega(x, \sigma(P)) = -\Omega(x, P). \tag{4.48}$$

Its zeros have the form

$$D + \sigma(D). \tag{4.49}$$

The proof of this theorem is similar to the proof of Theorem 4.2, and we shall omit it. We note that the equivalence classes of divisors described in this theorem form the Prym manifold of the two-sheeted covering $\Gamma \to \Gamma_0$.

5. Construction of (Complex) Finite-Zone Operators

In the preceding section we studied the spectral properties of matrix operators of the form (3.1) with periodic coefficients. In this section we shall take up the solution of the "inverse problem," i.e., reconstruction of the coefficients of a finite-zone operator on the basis of its spectrum, the Riemann surface Γ and the supplementary spectrum — the poles of the Bloch function. It will become evident below that, generally speaking, we hereby obtain quasiperiodic coefficients. This situation is characteristic for the solution of inverse problems in the theory of finite-zone operators [43].

Let Γ be an arbitrary Riemann surface of genus g. On it we fix n distinct points $P_1, \ldots, P_n$. Let $k_1^{-1}, \ldots, k_n^{-1}$ be local parameters in neighborhoods of these points. We choose an arbitrary nonspecial divisor D of degree $g + n - 1$. Let $\psi^j = \psi^j(\mathbf{x}, P)$ be the Baker–Akhiezer (BA) function with poles at the points of this divisor and with asymptotics as $P \to P_l$ of the form

$$\psi^j(\mathbf{x}, P) = \varepsilon_l^{-1} e^{ik_l x^l}\left(\delta_l{}^j + \frac{v_l{}^j}{k_l} + O(k_l^{-2})\right), \quad l = 1, \ldots, n \tag{5.1}$$

(we recall that such a function is uniquely determined on the basis of the divisor D). Here $\mathbf{x} = (x^1, \ldots, x^n)$, and $\varepsilon_1, \ldots, \varepsilon_n$ are numbers of the form (4.3). We have the following result.

LEMMA 5.1. For the functions ψ^j the following linear equations are satisfied:

$$\frac{\partial \psi^j}{\partial x^l} = i v_l{}^j \psi^l, \ l \neq j. \tag{5.2}$$

The proof of this lemma is altogether standard for the theory of BA functions (cf. [26]).

For $n = 2$ Eqs. (5.2) are always compatible. For $n > 2$ the compatibility conditions can be written in the form

$$i\frac{\partial v_j{}^l}{\partial x^k} + v_k{}^l v_j{}^k = 0, \ i \neq j, \ k \neq i, \ j. \tag{5.3}$$

This system is the simplest function connected with the n-point BA function with $n > 2$ (this is similar to the fact that the Kadomtsev–Petviashvili equation is the simplest in the theory of one-point BA functions). On imposing the additional condition

$$\sum_{k=1}^{n} \frac{\partial v_j{}^l}{\partial x^k} = 0 \tag{5.4}$$

the restriction of the matrix $V = (v_j^i)$ to a two-dimensional plane of the form

$$x^k = a_k x + b_k t, \quad k = 1, \ldots, n, \tag{5.5}$$

satisfies Eq. (3.8). To obtain the additional condition (5.4) it is necessary to require that on the surface Γ there exist a meromorphic function $\lambda = \lambda(P)$ with poles of first order at the points $P_1, \ldots, P_n$ (and having no other poles). Then the function λ realizes Γ as an n-sheeted covering of the complex plane. (All n-sheeted coverings are obtained in this way.) In this case for $k_1, \ldots, k_n$ it is possible to take the function λ itself. The functions ψ^j then satisfy the following equality is satisfied:

$$\sum_{k=1}^{n} \frac{\partial \psi^j}{\partial x^k} = i\lambda \psi^j, \quad j = 1, \ldots, n. \tag{5.6}$$

Conclusion. For n-sheeted surfaces Γ the restriction of the function ψ to the line $x^k = a_k x$, $k = 1, \ldots, n$, is an eigenfunction of the operator

$$L(\lambda)=i\partial_x+\lambda A-U,\quad U=[A,V],\quad L(\lambda)\psi=0. \tag{5.7}$$

Suppose that on the surface Γ, aside from the function λ, there exists another function $\mu = \mu(P)$ with poles of first order at the points $P_1,\dots,P_n$ and

$$\mu=c_k\lambda+d_k+O(\lambda^{-1}),\quad P\to P_k. \tag{5.8}$$

Then the vector ψ is an eigenvector for the matrix

$$M(\lambda)=\lambda C-[C,V]+D, \tag{5.9}$$

$$C=\operatorname{diag}(c_1,\dots,c_n),\quad D=\operatorname{diag}(d_1,\dots,d_n), \tag{5.10}$$

and the Riemann surface Γ is a plane algebraic curve of degree n given by the equation

$$\det(\mu-M(\lambda))=0. \tag{5.11}$$

All these facts are altogether standard for the theory of BA functions. We note that if the genus of the surface Γ is equal to $(n-1)(n-2)/2$, then the plane curve (5.11) is nonsingular. The matrix $V = (v^i_j)$ constructed on the basis of a plane curve $\Gamma\subset\mathbf{CP}^2$ of degree n satisfies the complex equation of Euler type

$$-i[C,V_x]+iD_x+[[A,V],[C,V]+D]=0, \tag{5.12}$$

where $x^k = a_k x$. The points $P_1,\dots,P_n$ are obtained as the intersection of the curve Γ with any line in $\mathbf{CP}^2$. The choice of "special parameter" λ on the curve Γ is equivalent to a choice of a pencil of lines of $\mathbf{CP}^2$ (one of the lines of this pencil also intersects the curve Γ at the points $P_1,\dots,P_n$). To prescribe such a pencil it is necessary to fix a point in $\mathbf{CP}^2$ not lying on Γ. Thus, each plane curve Γ of degree n and point in $\mathbf{CP}^2$ not lying on Γ gives a one-parameter family of equations of the form (5.12) (the coefficients C, D depend on the choice of secant in the pencil). In order that the matrix D vanish it is necessary that the tangents to Γ at the points $P_1,\dots,P_n$ pass through a single point. This imposes $n-1$ additional conditions on the plane curve Γ.

If, in addition, the potential V does not depend on x, then we obtain stationary solutions of Eqs. (5.12). In order to construct such solutions there must exist on Γ a third function with simple poles at $P_1,\dots,P_n$ which does not reduce to λ, μ. There are no such functions on nonsingular plane curves.

Conclusion. Singular curves Γ correspond to stationary solutions of Eqs. (5.12) of Euler type.

We return to the case of an arbitrary Riemann surface Γ. We define the formally dual BA function $\psi^+ = (\psi^+_1,\dots,\psi^+_n)$ by the following conditions:

1) the functions ψ^+_j have asymptotics of the form

$$\psi_j^+=\psi_j^+(x,P)=\varepsilon_l^{-1}e^{-ik_lx^l}\left(\delta_j^l+\frac{v_j^{l+}}{k_l}+O\left(k_l^{-2}\right)\right),\quad P\to P_l; \tag{5.13}$$

2) the functions ψ^+_j have poles at points of a divisor D^+ of degree $g+n-1$.

3) There exists a meromorphic differential Ω with poles of second order at the points $P_1,\dots,P_n$ of the form

$$\Omega=\sigma_l dk_l\left(1+O\left(k_l^{-1}\right)\right),\quad P\to P_l, \tag{5.14}$$

such that its zeros have the form $D + D^+$. In the case of general position the divisor D^+ is uniquely determined by this condition and is called the divisor dual to D.

