Exact solutions of the time-dependent Schrödinger equation with self-consistent potentials

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Fiz. Elem. Chastits At. Yadra **19**, 579–621 (May–June 1988)

A unified scheme is proposed for the construction of many-soliton solutions of a number of models associated with the time-dependent Schrödinger equation (vector nonlinear Schrödinger equation, vector model of Yajima–Oikawa type, and others). Cases of nontrivial condensate boundary conditions are investigated in detail.

INTRODUCTION

The main aim of the present paper is to describe integrable models associated with the time-dependent Schrödinger equation in a unified approach and to construct many-soliton solutions of such models. These include, for example, the vector nonlinear Schrödinger equation (NSE) with different internal symmetry groups, a vector model of Yajima–Oikawa type, and others. The "integrability" of some of these systems follows from the fact that they have commutation representations (*L–A* pair or *L–A–B* triplet). However, for noncompact internal symmetry groups, for which condensate boundary conditions are physically realistic, the standard technique of the inverse scattering problem is not effective.

Our proposed approach, which does not use commutation representations, actually arose in the heart of the algebraic-geometric theory of integrable systems. It is well known that this theory is used to construct periodic and quasiperiodic solutions of integrable systems; it is not nearly so well known that the technique also makes it possible to construct effectively all currently known explicit solutions of such systems (many-soliton, rational, and combinations of these). We wish to demonstrate this in a form accessible to nonmathematicians, considering examples of models described by a Schrödinger equation with a self-consistent potential.

The paper is arranged as follows. In Sec. 1, taking the example of the generalized Heisenberg magnet, we show how such models arise. In Sec. 2, we describe in general form the method of constructing and investigating explicit solutions of such models. In Sec. 3, we consider some specific examples and give corresponding formulas. At the end, we discuss the results.

1. PHYSICAL MODELS DESCRIBED BY A TIME-DEPENDENT SCHRÖDINGER EQUATION WITH A SELF-CONSISTENT POTENTIAL

The investigation of nonlinear wave phenomena in physics often leads to systems of differential equations that model the interaction of a finite number of waves or wave packets. The simplest such system is the scalar nonlinear Schrödinger equation (SNSE)

$$i\psi_t + \psi_{xx} + \varepsilon\,|\psi|^2\psi = 0 \quad (\varepsilon = \pm 1), \tag{1}$$

which describes the self-interaction of a packet of high-frequency waves (for $\varepsilon = -1$, it is also called the Gross–Pitaevskiĭ equation), in particular, the self-interaction of spin waves (magnons) in ferromagnets, excitons in molecular crystals, Langmuir waves in a plasma, etc. Equation (1) is, together with the KdV equation, the currently most popular and best studied nonlinear model of mathematical physics, being integrable at both the classical and the quantum level. Moreover, the quantum (or semiclassical) approach makes it possible to use not only wave language but also particle language. The simplest physical model described by (1) is a Bose gas with a point two-body interaction at zero temperature (see, for example, Ref. 14). This model provides a very transparent interpretation of the results, which, with the necessary redefinitions and renotation, can be used in the framework of other physical models as well.

A natural generalization of (1) is the system describing the interaction of a packet $\psi(x, t)$ of high-frequency waves with a low-frequency wave $U(x, t)$. In this case, the complex function $\psi(x, t)$ satisfies, as before, an SNSE,

$$i\psi_t + \psi_{xx} + U\psi + \lambda\,|\psi|^2\psi = 0, \tag{2}$$

in which the part of the potential U is played by the low-frequency wave, which is described by one of the following equations (of self-consistency):

$$\Box U = -|\psi|^2_{xx} \qquad (\text{Zakharov}^{16}), \tag{3a}$$

$$(\partial_t + \partial_x)\,U = |\psi|^2_x \qquad (\text{Yajima–Oikawa}^3), \tag{3b}$$

$$(\partial_t + \partial_x + \alpha\partial_x^3 + \beta U\partial_x)\,U = |\psi|^2_x \quad (\text{Nishikawa } et\ al.^4), \tag{3c}$$

$$(\Box + \alpha\partial_x^4)\,U + \beta\partial_x^2 U^2 = -|\psi|^2_{xx} \quad (\text{Makhan'kov}^5). \tag{3d}$$

The systems (2) and (3) with $\lambda = 0$ were obtained in plasma physics, in which they modeled the interaction of Langmuir and ion-acoustic waves. Later, it was shown that analogous equations arise in the investigation of the interaction of spin waves with phonons in ferromagnets[6] and excitons with phonons in molecular crystals.[7] However, in the general case we now have $\lambda \neq 0$.

Another natural generalization of the potential (1) is the transition from the scalar variant of the NSE to vector form, $\psi \to \psi = (\psi_1, \psi_2, \ldots, \psi_n)^{\text{tr}}$ (VNSE), with replacement of $|\psi|^2$ by the inner product

$$(\psi, \psi) = \sum_{ij=1}^{n} g_{ij}\bar{\psi}_i\psi_j, \tag{4}$$

where g_{ij} is the metric of the isotopic space. The Hamiltonian of the system is often found to be invariant with respect to some internal symmetry group, which is compact or noncompact, depending on whether or not the matrix $g_{ij} = \lambda_i\delta_{ij}$ is sign definite or indefinite; in the case of Hermitian Hamiltonians, the group is $U(p,q)$. Such models describe a Bose gas with internal quasispin ("color") degrees of freedom; they also arise in the case of propagation in a nonlinear medium of a high-frequency plane wave with circular polarization[1] in the description of spin waves in magnets with layered structure, in the derivation of the classical continuum analog of the Hubbard model, and elsewhere. Some of these models are integrable[2] and admit fairly complete investigation. Finally, combining both generalizations, we arrive at vector variants of the time-dependent Schrödinger equation with a self-consistent potential (low-frequency branch) in one of the forms (3) listed above (though others are also possible), i.e., we obtain a system of equations consisting of the equation

$$i\psi_t + \psi_{xx} + U\psi + \lambda\,(\psi, \psi)\,\psi = 0 \tag{5}$$

and one of the equations in Eqs. (3) with a right-hand side that depends on the invariant combination (ψ, ψ).

At the semiclassical level, all these models admit an interpretation in the language of a multicomponent Bose gas (with internal degrees of freedom) with, in the general case, different forms of interaction between the particles of the components and the background mode. In other words, Eqs. (5) and (3) describe a mixture of gases in which, for

$$g_{ik} = \operatorname{diag}(1, 1, \ldots, -1, -1 \ldots),\ i, k = 1, \ldots, n, \tag{6}$$

the particles of the different components attract or repel each other ($\lambda \neq 0$) and can also emit and absorb acoustic waves. Therefore, we shall, without specifying a definite physical problem and interpretation of the results, call equations like (5) Bose-gas models.

Nevertheless, since it is precisely in the theory of the condensed state that in recent years models of the type (5) have made their appearance and been studied, the behavior of the corresponding systems is of ever increasing interest in this branch of physics.

The experimental investigation of magnetic crystals shows that many of them possess a layered or multichain structure.[8] In the overwhelming majority of cases, the interaction between the layers or chains has a strong influence on the dynamical behavior of the crystal as a whole. Typical representatives are crystals of salts,[8] though analogous structures are also encountered in organic systems in the form of molecular chains.[9] The microscopic theory of such structures is usually based on a generalization of the Heisenberg spin model to the case of several components.[10] The interaction of "color" degrees of freedom for the interacting spins of a one-dimensional chain makes it possible to describe multilayer magnetic systems with weak coupling. Further, since the one-dimenisonal Hubbard model for a half-filled band corresponds to a two-component Heisenberg spin chain[11] with interaction between the components, a multicomponent spin chain corresponding to the generalized Hubbard model[12] can be used to describe collective excitations and their statistical properties in systems with several spin species.

In all these cases, we arrive at Bose gas models (5) that, strictly speaking, realize a dynamical description of the corresponding systems at zero temperature. Under experimental conditions, even at very low temperatures, one usually measures certain average characteristics such as the static or the dynamical structure factor. For the theoretical calculation of these, one sometimes uses a partition function, which is defined in terms of a functional integral [$Z = \int D\Phi, D\Phi \exp(-\beta H)$, $\beta = T^{-1}$, in the case of real fields]. However, for the Bose-gas models (5) such an approach involves certain difficulties,[14] and therefore the so-called phenomenological approach, first formulated by Krumhansl and Schrieffer,[13] has become popular. These authors noted that the partition function found by transfer-matrix methods from the functional integral and the partition function obtained in the approximation of an ideal gas of kinks in the Φ^4 model were almost identical. Subsequently, the phenomenological approach was widely used to find the structure factors in different models (see the review of Ref. 14 and the bibliography given there). We note that one of the most important directions in the use of the phenomenological approach is study of the stability of solitons and the elasticity (or quasielasticity) of their interaction. Such properties usually hold in the framework of integrable models with a sufficiently small number of interacting waves. In the cases when this is not so, the distribution function of the solitons with respect to both their velocities and amplitudes (or frequencies) must be found on the basis of other considerations [for example, in the study of Ref. 15, which was based on numerical experiments, Degtyarev *et al.* wrote down and solved an approximate (phenomenological!) kinetic equation for solitons in the framework of the system (2)–(3a) with $\lambda = 0$].

For integrable systems of the form (5) (with $n > 1$) it is therefore very important to know in analytic form not only the complete spectrum of single-soliton solutions but also two- and sometimes three-soliton formulas (particularly their asymptotic behaviors) in order to gauge the validity of using the phenomenological approach.

Generalized Heisenberg model and Bose-gas models

We consider the "color" generalization of a magnetic chain with Hamiltonian[17]

$$H = H_s + H_L, \tag{7}$$

where

$$H_s = -\frac{1}{2}\Big\{\sum_{ij,\,\alpha\beta}\Big[\frac{1}{2}J_{ij}^{\alpha\beta}(S_i^{+\alpha}S_j^{-\beta} + S_i^{-\alpha}S_j^{+\beta}) + R_{ij}^{\alpha\beta}S_i^{z\alpha}S_j^{z\beta}\Big]\Big\}, \tag{7a}$$

$$H_L = T + U_0,\ T = \frac{m}{2}\sum_j x_j^2,\ U_0 = \frac{mv_0^2}{2a_0^2}\sum_j (x_{j+1} - x_j - a_0)^2, \tag{7b}$$

which describes the interaction of spins of different "colors" (species) ($\alpha = 1, \ldots, n$). Ignoring the interaction between

0090-4759/88/030252-18$04.40

the color and spatial degrees of freedom in the exchange integrals and taking into account the interaction of only nearest neighbors, we obtain

$$J_{ij}^{\alpha\beta}=J_{jj+\sigma}K^{\alpha\beta},\quad R_{jj+\sigma}^{\alpha\beta}=L_1^{\alpha}L_2^{\beta}\tilde{J}_{ij+\sigma}, \tag{8}$$

where $J_{jj+\sigma}=J(|x_j-x_{j+\sigma}|)$ is the exchange integral of the nearest spins, and $S^{\pm}=S^x\pm iS^y$ and S^z are spin operators.

For sufficiently large values of the spins s^{α}, the Hamiltonian (7) can be rewritten in terms of Bose operators of creation, $a_j^{+\alpha}$, and annihilation, a_j^{α}, by means of the generalized Holstein–Primakoff representation: $S_j^{+\alpha}=\sqrt{2s^{\alpha}}(1-\hat{n}_i^{\alpha}/2s^{\alpha})^{1/2}a_j^{\alpha}$; $S_j^{-\alpha}=\sqrt{2s^{\alpha}}a_j^{+\alpha}(1-\hat{n}_j^{\alpha}/2s^{\alpha})^{1/2}$; $\hat{n}_j^{\alpha}=a_j^{+\alpha}a_j^{\alpha}$; $S_j^{z\alpha}=s^{\alpha}-\hat{n}_j^{\alpha}$;

$$H_s=\text{const}-\frac{1}{2}\sum_{j,\sigma}\Big\{sJ_{jj+\sigma}\sum_{\alpha\beta}K^{\alpha\beta}(a_j^{+\alpha}a_{j+\sigma}^{\beta}+a_{j+\sigma}^{+\beta}a_j^{\alpha})$$
$$-\tilde{J}_{ij+\sigma}\Big[s\sum_{\alpha}(l_2L_1^{\alpha}\hat{n}_j^{\alpha}+l_1L_2^{\alpha}\hat{n}_{j+\sigma}^{\alpha})+\sum_{\alpha\beta}L_1^{\alpha}L_2^{\beta}\hat{n}_j^{\alpha}\hat{n}_{j+\sigma}^{\beta}\Big]\Big\}, \tag{9}$$

where $l_i=\mathrm{Tr}\,L_i$, $s^{\alpha}=s$.

The evolution of the operator $a_j^{\alpha}(t)$ is determined by the Heisenberg equation $i\hbar\dot{a}_j^{\alpha}(t)=[a_j^{\alpha},H_s]$.

To go over from the quantum Hamiltonian (9) to the classical Hamiltonian, we apply a reduction procedure based on the use of coherent states of the Heisenberg–Weyl group[17]:

$$|\varphi^{\alpha}\rangle=\prod_j|\varphi_j^{\alpha}\rangle=\prod_j e^{\frac{1}{2}|\varphi_j^{\alpha}|^2}e^{\varphi_j^{\alpha}a_j^{+\alpha}}|0\rangle,$$

which in our cases possess an important property, namely, the operator

$$\hat{A}=\sum_{m,n}C_{mn}(a_j^{+\alpha})^m(a_j^{\alpha})^n,$$

expressed in normal (Wick) form and averaged over the states $|\varphi_j^{\alpha}\rangle$, gives

$$A\equiv\langle\varphi_j^{\alpha}|\hat{A}|\varphi_j^{\alpha}\rangle=\sum_{m,n}C_{mn}(\bar{\varphi}_j^{\alpha})^m(\varphi_j^{\alpha})^n. \tag{10}$$

We use this relation and go to the continuum limit by means of the standard procedure of expanding $\varphi^{\alpha}(\xi)=\varphi_j^{\alpha}$ in a Taylor series $\varphi_{j+1}^{\alpha}=\varphi^{\alpha}(\xi)+a_0\varphi_{\xi}^{\alpha}(\xi)+\frac{1}{2}a_0^2\varphi_{\xi\xi}^{\alpha}(\xi)+\ldots$ and representing the exchange integrals in the form $J(|x_{j+1}-x_j|)=J_0-J_1(x_{j+1}-x_j-a_0)$ (and similarly for $\tilde{J}$). As a result, we obtain the system

$$\ddot{x}=U_0^2x_{\xi\xi}+\frac{s}{m}\sum_{\alpha\beta}\tilde{T}^{\alpha\beta}(\bar{\varphi}^{\alpha}\varphi^{\beta})_{\xi}; \tag{11}$$

$$i\varphi_t^{\alpha}=-b\sum_{\beta}(K_{(\alpha,\beta)}\varphi_{\xi\xi}^{\beta}-sT_{\alpha\beta}\varphi^{\beta}+s\tilde{T}_{\alpha\beta}\varphi^{\beta}x_{\xi})$$
$$-J_0\varphi^{\alpha}\sum_{\beta}(L_1^{\beta}L_2^{\alpha}+L_2^{\alpha}L_1^{\beta})|\varphi^{\beta}|^2, \tag{12}$$

in which

$$T_{\alpha\beta}=J_0K_{(\alpha,\beta)}-\tilde{J}_0(l_1L_2^{\alpha}+l_2L_1^{\alpha})\delta_{\alpha\beta};$$
$$\tilde{T}_{\alpha\beta}=J_1K_{(\alpha,\beta)}-\tilde{J}_1(l_1L_2^{\alpha}+l_2L_1^{\alpha})\delta_{\alpha\beta},\quad b=J_0s/2,$$

and (α,β) denotes symmetrization with respect to the indices α and β. The further investigation of the system (11)–(12) is associated with the imposition of additional conditions (reductions) on the matrices of coefficients T and L.

