HAMILTONIAN FORMALISM OF ONE-DIMENSIONAL SYSTEMS OF HYDRODYNAMIC TYPE, AND THE BOGOLYUBOV–WHITMAN AVERAGING METHOD

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I. **One-dimensional systems of hydrodynamic type. Poisson brackets and Riemannian geometry.** From a purely mathematical point of view many systems such as ideal fluids including mixtures and systems with internal degrees of freedom are given in the one-dimensional case by equations of first order

$$u^i_t = v^i_j(u)u^j_x. \tag{1}$$

The field variables u^i are usually the density of momentum and energy (or mass) and possibly a number of others. The Euler equations for ideal fluids are Hamiltonian with respect to Poisson brackets of special form (see [1] and [2]) which in the one-dimensional case we shall generalize to general systems of the form (1) in connection with new applications.

Definition 1. A Poisson bracket on a space of fields $u^k(x)$ is called a *bracket of hydrodynamic type* if it has the form

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + b^{ij}_k(u(x))u^k_x\delta(x-y). \tag{2}$$

For any pair of functionals $I = \int P(u, u_x, \dots)dx$, $J = \int Q(u, u_x, \dots)dx$ we have

$$\{I, J\} = \int dx \frac{\delta I}{\delta u^i(x)} A^{ij} \frac{\delta J}{\delta u^j(x)}, \quad A^{ij} = g^{ij}\frac{\partial}{\partial x} + b^{ij}u^k_x. \tag{3}$$

Definition 2. *Hamiltonians of hydrodynamic type* are functionals of the form

$$H = \int h(u)\, dx, \tag{4}$$

which do not depend on the derivatives $u_x, u_{xx}, \dots$.

The class of hydrodynamic Hamiltonians (4) and the form of the Poisson bracket (2) are invariant relative to local transformations of the fields $u = u(w)$ which do not contain derivatives. We have the following result.

Theorem 1. 1) *Under local changes of the fields $u = u(w)$ the coefficient $g^{ij}(u)$ in the bracket* (2) *transforms like a bilinear form* (*a tensor with upper indices*); *if* $\det g^{ij} \neq 0$, *then the expression $b^{ij}_k(u) = g^{is}\Gamma^j_{sk}$ transforms in such a way that the Γ^j_{sk} are the Christoffel tymbols of a differential-geometric connection.*

2) *In order that the bracket* (2) *be skew-symmetric it is necessary and sufficient that the tensor $g^{ij}(u)$ be symmetric* (*i.e., that it define a pseudo-Riemannian metric*

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if $\det g^{ij} \neq 0$) *and the connection* Γ^j_{sk} *be consistent with the metric,* $g^{ij}{}_{;k} = \nabla_k g^{ij} = 0$.

3) *In order that the bracket* (2) *satisfy the Jacobi identity it is necessary and sufficient that the connection* Γ^j_{sk} *have no torsion and the curvature tensor vanish. In this case the connection is defined by the metric* $g^{ij}(u)$ *which can be reduced to constant form.*

Corollary. *Poisson brackets of hydwdynamic type with the condition of nondegeneracy* $\det g^{ij} \neq 0$ *are classified relative to local changes* $u = u(w)$ *by a single invariant—the signature of the metric* $g^{ij}(u)$*, i.e., the number* q *of negative squares and the number* l *of positive squares, with* $l + q = n$.

Theorem 1 is proved by direct verification. Only part 3) contains some technical difficulties.

For future applications it is also useful to introduce the following definition.

Definition 3. Coordinates $(u^1, \dots, u^n)$ in field space are called *Liouville coordinates* if the Poisson bracket (2) has the form

$$\{u^i(x), u^j(y)\} = [\gamma^{ji}(u(x)) + \gamma^{ij}(u(y))]\delta'(x-y), \tag{5}$$

or, equivalently,

$$g^{ij} = \gamma^{ji}(u) + \gamma^{ij}(u), \quad g^{is}\Gamma^j_{sk} = \partial\gamma^{ij}/\partial u^k. \tag{6}$$

Remark. There are important cases (for example, a compressible fluid with fields p, ρ and s; see [1]) where the rank of the matrix $g^{ij}(u)$ is less than the dimension for all u. Investigation of such brackets will be carried out in another paper.

II. **The Hamiltonian formalism and the Bogolyubov–Whitman averaging method.** The classical Bogolyubov–Krylov–Mitropol'skiĭ averaging method (see [3]) is constructed for finite-dimensional systems of classical mechanics and oscillation theory. It presupposes that a family of tori is known, i.e., exact periodic or quasiperiodic solutions (rapid oscillations) depending on several parameters, and it studies the average drift with respect to rapid oscillations of a particle along a given family of parameters. An analogue of the averaging method for partial differential equations having Lagrangian form was developed first by Whitman [4] and then by a number of other authors [5]–[7]. Our aim is to develop a purely Hamiltonian version of the averaging method and investigate the Hamiltonian formalism of the averaged equations. We shall consider only one-dimensional systems, i.e., (x,t)-systems.