We define the meromorphic differentials

$$\Omega_j^l(x,P)=\psi_j^+(x,P)\psi^l(x,P)\Omega,\quad i,j=1,\dots,n. \tag{5.15}$$

At infinity these differentials have poles (they have no other singularities). For $i \neq j$ the differential Ω^i_j has simple poles at the points P_i and P_j; for $i = j$ the differential Ω^i_i has a double pole at the point P_i. From (5.1), (5.13), and (5.14) it follows easily that

$$\Omega_j^l=\begin{cases}v_j^l\dfrac{dk}{k}+\dots, & k=k_l,\quad P\to P_l,\\ v_j^{l+}\dfrac{dk}{k}+\dots, & k=k_j,\quad P\to P_j,\end{cases} \tag{5.16}$$

$$\Omega_i^l = dk(1+O(k^{-1})), \quad k=k_l, \quad P\to P_l. \tag{5.17}$$

Applying the residue theorem to the differential Ω_j^i, we obtain

$$v_j^{l+} = -v_j^l. \tag{5.18}$$

Finally, we note that the rank of the matrix Ω_j^i of the form (5.15) is equal to 1. We shall obtain explicit formulas for the matrices Ω_j^i.

LEMMA 5.1'. The matrices of differentials $(\Omega_j^i(P))$ satisfying the conditions enumerated above form a family of dimension $g + n - 1$. Each such matrix is determined by a point of the Jacobian $J(\Gamma)$ of general position and by a collection of nonzero complex constants $\lambda_1,\ldots,\lambda_n$ defined up to a factor and has the form

$$\Omega_j^l(P) = \frac{\lambda_l}{\lambda_j}\,\frac{\theta(A(P)-A(P_l)-z)\,\theta(A(P)-A(P_j)+z)}{\theta^2(z)\,E(P_l,P)\,E(P,P_j)\sqrt{dk_l^{-1}}\sqrt{dk_j^{-1}}}. \tag{5.19}$$

The proof of the assertion of the lemma regarding the dimension can be easily deduced from the Riemann–Roch theorem. It is also easily verified that the matrix (5.19) has rank 1 and poles of the required form. Here the vector z is connected with the original divisor D by the relation on the Jacobian $J(\Gamma)$

$$z \equiv A\left(D - \sum_{j=1}^{n} P_j\right) + \mathcal{K}. \tag{5.20}$$

We note also that

$$-z \equiv A\left(D^+ - \sum_{j=1}^{n} P_j\right) + \mathcal{K}. \tag{5.20'}$$

In these formulas $\mathcal{K}$ are the Riemann constants. The quantities $\lambda_1,\ldots,\lambda_n$ have the form

$$\ln\lambda_j = \sum_{s=1}^{g+n-1}\int_{P_0}^{Q_s}\Omega_{Q_0P_j} + \varkappa_j, \quad D = \sum_{s=1}^{g+n-1} Q_s, \quad j=1,\ldots,n, \tag{5.20''}$$

where Q_0 is an arbitrary point of the surface Γ, $\Omega_{Q_0P_j}$ are normalized differentials of third kind with simple poles at the points Q_0, P_j, and $\varkappa_j$ are some constants.

Remark. A mapping $D \to (z, \lambda_1,\ldots,\lambda_n)$ of the form (5.20), (5.20''), where $\deg D = g + n - 1$, is an analogue of the Abel mapping (see the Appendix) defined with the help of differentials of first and third kinds on $\Gamma\setminus(P_1\cup\ldots\cup P_n)$ (cf. [46]). The collection of parameters $(z, \lambda_1,\ldots,\lambda_n)$, where $z\in\mathbf{C}^g$, is determined nonuniquely up to transformations

$$\begin{gathered} z\mapsto z+2\pi i N + BM, \\ \lambda_j \mapsto \lambda_j \exp\langle M, A(P_j)\rangle, \quad j=1,\ldots,n, \end{gathered} \tag{5.21}$$

where $N, M\in\mathbf{Z}^g$. This means that the collection of all matrices (Ω_j^i) of the form described above is fibered over the Jacobian $J(\Gamma)$ with fiber $(\mathbf{C}^*)^{n-1}$; the relations (5.21) are the transition formulas of this fibering. The origin of this fibering can be described in another way as follows: by replacing the divisor D by a linearly equivalent divisor $D' \sim D$, we obtain BA functions differing by a meromorphic factor. This leads to the replacement of the matrices (v_j^i) and (Ω_j^i) by similar matrices $(v_j^i) \to \Lambda^{-1}(v_j^i)\Lambda$, $(\Omega_j^i) \to \Lambda^{-1}(\Omega_j^i)\Lambda$, where $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$. The linear equivalence classes of divisors form the Jacobian $J(\Gamma)$.

We shall now consider explicitly the dependence on $\mathbf{x} = (x^1,\ldots,x^n)$.

LEMMA 5.2. Let $U^{(1)},\ldots,U^{(n)}$ be the vectors of periods of normalized meromorphic differentials $\Omega_{P_1},\ldots,\Omega_{P_n}$ with double poles at the points $P_1,\ldots,P_n$, respectively. Then the dependence of the parameters $(z, \lambda_1,\ldots,\lambda_n)$ on (5.19) on $x^1,\ldots,x^n$ is given by the following formulas:

$$z = i\sum_{j=1}^{n} x^j U^{(j)} + z_0, \tag{5.22}$$

(z_0 is an arbitrary g-dimensional vector),

$$\lambda_j=\lambda_j^0 \exp \sum_{m\neq j} i x^m \gamma_j^m, \tag{5.23}$$

where

$$\gamma_l^m=-\frac{d}{d\left(k_m^{-1}\right)} \ln E(P, P_l)|_{P=P_m}, \tag{5.24}$$

and λ_i^0, $i = 1,\ldots,n$, are arbitrary nonzero constants.

Proof. The divisor of zeros of the differential Ω_{ji} of the form (5.19) can be represented as the sum $D^i + D_j$, where D^i, D_j are divisors of degree g such that

$$A(D^i-P_i)=z-\mathcal{K}, \tag{5.25}$$

$$A(D_j-P_j)=-z-\mathcal{K} \tag{5.26}$$

($\mathcal{K}$ are the Riemann constants). For a point z of general position these divisors are nonspecial. On the basis of them we construct the BA functions φ^i and φ_j^+, respectively, where

$$\varphi^i(\mathbf{x}, P)=e^{ikx^j}(c_j^i+O(k^{-1})), \quad k=k_j, \quad P\to P_j, \tag{5.27}$$

$$\varphi_j^+(\mathbf{x}, P)=e^{-ikx^i}(c_j^{+i}+O(k^{-1})), \quad k=k_i, \quad P\to P_i, \tag{5.28}$$

where $c_i^i = c_j^{+j} = 1$. Then

$$\Omega_j^i(\mathbf{x}, P)=\varphi^i(\mathbf{x}, P)\varphi_j^+(\mathbf{x}, P)\Omega_j^i(P), \tag{5.29}$$

where $\Omega_j^i(\mathbf{x}, P)$ has the form (5.15), and $\Omega_j^i(P)$ has the form (5.19) with the change $(z, \lambda) \to (z_0, \lambda_0)$. For the functions φ^i, φ_j^+ we obtain

$$\varphi^i(\mathbf{x}, P)=\alpha^i(\mathbf{x})\exp\left(i\int_{P_0}^{P}\sum_{k=1}^{n} x_k\Omega_{P_k}\right)\frac{\theta(A(P)-A(P_i)+i\Sigma x_k U^{(k)}+z_0)}{\theta(A(P)-A(P_i)+z_0)} \tag{5.30}$$

$$\varphi_j^+(\mathbf{x}, P)=\alpha_j^+(\mathbf{x})\exp\left(-i\int_{P_0}^{P}\sum_{k=1}^{n} x_k\Omega_{P_k}\right)\frac{\theta(A(P)-A(P_j)-i\Sigma x_k U^{(k)}-z_0)}{\theta(A(P)-A(P_j)-z_0)}, \tag{5.31}$$