Some special reductions

Example 1. Suppose that the exchange integrals of the color degrees of freedom are proportional to each other and are diagonal:

$$K_{(\alpha,\beta)}=2b_1L_1^{\beta}\delta_{\alpha\beta}=2b_2L_2^{\beta}\delta_{\alpha\beta}\equiv\lambda_{\alpha}\delta_{\alpha\beta};$$

then the system (11)–(12) reduces to a system of the form (5) and (3a). In the quasistationary (inertialess) limit, when in Eq. (11) we can ignore the time derivative,

$$U(\xi,t)\equiv x_{\xi}=-\frac{s}{mU_0^2}\sum_{\alpha\beta}\tilde{T}_{\alpha\beta}(\bar{\varphi}^{\alpha}\varphi^{\beta})+c, \tag{13}$$

Eq. (12) reduces to the VNSE generated by the Hamiltonian

$$H=\int d\xi\,[b(\varphi_{\xi}^{*}K\varphi_{\xi})-d(\varphi^{*}K\varphi)^2-\tilde{\mu}(\varphi^{*}K\varphi)], \tag{14}$$

in which

$$(\varphi^{*}K\varphi)\equiv\sum_{\alpha\beta}\bar{\varphi}^{\alpha}K_{(\alpha,\beta)}\varphi^{\beta}=\sum_{\alpha}\lambda^{\alpha}|\varphi^{\alpha}|^2;$$

$$d=\frac{s^2\nu^2}{mv_0^2}+\frac{\tilde{J}_0}{2b_1b_2},\quad\tilde{\mu}=s(\mu-c\nu),$$

$$T_{\alpha\beta}=\mu\lambda^{\alpha}\delta_{\alpha\beta},\quad\tilde{T}_{\alpha\beta}=\nu\lambda^{\alpha}\delta_{\alpha\beta},$$

$$\mu=\Big[J_0-\frac{\tilde{J}_0}{2b_1b_2}(b_1l_1+b_2l_2)\Big],\quad\nu=\Big[J_1-\frac{\tilde{J}_1}{2b_1b_2}(b_1l_1+b_2l_2)\Big].$$

The variables $\varphi^{\alpha}(\xi,t)$ and $\bar{\varphi}^{\alpha}(\xi,t)$ are canonically conjugate variables,

$$\{\varphi^{\alpha}(x),\bar{\varphi}^{\beta}(y)\}=i\delta^{\alpha\beta}\delta(x-y), \tag{15}$$

with the usual Poisson brackets

$$\{A,B\}=i\sum_{\alpha=1}^{n}\int d\xi\left(\frac{\delta A}{\delta\varphi^{\alpha}}\frac{\delta B}{\delta\bar{\varphi}^{\alpha}}-\frac{\delta B}{\delta\varphi^{\alpha}}\frac{\delta A}{\delta\bar{\varphi}^{\alpha}}\right).$$

The further simplification of (14) is associated with the presence of internal symmetry in the system, when the strengths of the interactions for different "colors" are the same in modulus, $\lambda^{\alpha}=\varepsilon_{\alpha}$:

$$\varepsilon_{\alpha}=\begin{cases}1, & \alpha=1,\ldots,p\\ -1, & \alpha=p+1,\ldots,p+q\end{cases}\quad(p+q=n).$$

Introducing the notation

$$\psi_{\alpha}(\xi,t)=\begin{cases}\varphi_{\alpha}(\xi,t), & \alpha=1,\ldots,p;\\ \varepsilon_{\alpha}\varphi_{\alpha}(\xi,t), & \alpha=p+1,\ldots,p+q,\end{cases} \tag{16}$$

$$(\Gamma_0)_{\alpha\beta}\equiv K_{(\alpha,\beta)}=\varepsilon_{\alpha}\delta_{\alpha\beta},\quad\frac{d}{b}=\varkappa,\quad\frac{\mu'}{b}=\rho,\quad H\to H/b,$$

we obtain

$$H=\int d\xi\,[(\psi_{\xi},\psi)-\varkappa(\psi,\psi)^2-\rho(\psi,\psi)]; \tag{17}$$

$$\{\psi^{\alpha}(\xi),\psi^{+\beta}(\eta)\}=i\delta^{\alpha\beta}\delta(\xi-\eta), \tag{18}$$

where

$$\psi^{+}=\psi^{*}\Gamma_0,\quad\Gamma_0=\begin{pmatrix}I_p & 0\\ 0 & -I_p\end{pmatrix},$$

and

$$(\psi,\psi)=\sum_{\alpha=1}^{p}|\psi^{\alpha}|^2-\sum_{\alpha=p+1}^{n}|\psi^{\alpha}|^2\equiv(\psi^{*}\Gamma_0\psi) \tag{19}$$

is the $U(p,q)$ inner product. The corresponding equation of motion

$$i\psi_t+\psi_{\xi\xi}+2\varkappa(\psi,\psi)\psi+\rho\psi=0 \tag{20}$$

is the $U(p,q)$ VNSE describing an integrable system.[2]

An analogous reduction, applied to the system (11)–(12) in dimensionless variables, is (5) + (3a):

$$\partial_t^2U-\partial_x^2U-(\psi,\psi)_{xx}=0; \tag{21a}$$

$$i\psi_t+\psi_{xx}-U\psi+\lambda(\psi,\psi)\psi=0. \tag{21b}$$

Note that the last term in Eq. (21b) is related to the expression of $(S^z)^2$ in the Holstein–Primakoff representation, is proportional to the degree of anisotropy of the original chain (the ratio $\bar{J}_0/J_0$), and is still present in the absence of magnon–phonon interaction.

The generalized Yajima–Oikawa system can be readily obtained from (21) by means of the standard procedure of transition to the monodirected version of the wave equation,

$$\partial_t^2-\partial_{\xi}^2\simeq-2\partial_{\xi}(\partial_t+\partial_{\xi}), \tag{22}$$

and integration with respect to ξ.

Example 2. To take into account the weak interaction between the color components in the chains, which we ignored in the previous treatment, we assume that nearest neighbors make a contribution in the color space. Then the reduction has the form

$$R_{ij}^{\alpha\beta}=\rho J_{ij}^{\alpha\beta},\quad J_{ij}^{\alpha\beta}=(J_{jj+\sigma}M^{\alpha\beta}+J^1V_{ij}^{\alpha\beta});$$
$$M^{\alpha\beta}=\delta^{\alpha\beta}+\varepsilon\delta^{\beta,\alpha+\delta},\quad V_{ij}^{\alpha\beta}=\delta_{ij}\delta^{\beta,\alpha+\delta}, \tag{23}$$

where $J^1/J\ll1$, and J and J^1 are the intercell and interchain exchange integrals, respectively. Using (23), we can obtain equations of the type (5) and (8) with additional small terms that take into account the nondiagonality of the matrix of the intercolor interaction. The influence of these terms on the dynamics of the system can be investigated by the "standard" soliton perturbation theory, or by means of direct methods, or by means of the inverse scattering method.[18]

Application of the procedure described above to the Hubbard model, or rather to its multisublattice spin equivalent, also leads under certain assumptions to systems of the form (5) + (3) with the $U(n/2,n/2)$ internal symmetry group in the case of an antiferromagnetic ground state and $U(n,0)$ in the case of a ferromagnetic ground state (see Ref. 17).

Example 3. Allowance in the Hamiltonian H_L for anharmonicity

$$U_{\text{anh}}=\frac{U_{\text{III}}}{3!}\sum_j(x_{j+1}-x_j-a_0)^3$$

and phonon dispersion

$$x_{j\pm1}=x\pm a_0x_{\xi}+\frac{1}{2}a_0^2x_{\xi\xi}\pm\frac{1}{6}a_0^3x_{\xi\xi\xi}+\frac{1}{4!}a_0^4x_{\xi\xi\xi\xi}+\ldots$$

leads to replacement of the wave equation (11) by the inhomogeneous Boussinesq equation

$$\partial_t^2x-\partial_{\xi}^2(v_0^2-\alpha\partial_{\xi}^2-\beta x)x=g\partial_{\xi}^2(\psi,\psi), \tag{24}$$

where α, β, and g are coefficients determined by the parameters of the original system. By means of a scale transformation of the variables ξ, t, x, and ψ we arrive at the system (5) + (3d). The transition to the monodirected variant in Eq. (24) by means of (22) leads to (3c).

Considering the multicomponent spin system, we have found that under certain assumptions (long-wave limit, etc.) it can be reduced to various field models with internal ("color") symmetries. Some of these models are integrable. They include the NSE with $U(p,q)$ symmetry (obtained in the quasistatic limit), the color generalization of the Yajima–Oikawa system (obtained in the transonic limit), and, finally, the system (5) (3d) for $\lambda=0$. The remaining nonintegrable variants can often be regarded as systems that are nearly integrable. All the obtained equations contain not only linear (phonon and magnon) solutions but also essentially nonlinear (soliton) solutions; we shall discuss the properties of these solutions in the following sections. For it is these solutions, together with the linear modes, that describe the elementary excitations of the corresponding system at low temperatures.[20] Finally, we note that the models considered above arise in many branches of physics; in particular, many of them appear to have been found for the first time in plasma physics (see, for example, the reviews of Ref. 21).

2. GENERAL SCHEME OF THE METHOD

In this section we describe the method of constructing integrable models associated with the time-dependent Schrödinger equation and of finding explicit localized solutions.

Although this method arises from the general "finite-gap" (algebraic-geometric) scheme, its exposition can in fact be given in closed form without any use of the results of algebraic geometry. It seems to the authors that the "algebraic-geometric" approach to the construction of many-soliton solutions is one of the simplest and most elementary methods that enables one to obtain these solutions even in the cases when for the auxiliary linear problems there is no systematic solution of the direct and inverse spectral problems.

It should be noted that our method of constructing solutions of the time-dependent Schrödinger equation with self-consistency conditions differs from the standard scheme of the inverse scattering method. For all these equations, representations of Lax type or representations in the form of L, A, B triplets are known. The linear problems that then arise are very varied. It can be shown that the solutions of these equations can be obtained in the framework of a single scheme that uses just one linear operator,

$$L=i\partial_y-\partial_x^2+u(x,y),$$

and not several, as is required by the inverse scattering method. At the same time, the operator L plays not only an auxiliary role but actually occurs as a composite part in the original system of equations.

Such an approach to the NSE and its vector generalization was first proposed in the case of periodic solutions in Ref. 22. Periodic solutions for the remaining self-consistency conditions were constructed in Ref. 23, and it was this that mainly stimulated the present work.

Construction of "integrable" Potentials of the time-dependent Schrödinger equation that are associated with a rational curve

By integrable potentials of the time-dependent Schrödinger equation that are associated with a rational curve, we mean potentials $u(x,t)$ for which the equation

$$[i\partial_t - \partial_x^2 + u(x,t)]\psi(x,t,k) = 0 \qquad (25)$$

has a solution of the form

$$\psi(x,t,k) = Q_N(x,t,k)e^{ikx+ik^2t}, \qquad (26)$$

where

$$Q_N(x,t,k) = k^N + a_1(x,t)k^{N-1} + \dots + a_N(x,t) \qquad (27)$$

is a polynomial in k of some degree N.

The variant of construction of such potentials which is presented below may not be the most general. Nevertheless, it contains as special cases the many-soliton and rational solutions of the linear equations in which we are interested.

We first construct complex integrable potentials. We specify a set of distinct numbers $\varkappa_1,\dots,\varkappa_M$ (α_{ij}^s), where $i=1,\dots,N$, $j=1,\dots,M$, $s=0,\dots,m_j$, and $m_1+\dots+m_M+M\geqslant N$. These quantities are parameters of the construction. For given parameters, the function $\psi(x,t,k)$ can be uniquely determined by requiring that it satisfy the following system of linear conditions[1]:

$$\sum_{j=1}^{M}\sum_{s=0}^{m_j}\alpha_{ij}^s\partial_k^s\psi(x,t,k)|_{k=\varkappa_j}=0, \quad i=\overline{1,N}. \qquad (28)$$

The conditions (28) are equivalent to a system of N linear equations for the coefficients $a_1,\dots,a_N$. To give the explicit form of these equations, we introduce the polynomials

$$P_{r,s}(x,t,k) = e^{-ikx-ik^2t}\partial_k^s(k^re^{ikx+ik^2t}) = e^{-ikx-ik^2t}\left(\frac{1}{i}\partial_x\right)^r\partial_k^se^{ikx+ik^2t} = (\partial_k + ix + 2ikt)^sk^r.$$

We denote by $\omega_j = \omega_j(x,t)$ the linear functions

$$\omega_j(x,t) = \varkappa_jx + \varkappa_j^2t, \quad j=\overline{1,M}.$$

Then the conditions (28) can be written in the form of the system of equations

$$\sum_{h=1}^{N}a_h(x,t)\sum_{j=1}^{M}\sum_{s=0}^{m_j}\alpha_{ij}^sP_{N-h,s}(x,t,\varkappa_j)e^{i\omega_j} = -\sum_{j=1}^{M}\sum_{s=0}^{m_j}\alpha_{ij}^sP_{N,s}(x,t,\varkappa_j)e^{i\omega_j}, \quad i=\overline{1,N}. \qquad (29)$$

We denote by $A(x,t) = (A_{ik}(x,t))$ the $N\times N$ matrix formed from the coefficients of $a_k(x,t)$ in (29) and by $\hat A(x,t,k)$ the $(N+1)\times(N+1)$ matrix of the form

$$\hat A(x,t,k) = \begin{pmatrix} k^N & k^{N-1}\ \dots\ 1 \\ -\sum_{j=1}^{M}\sum_{s=0}^{m_j}\alpha_{1j}^sP_{N,s}(x,t,\varkappa_j)e^{i\omega_j} & \\ \vdots & A(x,t) \\ -\sum_{j=1}^{M}\sum_{s=0}^{m_j}\alpha_{Nj}^sP_{N,s}(x,t,\varkappa_j)e^{i\omega_j} & \end{pmatrix}. \qquad (30)$$

THEOREM 1. Suppose that the matrix $A(x,t)$ of the system (29) is not identically (with respect to x and t) degenerate. Then a function $\psi(x,t,k)$ of the form (26) determined by the conditions (28) satisfies Eq. (25) with potential $u(x,t)$ equal to

$$u(x,t) = 2i\partial_xa_1(x,t) = 2\partial_x^2\ln\det A(x,t). \qquad (31)$$

The proof is absolutely standard for the theory of "finite-gap" integration and uses only the form of ψ and the conditions that determine it.

If we set $u(x,t) = 2i\partial_xa_1(x,t)$, then substitution of (26) in (25) immediately shows that the left-hand side of this equation is equal to a function $\tilde\psi(x,t,k)$ of the form

$$\tilde\psi(x,t,k) = (\tilde a_1(x,t)k^{N-1}+\dots+\tilde a_N(x,t))e^{ikx+ik^2t},$$

which is completely analogous to (26) but without the term k^N in the preexponential factor. Since the conditions (27) are linear and do not depend on x and t, for any linear operator $\Lambda = \Lambda(\partial_x, \partial_t)$ we have the result that the function $\tilde\psi(x,t,k) = \Lambda\psi(x,t,k)$ will again satisfy the conditions (27). Hence, $\tilde a_1,\dots,\tilde a_N$ satisfy a system of linear equations with the same coefficients as for $a_1,\dots,a_N$. In contrast to the equations for these last quantities, the system of equations for $\tilde a_1,\dots,\tilde a_N$ is homogeneous. Hence, $\tilde a_1 = \dots = \tilde a_N = 0$, and Eq. (25) is proved.