Suppose there are given a collection of field ψ^α and a Hamiltonian system of the form

$$\psi^\alpha_t = A^{\alpha\beta}(\delta H/\delta\psi^\beta(x)), \quad H = \int dx'\, h(\psi, \psi_x, \dots), \tag{7}$$

where $A = (A^{\alpha\beta})$ is a linear differential operator of order m with coefficients depending on $\psi, \psi_x, \psi_{xx}, \dots$. It is assumed that formula (8) defines a correct Poisson bracket of functionals:

$$\{I, J\}_\psi = \int dx \frac{\delta I}{\delta\psi^\alpha(x)} A^{\alpha\beta} \frac{\delta J}{\delta\psi^\beta(x)} \tag{8}$$

It is required that the following properties hold:

1) Translation on x is Hamiltonian, i.e., there exists a Hamiltonian $H_1 = p$ defining a group of translations on x (the momentum).

2) There is a given family of tori, that is, exact quasiperiodic solutions of the system (7) of the form

$$\psi^\alpha = F^\alpha(U_1 x + U_0 t, u), \tag{9}$$

where $u = (u^j)$, $U_q = U_q^s(u)$, $j = 1, \dots, n$, $s = 1, \dots, N$, $q = 1, 0$, and $F^\alpha(\eta_1, \dots, \eta_N, u)$ are functions of N variables η_s and the parameters u^j which are 2π-periodic in each variable η_s.

3) There are given field integrals $I_1, \dots, I_n$ of the system (7) such that the constants $(u^1, \dots, u^n)$ are determined by their values

$$I_k = u^k, \quad \{I_k, H\}_\psi = 0, \quad I_k = \int P_k(\psi, \psi_x, \dots) dx. \tag{10}$$

The integrals I_k are Liouville, i.e., they commute pairwise; they include the momentum $p = I_1$ and the energy $E = H = I_2$,

$$\{I_k, I_l\}_\psi = 0. \tag{11}$$

Because of (11), we consider the Poisson brackets of densities of these integrals with them, and we write them as total derivatives

$$\{P_k(\psi(x), \psi_x(x), \dots), I_l\}_\psi = (\partial/\partial x) B_{kl}(\psi, \psi_x, \dots), \tag{12}$$

where the B_{kl} are polynomials if the densities P_k and the bracket are polynomials in $\psi, \psi_x, \dots$.

The averaging method considers asymptotic solutions which have the form (9) in the first approximation, where the remainder has zero mean over rapid oscillations and $(u^1, \dots, u^n)$ are slowly varying functions, i.e., they depend on the "slow" variables (X, T). The averaging over the rapid variables, we arrive at the following assertion.

Theorem 2. 1) *If conditions* 1)–3) *above are satisfied, the functions $u^j(X,T)$ satisfy the "equations of slow modulations of Whitham type" of first order in X and T of hydrodynamic type.*

2) *These equations are Hamiltonian with Hamiltonian $\bar{H} = \int E\, dX$, $E = I_2$, and Poisson bracket of the form* (2). *The Poisson bracket of the averaged momentum density has the form*

$$\{p(X), p(Y)\}_u = 2p(X)\delta'(X-Y) + p_X \delta(X-Y), \quad p = u^1. \tag{13}$$

The Poisson bracket of all remaining variables can be computed by the formula

$$\{u^i(X), u^j(Y)\}_u = \underline{[\gamma^{ij}(X) + \gamma^{ij}(Y)]\delta'(X-Y);} \tag{14}$$

$$\gamma^{ij}(u) = B_{ij}(\psi, \psi_x, \dots), \tag{14'}$$

i.e., the coordinates $(u^1, \dots, u^n)$ are Liouville coordinates. The momentum $\bar{p} = \int p(X)\, dX$ in this Poisson bracket is the generator of the group of translations. It is assumed that the metric $g^{ij} = \gamma^{ij} - \gamma^{ji}$ is nondegenerate.

3) *The action variables $J_j(u) = \oint_{a_j} p\, dq$ conjugate to the angles η_j on the tori in the original Poisson bracket* (8) *and the variables $U_1^j(u)$ if the collection (J, U_1) is*

independent have the following "averaged" Poisson brackets:

$$\{J^j(X), U_1^k(Y)\}_u = \delta^{jk}\delta'(X-Y), \tag{15}$$

$$\{J^j(X), J^k(Y)\}_u = \{U_1^j(X), U_1^k(Y)\}_u = 0. \tag{15$'$}$$

In this case the signature of the metric $g^{ij}(u)$ is such that $q \geqslant N$ and $l \geqslant N$, where (q,l) are the numbers of negative and positive squares.