where $\alpha^i(\mathbf{x})$, $\alpha_j^+(\mathbf{x})$ are normalizing factors having the form

$$(\alpha^i(\mathbf{x}))^{-1}=\exp\left(-i\sum_{k=1}^{n}\tilde{\gamma}_k^i x^k\right)\frac{\theta\left(i\sum_{k=1}^{n}x^k U^{(k)}+z_0\right)}{\theta(z_0)}, \tag{5.32}$$

$$(\alpha_j^+(\mathbf{x}))^{-1}=\exp\left(i\sum_{k=1}^{n}\tilde{\gamma}_k^j x^k\right)\frac{\theta\left(i\sum_{k=1}^{n}x^k U^{(k)}+z_0\right)}{\theta(z_0)}, \tag{5.33}$$

$$\tilde{\gamma}_k^i=\int_{P_0}^{P_i}\Omega_{P_k} \tag{5.34}$$

(for k = i the integral is understood in the sense of principal value). In the expression (5.29) only the difference

$$\tilde{\gamma}_k^i-\tilde{\gamma}_k^j=\int_{P_j}^{P_i}\Omega_{P_k} \tag{5.35}$$

make a contribution. Evaluating this integral in terms of the Prym form E(P, Q), we obtain the formulas (5.24). The proof of the lemma follows from this and (5.29).

Summarizing the lemmas proved, we obtain the following result.

THEOREM 5.1. a) The collection of complex finite-zone operators of the form (5.2) with given spectrum Γ forms an (n − 1)-dimensional bundle of the form (5.21) over the Jacobian J(Γ).

b) The functions

$$v_j^i(x)=\frac{\lambda_i(x)}{\lambda_j(x)}\cdot\frac{\theta\left(A(P_j)-A(P_i)+i\sum_{k=1}^{n}x^kU^{(k)}+z_0\right)}{\theta\left(i\sum_{k=1}^{n}x^kU^{(k)}+z_0\right)E(P_i,\ P_j)\sqrt{dk_i^{-1}}\sqrt{dk_j^{-1}}} \tag{5.36}$$

($i\neq j$), where the quantities $\lambda_i(x)$ are defined by equalities (5.23), are the coefficients of these finite-zone operators and are thus solutions of the system (5.3).

We recall that for n-sheeted Riemann surfaces Γ these functions for $x^k = a_k x$ give all complex finite-zone potentials of operators of the form (3.1), (3.3) and thus all finite-zone solutions of the system (3.8).

Remarks. To the changes of local parameters

$$k_i\mapsto c_ik_i+\ldots,\quad i=1,\ldots,n, \tag{5.37}$$

there corresponds the group of transformations of the original system of the form

$$v_j^i\mapsto c_jv_j^i,\quad x_j\mapsto c_j^{-1}x_j. \tag{5.38}$$

6. A Criterion That the Finite-Zone Operator Pencils Constructed Be J-Hermitian. Plane Real Curves Corresponding to Solutions of the Euler Equations

THEOREM 6.1. Let Γ be an arbitrary Riemann surface of genus g with antiinvolution τ of separating type. Suppose that the points $P_1,\ldots,P_n$ are fixed relative to τ. We choose local parameters $k_1^{-1},\ldots,k_n^{-1}$ in neighborhoods of these points which are symmetric relative to τ,

$$\tau^*k_j=\bar{k}_j, \tag{6.1}$$

whereby the sign of the differential dk_j^{-1} in a neighborhood of the point P_j is equal to $\sigma_j = \pm1$, $j = 1,\ldots,n$. Suppose that the parameters $(z_0,\ \lambda_1^0,\ldots,\lambda_n^0)$, which define the finite-zone potential $V = (v_j^i)$ in accordance with formulas (5.36), (5.23), (5.24), are subject to the following conditions: 1) the vector $z_0\in J(\Gamma)$ has the form (A.36) of the appendix [the basis of cycles in $H_1(\Gamma)$ is chosen in the form (A.31), (A.32)]; 2) the constants $\lambda_1^0,\ldots,\lambda_n$ are equal to one in modulus. Then formulas (5.36), (5.23), (5.24) determine a smooth finite-zone potential $V = (v_j^i)$ of the operator (5.2) with the J-Hermitian condition

$$\bar{v}_j^i=-\sigma_i\sigma_jv_i^j. \tag{6.2}$$

In the case where on Γ there exists a meromorphic function λ with simple poles at the points $P_1,\ldots,P_n$, where $k_j = \lambda$ in a neighborhood of the point P_j, $j = 1,\ldots,n$, the conditions listed on the Riemann surface Γ, the antiinvolution τ, the position on Γ of the points $P_1,\ldots,P_n$, the signs of $d\lambda$ in neighborhoods of the points P_j, and the values of the parameters $(z_0,\ \lambda_1^0,\ldots,\lambda_n^0)$ are also necessary.

Proof. We shall first prove sufficiency of the conditions of the theorem. Under the conditions listed we have for differentials $\tilde{\Omega}_j^i = \Omega_j^i\sqrt{dk_i^{-1}}\sqrt{dk_j^{-1}}$ of the form (5.19) a symmetry of the form

$$\overline{\tilde{\Omega}_j^i(\tau(P))}=\tilde{\Omega}_i^j(P),\ i,\ j=1,\ldots,n. \tag{6.3}$$

This follows immediately from the explicit formulas (5.19), (5.23), (5.24) if we use the symmetry (A.34) of the theta function and the Prym form (A.37). For the differentials Ω_j^i because of the condition on the signs of $dk_1^{-1},\ldots,dk_n^{-1}$, there is the relation

$$\overline{\Omega_j^i(\tau(P))}=\sigma_i\sigma_j\Omega_i^j(P). \tag{6.4}$$

The expression (6.2) follows from this and from the formulas (5.16) for the residues.

Suppose now that on Γ there exists a meromorphic function λ with simple poles at the points $P_1,\ldots,P_n$. From Theorem 4.1 there then follow the conditions on the Riemann surface Γ, the position of the points $P_1,\ldots,P_n$, and signs of $d\lambda$ in neighborhood of these points. Further, from Theorem 4.2 we obtain the linear equivalence

$$D+\tau(D)\sim K+2\sum_{j=1}^{n} P_j. \tag{6.5}$$

Because of (5.20), this implies that the vector z must lie on one of the imaginary components of the Jacobian $J(\Gamma)$. That this component has the form (A.36) follows from the smoothness of v_j^i (otherwise there would be singularities due to the zeros of $\theta(z)$; cf. [46]). We shall now show that the constants $\lambda_1^0,\ldots,\lambda_n^0$ are unimodular. Indeed, for $\lambda_i^0 = 1$, $i = 1,\ldots,n$ the symmetry (6.4) is clearly satisfied for differentials Ω_j^i of the form (5.19). The extent of nonuniqueness of the matrix (Ω_j^i) of rank 1 with respect to the quantity z_0 consists in transformations of the form $\Omega_j^i \mapsto \lambda_i^0 \Omega_j^i (\lambda_j^0)^{-1}$. These transformations preserve the symmetry (6.4) if and only if all the constants λ_i^0 are unimodular. The proof of the theorem is complete.

Sufficient conditions for the symmetry of the matrices v_j^i [the skew symmetry $L^+(-\lambda) = -L(\lambda)$, $J = 1$] and also their necessity for Riemann surfaces Γ which are n-sheeted coverings are obtained similarly.

THEOREM 6.2. In order to obtain symmetric matrices $V = (v_j^i)$, to the conditions listed on the Riemann surface Γ, the position of the points on it, the signs of the local parameters, and the local parameters $(z_0, \lambda_1^0,\ldots,\lambda_n^0)$ it is necessary to add the following condition: on the surface Γ there must be given an involution σ, where $\sigma\tau = \tau\sigma$, $\sigma(P_j) = P_j$, $j = 1,\ldots,n$. The vector z_0 must lie on the odd part of the Jacobian $J(\Gamma)$, $\sigma(z_0) \equiv -z_0$ (the Prym manifold), and $(\lambda_i^0)^2 = 1$, $i = 1,\ldots,n$.