The proof of the second equality in (31) follows directly from Cramer's rule for the solution $a_1(x,t)$ of the system (29) and from the obvious relation

$$\frac{1}{i}\partial_x[P_{N-1,s}(x,t,k)e^{ikx+ik^2t}] = P_{N,s}(x,t,k)e^{ikx+ik^2t}.$$

The theorem is proved.

Remark. For nondegeneracy of the matrix $A(x,t)$ it is obviously necessary that the matrix (α_{ij}^S) of coefficients of the system of linear conditions (28) have rank N. In what follows, we shall assume that this condition on the rank of the matrix (α_{ij}^S) is satisfied. We note also that the function $\psi(x,t,k)$ is not changed if the matrix (α_{ij}^S) is multiplied from the left by an arbitrary constant nondegenerate $N\times N$ matrix.

As we have already said, for arbitrary values of the parameters $\varkappa_j$, (α_{ij}^s) the obtained potentials $u(x,t)$ are *complex* and *meromorphic* functions of the variables x and t. We now describe the restrictions on these parameters that guarantee *reality* and *nonsingularity* (for all real x and t) of the potentials $u(x,t)$.

We consider here only the case $m_1 = \dots m_M = 0$ (an example that leads to rational solutions of the SNSE and in which $N=2$, $m_j\neq 0$ was analyzed in Ref. 24). In this case, we require that $M=2N$ and that the quantities $\varkappa_1,\dots,\varkappa_{2N}$ have nonzero imaginary parts and are arranged in complex-conjugate pairs:

$$\varkappa_{N+i} = \bar\varkappa_i, \quad i=\overline{1,N}.$$

Without loss of generality, it can be assumed that the minor of the matrix $(\alpha_{ij})\equiv(\alpha_{ij}^0)$ formed from the columns with numbers $j=N+1,\dots,2N$ is not degenerate. We shall assume that this minor is unity (see the remark above). Then the system of equations (28) can be rewritten in the form

$$\psi(\bar\varkappa_i) = -\sum_{j=1}^{N}\alpha_{ij}\psi(\varkappa_j), \quad i=\overline{1,N}, \qquad (32)$$

where α_{ij} is a constant $N\times N$ matrix. It is convenient to renormalize the function $\psi(x,t,k)$, setting

$$\Psi(x,t,k) = \frac{\psi(x,t,k)}{(k-\varkappa_1)\dots(k-\varkappa_N)}$$

$$= \left(1+\sum_{j=1}^{N}\frac{r_j(x,t)}{k-\varkappa_j}\right)e^{ikx+ik^2t}. \qquad (33)$$

For the renormalized function $\Psi(x,t,k)$ the conditions (32) are written in the form

$$\Psi(x,t,\bar\varkappa_i) = -\sum_{j=1}^{N}C_{ij}\operatorname*{res}_{k=\varkappa_j}\Psi(x,t,k), \qquad (34)$$

where the constant matrix (C_{ij}) is related to the matrix (α_{ij}) by the equations

$$C_{ij} = [R(\bar\varkappa_i)]^{-1}\alpha_{ij}R'(\varkappa_j), \quad i,j=\overline{1,N}, \qquad (35)$$

$R(k) = (k-\varkappa_1)\dots(k-\varkappa_N)$, and the prime denotes the derivative with respect to k.

THEOREM 2. Suppose that the parameters $\varkappa_1,\dots,\varkappa_N$, (C_{ij}), which specify through the conditions (34) a function $\Psi(x,t,k)$ of the form (33), satisfy the following requirements:

a) the matrix (C_{ij}) is anti-Hermitian, $C_{ij} = -\bar C_{ji}$;

b) we number the points $\varkappa_1,\dots,\varkappa_N$ in such a way that $\operatorname{Im}\varkappa_i>0$, $i=1,\dots,p$ and $\operatorname{Im}\varkappa_i<0$, $i=p+1,\dots,N$. It is required that the Hermitian matrix

$$(i^{-1}C_{kl}), \quad 1\leqslant k, \quad l\leqslant p,$$

be positive definite and that the Hermitian matrix

$$(i^{-1}C_{kl}), \quad p+1\leqslant k, \quad l\leqslant N,$$

be negative definite (these matrices need not be strictly definite). Then the function $\Psi(x,t,k)$ for $k\neq\varkappa_j$ depends smoothly on x,t for all real x,t and is an eigenfunction for the operator $L = i\partial_t - \partial_x^2 + u(x,t)$ with a smooth real potential (and zero eigenvalue). For these functions,

$$\Psi(x,t,k) = \frac{\det\hat M(x,t,k)}{\det M(x,t)}e^{ik(x+kt)}; \qquad (36)$$

$$u(x,t) = 2\partial_x^2\ln\det M(x,t), \qquad (37)$$

where

$$M_{ij}(x,t) = C_{ij} + \frac{e^{i(\bar\omega_i-\omega_j)}}{\bar\varkappa_i-\varkappa_j}, \quad \omega_i = \varkappa_i(x+\varkappa_it), \quad i,j=\overline{1,N}; \qquad (38)$$

$$\left.\begin{aligned}&\hat M_{ij} = M_{ij} \quad \text{for} \quad i,j=\overline{1,N},\ \hat M_{00}=1,\ \hat M_{i0}=e^{i\bar\omega_i},\\ &\hat M_{0i} = (k-\varkappa_i)^{-1}e^{-i\omega_i}, \quad i=\overline{1,N}.\end{aligned}\right\} \qquad (39)$$

Proof. We consider the rational function

$$\Omega(x,t,k) = \Psi(x,t,k)\overline{\Psi(x,t,\bar k)}. \qquad (40)$$

The residue of this function at the point $k=\infty$ is $-(a_1(x,t)+\overline{a_1(x,t)})$. In addition, this function has simple poles at the points $k=\varkappa_i$, $k=\bar\varkappa_i$ with residues

$$\operatorname{res}_{\varkappa_i}\Omega(x,t,k) = \operatorname{res}_{\varkappa_i}\Psi(x,t,k)\overline{\Psi(x,t,\bar k)} = -\sum_{j=1}^{N}\bar C_{ij}\Psi_i\bar\Psi_j, \qquad (41)$$

where we have set

$$\Psi_i = \Psi_i(x,t) = \operatorname{res}_{\varkappa_i}\Psi(x,t,k) = r_i(x,t)e^{i\omega_i}, \quad i=\overline{1,N}. \qquad (42)$$

Similarly,

$$\operatorname{res}_{\bar\varkappa_i}\Omega(x,t,k) = \Psi(x,t,\bar\varkappa_i)\operatorname{res}_{\bar\varkappa_i}\overline{\Psi(x,t,\bar k)} = -\sum_{j=1}^{N}C_{ij}\Psi_j\bar\Psi_i. \qquad (43)$$

It therefore follows from the condition of anti-Hermiticity that the sum of the residues of the function $\Omega(x,t,k)$ at all finite poles is zero. From this we conclude that the residue at infinity is zero, i.e., $\bar a_1 = -a_1$. This then entails reality of the potential $u(x,t)$ by virtue of the formula $u = 2i\partial_xa_1$.

Smoothness of $u(x,t)$ and $\Psi(x,t,k)$ for $k\neq\varkappa_j$ is equivalent, as follows from (36) and (37), to nondegeneracy of the matrix $M(x,t)$, which is the matrix of coefficients of Ψ_j that are determined by Eq. (42) in the system of equations (44), which is equivalent to the original equation (34). We shall show that the system of conditions (34) has a unique solution for all real x,t. We rewrite this system in the form of a system of linear equations for the residues $\Psi_i(x,t)$, which are determined by Eq. (42):

$$\sum_{j=1}^{N}\left(C_{ij}+\frac{e^{i(\bar\omega_i-\omega_j)}}{\bar\varkappa_i-\varkappa_j}\right)\Psi_j = -e^{i\bar\omega_i}, \quad i=\overline{1,N}. \qquad (44)$$

Degeneracy at certain x,t of the matrix of coefficients of this system signifies solvability of the corresponding homogeneous system (with zeros on the right-hand sides). This last condition is equivalent to the existence of a nonvanishing function $\Psi'(x,t,k)$ of the form

$$\Psi'(x,t,k) = \sum_{j=1}^{N}\frac{\tilde r_j(x,t)}{k-\varkappa_j}e^{ik(x+kt)},$$

which satisfies the conditions (34). We shall show that this is impossible. We consider the integral over the real axis

$$0<\int_{-\infty}^{\infty}|\Psi'(x,t,k)|^2dk = \int_{-\infty}^{\infty}\Omega'(x,t,k)dk = I,$$

where $\Omega'(x,t,k)$ is constructed from the function $\Psi'(x,t,k)$ by means of Eq. (40). We calculate the integral I in terms of the residues of the function $\Omega'(x,t,k)$ lying in the upper half-plane. For these residues, Eqs. (41) and (43) hold if Ψ_i is replaced by $\Psi_i' = \operatorname{res}_{\varkappa_i}\Psi'$. Therefore, we have

$$\begin{aligned}I &= 2\pi i\left(\sum_{i=1}^{p}\sum_{j=1}^{N}C_{ij}\Psi_i'\bar\Psi_j' - \sum_{i=p+1}^{N}\sum_{j=1}^{N}C_{ij}\Psi_j'\bar\Psi_i'\right)\\ &= 2\pi i\left(\sum_{i,j=1}^{p}C_{ji}\Psi_i'\bar\Psi_j' - \sum_{i,j=p+1}^{N}C_{ij}\Psi_j'\bar\Psi_i'\right.\\ &\left.+\sum_{i=1}^{p}\sum_{j=p+1}^{N}C_{ji}\Psi_i'\bar\Psi_j' - \sum_{i=p+1}^{N}\sum_{j=1}^{p}C_{ij}\Psi_j'\bar\Psi_i'\right).\end{aligned}$$

Redenoting in the first and third sums the indices of summation, we finally obtain

$$I = 2\pi i\left(\sum_{i,j=1}^{p}\bar\Psi_i'C_{ij}\Psi_j' - \sum_{i,j=p+1}^{N}\bar\Psi_i'C_{ij}\Psi_j'\right)\leqslant 0$$

by virtue of condition (b) of the theorem. The resulting contradiction proves the smoothness for all real x,t, of the functions $\Psi(x,t,k)$ and $u(x,t)$. Equations (36) and (37) can be derived in the same way as (30) and (31). The theorem is proved.

DEFINITION 1. We shall call the integrable potential $u(x,t)$ specified in the framework of our construction by the

N parameters $\varkappa_1,\ldots,\varkappa_N$ together with the $N\times N$ matrix (C_{ij}) an N-soliton potential.

For the corresponding solutions of the scalar NSE (see Sec. 3), such a definition of the number of solitons agrees with the standard definition. In vector models, our definition of the number of solitons does not always agree with the intuitive definition.[25]

We now establish the circumstances under which two sets of "spectral data" $(\varkappa_i)$, (C_{ij}) and $(\varkappa_i')$, (C_{ij}') determine the same Schrödinger operator with the same family of eigenfunctions $\Psi(x,t,k)$. For this, it is convenient to use the relations (32). We represent the matrix (α_{ij}), which is related to (C_{ij}) by Eq. (35), in the block form

$$(\alpha_{ij})=\begin{pmatrix}\alpha_+ & \beta\\ \gamma & \alpha_-\end{pmatrix},$$

where the square matrices α_+ and α_- have dimensions $p\times p$ and $(N-p)\times(N-p)$ respectively. We assume that the matrix α_- is nondegenerate. Then a transformation of the form $[\varkappa_i,(\alpha_{ij})]\to[\varkappa_i'(\alpha_{ij}'),]$ where

$$\varkappa_i'=\begin{cases}\varkappa_i, & i=\overline{1,\ p}\\ \bar\varkappa_i, & i=\overline{p+1,\ N}\end{cases}; \tag{45}$$

$$(\alpha_{ij}')=\begin{pmatrix}\alpha_+-\beta\alpha_-^{-1}\gamma & -\beta\alpha_-^{-1}\\ \alpha_-^{-1}\gamma & \alpha_-^{-1}\end{pmatrix}, \tag{46}$$

does not change the relations (32), which determine the functions $\Psi(x,t,k)$. Thus, in the case of nondegeneracy of the minor α_- the points $\varkappa_{p+1},\ldots,\varkappa_N$ can be "pushed" from the lower into the upper half-plane [the matrix (C_{ij}) having been also changed, in accordance with Eqs. (46) and (35)] without change of the Schrödinger operator and its eigenfunction.

We also mention a case of "trivialization" of the basic system (34), for which there is in the matrix (C_{ij}) a cross of zeros, i.e.,

$$C_{i_0 j}=C_{j i_0}=0,\quad j=\overline{1,\ N}.$$

In this case, the eigenfunction $\Psi(x,t,k)$ has the form

$$\Psi(x,\ t,\ k)=\frac{k-\bar\varkappa_{i0}}{k-\varkappa_{i0}}\tilde\Psi(x,\ t,\ k),$$

where the function $\tilde\Psi(x,t,k)$ does not depend on $\varkappa_{i0}$ and is determined by a system of the form (34), from which the row with the number i_0 has been removed. The potential $u(x,t)$ will also not depend on $\varkappa_{i0}$.

We shall now also give the laws of transformation of the spectral data corresponding to Galileo, scale, and other simple transformations of the potential of the Schrödinger operator.

a) The Galileo transformation

$$x'=x+vt;\quad t'=t.$$

For this,

$$\left.\begin{aligned}&\varkappa_i'=\varkappa_i-v/2,\quad i=\overline{1,\ N};\\ &(C_{ij}')=(C_{ij}).\end{aligned}\right\} \tag{47}$$

We have

$$\left.\begin{aligned}&u(x,\ t)=u'(x',\ t'),\\ &\Psi'(x',\ t',\ k')\\ &=\Psi(x,\ t,\ k)\,e^{-i\frac{v}{2}\left(x+\frac{v}{2}t\right)},\ \text{where } k'=k-v/2.\end{aligned}\right\} \tag{48}$$

b) The translations

$$x'=x+x_0;\quad t'=t+t_0.$$

Under such transformations, the spectral data transform in accordance with the law

$$\varkappa_i'=\varkappa_i,\quad i=\overline{1,\ N};$$

$$C_{ij}'=C_{ij}\exp\{i[(\bar\varkappa_i-\varkappa_j)x_0+(\bar\varkappa_i^2-\varkappa_j^2)t_0]\},\quad i,\ j=\overline{1,\ N}. \tag{49}$$

Obviously,

$$u'(x',\ t')=u(x,\ t);$$

$$\Psi'(x',\ t',\ k)=\Psi(x,\ t,\ k)\,e^{ik(x_0+kt_0)}.$$

c) The scale transformations

$$x'=\lambda x;\quad t'=\lambda^2 t.$$

The transformation law for the spectral data is

$$\varkappa_i'=\lambda^{-1}\varkappa_i,\quad i=\overline{1,\ N};$$

$$(C_{ij}')=(C_{ij}).$$

The potential and eigenfunctions transform as

$$u'(x',\ t')=\lambda^{-2}u(x,\ t);\quad \Psi'(x',\ t',\ k')=\Psi(x,\ t,\ k),$$

where $k'=\lambda' k$.

d) Spatial and time reflections:

$$x'=-x;\quad t'=-t.$$

Then

$$\varkappa_i'=\bar\varkappa_i,\quad i=\overline{1,\ N};$$

$$C_{ij}'=\bar C_{ij},\quad i,\ j=\overline{1,\ N}.$$

At the same time,

$$u'(x',\ t')=u(x,\ t);$$

$$\Psi'(x',\ t',\ k')=\overline{\Psi(x,\ t,\ k)},$$

where $k'=\bar k$.

