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Differential Geometry of Moduli Spaces and its Applications to Soliton Equations and to Topological Conformal Field Theory

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Abstract: We construct flat Riemann metrics on moduli spaces of algebraic curves with marked meromorphic function. This gives a new class of exact algebraic-geometry solutions of some non-linear equations in terms of functions n moduli spaces. We show that the Riemann metrics on moduli spaces coincide with two-point correlators in topological conformal field theory and calculate the partition function for An-model for arbitrary genus. A universal method for constructing complete families of conservation laws for Whitham-type hierarchies of PDE also is proponed.

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Introduction

The recent progress in the study of matrix models[1] of QFT revealed a remarkable connection with hierarchies of integrable equations of the KdV-type. It was shown also[2-4] that so-called topological conformal field theories (TCFT) are very important in the study of the low-dimensional string theories and of the matrix models (the general notion of topological field theory was introduced by E. Witten[5]).

The Landau-Ginsburg potentials machinary[6,7] (see below sect.4) in TCFT was analyzed from different points of view. The relation of it with the singularity theory was investigated in refs.[6,7,8] (see also ref.[9]). Very recently Krichever[10] has observed the relation of this machinery with the so-called averaged KdV-type hierarchy[11-15] (or *Whitham-type hierarchy*). He showed that the target space for this Whitham-type hierarchy coincides with the coupling space of zero genus TCFT and the dependence of the Landau-Ginsburg potential on the coupling constants is determined via solving the equation of this hierarchy (in fact, a very particular solution proved to be involved).

Our main observation is that the flat metric on the target space of Whitham-type hierarchy being involved in the Hamiltonian description of it (see refs.[11,12,13,15]) coincides with the two-point correlation of the corresponding TCFT. Starting from this point we have found a very general construction of flat Riemann metrics on moduli spaces M of algebraic curves of given genus with marked meromorphic function. This function in TCFT plays the role of Landau-Ginsburg potential (we consider only the A_{n-1}-theories) and the relevant moduli space M being the coupling space. It turns out that the equations of flatness of these Riemann metrics coincide with well-known in the soliton theory N-wave interaction system. We obtain therefore a new class of exact solutions of the N-wave system in terms of some special functions on moduli spaces M (the simplest solution of this class has been found in ref.[16]). Some global properties of moduli spaces of the type being described above also follow from our condiderations. We construct also the general class of Whitham-type hierarchies of dynamical systems in the loop spaces $\mathcal{L}M$. We describe the bi-Hamiltonian structure and recurrence operator for this hierarchy and construct explicity the complete family of conservation laws. As a result of these considerations the explicit formula for the non-zero genus TCFT partition function is obtained.
In the appendix we discuss the relation of TCFT to the theory of Frobenius algebras.

1. Orthogonal systems of curvilinear co-ordinates, integrable equations and Hamiltonian formalism.

We start with some information on the geometry of curvilinear orthogonal co-ordinate systems.

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