Sufficiency of these conditions can easily be obtained from the explicit formulas (5.36) and elementary properties [46] of Riemann surfaces with an involution and their theta functions. Necessity (for n-sheeted coverings) follows easily from the results of Sec. 4.

For $J = 1$ the conditions obtained can be made still more effective by describing necessary and sufficient conditions on the Riemann surfaces Γ giving Hermitian pencils of operators $L(\lambda)$. As an example, we shall here treat the simplest finite-zone operators $L(\lambda)$ for which the matrix $M(\lambda)$, commuting with $L(\lambda)$, has the form

$$M(\lambda)=\lambda C-[C, V]+D, \quad V^*=-V, \tag{6.6}$$

where $C = \mathrm{diag}(c_1,\ldots,c_n)$, $D = \mathrm{diag}(d_1,\ldots,d_n)$ are real matrices and $c_i \neq c_j$ for $i \neq j$. As already mentioned, the coefficients $V = (v_j^i)$ of the operator $L(\lambda)$ satisfy equations of the form

$$i[C, V_x]-iD_x=[[A, V], [C, V]-D]. \tag{6.7}$$

For $D = 0$ we obtain the integrable Euler equations found by S. V. Manakov.

In this case the Riemann surface Γ of the spectrum of the operator $L(\lambda)$ has the form

$$R(\lambda, \nu)=\det(\nu-M(\lambda))=0. \tag{6.8}$$

This is a plane algebraic curve of degree n. We have the following result.

LEMMA 6.1. For a skew-Hermitian matrix V of general position all branch points of the Riemann surface Γ of the form (6.8) are nonreal and pairwise distinct.

Proof. Nonrealness of the branch points has been proved above (see Sec. 4). Noncoincidence of the branch points follows from the fact that the coefficients of the discriminant $\Delta(\lambda)$ of the polynomial $R(\lambda, \nu)$ ($\deg \Delta(\lambda) = n(n-1)$) are independent as functions of V.

COROLLARY. For a skew-Hermitian matrix V of general position a plane real curve Γ of the form (6.8) is nonsingular. The real plane nonsingular curves of degree n thus obtained belong to a single isotopy class.

Proof. Because of Lemma 6.1, an n-sheeted Riemann surface Γ of the form (6.5) has $n(n-1)$ branch points in the case of general position. Its genus g is then $g = (n-1)(n-2)/2$ according to the Riemann–Hurwitz formula. Now a plane curve of degree n of this genus is necessarily nonsingular (singularities would reduce the genus). Further, appearance of a singularity on the real axis can occur only by the coalescing of a pair of eigenvalues ν_i, ν_j of a matrix $M(\lambda)$ of the form (6.6). But for real λ this matrix is Hermitian. Hermitian matrices with coincident eigenvalues have codimension 3 in the space of all Hermitian matrices. Therefore, for a general one-parameter deformation of the matrix V coalescence of the eigenvalues does not occur for a family of matrices $M = M(\lambda)$ of the form (6.6) (λ is real). Coalescence of complex conjugate pairs of branch points occurs on a submanifold of codimension 2 in the space of curves of the form (6.5). This implies that a general one-parameter deformation of one curve of the form (6.8) into another such curve gives nonsingular curves. The proof of the corollary is complete.

We shall now study the question of the position on $\mathbf{RP}^2$ of real components of curves of the form (6.8) with $V^* = -V$ (according to the corollary, this position is the same for all curves of this form). We have the following result.

THEOREM 6.3. For $n = 2m$ the real components of nonsingular curves $\Gamma\subset\mathbf{RP}^2$ of the form (6.8) consist of m ovals imbedded in one another (a "nest" of m ovals). For $n = 2m + 1$, aside from the nest of m ovals, there is still one one-sided cycle in $\mathbf{RP}^2$ (the "projective line").

Proof. The curve (6.8) intersects the infinitely distant line in $\mathbf{RP}^2$ at n distinct points $P_1,\ldots,P_n$, $P_j = \{\lambda = \infty,\ \nu/\lambda = c_j\}$. Considering the absence of real branch points, we obtain the following: each line λ = const on $\mathbf{RP}^2$ intersects the curve Γ in n distinct points. We shall assume that the numbers c_j are arranged in decreasing order: $c_1 > c_2 > \ldots > c_n$. From what has been said it follows that the branch of the curve Γ having as asymptote as $\lambda \to +\infty$ the line $\nu = c_i\lambda + d_i$ for $\lambda \to -\infty$ has as asymptotic the line $\nu = c_j\lambda + d_j$, where $j = n - i + 1$ (see Fig. 4). The assertion of the theorem obviously follows from this.

From the proof it also follows that the points $P_1,\ldots,P_n$ are situated on the real components of the curve Γ in the following manner: for $n = 2m$ the pairs of points P_j and P_{n-j+1} lie on the j-th oval (counting from the outside). For $n = 2m + 1$ the position of the pairs P_j and P_{n-j+1} for $j \neq m + 1$ is the same as for even n, while the point P_{m+1} alone lies on the one-sided component of the curve Γ. (We recall that we consider the numbers $c_1,\ldots,c_n$ to be ordered; this determines the order of the points $P_1,\ldots,P_n$.)

Conversely, suppose that Γ is a plane real nonsingular curve of degree n with $[(n + 1)/2]$ components arranged as described in Theorem 6.1. Then the corresponding complex Riemann surface Γ belongs to separating type [36]. We have seen in Sec. 5 that to represent a plane curve of degree n in the form (6.8) it is necessary to choose a "spectral parameter" λ on it, i.e., a pencil of lines passing through some point in $\mathbf{CP}^2$. We choose this point in $\mathbf{RP}^2\subset\mathbf{CP}^2$ lying strictly within the innermost oval of the curve Γ. We choose coordinates λ, ν in $\mathbf{RP}^2$ so that this point lie on a nonsingular line. Let $P_1,\ldots,P_n$ be nonsingular points of the curve Γ, where for $P \to P_j$, $\lambda \to \infty$, $\nu = c_j\lambda + d_j + O(\lambda^{-1})$. We choose a basis of cycles a_i, b_i on the Riemann surface Γ as described in the Appendix for curves of separating type. From Theorem 6.1 we immediately obtain the following result.

THEOREM 6.4. Under the conditions listed on the Riemann surface Γ, the choice of spectral parameter λ on it and of points $P_1,\ldots,P_n$, and also the basis of cycles of type (A.30) formulas (5.36) give the coefficients of a Hermitian pencil of operators $L(\lambda)$ of the form (5.7) for $x^k = a_k x$ where the constants $\ln\lambda_1^0,\ldots,\ln\lambda_n^0$ are purely imaginary, and the vector z_0 has the form

$$z_0=(\zeta,\ r,\ \bar{\zeta}),\ \zeta\in\mathbf{C}^{\rho},\ r\in\mathbf{R}^{g-2\rho},\quad \rho=\begin{cases}(k-1)^2, & n=2k,\\ k(k-1), & n=2k+1.\end{cases} \tag{6.9}$$

We recall that if the points $P_1,\ldots,P_n$ on the curve Γ are such that the tangents at them intersect at one point, then the matrix D can be made zero by an appropriate choice of the coordinate ν.