Asymptotic properties of the potentials and eigenfunctions of the constructed operators

The case $N=1$

The system (44) reduces to the single equation

$$\left(C+\frac{e^{i(\bar\omega-\omega)}}{\bar\varkappa-\varkappa}\right)\Psi_1(x,\ t)=-e^{i\bar\omega}.$$

Here $\varkappa\equiv\varkappa_1$ (we suppose $\mathrm{Im}\,\varkappa>0$), $C=C_{11}$, $\mathrm{Re}\,C_{11}=0$, $\mathrm{Im}\,C>0$ (for $C=0$, everything becomes trivial), $\omega=\varkappa(x+\varkappa t)$. Hence

$$\Psi_1(x,\ t)=-e^{i\bar\omega}\left(C+\frac{e^{i(\bar\omega-\omega)}}{\bar\varkappa-\varkappa}\right)^{-1}.$$

We write $\varkappa=\alpha+i\beta$. We then obtain[2)]

$$\Psi_1(x,\ t)=\frac{i\beta}{\sqrt{-i\beta C}}\,\frac{e^{i\alpha x+i(\alpha^2-\beta^2)t}}{\mathrm{ch}\,[\beta(x-x_0)+2\alpha\beta t]}, \tag{50}$$

where x_0 has the form

$$x_0=\frac{1}{\beta}\ln\sqrt{(\bar\varkappa-\varkappa)\,C}. \tag{51}$$

For $r=r_1(x,t)$, we obtain

$$r(x,\ t)=i\beta\{1+\mathrm{th}\,[\beta(x-x_0)+2\alpha\beta t]\}. \tag{52}$$

Thus, to the case $N=1$ there corresponds the well-known "single-soliton" potential of the Schrödinger operator which decreases in all directions except directions of the form $x=-2\alpha t+\mathrm{const}$:

$$u(x,\ t)=2i\,\partial_x r(x,\ t)=-2\beta^2\,\mathrm{ch}^{-2}\,[\beta(x-x_0)+2\alpha\beta t]. \tag{53}$$

The eigenfunction of the corresponding Schrödinger operator has the form

$$\Psi(x,\ t,\ k)=\left[1+i\beta\,\frac{1+\mathrm{th}\,[(x-x_0)+2\alpha\beta t]}{k-\varkappa}\right]e^{ik(x+kt)}. \tag{54}$$

The case $N>1$

The asymptotic behavior of the functions $\Psi(x,t,k)$ cannot be readily described for arbitrary values of the parameters $(\varkappa_i)$, (C_{ij}). We shall consider here only the simplest case when $\mathrm{Im}\,\varkappa_i>0$, $i=1,\ldots,N$:

$$\det(C_{ij})\neq 0.$$

Some other examples of calculation of the asymptotic behaviors are considered below.

We consider first the asymptotic behavior at large x and fixed t. As $x\to-\infty$,

$$e^{i\bar\omega_j}=e^{i\bar\varkappa_j(x+\bar\varkappa_j t)}\to 0;$$

$$e^{-i\omega_j}\to 0,\quad j=\overline{1,\ N}.$$

Therefore, the system (44) for the residues $\Psi_1,\ldots,\Psi_N$ of the function $\Psi(x,t,k)$ goes over into the system

$$\sum_{j=1}^{N}C_{ij}\Psi_j=0,\quad i=\overline{1,\ N}.$$

Therefore, $\Psi_j\to 0$ for all j. It is easy to see that this decrease is exponential, i.e., as $x\to-\infty$

$$\Psi_j(x,\ t)\to\Psi_j^0(t)\,e^{\beta x},\quad j=\overline{1,\ N},$$

where $\Psi_j^0(t)$ are certain functions of t, and

$$\beta=\min_j|\mathrm{Im}\,\varkappa_j|. \tag{55a}$$

We also have

$$r_j(x,\ t)\to 0,\quad j=\overline{1,\ N}, \tag{55b}$$

since $r_j=\Psi_j e^{-i\omega_j}$. Therefore, the potential $u(x,t)$ also decreases exponentially; it is readily seen that

$$u(x,\ t)=O(e^{2\beta x}),\quad x\to-\infty,$$

where β is determined by Eq. (55a). From (55b) we obviously find that

$$\Psi(x,\ t,\ k)\to e^{ik(x+kt)},\quad x\to-\infty.$$

In the limit $x\to+\infty$, the calculation is somewhat more complicated. Here the matrix of the system (44) tends to infinity. Going over to the variables $r_j=\Psi_j\exp(-i\omega_j)$, we rewrite the system (44) in the form

$$\sum_{j=1}^{N}\left(C_{ij}e^{-i\bar\omega_i}+\frac{1}{\bar\varkappa_i-\varkappa_j}\right)r_j=-1,\quad i=\overline{1,\ N}.$$

As $x\to+\infty$, this system is transformed into

$$1+\sum_{j=1}^{N}\frac{r_j^0}{\bar\varkappa_i-\varkappa_j}=0,\quad i=\overline{1,\ N}, \tag{56}$$

where by $r_1^0,\ldots,r_N^0$ we have denoted the limiting values.

The rational function

$$f=1+\sum_{i=1}^{N}\frac{r_i^0}{k-\varkappa_i}$$

can be uniquely represented in the form

$$f=\frac{P(k)}{\prod_i(k-\varkappa_i)},$$

where $P(k)$ is a polynomial of degree N with unity as the leading coefficient. It follows from (56) that $f(\bar\varkappa_i)=0$ and, hence, $P(k)=\Pi_i(k-\bar\varkappa_i)$. Hence

$$\Psi(x,\ t,\ k)\to\prod_{j=1}^{N}\frac{k-\bar\varkappa_j}{k-\varkappa_j}\,e^{ik(x+kt)},\quad x\to+\infty. \tag{57}$$

The functions Ψ_j decrease exponentially, and

$$\Psi_j(x,\ t)\to r_j^0 e^{i\omega_j}\ \text{as}\ x\to+\infty,\ j=\overline{1,\ N}. \tag{58}$$

One can show that

$$u(x,\ t)=O(e^{-2\beta x}),\quad x\to+\infty, \tag{59}$$

where β is determined by Eq. (55a).

For a degenerate matrix (C_{ij}), the asymptotic behaviors at large x are more complicated. We note here the following property: Suppose that all the points $\varkappa_1,\ldots,\varkappa_N$ lie in the upper half-plane but the matrix (C_{ij}) is degenerate. If the imaginary parts $\mathrm{Im}\,\varkappa_1,\ldots,\mathrm{Im}\,\varkappa_N$ are all pairwise different, then at least one of the functions $\Psi_1(x,t),\ldots,\Psi_N(x,t)$ as $x\to-\infty$ increases unboundedly. For suppose the constants $\lambda_1,\ldots,\lambda_N$ (not all zero) satisfy the conditions

$$\sum_{i=1}^{N}\lambda_i C_{ij}=0,\quad j=\overline{1,\ N}.$$

Multiplying the i-th equation of the system (44) by λ_i and then adding all the equations, we obtain

$$\sum_{i,\ j=1}^{N}\lambda_i\frac{e^{i(\bar\omega_i-\omega_j)}}{\bar\varkappa_i-\varkappa_j}\Psi_j=-\sum_{i=1}^{N}\lambda_i e^{i\bar\omega_i}.$$

Going to the limit $x\to-\infty$ in this relation under the assumption that the functions $\Psi_1,\ldots,\Psi_N$ are bounded, we obtain $\lambda_1=\ldots=\lambda_N=0$. The resulting contradiction proves the unboundedness of the functions Ψ_j.

Note also that for a special choice of the spectral data $(\varkappa_i)$, (C_{ij}) the potential $u(x,t)$ is a periodic or quasiperiodic function of x. For periodicity with respect to x with period T, we obviously require

$$C_{ij}\exp[i(\bar\varkappa_i-\varkappa_j)T]=C_{ij},\quad i,\ j=\overline{1,\ N}. \tag{60}$$

This follows from (49). If we set $\varkappa_j=\alpha_j+i\beta_j$, then (60) can be rewritten in the form

$$\left.\begin{array}{l}\alpha_i-\alpha_j=T^{-1}2\pi n_{ij};\\ \beta_i+\beta_j=0\end{array}\right\}\quad \text{for}\quad C_{ij}\neq 0,$$

where n_{ij} are certain integers.

If we replace this condition by

$$\beta_i+\beta_j=0\quad \text{for}\quad C_{ij}\neq 0,$$

then the potential $u(x,t)$ will be a quasiperiodic function with respect to x. It is obvious that these relations can be solved if the matrix (C_{ij}) satisfies the requirements

$$C_{ii}=0,\quad C_{ij}C_{jk}C_{ki}=0,\quad i,\ j,\ k=\overline{1,\ N}.$$

The conditions of periodicity and quasiperiodicity with respect to t are similar.

We now turn to the study of the asymptotic behavior with respect to t at large t for fixed x. We shall again assume that $\operatorname{Im}\varkappa_i>0$, $i=1,\dots,N$, and $\det(C_{ij})\neq 0$. Two essentially different cases are possible. The first is when $\operatorname{Im}\varkappa_j^2\equiv 0$ for all $j=1,\dots,N$. Calculations similar to the ones just made show that the functions $\Psi_1(x,t),\dots,\Psi_N(x,t)$ and $u(x,t)$ are exponentially damped as $t\to\infty$, the rate of damping being determined by the number $\min_j[\operatorname{Im}\varkappa_j^2]$.

The second case is when at least one of the imaginary parts $\operatorname{Im}\varkappa_j^2$ is zero. We choose the simplest situation:

$$\operatorname{Im}\varkappa_1^2>\dots>\operatorname{Im}\varkappa_{N-1}^2>\operatorname{Im}\varkappa_N^2=0. \tag{61}$$

As $t\to-\infty$, the system (48) can be rewritten asymptotically in the form

$$\begin{gathered}\sum_{j=1}^{N}C_{ij}\Psi_j=0,\quad i=\overline{1,\ N-1}, \\ \sum_{j=1}^{N-1}C_{Nj}\Psi_j+\left(C_{NN}+\frac{e^{i(\overline{\omega}_N-\omega_N)}}{\overline{\varkappa}_N-\varkappa_N}\right)\Psi_N=-e^{i\overline{\omega}_N}.\end{gathered} \tag{62}$$

We denote by (C^{ij}) the matrix that is the inverse of (C_{ij}). Making in (62) a change of variables in accordance with the formula

$$\Psi_j=\sum_{l=1}^{N}C^{jl}\Phi_l,\quad j=\overline{1,\ N},$$

we obtain

$$\Phi_j=0\quad \text{for}\quad j=\overline{1,\ N-1},$$

$$\Phi_N=e^{i\overline{\omega}_N}\left[1+\frac{e^{i(\overline{\omega}_N-\omega_N)}}{\overline{\varkappa}_N-\varkappa_N}C^{NN}\right]^{-1}.$$

From this we obtain the form of the oscillating asymptotic behavior of the function Ψ_j as $t\to-\infty$:

$$\Psi_j\to-\frac{C^{jN}}{C^{NN}}\frac{e^{i\overline{\omega}_N}}{(C^{NN})^{-1}+\dfrac{e^{i(\overline{\omega}_N-\omega_N)}}{\overline{\varkappa}_N-\varkappa_N}}$$

$$=\frac{C^{jN}}{C^{NN}}\sqrt{\frac{C^{NN}}{-2i\beta_N}}\,i\beta_N\frac{e^{-i\beta_N^2t}}{\operatorname{ch}[\beta_N(x-x_0^-)]},\quad j=\overline{1,\ N}, \tag{63}$$

where $\varkappa_N=i\beta_N$, and

$$x_0^-=\beta_N^{-1}\ln\sqrt{\frac{-2i\beta_N}{C^{NN}}}. \tag{64}$$

From this it is readily deduced that the potential $u(x,t)$ has a soliton asymptotic behavior (fixed soliton) of the form

$$u(x,\ t)\to-2\beta_N^2\operatorname{ch}^{-2}[\beta_N(x-x_0^-)],\quad t\to-\infty.$$

We now calculate the asymptotic behavior as $t\to+\infty$. It is here convenient to go over to the variables $r_1(x,t),\dots,r_N(x,t)$. The system (44) can be rewritten asymptotically in the form

$$\begin{gathered}1+\sum_{j=1}^{N}\frac{r_j}{\overline{\varkappa}_i-\varkappa_j}=0,\quad i=\overline{1,\ N-1}, \\ \sum_{j=1}^{N-1}\left(C_{Nj}e^{-i\overline{\omega}_N}+\frac{1}{\overline{\varkappa}_N-\varkappa_j}\right)r_j \\ +\left(e^{-i(\overline{\omega}_N-\omega_N)}+\frac{1}{\overline{\varkappa}_N-\varkappa_N}\right)r_N=-1.\end{gathered} \tag{65}$$

The first equations of the system (65) gives the following result: $N-1$ zeros of the rational function

$$R(k)=1+\sum_{j=1}^{N}\frac{r_j}{k-\varkappa_j}$$

lie at the points $\overline{\varkappa}_1,\dots,\overline{\varkappa}_{N-1}$. Setting

$$R(k)=(k-a)\prod_{i=1}^{N-1}(k-\overline{\varkappa}_i)\left(\prod_{i=1}^{N}(k-\varkappa_i)\right)^{-1},$$

where a is an unknown quantity, we obtain

$$r_j=\frac{(\varkappa_j-a)\prod_{i=1}^{N-1}(\varkappa_j-\overline{\varkappa}_i)}{\prod_{i\neq j}(\varkappa_j-\varkappa_i)},\quad j=\overline{1,\ N}.$$

Substituting in the last equation of (65), we determine a. After some simple calculations we then obtain in the limit $t\to+\infty$:

$$\Psi_N(x,\ t)\to\frac{i\beta_N}{\sqrt{-2i\beta_NC_{NN}}}\frac{e^{-i\beta_N^2t+i\varphi_0^+}}{\operatorname{ch}[\beta_N(x-x_0^+)]},$$

where

$$\left.\begin{array}{l}\varphi_0^+=\arg Z_N,\quad x_0^+=\dfrac{1}{\beta_N}\ln[\sqrt{-2i\beta_NC_{NN}}\,|Z_N|];\\ Z_N=\prod\limits_{i\neq N}\dfrac{\varkappa_N-\overline{\varkappa}_i}{\varkappa_N-\varkappa_i}.\end{array}\right\} \tag{66}$$

For the complete function $\Psi(x,t,k)$ we obtain the asymptotic behavior

$$\Psi(x,\ t,\ k)\to\prod_{j=1}^{N-1}\frac{k-\overline{\varkappa}_j}{k-\varkappa_j}\left\{1+\frac{1}{2}\frac{\varkappa_N-\overline{\varkappa}_N}{k-\varkappa_N}\times[1+\operatorname{th}(\beta_N(x-x_0^+))]\right\}e^{ik(x+kt)}.$$

For the potential $u(x,t)$ we again obtain a soliton asymptotic behavior:

$$u(x,\ t)\to 2\beta_N^2\operatorname{ch}^{-2}[\beta_N(x-x_0^+)],\quad t\to+\infty.$$

The phase shift of the fixed soliton corresponding to the transition from $t\to-\infty$ to $t\to+\infty$ can thus be calculated in accordance with the formula

$$x_0^+-x_0^-=\frac{1}{\beta_N}\ln\left(\sqrt{C_{NN}C^{NN}}\left|\prod_{i\neq N}\frac{\varkappa_N-\overline{\varkappa}_i}{\varkappa_N-\varkappa_i}\right|\right). \tag{67}$$

The $t\to\pm\infty$ asymptotic behaviors on the straight lines $x=vt+x_0$ can be calculated similarly (one can directly use Eqs. (47) and (48) for a Galileo transformation]. We find that the potential $u(x,t)$ decomposes asymptotica into solitons of the type (53), moving with velocities $v_j=-(\operatorname{Im}\varkappa_j^2/\operatorname{Im}\varkappa_j$, $j=1,\dots,N$. The phase shifts of these solitons can be calculated in accordance with formulas of the type (67). We note that the interaction of these solitons does not reduce to a binary interaction, owing to the presence of the term $(1/\beta_N\ln\sqrt{C_{NN}C^{NN}}$, in (67).