III. **The most important examples.**

Example 1. *The nonlinear Klein–Gordon equation $\Box\psi = -V'(\psi)$ with the natural Hamiltonian formalism.* The Hamiltonian formalism of single-phase equations of Whitham type have been considered only in the variables J, U_1 (see Theorem 2) by Hayes [8]. Maslov [9] first discovered that the averaged equations are equivalent to relativisitic hydrodynamics (RH). The physical Liouville variables are $u^1 = p$, $u^0 = E$; the Poisson bracket, the equations, and the energy-momentum tensor have the form

$$\gamma^{ij}(u) = \begin{pmatrix} p & E-2s \\ E & p \end{pmatrix}, \quad T^{ij} = \begin{pmatrix} E & p \\ p & E-2s \end{pmatrix}, \quad i,j = 0,1; \tag{16}$$

$$\begin{gathered} \psi = \psi(x-ct), \quad dx\sqrt{2(\varepsilon - V(\psi))} = d\psi\sqrt{1-c^2}; \\ p_T = -(\partial/\partial X)(E-2s), \quad E_T = (\partial p/\partial X), \quad (\partial T^{ij}/\partial X^j) = 0; \end{gathered} \tag{17}$$

$$u_T^j = \{u^j, \bar{H}\}_u, \quad \bar{H} = \int E\,dX. \tag{18}$$

The energy-momentum tensor T^{ij} for RH can be reduced by Lorentz transformations to diagonal form with eigenvalues ε and P, where $2s = \varepsilon - P$ and $\det T = \varepsilon P$ are invariants. The Galilean principle is that a constraint on the tensor T^{ij} (the equation of state) is imposed only on the invariants, $P = P(\varepsilon)$.[1]

Assertion. *The bracket* (16) *generates a connection without torsion if and only if the invariants of the tensor T^{ij} of the form* (16) *are related by a functional dependence.*

Example 2. General systems of Korteweg–de Vries (KdV) type are of the form

$$\psi_t = (\delta H/\delta\psi(x))_x, \quad H = \int (\psi_x^2/2 + V(\psi))\,dx \tag{19}$$

with bracket $A_0 = \partial/\partial x$. The family of periodic solutions $\psi = F(U_1 x + U_2 t, u^0, u^1, u^2)$ depends on the parameters $\bar{\psi} = u^0$, $\bar{\psi}^2/2 = p = u^1$ and $E = u^2$. After averaging, the Poisson bracket has the form

$$\gamma_0^{ij} = \begin{pmatrix} 1 & u^0 & -cu^0 - d \\ 0 & u^1 & -cu^1 + 2f \\ 0 & u^2 & -cu^2 - cf + d^2 \end{pmatrix}, \tag{20}$$

where the averaging goes over a family of solutions of the form $u = u(x-ct)$, $(u')^2 = 2V(u) + cu^2 + 2du + 2f$. The metric $g^{ij}(u)$ has signature $(2,1)$. The quantities $p_+ = \bar{\psi}^2/2$ and $p_- = \frac{1}{2}\overline{(\psi - \bar{\psi})^2}$ generate two independent variables with the Poisson brackets of momentum density which have zero brackets with one another. We thus obtain a "two-fluid system".

[1] I. M. Khalatnikov has indicated a method of integrating these equations (see [10], §4).

Example 3. *Integrable systems. The multiphase case.* For the usual KdV equation ($V(\psi) = \psi^3$) the averaged "single-phase" equation was first studied by Gurevich and Pitaevskiĭ (see [11], Chapter IV), and the multiphase equation in [6] and [7].[2] The usual KdV equation is Hamiltonian also in the second bracket, where $A_1 = -\partial^3 + 2(\psi\partial + \partial\psi)$, $\partial = \partial/\partial x$. After averaging in these same variables $u^j = I_j$, where I_j is the Kruskal integral, $j = 0, 1, \dots$; $I_0 = \bar\psi$, $I_1 = \bar\psi^2/2$, $I_2 = \overline{(\psi_x^2/2 + \psi^3)}$, ... (for any number of phases), we obtain a new bracket of hydrodynamic type, where $p_1 = \bar\psi$ and $E_1 = \bar\psi^2/2$. For a single phase we have

$$\gamma_1^{ij}(u) = \begin{pmatrix} 2u^0 & -cu^0 - d & 10u^2 \\ 2u^1 & -cu^1 + 2f & \frac{5}{3}(-cu^2 - cf + d^2) \\ 2u^2 & -cu^2 - cf + d^2 & (c^2 - 2d)u^2 + 2c^2 f - cd^2 - 2df \end{pmatrix}. \tag{21}$$

The densities of the Kruskal integrals averaged over "finite-zone" tori become functions of the branch points $I_j(\lambda_0, \dots, \lambda_{2N})$, where N is the number of phases. They all generate Hamiltonians of hydrodynamic type (see above) and are involutive in both averaged Poisson brackets. The metric g^{ij} has signature $(N, N+1)$.

A theory of integrating systems of hydrodynamic type possessing infinite series of "higher integrals" of this same type has not yet been constructed. So far we know of no analogues of the method of the inverse problem.

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