We now add to the condition that V be skew-Hermitian the condition that V be symmetric. On the plane curve Γ there then arises an involution $\sigma:(\lambda,\ \nu)\ (-\lambda,\ -\nu)$. The diagonal matrix D is hereby automatically equal to zero if the points $P_1,\ldots,P_n$ are fixed under σ. Plane real curves with an involution σ having n branch points for even n and $n + 1$ branch points for odd n can be classified according to the number of purely imaginary branch points. For $n = 4k + 2$, $4k + 3$ there is always one pair of imaginary branch points W_0^+, $W_0^- = \tau(W_0)$. Moreover, there are k purely imaginary quadruplets of branch points situated symmetrically relative to τ, $0 \leqslant 4k \leqslant n^2 - n - 2$. For $n = 4k$, $4k + 1$ there are k purely imaginary quadruplets of branch points $0 \leqslant 4k \leqslant n^2 - n$. In all cases the number k is a topological invariant of the triplet $(\Gamma,\ \tau,\ \sigma)$, where τ is an antiinvolution on the surface Γ and σ is an involution. There are no other topological invariants (except the degree) for plane curves Γ of degree n with nonreal branch points.

Plane curves Γ of degree n with an involution of the type described above make it possible to construct the simplest Hermitian finite-zone operators $L(\lambda)$ with the condition of skew-symmetry $L^{+}(-\lambda) = -L(\lambda)$. The coefficients of these operators, as already repeatedly mentioned in the present survey, are solutions of the Euler equations (3.15) for the right-invariant metrices on the group $SO(n)$ found by Manakov [30]. Each invariant manifold (an invariant torus) of such systems determines a real Riemann surface Γ with an involution σ of the form (6.5) where the coefficients of the polynomial $R(\lambda, \nu)$ are integrals of the system (3.15). Each such invariant manifold is isomorphic to a collection of finite-zone operators $L(\lambda)$ with spectrum Γ. This makes it possible to prove that the invariant tori of the Euler equations on the group $SO(n)$ of the form (3.15) corresponding to Riemann surfaces of the form (6.8) are isomorphic to coverings over the Prym manifold of the surface Γ with involution σ. The proof of this assertion and also an analysis of the explicit formulas for solutions of the Euler equations on $SO(n)$ we shall present in a subsequent work.

APPENDIX

Some Facts from the Theory of Riemann Surfaces and Theta Functions

Let Γ be a compact Riemann surface of genus $g \geqslant 1$.* If Γ is the Riemann surface of an algebraic function $w = w(z)$ given by the equation

$$R(z, w) = w^n + a_1(z)\, w^{n-1} + \ldots + a_n(z) = 0, \tag{A.1}$$

where $R(z, w)$ is a polynomial, then the affine part of Γ coincides with the complex algebraic curve (A.1) in $\mathbf{C}^2$ in the case where this curve is nonsingular (smooth). An important example for us are plane curves of degree n where the degree of the polynomials $a_i(z)$ are equal to i $(i = 1,\ldots,n)$. The surface (A.1) covers the z plane in n-sheeted fashion under the natural projection $(z, w) \to z$, i.e., to a given value of z, generally speaking, there correspond n distinct values $w_1(z),\ldots,w_n(z)$ of the algebraic function $w(z)$ defined by Eq. (A.1). Branch points are formed on the surface Γ when several of these branches coalesce. They are defined from the following system:

$$R(z, w) = 0, \quad R_w(z, w) = 0. \tag{A.2}$$

For a plane nonsingular curve of degree n we obtain $n(n - 1)$ branch points. The genus (number of handles) of such a Riemann surface is equal to g $(n - 1)(n - 2)/2$. In the one-dimensional homology group $H_1(\Gamma) = \mathbf{Z} + \ldots + \mathbf{Z}$ (2g terms) it is possible to choose a basis of cycles 'close (closed contours) $a_1,\ldots,a_g$, $b_1,\ldots,b_g$ with the following intersection indices:

$$a_i \circ a_j = b_i \circ b_j = 0, \; a_i \circ b_j = \delta_{ij}; \; i, \; j = 1, \ldots, g. \tag{A.3}$$

A basis of the holomorphic differentials (of first kind) on a plane nonsingular curve of degree n has the form

$$\eta_{ij} = \frac{z^i w^j}{R_w(z, w)}\, dz, \quad i + j \leqslant n - 3. \tag{A.4}$$

Chosing suitable linear combinations $\sum c_{ij}\eta_{ij}$, we obtain the canonical basis of holomorphic differentials

$$\omega_1, \ldots, \omega_g, \tag{A.5}$$

normalized by the conditions

$$\oint_{a_k} \omega_j = 2\pi i \delta_{jk}; \; j, \; k = 1, \ldots, g. \tag{A.6}$$

The matrix $B = (B_{jk})$

$$B_{jk} = \oint_{b_k} \omega_j; \; j, \; k = 1, \ldots, g, \tag{A.7}$$

is called the period matrix of the Riemann surface Γ. This matrix is symmetric and has negative-definite real part. On the basis of this matrix we construct a 2g-dimensional torus $T^{2g} = J(\Gamma)$ called the Jacobi manifold (or Jacobian) of the Riemann surface Γ: we set

$$J(\Gamma) = \mathbf{C}^g / \{2\pi i N + BM \mid N, M \in \mathbf{Z}^g\}. \tag{A.8}$$

*Only compact Riemann surfaces will be encountered below, and we shall not mention this each time.

The (Riemann) theta function of the surface Γ is constructed on the period matrix $B = (B_{jk})$:

$$\theta(z) = \sum_{N \in Z^g} \exp\left\{\frac{1}{2}\langle BN, N\rangle + \langle N, z\rangle\right\};$$
$$z = (z_1, \ldots, z_g),$$
$$N = (N_1, \ldots, N_g), \quad \langle N, z\rangle = \sum N_j z_j, \tag{A.9}$$
$$\langle BN, N\rangle = \sum B_{ij} N_i N_j.$$

Under translation of the argument z by a vector of the period lattice the theta function transforms according to the following law:

$$\theta(z + 2\pi i N + BM) = \exp\left\{-\frac{1}{2}\langle BM, M\rangle - \langle z, M\rangle\right\}\theta(z); \quad N, M \in \mathbf{Z}^g. \tag{A.10}$$

Theta functions with the following characteristics are also frequently used:

$$\theta[\alpha, \beta](z) = \exp\left\{\frac{1}{2}\langle B\alpha, \alpha\rangle + \langle z + 2\pi i\beta, \alpha\rangle\right\}\theta(z + 2\pi i\beta + B\alpha); \quad \alpha, \beta \in R^g. \tag{A.11}$$

Characteristics $[\alpha, \beta]$ for which all coordinates are equal to 0 or 1/2 [we write this as follows: $2\alpha \in ((\mathbf{Z}_2)^g$, $2\beta \in (\mathbf{Z}_2)^g)$] are called half periods. A half period $[\alpha, \beta]$ is even if $4\langle\alpha, \beta\rangle \equiv 0 \pmod 2$ and is odd otherwise.

The Abel mapping of the Riemann surface Γ into its Jacobi manifold $J(\Gamma)$, $A(P) = (A_1 \times (P), \ldots, A_g(P))$ has the form

$$A_k(P) = \int_{P_0}^{P} \omega_k, \quad k = 1, \ldots, g, \tag{A.12}$$

where P_0 is a fixed point on Γ.

A divisor on Γ is a formal integral linear combination of points of Γ:

$$D = \sum_{i=1}^{N} n_i P_i, \quad n_i \in \mathbf{Z}. \tag{A.13}$$

For example, for any meromorphic function f on Γ we define the divisor (f) of its zeros $P_1, \ldots, P_n$ and poles $Q_1, \ldots, Q_m$ of multiplicities $p_1, \ldots, p_n$ and $q_1, \ldots, q_m$, respectively ($p_1 + \ldots + p_n = q_1 + \ldots + q_m$) by

$$(f) = p_1 P_1 + \ldots + p_n P_n - q_1 Q_1 - \ldots - q_m Q_m \tag{A.14}$$

(such divisors are called principal divisors). The divisors form an Abelian group in the obvious way. The degree of the divisor $D = \Sigma n_i P_i$ is the number

$$\deg D = \sum n_i. \tag{A.15}$$

The Abel mapping (A.12) extends linearly to the group of all divisors.