We emphasize that the asymptotic decay into solitons that we have just described takes place under the condition that the imaginary parts of $\varkappa_1^2,\dots,\varkappa_N^2$ are different. If some of these imaginary parts are the same, bound states of solitons arise. We give the form of the corresponding asymptotic behaviors for the case

$$\operatorname{Im}\varkappa_1^2\geqslant\operatorname{Im}\varkappa_2^2\geqslant\dots>\operatorname{Im}\varkappa_{N-m+1}^2=\dots=\operatorname{Im}\varkappa_N^2=0,$$

assuming, as above, that $\operatorname{Im}\varkappa_j>0$, $j=1,\dots,N$, $\det(C_{ij})\neq 0$.

a) The limit $t\to-\infty$. We denote by (C^{ij}) the matrix that is the inverse of (C_{ij}); by $(C_{ij}^-)_{N-m+1\leqslant i,j\leqslant N}$ we denote the matrix that is the inverse of $(C^{ij})_{N-m+1\leqslant i,j\leqslant N}$,

$$\sum_{s=N-m+1}^{N}C_{is}^-C^{sj}=\delta_i^j,\quad i,\ j=\overline{N-m+1,\ N}.$$

Then in the limit $t\to-\infty$ the function $\Psi(x,t,k)$ has an oscillating (i.e., quasiperiodic with respect to t for real k) asymptotic behavior of the form

$$\Psi(x,\ t,\ k)\to\Psi^-(x,\ t,\ k),$$

where the function

$$\Psi^-(x,\ t,\ k)=\left(1+\sum_{j=N-m+1}^{N}\frac{r_j^-}{k-\varkappa_j}\right)e^{ik(x+kt)}$$

is specified by the m points of the discrete spectrum $\varkappa_{N-m+1},\dots\varkappa_N$ and the $m\times m$ matrix (C_{ji}^-) in accordance with the basic construction (see the beginning of Sec. 2).

Let

$$\Psi_j^-(x,\ t)=r_j^-e^{i\omega_j},\quad j=\overline{N-m+1,\ N},$$

be the residues of this function. Then the residues $\Psi_1,\dots,\Psi_N$ of the function $\Psi(x,t,k)$ have as $t\to-\infty$ an oscillating asymptotic behavior of the form

$$\Psi_j(x,\ t)\to\Psi_j^-(x,\ t),\quad j=\overline{N-m+1,\ N},$$

$$\Psi_j(x,\ t)\to\sum_{l,\ s=N-m+1}^{N}C^{jl}C_{ls}^-\Psi_s^-(x,\ t),\quad j=\overline{1,\ N-m}.$$

The corresponding asymptotic behavior of the potential $u(x,t)\to u^-(x,t)$ is an m-soliton behavior, determined by the discrete spectrum $\varkappa_{N-m+1},\dots,\varkappa_N$ and the matrix (C_{ij}^-). The function $u^-(x,t)$ depends quasiperiodically on t.

b) The limit $t\to+\infty$. We denote by $R(k)$ a rational function of the form

$$R(k)=\prod_{i=1}^{N-m}\frac{k-\overline{\varkappa}_i}{k-\varkappa_i},$$

and by (C_{ij}^+) an $m\times m$ matrix of the form

$$C_{ij}^+=R^{-1}(\overline{\varkappa}_i)C_{ij}R(\varkappa_j),\quad i,\ j=\overline{N-m+1,\ N}.$$

Then in the limit $t\to+\infty$ the asymptotic behavior of the function $\Psi(x,t,k)$ is

$$\Psi(x,\ t,\ k)\to R(k)\Psi^+(x,\ t,\ k), \tag{68}$$

where the function $\Psi^+(x,t,k)$ is determined by the m points of the discrete spectrum $\varkappa_{N-m+1},\dots,\varkappa_N$ and the $m\times m$ matrix (C_{ij}^+) in accordance with the basic construction. The form of the asymptotic behaviors for the functions $\Psi_j(x,t)$ can readily be directly extracted from formula (68). The potential has a t-quasiperiodic m-soliton asymptotic behavior $u(x,t)\to u^+(x,t)$ determined by the discrete spectrum $\varkappa_{N-m+1},\dots,\varkappa_N$ and the matrix (C_{ij}^+). The transition from the matrix (C_{ij}^-) to the matrix (C_{ij}^+) determines the law of interaction of the bound state of m solitons with the remaining components of the N-soliton solution.

Self-consistency conditions

In the neighborhood of $k=\infty$ the function $\Psi(x,t,k)$ defined at the beginning of the section can be represented in the form

$$\Psi(x,\ t,\ k)=\left(1+\sum_{l=1}^{\infty}\xi_l(x,\ t)k^{-l}\right)e^{ik(x+kt)}. \tag{69}$$

The first factor is the expansion with respect to k^{-1} of the pre-exponential factor in (33). In particular,

$$\xi_1=a_1=\sum_{j=1}^{N}r_j;\quad \xi_1+\bar{\xi}_1=0.$$

Substitution of the expansion (69) in Eq. (25) leads to the following equations for the coefficients $\xi_1(x,t)$:

$$i\dot{\xi}_l-2i\xi_{l+1}'-\xi_l''+u\xi_l=0,\quad l=0,\ 1\ \dots;\quad \xi_0=1. \tag{70}$$

The dot denotes the derivative with respect to t, and the prime denotes the derivative with respect to x.

We consider again the meromorphic function

$$\Omega(x,\ t,\ k)=\Psi(x,\ t,\ k)\overline{\Psi(x,\ t,\ \bar{k})}.$$

Its expansion in the neighborhood of infinity has the form

$$\Omega(x,\ t,\ k)=1+\sum_{l=2}^{\infty}J_l(x,\ t)k^{-l}.$$

The first few coefficients of the expansion have the form

$$\left.\begin{array}{l}J_2=\xi_2+\bar{\xi}_2-\xi_1^2,\quad J_3=\xi_3+\bar{\xi}_3+\xi_1(\bar{\xi}_2-\xi_2),\\ J_4=\xi_4+\bar{\xi}_4+\xi_1(\bar{\xi}_3-\xi_3)+|\xi_2|^2.\end{array}\right\} \tag{71}$$

Using the relations (70), we can readily find the connection between the expressions (71) and the potential $u(x,t)$ of the time-dependent Schrödinger equation.

LEMMA. For any formal solution $\Psi(x,t,k)$ of the form (26) of Eq. (25) we have

$$J_2(x,\ t)=\frac{1}{2}u(x,\ t)+c_2,\quad c_2=\text{const}; \tag{72}$$

$$\partial_xJ_3(x,\ t)=\frac{1}{2}\dot{u}(x,\ t); \tag{73}$$

$$\partial_x^2J_4(x,\ t)=\frac{3}{8}\ddot{u}-\frac{1}{8}(u_{xxx}-6uu_x)_x. \tag{74}$$

The relation (72) was obtained in Ref. 22, and the expressions (73) and (74) in Ref. 23. Note that the constant c_2

in (72) can be found from the asymptotic behavior of $\Omega(x,t,k)$ as $x\to\infty$. For example, in the situation considered above with $\operatorname{Im}\varkappa_i>0$ and a nondegenerate matrix (C_{ij}^{+}) we have $\Omega(x,t,k)\to 1$, $u(x,t)\to 0$, $x\to\pm\infty$, whence $c_2=0$.

The obtained relations serve as the basic construction of solutions with all forms of self-consistency conditions [see (79), (81), and (82) below]. Let $E(k)$ be a rational function of one of the three forms

$$E(k)=k+\sum_{i=1}^{n}\varepsilon_i\frac{b_i^2}{k-k_i}; \tag{75}$$

$$E(k)=k^2+\alpha k+\sum_{i=1}^{n}\varepsilon_i\frac{b_i^2}{k-k_i}; \tag{76}$$

$$E(k)=k^3+\beta k^2+\gamma k+\sum_{i=1}^{n}\varepsilon_i\frac{b_i^2}{k-k_i}. \tag{77}$$

The constants α,β,γ,k_i, and b_i are arbitrary real quantities. The coefficients are $\epsilon_i=\pm 1$. We denote by $\Phi_i(x,t)$ the functions

$$\Phi_i(x,t)=b_i\Psi(x,t,k_i),\quad j=\overline{1,n}.$$

By definition, the functions Φ_i satisfy the equations

$$i\dot{\Phi}_j-\Phi_j''+u(x,t)\Phi_j=0,\quad j=\overline{1,n}. \tag{78}$$

The functions $\Psi_j(x,t), j=1,\dots,N$, defined by (42) also satisfy the same equation:

$$i\dot{\Psi}_j-\Psi_j''+u(x,t)\Psi_j=0,\quad j=\overline{1,N}.$$

THEOREM 3. Suppose the functions $\Phi_i(x,t)$ and $\Psi_j(x,t)$ are constructed from the set of parameters $\varkappa_1,\dots,\varkappa_N(C_{ij})$ and from a rational function $E(k)$ of one of the three types (75)–(77). Then the following self-consistency conditions hold.

1. If $E(k)$ has the form (75), then

$$\frac{u}{2}+\sum_{i=1}^{n}\varepsilon_i b_i^2+c_2=\sum_{i=1}^{n}\varepsilon_i|\Phi_i(x,t)|^2-\sum_{i,j=1}^{N}\overline{\Psi_i(x,t)}E_{ij}\Psi_j(x,t), \tag{79}$$

where

$$E_{ij}=iC_{ij}(\overline{E(\varkappa_i)}-E(\varkappa_j)),\quad i,j=\overline{1,N}. \tag{80}$$

2. If $E(k)$ has the form (76), then

$$\frac{\dot{u}}{2}+\alpha\frac{u'}{2}=\sum_{i=1}^{n}\varepsilon_i|\Phi_i(x,t)|_x^2-\Big(\sum_{i,j=1}^{N}\overline{\Psi}_iE_{ij}\Psi_j\Big)_x, \tag{81}$$

and the matrix (E_{ij}) has the form (80).

3. If $E(k)$ has the form (77), then

$$\frac{3}{8}\ddot{u}-\frac{1}{8}(u_{xxx}-6uu_x)_x+\beta\frac{\dot{u}_x}{2}+\gamma\frac{u_{xx}}{2}=\sum_{i=1}^{n}\varepsilon_i|\Phi_i|_{xx}^2-\Big(\sum_{i,j=1}^{N}\overline{\Psi}_iE_{ij}\Psi_j\Big)_{xx}, \tag{82}$$

and the matrix (E_{ij}) has the form (80).

Proof. Consider the rational function

$$\hat{\Omega}(x,t,k)=E(k)\Psi(x,t,k)\overline{\Psi(x,t,\bar{k})}.$$

Applying to this function the residue theorem, and bearing in mind (41) and (43), we obtain the self-consistency conditions (79)–(82). The theorem is proved.

We note that a matrix (E_{ij}) of the form (80) is Hermitian. Therefore, by a linear transformation of the functions $\Psi_1,\dots,\Psi_N$ this matrix can be made diagonal, and on the diagonals there will be ± 1 or 0.

For common values of the parameters $\varkappa_j(C_{ij})$ and also an arbitrary rational function $E(k)$ of one of the indicated types, the Hermitian quadratic form on the right-hand sides of the self-consistency conditions will have a high rank equal to $N+n$. For the self-consistency conditions (79), for example this will mean that the functions $\Phi_1,\dots,\Phi_n,\Psi_1,\dots,\Psi_N$ give a solution of an $(n+N)$-component vector NSE whose symmetry type is determined by the signature of the Hermitian matrix

$$\begin{pmatrix}-\varepsilon_1 & & & 0\\ & \ddots & & \\ & & -\varepsilon_n & \\ 0 & & & \boxed{E_{ij}}\end{pmatrix}. \tag{83}$$

When special conditions are imposed on the parameters of the construction, the rank of this matrix may decrease, this corresponding to a decrease in the number of components of the vector NSE (the same applies to the other self-consistency conditions).

What is the difference between the solutions (Ψ,Φ) of the self-consistency equations corresponding to matrices of the form (83) with the same rank and signature but different numbers n of finite poles of the function $E(k)$? It follows from the foregoing results that the functions $\Psi_1,\dots,\Psi_N$ and $\Phi_1,\dots,\Phi_n$ have different asymptotic behaviors at large x—in general, the functions $\Phi_i(x,t)$ have an oscillating asymptotic behavior as $x\to\infty$, while the functions $\Psi_j(x,t)$ decrease exponentially at large x. This cirumstance must be taken into account in the construction of the many-soliton solutions by choosing the function $E(k)$ in accordance with the required boundary conditions.

3. SOME EXAMPLES

Scalar models

Example 1. We shall show how in the framework of our construction we can obtain the well known[30] many-soliton solutions of the scalar NSE (with attraction). We take $E(k)=k$. The matrix (E_{ij}) of the form (80) must have rank 1 for the scalar case. Therefore, the matrix (C_{ij}) must have the form

$$C_{ij}=\lambda\frac{\bar{\gamma}_i\gamma_j}{\bar{\varkappa}_i-\varkappa_j},\quad i,j=\overline{1,N}, \tag{84}$$

where $\gamma_1,\dots,\gamma_N$ are arbitrary complex constants, and λ is a real number.

It can be assumed that these constants are nonzero and are normalized by the condition

$$\sum_{j=1}^{N}|\gamma_j|^2=1.$$

[If $\gamma_j=0$ for some j, then in the matrix (C_{ij}) we obtain a

vanishing row and a vanishing column, i.e., the system (34) becomes trivial (see Sec. 2).] Then it can be assumed that all the $\bar{\varkappa}_1,\dots,\bar{\varkappa}_N$ lie in the upper half-plane. Indeed, this follows from the results of Sec. 2 by virtue of the nondegeneracy of matrices of the form $[(\varkappa_i-\bar{\varkappa}_j)^{-1}]$.[3] The condition for positive definiteness of the matrix $i^{-1}(C_{ij})$, where (C_{ij}) has the form (84), is equivalent to the inequality $\lambda>0$ (we assume that $\operatorname{Im}\varkappa_i>0, i=1,\dots,N$). The Hermitian form of the type (80) reduces in this case to

$$-\lambda\sum_{i,j=1}^{N}\bar{\gamma}_i\gamma_j\overline{\Psi}_i\Psi_j=-\lambda\Big|\sum_{i=1}^{N}\gamma_i\Psi_i(x,t)\Big|^2.$$

In our case, the constant c_2 in (79) is zero. We finally obtain the result that the function

$$\varphi(x,t)=\sqrt{|\lambda|}\sum_{i=1}^{N}\gamma_i\Psi_i(x,t)=\sqrt{|\lambda|}\frac{\det\hat{M}(x,t)}{\det M(x,t)}, \tag{85}$$

where the $N\times N$ matrix $M(x,t)$ has the form

$$M_{ij}=\frac{\lambda\bar{\gamma}_i\gamma_j+e^{i(\bar{\omega}_i-\omega_j)}}{\bar{\varkappa}_i-\varkappa_j},$$

$$\hat{M}_{ij}(x,t)=M(x,t)\quad\text{for}\quad i,j=\overline{1,N},$$
$$\hat{M}_{00}=0,\quad \hat{M}_{i0}=e^{i\bar{\omega}_i},\quad \hat{M}_{0i}=\gamma_i,\quad i=\overline{1,N}, \tag{86}$$

is a decreasing, as $|x|\to\infty$, solution of the NSE

$$i\varphi_t=\varphi_{xx}+2|\varphi|^2\varphi.$$

Example 2. One can construct similarly decreasing solutions of the time-dependent (scalar) Schrödinger equation

$$i\varphi_t=\varphi_{xx}-u\varphi \tag{87}$$

with self-consistency conditions of the form

$$\frac{1}{2}u_t=-|\varphi|_x^2 \tag{88}$$

or

$$3\ddot{u}-(u_{xxx}-6uu_x)_x=-8|\varphi|_{xx}^2 \tag{89}$$

[we have set $\alpha=\beta=\gamma=0$ in Eqs. (81) and (82)]. For these conditions, the solution has the form (85), where the matrix $M(x,t)$ is given by

$$M_{ij}=\frac{\lambda\bar{\gamma}_i\gamma_j}{\bar{\varkappa}_i^q-\varkappa_j^q}+\frac{e^{i(\bar{\omega}_i-\omega_j)}}{\bar{\varkappa}_i-\varkappa_j},\quad q=2,3,$$

the matrix $(\hat{M}_{ij})$ is given by formulas (86), and $\lambda>0$. Here, $q=2$ for Eq. (88); in this case, the $\varkappa_1,\dots,\varkappa_N$ must be taken to lie in the first quadrant of the complex plane, i.e.,

$$\operatorname{Im}\varkappa_i>0,\quad \operatorname{Re}\varkappa_i>0,\quad i=\overline{1,N}.$$

For Eq. (89), $q=3$; the $\varkappa_1,\dots,\varkappa_N$ lie in the sectors

$$0<\arg\varkappa_i<\frac{\pi}{3},\quad \frac{2\pi}{3}\arg\varkappa_i<\pi,\quad i=\overline{1,N},$$

and $\varkappa_i^3\neq\varkappa_j^3$ for $i\neq j$.