Two divisors are called linearly equivalent if their difference is a principal divisor. According to the classical Abel theorem, the necessary and sufficient conditions for linear equivalence of divisors D and D' have the form

$$1)\ \deg D = \deg D', \quad 2)\ A(D) \equiv A(D') \tag{A.16}$$

Here and henceforth the symbol $\equiv$ is used for equality on the Jacobi manifold, i.e., equality modulo the period lattice.

Example. The divisors of the zeros and poles of two differentials ω, ω' meromorphic on Γ are linearly equivalent. This class of divisors is called the canonical class of the surface Γ and is denoted by K ($\deg K = 2g - 2$).

A divisor $D = \Sigma n_i P_i$ for which all the multiplicities n_i are positive is called positive (or effective). Two divisors D, D' are connected by the inequality $D \geqslant D'$, by definition, if their difference $D - D'$ is a positive divisor. We note a useful property of divisors of degree $\geqslant g$: any such divisor is linearly equivalent to a positive divisor. For positive divisors $D = \Sigma n_i P_i$ the dimension $l(D)$ of the space of meromorphic functions f on Γ for which the following inequality is satisfied is of interest:

$$(f) \geqslant -D. \tag{A.17}$$

This space consists of those meromorphic functions which can have poles only at points P_i of multiplicity no greater than n_i (constants are also contained in this space). For a positive divisor of general position the dimension $l(D)$ has the form

$$l(D) = \begin{cases} 1, & \deg D \leqslant g, \\ \deg D - g + 1, & \deg D \geqslant g. \end{cases} \tag{A.18}$$

(the second formula is also valid for any divisor D if $\deg D > 2g - 2$). Divisors D for which $l(D) = \deg D - g + 1$ are also called nonspecial.

Example. For an n-sheeted Riemann surface of the form (A.1) divisors D_{z_0} of the form $D_{z_0} = P_1 + \ldots + P_g$, where $P_i = (z_0, w_i(z_0))$, $i = 1,\ldots,n$, are preimages of the point z_0, are all linearly equivalent to one another for different z_0. The function $f(z) = (z - z_0)^{-1}$ has poles of first order at points of the divisor D_{z_0}. Therefore, $l(D_{z_0}) \geqslant 2$. For a plane curve of degree n we have $l(D_{z_0}) \geqslant 3$ (actually, in the nonsingular case equality always holds). Indeed, we take $z_0 = \infty$, then the functions z and w have poles of no more than first order at the "infinitely distant" points of the surface Γ. The inequality $l(D) \geqslant 3$ for some divisor D of degree n is characteristic for plane curves of degree n.

On the manifold Γ we fix a point P_0 and consider the Abel mapping defined on unordered collections $(P_1,\ldots,P_g)$ of points of Γ, i.e., on the g-th symmetric power $S^g\Gamma$. The Abel mapping also defines the mapping

$$A: S^g\Gamma \to J(\Gamma); \quad A(P_1, \ldots, P_g) = \sum_{j=1}^{g} A(P_j). \tag{A.19}$$

The problem of inverting this mapping is known as the Jacobi inversion problem. Its solution (Riemann) can be given in the language of theta functions. Namely, if for a vector $\zeta = (\zeta_1,\ldots,\zeta_g)$ the function $\theta(A(P) - \zeta)$ is not identically zero on the Riemann surface Γ, then on Γ it has exactly n zeros $P_1,\ldots,P_g$ giving the solution of the inversion problem

$$A(P_1) + \ldots + A(P_g) \equiv \zeta - \mathcal{K}, \tag{A.20}$$

where $\mathcal{K} = (\mathcal{K}_1, \ldots, \mathcal{K}_g)$ is the so-called vector of Riemann constants [13] which depends only on the Riemann surface, the choice of a basis of cycles on it, and the initial point P_0. In this case the divisor $D = P_1 + \ldots + P_g$ is nonspecial, and the points $P_1,\ldots,P_g$ are determined from the system (A.20) uniquely up to permutation.

We now note some properties of meromorphic differentials on a Riemann surface. For any meromorphic differential ω there is the important relation (the residue theorem)

$$\sum_{\omega(P)=\infty} \operatorname{Res}_P \omega = 0 \tag{A.21}$$

(the summation gives overall poles of ω).

We call a meromorphic differential normalized if

$$\oint_{a_i} \omega = 0, \quad i = 1, \ldots, g. \tag{A.22}$$

This normalization together with the prescription of the poles and corresponding principal parts uniquely determine the meromorphic differential. Any meromorphic differential can be represented as the sum of some holomorphic differential and a linear combination of the following meromorphic basis differentials:

a) Abelian differentials of second kind having one pole $\Omega_Q^{(n)}$ of multiplicity $n+1$ at the point Q and a principal part of the form $z^{-n-1}dz$ $(n \geqslant 1)$;

b) Abelian differentials of third kind Ω_{PQ} having a pair of simple poles at the points P, Q with residues +1, −1, respectively.

In computations with meromorphic differentials their "generating function" are useful — the Prym form of the surface Γ [46]. If P, Q are points of Γ and z, w are local parameters in neighborhoods of these points, we set

$$E(P, Q)=\varepsilon(P, Q)(dz)^{-\frac{1}{2}}(dw)^{-\frac{1}{2}}=\frac{\theta[\nu](A(P)-A(Q))}{\sqrt{\sum \omega_i(P)\theta_i[\nu](0)}\sqrt{\sum \omega_j(Q)\theta_j[\nu](0)}}. \tag{A.23}$$

Here $\nu\in\frac{1}{2}(\mathbf{Z}_2)^{2g}$ is any odd (see above) nondegenerate (i.e., $\operatorname{grad}\theta[\nu](0)\neq 0$) half period (cf. [46]); the indices i, j of the theta functions denote differentiation with respect to the corresponding variables z_i, z_j; $\omega_1,\ldots,\omega_g$ are the basis holomorphic differentials. The quantity $E(P, Q) = -E(Q, P)$ is single-valued (in each variable) on the Riemann surface Γ dissected along the cycles a_i and vanishes only for $P = Q$; on passing about a cycle b_j it acquires the factor

$$\exp\left(-\frac{1}{2}B_{jj}-\int_Q^P\omega_j\right). \tag{A.24}$$

For a differential of third kind Ω_{PQ} we obtain the expression

$$\Omega_{PQ}(X)=d\ln\frac{E(X, P)}{E(X, Q)}\quad (X\in\Gamma). \tag{A.25}$$

The differentials of second kind can be obtained from the bilinear differential

$$\omega(P, Q)=\omega(Q, P)=\left(\frac{d}{dz}\frac{d}{dw}\ln E(P, Q)\right)dz\,dw \tag{A.26}$$

(z and w are local parameters in neighborhoods of the points P and Q). Then, for example,

$$\Omega_Q^{(1)}(P)=\frac{\omega(P, Q)}{dw} \tag{A.27}$$

etc. We shall indicate a useful expression for an arbitrary (not normalized) differential of third kind ω_{PQ} with simple poles at the points P, Q:

$$\omega_{PQ}=\frac{\theta(A(X)-A(P)-\zeta)\,\theta(A(X)-A(Q)+\zeta)}{E(X, P)\,E(Q, X)\sqrt{dz}\sqrt{dw}}, \tag{A.28}$$

where $\zeta\in\mathbf{C}^g$ is an arbitrary vector of general position [if $\theta(\zeta) = 0$ then the differential (A.28) becomes a holomorphic differential]. If the points P and Q coalesce, then (A.28) becomes a differential of second kind with a double pole at the point $P = Q$.