For self-consistency conditions of the form

$$\frac{1}{2}\dot{u}=|\varphi|_x^2 \tag{90}$$

or

$$3\ddot{u}-(u_{xxx}-6uu_x)_x=8|\varphi|_{xx}^2 \tag{91}$$

the solution can be expressed in the same form, but $\lambda<0$, and the restrictions on $\varkappa_i$ are different, namely, for Eq. (90) $(q=2)$ the following inequalities must hold:

$$\operatorname{Im}\varkappa_i>0,\quad \operatorname{Re}\varkappa_i<0,\quad i=\overline{1,N}.$$

For Eq. (91) $(q=3)$, the restrictions on $\varkappa_1,\dots,\varkappa_N$ are

$$\frac{\pi}{3}<\arg\varkappa_i<\frac{2\pi}{3},\quad i=\overline{1,N}.$$

Example 3. We shall explain the technique for constructing nondecreasing solutions for different self-consistency equations initially for the simple example of scalar nonlinear Schrödinger equations. For the construction of solutions of the NSE with attraction that are nondecreasing (oscillating) as $|x|\to\infty$, we can take the function $E(k)$ in the form

$$E(k)=k-\frac{b_1^2}{k-k_1}.$$

The Hermitian form (E_{ij}) of the form (80) must be zero (since otherwise we do not obtain the scalar NSE). In other words, we must have fulfillment of the matching conditions

$$E(\bar{\varkappa}_i)=E(\varkappa_j)\quad\text{for}\quad C_{ij}\neq 0 \tag{92}$$

(we have used the fact that the coefficients b_1 and k_1 are real). Since the degree of the rational function $E(k)$ is 2, we obtain the result that the equation $E(\bar{\varkappa}_i)=E(\varkappa_j)$ for each i can be satisfied for precisely one value of j. Therefore, for each i there is precisely one value of the index $j=\nu(i)$ such that $C_{ij}\neq 0$ [we recall that we are not considering matrices (C_{ij}) with vanishing rows]. It follows from the Hermiticity of the matrix (C_{ij}) that ν is an involution on the set of indices $(1,\dots,N)$. Since the relation $E(\bar{\varkappa}_i)=E(\varkappa_i)$ cannot be satisfied for nonreal $\varkappa_i$, the involution ν does not have fixed points. Thus, N is even and the points $\varkappa_i$ can be numbered in such a way that

$$E(\varkappa_{N-i+1})=E(\bar{\varkappa}_i),\quad i=\overline{1,N/2}.$$

The matrix (C_{ij}) is antidiagonal. It is readily seen that the points $\varkappa_1,\dots,\varkappa_{N/2}$ can be assumed to lie in the upper half-plane; then the points

$$\varkappa_{N-i+1}=k_1-\frac{b_1^2}{\bar{\varkappa}_i-k_1},\quad i=\overline{1,N/2} \tag{93}$$

will lie in the lower half-plane. Finally, we obtain the following formulas for nondecreasing solutions of the NSE of the form

$$i\varphi_t=\varphi_{xx}+2(|\varphi|^2-b_1^2)\varphi,$$

$$\varphi(x,t)=b_1e^{i\eta}\frac{\det\hat{M}(x,t)}{\det M(x,t)}e^{ik_1(x+k_1t)}, \tag{94}$$

where η is an arbitrary real constant and the $N\times N$ matrix $M(x,t)$ has the form

$$M_{ij}(x,t)=C_i\delta_{i,N-j+1}+\frac{e^{i(\bar{\omega}_i-\omega_j)}}{\bar{\varkappa}_i-\varkappa_j},$$

where $C_1,\dots,C_N$ are any nonvanishing complex numbers satisfying the relation of skew-Hermiticity

$$C_{N-i+1}=-\bar{C}_i,\quad i=\overline{1,N/2},$$

and $\varkappa_1,\dots,\varkappa_N$ satisfy the conditions (93); the matrix $\hat{M}(x,t)$ has the form

$$\left.\begin{aligned} &\hat{M}_{ij} = M_{ij} \quad \text{for} \quad 1 \leqslant i,\ j \leqslant N; \\ &\hat{M}_{00} = 1, \quad \hat{M}_{i0} = e^{i\bar{\omega}_i}, \quad \hat{M}_{0i} = \frac{e^{-i\omega_i}}{k_1 - \varkappa_i}, \quad i = \overline{1,\ N}. \end{aligned}\right\} \tag{95}$$

Note that if

$$\mathrm{Im}\,(\varkappa_i + \varkappa_{N-i+1}) = 0, \quad i = \overline{1,\ N/2} \tag{96}$$

[by virtue of (93), the points $\varkappa_i$ and $\varkappa_{N-i+1} = 2k_1 - \varkappa_i$ lie in this case on a circle of radius b_1 with center at the point k_1], then the solution $\varphi(x,t)$ will be a quasiperiodic function of x (see Sec. 2). But if the conditions (96) are not satisfied for any value of i, then the asymptotic behavior of solutions $\varphi(x,t)$ of the form (94) for constant t will be

$$\varphi(x,\ t) \to b_1 e^{i\eta} \prod_j{}' \left(\frac{k_1 - \bar{\varkappa}_j}{k_1 - \varkappa_j}\right) e^{ik_1(x+k_1t)}, \quad x \to -\infty,$$

where the product is taken over all j for which $\mathrm{Im}(\varkappa_j + \varkappa_{N-j+1}) < 0$;

$$\varphi(x,\ t) \to b_1 e^{i\eta} \prod_j{}'' \left(\frac{k_1 - \bar{\varkappa}_j}{k_1 - \varkappa_j}\right) e^{ik_1(x+k_1t)}, \quad x \to +\infty,$$

in which the product is over the remaining values of j. We omit the derivation of these formulas. The asymptotic behavior with respect to t can be calculated similarly, but depends on the relations between the $\mathrm{Im}\,\varkappa_j^2$.

We now turn to study of the NSE with repulsion:

$$i\varphi_t = \varphi_{xx} - 2\,(|\varphi|^2 - b_1^2)\,\varphi.$$

Here, the function $E(k)$ must have the form

$$E(k) = k + \frac{b_1^2}{k - k_1},$$

while the matching conditions (92) must still hold. As above, we obtain an involution ν on the set of indices $(1, 2, \ldots, N)$ such that $C_{ij} \neq 0$ only for $j = \nu(i)$. However, this involution may have fixed points. Suppose that there are 1 fixed points y of the involution ν; then $N = l + 2m$. One can choose a numbering of the points $\varkappa_1, \ldots, \varkappa_N$, such that the points $\varkappa_1, \ldots, \varkappa_l$ lie on a circle of radius b_1 with center at k_1, while the remaining points are distributed in pairs symmetric with respect to this circle, i.e.,

$$|\varkappa_i - k_1| = b_1, \quad i = \overline{1,\ l};$$

$$\varkappa_{N-i+l+1} = k_1 + \frac{b_1^2}{\bar{\varkappa}_i - k_1}, \quad i = \overline{l+1,\ l+m}.$$

The matrix (C_{ij}) must have the form

$$(C_{ij}) = \begin{pmatrix} \begin{matrix} C_1 & & \\ & \ddots & \\ & & C_l \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & & & d_1 \\ & \ddots & d_2 & \\ & -\bar{d}_2 & \ddots & \\ -\bar{d}_1 & & & 0 \end{matrix} \end{pmatrix},$$

where the numbers $C_1, \ldots, C_1$ are purely imaginary and the numbers $d_1, \ldots, d_m$ are arbitrary; all these quantities are nonzero. The points $\varkappa_1, \ldots, \varkappa_l$ can be assumed to lie in the upper half-plane. Note that the points $\varkappa_i$ and $\varkappa_{N-i+l+1}$ for each $i > l$ lie in one of the half-planes. Therefore, from the condition of definiteness of the matrix $(i^{-1}C_{ij})$ we obtain $d_1 = \cdots = d_m = 0$. Thus, it can be assumed that $m = 0$ and that the matrix (C_{ij}) is diagonal. We finally find that the solutions of the NSE with repulsion can be expressed in the form (94), where the matrix $(M_{ij}(x,t))$ has the form

$$M_{ij}(x,\ t) = i\tilde{C}_i\delta_{ij} + \frac{e^{i(\bar{\omega}_i - \omega_j)}}{\bar{\varkappa}_i - \varkappa_j},$$

the numbers $\varkappa_1, \ldots, \varkappa_N$ lie in the upper half-plane, with $|\varkappa_i - k_1| = b_1$, $i = 1, \ldots, N$, and the numbers $\tilde{C}_1, \ldots, \tilde{C}_N$ are real and positive; the matrix $\hat{M}$ is constructed from the matrix M in accordance with (95). The simplest of these solutions ($N = 1$) has the step form

$$\varphi(x,\ t) = b_1\left[1 + i\beta\,\frac{1 + \mathrm{th}\,\tau}{k_1 - \varkappa}\right] e^{ik_1(x+k_1t)+i\eta},$$

where $x = \alpha + i\beta = k_1 + b_1(\cos\xi + i\sin\xi)$, $\xi \neq 0, \pi$ is an arbitrary parameter, and

$$\tau = b_1\sin\xi\,[(x - x_0) + 2\,(k_1 + b_1\cos\xi)\,t], \quad x_0 = \frac{1}{\beta}\sqrt{2\beta\tilde{C}}$$

[see Eqs. (51)–(54)]. For $N > 1$, the constructed solution is a nonlinear superposition of steps.

Vector models

Example 1. We shall construct solutions of the vector NSE with $U(n, 0)$ symmetry that decrease as $x \to \infty$. Suppose first that $n \leqslant N$. To obtain decreasing solutions, we must take the function $E(k)$ in the form $E(k) = k$. The Hermitian matrix (E_{ij}) will have the form

$$E_{ij} = (\bar{\varkappa}_i - \varkappa_j)\,C_{ij}, \quad i,\ j = \overline{1,\ N}.$$

This matrix must be non-negative definite of rank n. We represent it in the form $E = \Gamma^+\Gamma$, where Γ is a rectangular matrix of rank n, i.e.,

$$E_{ij} = \sum_{q=1}^{n} \bar{\gamma}_{qi}\gamma_{qj}. \tag{97}$$

We obtain the form of the matrix (C_{ij}):

$$C_{ij} = \frac{\sum_{q=1}^{n} \bar{\gamma}_{qi}\gamma_{qj}}{\bar{\varkappa}_i - \varkappa_j}, \quad i,\ j = \overline{1,\ N}. \tag{98}$$

It is readily seen that if the matrix $\Gamma = (\gamma_{ij})$ does not have vanishing columns, then the matrix (C_{ij}) of the form (98) will be positive definite if all the numbers $\varkappa_1, \ldots, \varkappa_N$ lie in the upper half-plane. But if in the matrix Γ there are vanishing columns, then in the matrix (C_{ij}) there will be vanishing columns (and rows), and this, as we know, corresponds to a decrease in the number of the parameters $\varkappa_1, \ldots, \varkappa_N$.

We finally conclude that if the numbers $\varkappa_1, \ldots, \varkappa_N$ lie in the upper half-plane and $\Gamma = (\gamma_{ij})$ is any rectangular $n \times N$ matrix, then functions $\Phi_1(x,t), \ldots, \Phi_n(x,t)$ of the form

$$\Phi_q(x,\ t) = \frac{\det M^{(q)}(x,\ t)}{\det M(x,\ t)}, \quad q = \overline{1,\ n}, \tag{99}$$

where the $N \times N$ matrix $M(x,t) = (M_{ij})$ has the form

$$M_{ij}(x,\ t) = \frac{\sum_{q=1}^{n} \bar{\gamma}_{qi}\gamma_{qj} + e^{i(\bar{\omega}_i - \omega_j)}}{\bar{\varkappa}_i - \varkappa_j}, \quad i,\ j = \overline{1,\ N}, \tag{100}$$

and the $(N+1) \times (N+1)$ matrices $M^{(q)}(x,t) = (M_{ij}^{(q)}(x,t))$ have the form

$$\left.\begin{aligned} &M_{ij}^{(q)} = M_{ij} \quad \text{for} \quad i,\ j = \overline{1,\ N}; \quad M_{00}^{(q)} = 0; \\ &M_{0j}^{(q)} = \gamma_{qj};\ M_{j0}^{(q)} = e^{i\bar{\omega}_j}, \quad j = \overline{1,\ N}, \end{aligned}\right\} \tag{101}$$

are solutions of the system of equations

$$i\dot{\Phi}_k = \Phi_k'' + 2\left(\sum_{q=1}^{n} |\Phi_q|^2\right)\Phi_k, \quad k = \overline{1,\ n}. \tag{102}$$

These solutions are exponentially damped for $|x| \to \infty$ and fixed t by virtue of the results of Sec. 2 [the matrix (C_{ij}) is here nondegenerate]. We shall describe the asymptotic behavior with respect to t below; it will be seen from this description that these solutions are a nonlinear superposition of N single-soliton solutions of the form

$$\Phi_{k,q}(x,\ t) = \Phi_{k,q}^{\pm}(\varkappa_q - \bar{\varkappa}_q)\,\frac{\exp\left[i\left(\alpha_q x + (\alpha_q^2 - \beta_q^2)\,t\right)\right]}{2\,\mathrm{ch}\,[\beta_q(x - x_0^{\pm}) + 2\alpha_q\beta_q t]}; \tag{103}$$

$$\varkappa_q = \alpha_q + i\beta_q, \quad q = \overline{1,\ N},$$

where $\Phi_{k,q}^{\pm}$ are certain constant vectors (different for $t \to +\infty$ and $t \to -\infty$) of length 1. [We recall that asymptotic decay into solitons and, hence, asymptotic behaviors of the form (103) are obtained only in the generic case in which the quantities $\mathrm{Im}\,\varkappa_j^2$ are pairwise different. The general N-soliton solution (in which the $\mathrm{Im}\,\varkappa_j^2$ may be equal) will be a conglomerate formed from solitons and bound states of them.]