The Baker–Akhiezer (BA) functions are the basic algebrogeometric tool in the theory of finite-zone operators and the solutions of nonlinear equations connected with them. These functions were introduced by Krichever [27] on the basis of a generalization of the analytic properties of the Bloch eigenfunctions of operators with periodic and almost periodic coefficients. We shall give the general definition of them and list their simplest properties.

Definition. Let $P_1,\ldots,P_n$ be points on a Riemann surface Γ, let $k_1^{-1},\ldots,k_n^{-1}$ be local parameters in neighborhoods of these points [where $k_i(P_i) = \infty$], let $q_1(k),\ldots,q_n(k)$ be a collection of polynomials, and let D be a divisor on Γ. The n-point (scalar of rank 1; see [29]) BA function given by these data is a meromorphic function $\psi = \psi(P)$ on $\Gamma\setminus(P_1\cup\cdots\cup P_n)$ such that a) the divisor $\tilde{\psi} \geqslant -D$; b) as $P \to P_i$ the product $\psi(P)\exp(-q_i(k_i(P)))$ is analytic $(i = 1,\ldots,n)$.

If D is a nonspecial divisor of degree N, then the dimension of the space of n-point BA functions with the given form of essential singularities $(q_1,\ldots,q_n)$ is equal to $N - g + 1$ [26] for almost all polynomials $q_1,\ldots,q_n$. In particular, if $D = Q_1 + \ldots + \ldots + Q_g$ is a nonspecial divisor of degree g, then the corresponding n-point BA function exists and is uniquely determined up to a factor. It has the form

$$\psi(P)=c\exp\left(\sum_{j=1}^n\Omega_{q_j}\right)\frac{\theta(A(P)+\sum_{j=1}^n U^{(q_j)}-\zeta)}{\theta(A(P)-\zeta)}, \tag{A.29}$$

where Ω_{qj} is the normalized Abelian differential of second kind with principal part at the point P_j of the form $dq_j(k_j)$ and $U^{(q_j)}$ is its vector of b-periods; $A(P)=\left(\int_{P_0}^{P}\omega_1,\ldots,\int_{P_0}^{P}\omega_g\right)$ is the Abel mapping (A.12); $\zeta=A(D)+\mathcal{K}$, where $\mathcal{K}$ is the vector of Riemann constants; c is an arbitrary factor.

We shall now list the most important properties of real Riemann surfaces. A Riemann surface Γ is called real if on it there is given an antiholomorphic involution (or, more briefly, an antiinvolution) $\tau:\Gamma\to\Gamma$, $\tau^2=1$. Suppose that the antiinvolution τ has on Γ n fixed components (ovals), $0\leqslant n\leqslant g+1$. There are two possible cases: I) the union of real ovals decomposes Γ into two components Γ^+ and $\Gamma^-=\tau\Gamma^+$; II) the union of ovals does not decompose Γ. Surfaces of type I we call surfaces of separating type, while those of type II we call surfaces of nonseparating type.

On a surface of type I (where $0<n$) it is possible to choose a basis of cycles

$$a_1,\ b_1,\ldots,\ a_\rho,\ b_\rho;\ a_{\rho+1},\ b_{\rho+1},\ldots,\ a_{\rho+n-1},\ b_{\rho+n-1};\\ a_1',\ b_1',\ldots,\ a_\rho',\ b_\rho', \tag{A.30}$$

where $g=2\rho+n-1$, such that $a_{\rho+k}$, $k=1,\ldots,n-1$, are real ovals,

$$a_i,\ b_i\in\Gamma^+,\ \tau(a_i)=a_i',\ \tau(b_i)=-b_i',\ (i=1,\ldots,\rho), \tag{A.31}$$

$$\tau(a_{\rho+k})=a_{\rho+k},\ \tau(b_{\rho+k})=-b_{\rho+k}\ (k=1,\ldots,n-1), \tag{A.32}$$

The antiinvolution τ generates an antiinvolution on the Jacobian $J(\Gamma)$ which we denote by the same letter τ. In natural coordinates $z_1,\ldots,z_\rho$; $z_{\rho+1},\ldots,z_{\rho+n-1}$; $z_1',\ldots,z_\rho'$ the action of τ on the Jacobian has the form

$$\tau(z_1,\ldots,z_\rho;\ z_{\rho+1},\ldots,z_{\rho+n-1};\ z_1',\ldots,z_\rho')=-(\bar z_1',\ldots,\bar z_\rho';\ \bar z_{\rho+1},\ldots,\bar z_{\rho+n-1};\ \bar z_1,\ldots,\bar z_\rho). \tag{A.33}$$

The theta function of the surface Γ possesses the symmetry

$$\theta(\tau(z))=\overline{\theta(z)}. \tag{A.34}$$

The imaginary components of the Jacobian $J(\Gamma)$ are defined by the condition

$$\tau(z)=-z. \tag{A.35}$$

These are 2^{n-1} nonintersecting g-dimensional real tori. On one such torus of the form

$$z=(\zeta;\ \eta;\ \bar\zeta),\ \zeta\in\mathbf{C}^\rho,\ \eta\in\mathbf{R}^{n-1} \tag{A.36}$$

the function $\theta(z)$ is always positive [46]. The Prym form $E(P, Q)$ possesses the symmetry

$$E(\tau(P),\ \tau(Q))=E(P,\ Q). \tag{A.37}$$

The vector of Riemann constants is real if the initial point of the Abel mapping is chosen to be real.

On a surface of nonseparating type with n ovals $(0\leqslant n\leqslant g)$ it is always possible to choose a basis of cycles a_i, b_j which transform under the action of the antiinvolution according to the law

$$\tau(a_i)=a_i,\ i=1,\ldots,g,\ \tau(b_i)=\begin{cases}-b_i, & 1\leqslant i\leqslant n,\\ -b_i+a_i, & n+1\leqslant i\leqslant g.\end{cases} \tag{A.38}$$

The theta function $\theta(z)$ of such Riemann surfaces posesses the symmetry [16]

$$\overline{\theta(z)}=\theta(\bar z+\lambda), \tag{A.39}$$

where the half period λ has the form

$$\lambda=\frac{1}{2}(0,\ldots,0,1,\ldots,1) \tag{A.40}$$

(zeros at the first n places).