So far we have constructed N-soliton solutions of Eq. (102) with $N \geqslant n$. For $N < n$, all N-soliton solutions of the n-component NSE can be obtained from the N-soliton solutions of the N-component NSE [with $U(N,0)$ symmetry] by multiplication by means of the action of the group $U(n)$.

Remark. By virtue of the definition of the N-soliton potential given in Sec. 2, an N-soliton solution of the vector NSE is specified by N poles $\varkappa_1, \ldots, \varkappa_N$. In particular, we shall, irrespective of the vector dimension, use the term "single-soliton solution" for one that is determined by a single pole $\varkappa = \varkappa_1$. It can always be obtained from a solution of the scalar NSE by means of an isorotation.

We note also that solutions of the form (99)–(101), which for given $\varkappa_1, \ldots, \varkappa_N$ corresponds to different matrices $\Gamma = (\gamma_{ij})$ with one and the same Hermitian form (E_{ij}) of the form (97), can be obtained from each other by the action of the unitary group $U(n)$.

The asymptotic behavior as $|t| \to \infty$ (x fixed) can be found from Eq. (61)–(66) by using the connection between the components $\Phi_1, \ldots, \Phi_n$ of the solution of the vector **NSE** and the residues $\Psi_1, \ldots, \Psi_N$ of the function $\Psi(x,t,k)$:

$$\Phi_k(x,\ t) = \sum_{j=1}^{N} \gamma_{kj}\Psi_j(x,\ t), \quad k = \overline{1,\ n}.$$

Suppose, as before, that the conditions (61) are satisfied, i.e., the N-th soliton is fixed while the others move from right to left). Then in the limit $t \to -\infty$ we find from (63) that the asymptotic behavior of the functions $\Phi_k(x,t)$ is

$$\Phi_k(x,\ t) \to \sum_{j=1}^{N} \frac{\gamma_{kj}C^{jN}(-2i\beta_N)\,e^{-i\beta_N^2 t}}{\sqrt{-2i\beta_N C^{NN}}\;2\,\mathrm{ch}\,\beta_N(x - x_0^-)}, \quad k = \overline{1,\ n},$$

$$\varkappa_N = i\beta_N.$$

The phase x_0^- has the form (64).

In the limit $t \to +\infty$, we obtain the asymptotic behavior

$$\Phi_k(x,\ t) \to \gamma_{kN}\left(\frac{|Z_N|}{-2i\beta_N C_{NN}}\right)^{1/2} i\beta_N\,\frac{e^{-i\beta_N^2 t + i\varphi_0^+}}{\mathrm{ch}\,\beta_N(x - x_0^+)},$$

$$k = \overline{1,\ n},$$

where Z_n, x_0^+, and φ_0^+ are given by Eq. (66). Thus, the phase shift $x_0^+ - x_0^-$ can be calculated in accordance with (64); the unit vectors $\Phi_{k,N}^{\pm}$ which occur in (103) have the form

$$\Phi_{k,N}^- = \sum_{j=1}^{N} \gamma_{kj}C^{jN}\,(-C^{NN}\cdot 2i\beta_N)^{-1/2};$$

$$\Phi_{k,N}^+ = \gamma_{k,N}\left(\frac{|Z_N|}{-2i\beta_N C_{NN}}\right)^{1/2} e^{i\varphi_0^+}.$$

Example 2. We construct solutions of the two-component NSE with oscillating asymptotic behavior. We analyze in detail only the two-soliton solutions.

Case 1. Both components oscillate as $|x| \to \infty$. The function $E(k)$ must be taken in the form

$$E(k) = k + \varepsilon_1\frac{b_1^2}{k - k_1} + \varepsilon_2\frac{b_2^2}{k - k_2}.$$

Here $\varepsilon_1, \varepsilon_2 = \pm 1$; these signs correspond to the symmetry type of the vector NSE. The Hermitian form (E_{ij}) of the form (80) must be zero, i.e., the following matching conditions must be satisfied:

$$E(\bar{\varkappa}_i) = E(\varkappa_j) \quad \text{for} \quad C_{ij} \neq 0, \quad i,\ j = \overline{1,\ N}. \tag{104}$$

If the conditions (104) on the set of parameters $(\varkappa_j)$, (C_{ij}) are satisfied for the given function $E(k)$, then the function $\Psi(x,t,k)$ constructed from these parameters in accordance with Eq. (36), (38), and (39) gives a solution $\Phi = (\Phi_1, \Phi_2)$ of the vector NSE of the form

$$i\dot{\Phi}_j = \Phi_j'' - 2\,[\varepsilon_1|\Phi_1|^2 + \varepsilon_2|\Phi_2|^2 - \varepsilon_1 b_1^2 - \varepsilon_2 b_2^2]\,\Phi_j \tag{105}$$

in accordance with the formulas

$$\Phi_j(x,\ t) = b_j\Psi(x,\ t,\ k_j), \quad j = 1,\ 2. \tag{106}$$

For $N = 1$, we obtain the single-soliton solution

$$\Phi_j(x,\ t) = b_j\left\{1 + \frac{1}{2}\,\frac{\varkappa - \bar{\varkappa}}{k_j - \varkappa}\,[1 + \mathrm{th}\,(\beta\,(x - x_0) + 2\alpha\beta t)]\right\} e^{ik_j(x+k_jt)},$$

$$j = 1,\ 2, \tag{107}$$

where the relationship between $\varkappa \equiv \varkappa_1 = \alpha + i\beta$ and the parameters k_1, k_2, b_1, b_2 is determined by the matching condition

$$E(\bar{\varkappa}) = E(\varkappa). \tag{108}$$

The signs ε_1 and ε_2 in (105) can take any values apart from $\varepsilon_1 = \varepsilon_2 = -1$ in this case, the relation (108) does not have solutions].

In the two-soliton case ($N = 2$) there are two types of matrix (C_{ij}) for which the conditions (104) have solutions. The first type is diagonal matrices (C_{ij}), i.e., $C_{12} = 0$; the second type is antidiagonal matrices, i.e., $C_{11} = C_{22} = 0$. For if $C_{11} \neq 0$ and $C_{12} \neq 0$, then we must have the conditions

$$E(\bar\varkappa_1)=E(\varkappa_1),\quad E(\bar\varkappa_1)=E(\varkappa_2).$$

The first of these shows that the number $r=E(\varkappa_1)$ is real. From this we conclude that the numbers $\varkappa_1$, $\bar\varkappa_1$, $\varkappa_2$ are the three roots of the cubic equation $E(k)=r$ with real coefficients. But this is impossible, since all three numbers $\varkappa_1,\bar\varkappa_1,\varkappa_2$ are nonreal (and distinct).

We consider in more detail both types of two-soliton solution.

Type 1. $C_{12}=0, C_{11}\neq 0, C_{22}\neq 0$. It can be assumed that $\operatorname{Im}\varkappa_1>0$, $\operatorname{Im}\varkappa_2>0$. The matching conditions have the form

$$E(\bar\varkappa_1)=E(\varkappa_1),\quad E(\bar\varkappa_2)=E(\varkappa_2).$$

For $\varepsilon_1=\varepsilon_2=-1$, these conditions cannot be solved. For other signs $(\varepsilon_1,\varepsilon_2)$, inequality restrictions arise. It can be shown that these restrictions can be described in terms of the position of the point of intersection with the real axis of the central perpendicular to the segment $[\varkappa_1,\varkappa_2]$

$$a=\frac{|\varkappa_2|^2-|\varkappa_1|^2}{2(\varkappa_2+\bar\varkappa_2-\varkappa_1-\bar\varkappa_1)},\qquad (109)$$

with respect to the segment $[k_1,k_2]$. Thus, for $U(0,2)$ symmetry, for which $\varepsilon_1=\varepsilon_2=1$, the quantity a of the form (109) must lie within the segment $[k_1,k_2]$, but for $U(1,1)$ symmetry (different signs ε_1, ε_2) outside the segment $[k_1,k_2]$ (included here is also the limiting case in which the segment $[\varkappa_1,\varkappa_2]$ is vertical).

The asymptotic behavior of these solutions for $|x|\to\infty$, and fixed t can be calculated as in Sec. 2. We have

$$\Phi(x,t)\to\begin{pmatrix}b_1e^{ik_1(x+k_1t)}\\ b_2e^{ik_2(x+k_2t)}\end{pmatrix},\quad x\to-\infty;\qquad (110)$$

$$\Phi(x,t)\to\begin{pmatrix}b_1\dfrac{(k_1-\bar\varkappa_1)(k_1-\bar\varkappa_2)}{(k_1-\varkappa_1)(k_1-\varkappa_2)}e^{ik_1(x+k_1t)}\\ b_2\dfrac{(k_2-\bar\varkappa_1)(k_2-\bar\varkappa_2)}{(k_2-\varkappa_1)(k_2-\varkappa_2)}e^{ik_2(x+k_2t)}\end{pmatrix},\quad x\to+\infty.\qquad (111)$$

The asymptotic behaviors of the constructed solutions for $|t|\to\infty$ with fixed x can be calculated similarly. Omitting the calculations, we give the form of the asymptotic behaviors for the case $\operatorname{Im}\varkappa_1^2>0$, $\operatorname{Im}\varkappa_2^2=0$. In the limit $t\to-\infty$,

$$\Phi_j(x,t)\to b_j\left\{1+\frac12\frac{\varkappa_2-\bar\varkappa_2}{k_j-\varkappa_2}[1+\operatorname{th}\beta_2(x-x_0^-)]\right\}e^{ik_j(x+k_jt)},$$

$j=1,2,$

where

$$\varkappa_2=i\beta_2,\quad \beta_2>0,\quad x_0^-=\frac{1}{\beta_2}\ln\sqrt{-2i\beta_2C_{22}}.$$

In the limit $t\to+\infty$, the asymptotic behavior is

$$\Phi_j(x,t)\to b_j\left\{1+\frac12\frac{\varkappa_2-\bar\varkappa_2}{k_j-\varkappa_2}[1+\operatorname{th}\beta_2(x-x_0^+)]\right\}e^{ik_j(x+k_jt)+i\eta_j},$$

$j=1,2,$

where

$$x_0^+-x_0^-=\frac{1}{\beta_2}\ln\left|\frac{\varkappa_2-\bar\varkappa_1}{\varkappa_2-\varkappa_1}\right|,\quad \eta_j=\arg\frac{k_j-\bar\varkappa_1}{k_j-\varkappa_1},\quad j=1,2.$$

The asymptotic behaviors as $|t|\to\infty$ on the straight lines $x=-2\alpha_1t+x_0$ have a similar form. We have obtained a nonlinear superposition of single-soliton solutions of the form (107).

Type 2. $C_{11}=C_{22}=0$, $C_{12}\neq 0$. It can be assumed that $\operatorname{Im}\varkappa_1>0$, $\operatorname{Im}\varkappa_2<0$. The matching conditions have the form

$$E(\bar\varkappa_1)=E(\varkappa_2).\qquad (112)$$

We shall discuss the solvability of these conditions (here, the signs ε_1, ε_2 can be arbitrary). Let

$$b=\frac{\bar\varkappa_2\varkappa_1-\varkappa_2\bar\varkappa_1}{(\varkappa_1-\bar\varkappa_1)-(\varkappa_2-\bar\varkappa_2)}$$

be the point of intersection of the segment $[\varkappa_1,\varkappa_2]$ with the real axis. Further, we set

$$d(k)=b-\frac{(\varkappa_1-\bar\varkappa_1)(\bar\varkappa_2-\varkappa_2)(\varkappa_1-\varkappa_2)(\bar\varkappa_2-\bar\varkappa_1)}{(b-k)(\varkappa_1+\bar\varkappa_2-\bar\varkappa_1-\varkappa_2)^2}.$$

If the conditions (112) are to have a solution, the points k_1, k_2, $\varkappa_1$, $\varkappa_2$ must not lie on one circle, i.e.,

$$k_2\neq d(k_1).$$

At the same time, the possibilities for the signs ϵ_1, ϵ_2 depend on k_1, k_2, $\varkappa_1$, $\varkappa_2$ as follows:

$$k_1<k_2<b\Rightarrow\varepsilon_1=-\varepsilon_2=1;$$
$$b<k_1<k_2\Rightarrow\varepsilon_1=-\varepsilon_2=-1;$$
$$k_2=b\Rightarrow b_1=0;$$
$$k_1=b\Rightarrow b_2=0;$$
$$k_1<b<k_2<d(k_1)\Rightarrow\varepsilon_1=\varepsilon_2=1;$$
$$k_1<b<d(k_1)<k_2\Rightarrow\varepsilon_1=\varepsilon_2=-1.$$

The asymptotic behaviors of the solutions $\Phi_j(x,t)$ for $|x|\to\infty$ with fixed t depend on the relationships between $\operatorname{Im}\varkappa_1$ and $\operatorname{Im}\varkappa_2$. Namely, for $\operatorname{Im}(\varkappa_1+\varkappa_2)>0$ the asymptotic behaviors have the form (110) and (111). For $\operatorname{Im}(\varkappa_1+\varkappa_2)=0$, the solution $\Phi(x,t)$ is quasiperiodic with respect to x. Finally, for $\operatorname{Im}(\varkappa_1+\varkappa_2)<0$ the asymptotic behaviors as $x\to\pm\infty$ in Eqs. (110) and (111) are interchanged.

The asymptotic behaviors for $|t|\to\infty$ at fixed x can be calculated very easily. Under the condition $\operatorname{Im}\varkappa_1^2>0$, $\operatorname{Im}\varkappa_2^2=0$ we shall have:

As $t\to-\infty$

$$\Phi_j(x,t)\to b_je^{ik_j(x+k_jt)},\quad j=1,2;$$

as $t\to+\infty$

$$\Phi_j(x,t)\to b_j\frac{(k_j-\bar\varkappa_1)(k_j-\bar\varkappa_2)}{(k_j-\varkappa_1)(k_j-\varkappa_2)}e^{ik_j(x+k_jt)},\quad j=1,2.$$

Thus, for solutions of this type the asymptotic behavior is purely exponential. These solutions do not reduce to a superposition of single-soliton solutions, and it is therefore natural to call them double solitons.

We note that for arbitrary N solutions of the form (106) of Eq. (105) reduce to a nonlinear superposition of solitons and double solitons. For the $U(2,0)$ case, triple solitons are also added. A triple soliton arises for $N=3$, when the matrix (C_{ij}) has the form

$$(C_{ij})=\begin{pmatrix}0 & C_{12} & C_{13}\\ C_{21} & 0 & 0\\ C_{31} & 0 & 0\end{pmatrix},$$

and the matching conditions are

$$E(\bar\varkappa_1)=E(\varkappa_2)=E(\varkappa_3)$$

$(\varepsilon_1=\varepsilon_2=-1)$, and the points $\varkappa_2$, $\varkappa_3$ lie in the upper half-plane and $\varkappa_1$ in the lower. We omit the proof of these assertions.

Case 2. The component $\Phi_1(x,t)$ oscillates as $x\to\infty$, while the component $\Phi_2(x,t)$ is damped as $x\to\infty$. We take the function $E(k)$ in the form

$$E(k)=k+\varepsilon_1\frac{b_1^2}{k-k_1}.$$

The simplest solution of this form—a single-soliton solution—arises in the framework of our construction for $N=1$. It is specified by the parameters $\varkappa\equiv\varkappa_1=\alpha+i\beta$ (let $\beta>0$) and $C_{11}=i\tilde C_{11}$, $\tilde C_{11}>0$, and has the form (obtained in Refs. 2 and 31)

$$\Phi_1(x,t)=b_1\left\{1+\frac12\frac{\varkappa-\bar\varkappa}{k_1-\varkappa}[1+\operatorname{th}\beta(x-x_0+2\alpha t)]\right\}e^{ik_1(x+k_1t)};$$

$$\Phi_2(x,t)=\frac{(\varkappa-\bar\varkappa)(|\varkappa-k_1|^2-\varepsilon_1b_1^2)^{1/2}}{|\varkappa-k_1|}\,\frac{\exp i(\alpha x+(\alpha^2-\beta^2)t)}{2\operatorname{ch}\beta(x-x_0+2\alpha t)}.\qquad (113)$$

Here

$$x_0=\frac{1}{\beta}\ln\sqrt{-2i\beta\tilde C_{11}}.\qquad (114)$$

The vector function $\Phi=(\Phi_1,\Phi_2)$ of the form (113) is a solution of the equation

$$i\dot\Phi_j=\Phi_j''-2[\varepsilon_1|\Phi_1|^2+\varepsilon_2|\Phi_2|^2-\varepsilon_1b_1^2]\Phi_j,\quad j=1,2.\qquad (115)$$

Here, the sign ε_2 is determined subject to the condition $|\varkappa-k_1|^2\neq\varepsilon_1b_1^2$ and has the form

$$\varepsilon_2=\operatorname{sgn}[\varepsilon_1b_1^2-|\varkappa-k_1|^2].$$

Thus, for $\varepsilon_1=-1$ we also have $\varepsilon_2=-1$ and (113) is a single-soliton solution of the vector NSE with $U(2,0)$ symmetry. For

$$\varepsilon_1=1,\quad|\varkappa-k_1|>b_1$$

we have $\varepsilon_2=-1$; we obtain a solution of the NSE with $U(1,1)$ symmetry. For

$$\varepsilon_1=1,\quad|\varkappa-k_1|<b_1$$

we obtain a solution the NSE with $U(0,2)$ symmetry.

If the relation

$$|\varkappa-k_1|^2=\varepsilon_1b_1^2\Leftrightarrow E(\bar\varkappa)=E(\varkappa)$$

holds (this is possible only for $\varepsilon_1=1$), the component Φ_2 vanishes identically, and the solution (113) reduces to the single-soliton solution (114) of the NSE with repulsion.

We shall show that for $\varepsilon_1=1$ the many-soliton solutions reduce to a nonlinear superposition of solitons. We must have the following conditions: The Hermitian form (E_{ij}) of the form (80) must have rank 1. We assume first that there are no matching conditions on the points $\varkappa_1,\ldots,\varkappa_N$, i.e.,

$$E(\bar\varkappa_i)\neq E(\varkappa_j),\quad i,j=\overline{1,N}.$$

Then the matrix (C_{ij}) which determines the solution must have the form

$$C_{ij}=\lambda\frac{\gamma_i\bar\gamma_j}{E(\bar\varkappa_i)-E(\varkappa_j)},\quad i,j=\overline{1,N}.\qquad (116)$$

Here, $\gamma_1,\ldots,\gamma_N$ are arbitrary complex constants which satisfy

$$\sum_i^N|\gamma_i|^2=1,\qquad (117)$$

and λ is a real number. Assuming that $\gamma_1,\ldots,\gamma_N\neq 0$ (see Example 1 above), we obtain nondegeneracy of the matrix (C_{ij}). It may therefore be assumed that all the points $\varkappa_1,\ldots,\varkappa_N$ lie in the upper half-plane. It follows from the condition of non-negative definiteness of the matrix $(i^{-1}C_{ij})$ that the points $\varkappa_1,\ldots,\varkappa_N$ and the number λ must satisfy one of the following conditions:

$$\left.\begin{matrix}1)\ \lambda>0,\quad \operatorname{Im}E(\varkappa_i)>0,\quad i=\overline{1,N},\\ \Updownarrow\\ |\varkappa_i-k_1|>b_1;\end{matrix}\right\}\qquad (118)$$

$$\left.\begin{matrix}2)\ \lambda<0,\quad \operatorname{Im}E(\varkappa_i)<0,\quad i=\overline{1,N},\\ \Updownarrow\\ |\varkappa_i-k_1|<b_1.\end{matrix}\right\}\qquad (119)$$

The first possibility corresponds to the $U(1,1)$ NSE, the second to the $U(0,2)$ NSE. Thus, determining from the points $\varkappa_1,\ldots,\varkappa_N$ which satisfy (118) or (119) (and lie in the upper half-plane), and the matrix (C_{ij}) of the form (116) [in these formulas, $E(k)=k+b_1^2(k-k_1)^{-1}$] the function $\Psi(x,t,k)$ by means of Eq. (36), (38), and (39), we obtain solutions of the vector NSE with $U(1,1)$ or $U(0,2)$, respectively, setting

$$\left.\begin{aligned}&\Phi_1(x,t)=b_1\Psi(x,t,k_1);\\ &\Phi_2(x,t)=\sqrt{|\lambda|}\sum_{i=1}^N\gamma_i\operatorname*{res}_{\varkappa_i}\Psi(x,t,k).\end{aligned}\right\}\qquad (120)$$

We shall now show that these solutions do indeed describe a nonlinear superposition of single-soliton solutions of the form (113). But first we consider the matching conditions. What happens if for certain i,j the condition $E(\bar\varkappa_i)=E(\varkappa_j)$ is satisfied? In this case, the i-th and j-th rows (and columns) of the matrix (E_{ij}) must be zero. Therefore, in the i-th row of the matrix (C_{ij}) only the element C_{ij} can be nonzero. But the numbers $\varkappa_i$, $\varkappa_j$ satisfying the matching condition lie in one half-plane. Therefore, the corresponding block of the matrix (C_{ij}) can remain sign definite only if $j=i$, i.e., the matching condition has the form

$$E(\bar\varkappa_i)=E(\varkappa_i)\Leftrightarrow|\varkappa_i-k_1|=b_1.$$

Thus, the general form of the matrix (C_{ij}) determining the solution of the described type of two-component NSE with $U(1,1)$ or $U(0,2)$ symmetry is

$$C_{ij}=\begin{cases}\dfrac{\lambda\gamma_i\bar\gamma_j}{E(\bar\varkappa_i)-E(\varkappa_j)} & \text{for}\quad\lambda(|\varkappa_i-k_1|-b_1)>0;\\ C_{ii}\delta_{ij}, & |\varkappa_i-k_1|=b_1.\end{cases}$$

We recall that for $\lambda>0$ we obtain $U(1,1)$ symmetry, while for $\lambda<0$ we have $U(0,2)$ symmetry. At the same time, the formulas (120) for the solution remain valid. The constants $\gamma_1, \ldots, \gamma_N$ satisfy (117).

We now turn to the calculation of the asymptotic behaviors. Here, all points $\{\varkappa_i\}$ lie in the upper half-plane and the matrix (C_{ij}) is nondegenerate. We can therefore use the asymptotic formulas of Sec. 2. Suppose that t is fixed. Then in the limit $x \to -\infty$

$$\Phi(x, t) \to \begin{pmatrix} b_1 e^{ik_1(x+k_1 t)} \\ 0 \end{pmatrix}.$$

In the limit $x \to +\infty$,

$$\Phi(x, t) \to \begin{pmatrix} b_1 \prod_i \frac{k_1 - \bar{\varkappa}_i}{k_1 - \varkappa_i} e^{ik_1(x+k_1 t)} \\ 0 \end{pmatrix}.$$

We calculate the asymptotic behaviors for $|t| \to \infty$ and fixed x in the same way as above under the assumption $\operatorname{Im} \varkappa_i^2 > 0$ for $i = 1, \ldots, N-1$, $\operatorname{Im} \varkappa_N^2 = 0$. Using the formulas of Sec. 2, we have in the limit $t \to -\infty$

$$\Phi_1(x, t) \to b_1 \left\{1 + \frac{1}{2} \frac{\bar{\varkappa}_N - \varkappa_N}{k_1 - \varkappa_N} [1 + \operatorname{th} \beta_N (x - x_0^-)]\right\} e^{ik_1(x+k_1 t)};$$

$$\Phi_2(x, t) \to \frac{-i\beta_N}{|\varkappa_N - k_1|} (||\varkappa_N - k_1|^2 - b_1^2|)^{1/2} \frac{e^{i(-\beta_N^2 t + \varphi_2^-)}}{\operatorname{ch} \beta_N (x - x_0^-)}.$$

Here, x_0^- has the form (64), and

$$\varphi_2^- = \arg \gamma_N + \arg \prod_{j \neq N} \frac{E_N - \bar{E}_j}{E_N - E_j},$$

where we have introduced the notation

$$E_j = E(\varkappa_j) = \varkappa_j + \frac{b_1^2}{\varkappa_j - k_1}.$$

As $t \to +\infty$, the asymptotic behavior is

$$\Phi_1(x, t) \to b_1 \left\{1 + \frac{1}{2} \frac{\bar{\varkappa}_N - \varkappa_N}{k_1 - \varkappa_N} [1 + \operatorname{th} \beta_N (x - x_0^+)]\right\} e^{ik_1(x+k_1 t) + i\varphi_1^+};$$

$$\Phi_2(x, t) \to \frac{-i\beta_N}{|\varkappa_N - k_1|} (||\varkappa_N - k_1|^2 - b_1^2|)^{1/2} \frac{e^{i(-\beta_N^2 t + \varphi_2^+)}}{\operatorname{ch} \beta_N (x - x_0^+)}$$

The phase x_0^+ has the form (66) and

$$\varphi_1^+ = \arg \prod_{j \neq N} \frac{k_1 - \bar{\varkappa}_j}{k_1 - \varkappa_j},$$

$$\varphi_2^+ = \arg \gamma_N + \arg \prod_{j \neq N} \frac{\varkappa_N - \bar{\varkappa}_j}{\varkappa_N - \varkappa_j}.$$

We have obtained single-soliton asymptotic behavior. A simple calculation shows that the interaction between solitons reduces to a binary interaction. This is a consequence of the following expressions for the phase shifts of the N-th soliton:

$$\Delta x_0 = x_0^+ - x_0^- = \sum_{j \neq N} \frac{1}{\beta_{Nj}} \ln \left\{ \left| \frac{E_N - \bar{E}_j}{E_N - E_j} \right| \left| \frac{\varkappa_N - \bar{\varkappa}_j}{\varkappa_N - \varkappa_j} \right| \right\};$$

$$\Delta\varphi_1 = \varphi_1^+ - \varphi_1^- = \sum_{j \neq N} \arg \frac{k_1 - \bar{\varkappa}_j}{k_1 - \varkappa_j};$$

$$\Delta\varphi_2 = \varphi_2^+ - \varphi_2^- = \sum_{j \neq N} \arg \frac{(\varkappa_N - \bar{\varkappa}_j)(E_N - \bar{E}_j)}{(\varkappa_N - \varkappa_j)(E_N - E_j)}.$$

In conclusion, we note that in the case of $U(2,0)$ symmetry, for which $\varepsilon_1 = -1$, the many-soliton solutions are a nonlinear superposition of single-soliton solutions, and also double solitons. The simplest double soliton corresponds to the case $N = 2$, the points $\varkappa_1, \varkappa_2$ satisfy the matching condition

$$\bar{\varkappa}_1 - \frac{b_1^2}{\bar{\varkappa}_1 - k_1} = \varkappa_2 - \frac{b_1^2}{\varkappa_2 - k_1},$$

and the matrix (C_{ij}) has the form

$$C_{ij} = \begin{pmatrix} 0 & C_{12} \\ C_{21} & 0 \end{pmatrix}.$$

We shall not analyze the properties of such solutions here.

CONCLUSIONS

We have presented above the present status of problems that arise from the study of a class of models which we have called models of a nonideal Bose gas. From the point of view of the theory of the condensed state the important question is that of the existence in some ordered system (crystal, magnet, etc.) of localized excitations of soliton (or solitonlike) type. For the understanding of the statistical properties of such excitations (if they exist), it is necessary to solve the problem of the stability of individual solitonlike objects, and also their interaction with each other. In the framework of the models considered here and related to the time-dependent Schrödinger equation some of these problems have been solved constructively. Namely, we have used the general method developed in Sec. 2 to obtain and investigate the asymptotic behaviors of the many-soliton solutions of some integrable versions of the NSE with self-consistent potentials. Such many-soliton solutions describe fairly well a rarefied gas of solitons. Moreover, depending on the effect of the interaction, one can speak of ideal, weakly nonideal gases, etc.

We discuss first the formulas obtained in Sec. 3 from the point of view of stability. It is known that in the framework of compact versions of the vector NSE with attraction, $U(p,0)$, solutions of plane-wave type (condensate solutions) and also those obtained from them by a local modification are unstable (instability of gravitational type). The condensate solutions obtained in the framework of compact versions of the vector NSE with repulsion, $U(0,q)$, are stable.[2,26] The stability of the localized solutions with zero-value boundary conditions for $U(p,0)$ and condensate conditions for $U(0,q)$ has been rigorously established only for some of the simplest (single-soliton) solutions.[26,27] The question of the stability of arbitrary N-soliton solutions of the $U(p,0)$ NSE is still open and on the answer must be expected to depend both on the type of equation and on the type of solution. At the least, it appears that the single-soliton droplet solutions of the $U(p,0)$ vector NSE are stable, as is indicated by qualitative arguments based on the inverse scattering method (see also the generalization of the Q theorem in Ref. 27).

For models with noncompact symmetry $U(p,q)$ stability of the condensate is ensured by the condition[2]

$$(\Psi_c, \Psi_c) = \sum_{j=1}^{p} |\Psi_c^{(j)}|^2 - \sum_{j=1}^{q} |\Psi_c^{(j)}|^2 > 0.$$

Only when this condition is satisfied do the many-soliton formulas have meaning in the condensate formulation of the problem. The stability of the two-soliton solutions (in our definition or single-soliton solutions in the naive definition) in the framework of the simplest noncompact $U(1,1)$ NSE was investigated at the Laboratory for Computation Techniques and Automation at the Joint Institute for Nuclear Research at Dubna by means of a numerical experiment. The results favor stability of such solitons. The many-soliton asymptotic behaviors obtained above enable us to assert that in the framework of the compact models with arbitrary signature [$U(p,0)$ or $U(0,q)$; in the first case see also Ref. 28], and also for the $U(1,1)$ NSE, the interaction between the solitons reduces to a binary elastic interaction, this interaction leading to a change in the phases of the solitons in the configuration and color spaces. There can also be color exchange, a result that was first established in Ref. 1.

All this means that a gas of solitonlike excitations can, depending on the physical formulation of the problem (i.e., what quantities are taken into account, what correlation functions are calculated, etc.), be regarded within the framework of one and the same model as ideal (soliton number density appreciably less than unity) or nonideal, if one is interested, for example, in its color. In the physical situations in which the soliton gas can be regarded to sufficient accuracy as ideal it is sensible to use the phenomenological approach of Ref. 13 to calculate, for example, the dynamical structure factors of scattering[14] in vector models, with, moreover, any signature of the metric of the "color" space when $N \geqslant 2$. In this sense, the method which we have proposed for investigating the vector equations and their solutions can be regarded as a tool for the further study of corresponding models of, for example, the physics of the condensed state (see Sec. 1).

[1)] *Translator's Note.* The Russian use of $i = \sqrt{1,N}$ to denote $i = 1, \ldots, N$ is retained in the displayed equations in this paper.

[2)] *Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic functions, etc., is retained here and throughout the article in the displayed equations.

[3)] In what follows, we shall also find it convenient to use this simple proposition: If all the numbers $\varkappa_1, \ldots, \varkappa_N, \bar{\varkappa}_1, \ldots, \bar{\varkappa}_N$ are different, and $\operatorname{Im} \varkappa_i > 0$, $i = \sqrt{1,p}$, $\operatorname{Im} \varkappa_j < 0$, $j = p+1, \ldots, N$, then the Hermitian matrix $[i(\bar{\varkappa}_i - \varkappa_j)]^{-1}$ has signature $(p, N-p)$.

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Translated by Julian B. Barbour