LITERATURE CITED

1. V. I. Arnol'd, Mathematical Methods of Classical Mechanics [in Russian], Nauka, Moscow (1974).
2. E. D. Belokolos and V. Z. Enol'skii, "The generalized Lamb Ansatz," Teor. Mat. Fiz., 53, No. 2, 271-282 (1982).
3. A. V. Brailov, "Complete integrability of some Euler equations and applications," Dokl. Akad. Nauk SSSR, 268, No. 5, 1043-1046 (1983).
4. A. P. Veselov and S. P. Novikov, "On Poisson brackets consistent with algebraic geometry and the Korteweg—de Vries dynamics on the set of finite-zone potentials," Dokl. Akad. Nauk SSSR, 266, No. 3, 533-537 (1982).
5. I. M. Gel'fand and L. A. Dikii, "The resolvent and Hamiltonian systems," Funkts. Anal. Prilozhen., 11, No. 2, 11-27 (1977).
6. E. B. Gledzero, F. V. Dolzhanskii, and A. M. Obukhov, Systems of Hydrodynamic Type and Their Application [in Russian], Nauka, Moscow (1981).
7. V. V. Golubev, Lectures on the Integration of the Equations of Motion of a Heavy Body near a Fixed Point [in Russian], Gostekhizdat, Moscow (1953).
8. P. Griffiths and J. Harris, Principles of Algebraic Geometry [Russian translation], Vol. 1, Mir, Moscow (1982).
9. L. A. Dikii, "Remarks on Hamiltonian systems connected with the rotation group," Funkts. Anal. Prilozhen., 6, No. 4, 83-84 (1972).
10. B. A. Dubrovin, "Analytic properties of spectral data for non-self-adjoint linear operators connected with real periodic solutions of the sine-Gordon equation," Dokl. Akad. Nauk SSSR, 265, No. 4, 789-793 (1982).
11. B. A. Dubrovin, "Completely integrable Hamiltonian systems connected with matrix operators and Abelian manifolds," Funkts. Anal. Prilozhen., 11, No. 4, 28-41 (1977).
12. B. A. Dubrovin, "The periodic problem for the Korteweg—de Vries equation in the class of finite-zone potentials," Funkts. Anal. Prilozhen., 9, No. 3, 41-51 (1975).
13. B. A. Dubrovin, "Theta functions and nonlinear equations," Usp. Mat. Nauk, 36, No. 2, 11-80 (1981).
14. B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, "The Schrödinger equation in a periodic field and Riemann surfaces," Dokl. Akad. Nauk SSSR, 229, No. 1, 15-18 (1976).
15. B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, "Nonlinear equations of Korteweg—de Vries type, finite-zone linear operators, and Abelian manifolds," Usp. Mat. Nauk, 31, No. 1, 55-136 (1976).
16. B. A. Dubrovin and S. M. Natanzon, "Real two-zone solutions of the sine-Gordon equation," Funkts. Anal. Prilozhen., 16, No. 1, 27-43 (1982).
17. B. A. Dubrovin and S. P. Novikov, "Algebrogeometric Poisson brackets for real finite-zone solutions of the sine-Gordon equation and the nonlinear Schrödinger equation," Dokl. Akad. Nauk SSSR, 267, No. 6, 1295-1300 (1982).
18. B. A. Durbovin and S. P. Novikov, "Periodic and conditionally periodic analogues of multisoliton solutions of the Korteweg—de Vries equation," Zh. Eksp. Teor. Fiz., 67, No. 12, 2131-2143 (1974).
19. B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, Modern Geometry. Methods and Applications [in Russian], Nauka, Moscow (1979).
20. V. E. Zakharov, L. A. Takhtadzhyan, and L. D. Faddeev, "A complete description of solutions of the sine-Gordon equation," Dokl. Akad. Nauk SSSR, 219, No. 6, 1334-1337 (1974).
21. A. R. Its, "inversion of hyperelliptic integrals and integration of nonlinear differential equations," Vestn. Leningr. Univ., No. 7, 39-46 (1976).
22. A. R. Its and V. P. Kotlyarov, "Explicit formulas for solutions of the nonlinear Schrödinger equation," Dokl. Akad. Nauk Ukr. SSR, Ser. A, No. 11, 965-968 (1976).
23. A. R. Its and V. B. Matveev, "Schrodinger operators with finite-zone spectrum and N-soliton solutions of the Korteweg—de Vries equation," Teor. Mat. Fiz., 23, No. 1, 51-68 (1975).
24. V. A. Kozel and V. P. Kotlyarov, "Almost-periodic solutions of the equation $u_{tt} - u_{xx} + \sin u = 0$," Dokl. Akad. Nauk Ukr. SSR, A, No. 10, 878-881 (1976).
25. V. V. Kozlov, "Integrability and nonintegrability in Hamiltonian mechanics," Usp. Mat. Nauk, 38, No. 1, 3-67 (1983).
26. I. M. Krichever, "Algebraic curves and commuting matrix differential operators," Funkts. Anal. Prilozhen., 10, No. 2, 75-76 (1976).
27. I. M. Krichever, "Algebrogeometric construction of the Zakharov—Shabat equations and their periodic solutions," Dokl. Akad. Nauk SSSR, 227, No. 2, 291-294 (1976).

28. I. M. Krichever, "Methods of algebraic geometry in the theory of nonlinear equations," Usp. Mat. Nauk, 32, No. 6, 183-208 (1977).
29. I. M. Krichever and S. P. Novikov, "Holomorphic bundles over algebraic curves and nonlinear equations," Usp. Mat. Nauk, 35, No. 6, 47-68 (1980).
30. S. V. Manakov, "A remark on the integration of the Euler equations of the dynamics of an n-dimensional solid body," Funkts. Anal. Prilozhen., 10, No. 4, 93-94 (1976).
31. V. A. Marchenko, Sturm-Liouville Operators and Their Applications [in Russian], Naukova Dumka, Kiev (1977).
32. A. S. Mishchenko, "Integrals of geodesic flows on Lie groups," Funkts. Anal. Prilozhen., 4, No. 3, 73-77 (1970).
33. A. S. Mishchenko and A. T. Fomenko, "The Euler equations on finite-dimensional Lie groups," Izv. Akad. Nauk SSSR, Ser. Mat., 42, 396-415 (1978).
34. S. P. Novikov, "The periodic problem for the Korteweg-de Vries equation," Funkts. Anal. Prilozhen., 8, No. 3, 54-66 (1974).
35. S. P. Novikov (ed.), Theory of Solitons. The Method of the Inverse Problem [in Russian], Nauka, Moscow (1980).
36. V. A. Rokhlin, "Complex topological characteristics of real algebraic curves," Usp. Mat. Nauk, 33, No. 5, 77-89 (1978).
37. V. V. Trofimov, "The Euler equations on Borel subalgebras of semisimple Lie algebras," Izv. Akad. Nauk SSSR, Ser. Mat., 43, No. 3, 714-732 (1979).
38. I. V. Cherednik, "Algebraic aspects of two-dimensional chiral fields. II," Itogi Nauki i Tekh. VINITI. Algebra. Topologiya. Geometriya, 18, 1981, pp. 73-150.
39. I. V. Cherednik, "Differential equations for the Baker-Akhiezer functions of algebraic curves," Funkts. Anal. Prilozhen., 12, No. 3, 45-54 (1978).
40. I. V. Cherednik, "On realness conditions in 'finite-zone integration,'" Dokl. Akad. Nauk SSSR, 252, No. 5, 1104-1108 (1980).
41. I. V. Cherednik, "On regularity of 'finite-zone' solutions of integrable matrix differential equations," Dokl. Akad. Nauk SSSR, 266, No. 3, 593-597 (1982).
42. M. Adler and P. van Moerbeke, "Completely integrable systems, Euclidean Lie algebras, and curves," Adv. Math., 38, No. 3, 267-317 (1980).
43. M. Adler and P. van Moerbeke, "Linearization of Hamiltonian systems, Jacobi varieties, and representation theory," Adv. Math., 38, No. 3, 318-379 (1980).
44. M. Adler and P. van Moerbeke, "The algebraic integrability of a geodesic flow on $SO(4)$," Invent. Math., 67, No. 2, 297-331 (1982).
45. B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, "Topological and algebraic geometrical methods in modern mathematical physics. II," in: Soviet Scientific Reviews, Vol. 3, Harwood Acad. Publisher, New York (1982).
46. J. Fay, "Theta-functions on Riemann surfaces," Lect. Notes Math., Springer, 352 (1973).
47. G. Forest and D. W. McLaughlin, "Spectral theory for the periodic sine-Gordon equation: a concrete viewpoint," J. Math. Phys., 23, No. 7, 1248 (1982).
48. P. D. Lax, "Periodic solutions of the Korgeweg-de Vries equation," Commun. Pure Appl. Math., 28, 141-188 (1975).
49. P. D. Lax, "Periodic solutions of the KdV equation," Lect. Appl. Math., 15, 85-96 (1974).
50. V. B. Matveev, "Abelian functions and solitons," Preprint No. 373, Inst. Teor. Phys., Wroclaw (1976).
51. H. P. McKean, "The sine-Gordon and sinh-Gordon equations on the circle," Commun. Pure Appl. Math., 34, No. 2, 197-257 (1981).
52. H. P. McKean and P. van Moerbeke, "The spectrum of Hill's equation," Invent. Math., 30, No. 3, 217-274 (1975).