TOPOLOGICAL CONFORMAL FIELD THEORY FROM THE POINT OF VIEW OF INTEGRABLE SYSTEMS

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ABSTRACT

Recent results on classification of massive topological conformal field theories (TCFT) in terms of monodromy data of auxillary linear operators with rational coefficients are presented. Procedure of coupling of a TCFT to topological gravity is described (at tree level) via certain integrable bihamiltonian hierarchies of hydrodynamic type and their τ-functions. It is explained how the calculation of the ground state metric on TCFT can be interpreted in terms of the theory of harmonic maps. Also a construction of some TCFT models via Coxeter groups is described.

Introduction

It is known, due to Kontsevich - Witten [24, 25, 31] that 2D topological gravity (coinciding with the intersection theory on the moduli space of stable algebraic curves) is described by the KdV hierarchy. For the examples of 2D topological field theories (TFT's) related to intersection theory on certain coverings over the moduli space it was conjectured descriptions of the models in terms of certain integrable hierarchies [30, 39, 40]. There are many other examples of 2D TFT's (e.g., topological sigma-models for any Käler manifold as the target space and Landau - Ginsburg topological models [1-3, 28-31, 36]), and it is unknown if it is possible to describe these theories by appropriate integrable hierarchies.

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(At least, an "experimental fact" is that the number of known integrable hierarchies is much less than the number of known 2D TFT's.) My aim is to try to construct these unknown integrable hierarchies.

I start with classification of massive 2D topological conformal field theories (TCFT's). The classification is based on the following system of nonlinear PDE for the unknown function $F(t)$, $t = (t^1, ..., t^n)$,

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\sigma} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\gamma \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\beta \partial t^\sigma} \tag{0.1}$$

with the constraint

$$\frac{\partial^3 F(t)}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}. \tag{0.2}$$

All the Greek indices run from 1 up to n, $\eta_{\alpha\beta}$ is a nondegenerate symmetric matrix, the matrix $\eta^{\alpha\beta}$ is its inverse. These equations were called in [21] *Witten - Dijkgraaf - E. Verlinde - H. Verlinde* (WDVV) equations. In fact in TCFT the function $F(t)$ should be a quasihomogeneous one of a degree $3 - d$ where degrees of the variables t^α equal $1 - q_\alpha$, $q_1 = 0$. In TCFT the function $F(t)$ is the primary free energy as a function on the coupling constants (moduli of the given TCFT model, see [4, 20]). The numbers d and q_α coincide resp. with the dimension of the model and with the charges of the primary fields.

My program is:

1. To classify 2D TCFT as quasihomogeneous solutions of WDVV equations, and
2. For any solution of WDVV (I recall that this describes the matter sector of a TCFT model) to construct (i.e., to calculate the partition function) a complete TCFT model (coupling of the given matter sector to topological gravity).

It turns out that TCFT models are parametrized by the monodromy data (Stokes matrices) of a certain linear differential operator with rational coefficients. Correlators of the primary fields of the TCFT with given Stokes matrix proves out to be high-order analogues of the Painlevé-VI transcendents being expressed via isomonodromy deformations of the linear operator. For any solution of WDVV an integrable hierarchy is constructed such that the tau-function of a particular solution of the hierarchy coincides with the tree-level partition function of the theory. The hierarchy proves out to carry a bi-hamiltonian structure (under certain nonresonancy conditions for the charges and the dimension of the theory). In Section 4 I discuss integrability of the tt^* equations [23] for the ground state metric of the TCFT and their relations to the theory of harmonic maps. In the last section the problem of selection solutions of WDVV is discussed.

1. Geometry of coupling space of a TCFT: Frobenius manifolds

I recall that A is called a Frobenius algebra (over **R** or **C**) if it is a commutative associative algebra with a unity and with a nondegenerate invariant symmetric inner product

$$< ab, c >=< a, bc > . \tag{1.1}$$

If e_i, $i = 1, \ldots, n$ is a basis in A then the structure of Frobenius algebra is specified by the coefficients η_{ij}, c_{ij}^k where

$$< e_i, e_j >= \eta_{ij} \tag{1.2a}$$

$$e_i e_j = c_{ij}^k e_k \tag{1.2b}$$

(summation over repeated indices will be assumed). The matrix η_{ij} and the structure constants c_{ij}^k satisfy the following conditions:

$$\eta_{ji} = \eta_{ij}, \quad \det(\eta_{ij}) \neq 0 \tag{1.3a}$$

$$c_{ij}^s c_{sk}^l = c_{is}^l c_{jk}^s \tag{1.3b}$$

(associativity),

$$c_{ijk} = \eta_{is}c^s_{jk} = c_{jik} = c_{ikj} \tag{1.3c}$$

(commutativity and invariance of the inner product). If $e = (e^i)$ is the unity of A then

$$e^s c^i_{sj} = \delta^i_j \tag{1.3d}$$

(the Kronecker delta).

1-dimensional Frobenius algebras are parametrized by 1 number (length of the unity). Any semisimple n-dimensional Frobenius algebra is isomorphic to the direct sum of n one-dimensional Frobenius algebras

$$f_i f_j = \delta_{ij} f_i, \quad < f_i, f_j >= \eta_{ii}\delta_{ij}. \tag{1.4}$$

Moreover, any Frobenius algebra without nilpotents is a semisimple one.

Definition 1.1. A manifold M is called ***Frobenius*** if it is equipped with three tensors $c = (c^k_{ij}(x))$, $\eta = (\eta_{ij}(x))$, $e = (e^i(x))$ satisfying (1.3) for any $x \in M$. We need also the invariant metric

$$ds^2 = \eta_{ij}(x)dx^i dx^j \tag{1.5a}$$

to be flat, the unity vector field e to be covariantly constant

$$\nabla e = 0 \tag{1.5b}$$

(here ∇ is the Levi-Civitá connection for ds^2) and the tensor

$$\nabla_z < u \cdot v, w > \tag{1.5c}$$

to be symmetric in the vectors u, ..., z.

The three tensors provide a structure of Frobenius algebra in the space of smooth vector fields $Vect(M)$ over the ring $\mathcal{F}(M)$ of smooth functions on M:

$$[v \cdot w]^k(x) = c^k_{ij}(x)v^i(x)w^j(x), \quad v \cdot e = v, \tag{1.6a}$$

$$< v, w > (x) = \eta_{ij}(x)v^i(x)w^j(x) \tag{1.6b}$$

for any $v,\ w \in Vect(M)$.

Informaly speaking, n-dimensional Frobenius manifolds are n-parameter deformations of n-dimensional Frobenius algebras. For any $x \in M$ the tangent space $T_x M$ is a Frobenius algebra with the structure constants $c^k_{ij}(x)$, invariant inner product $\eta_{ij}(x)$, and unity $e^i(x)$.

Localy Frobenius manifolds are in 1-1 correspondence with solutions of WDVV equations (i.e., with 2D TFTs). Indeed, for the flat metric (1.5a) localy flat coordinates t^α exist such that the metric is constant in these coordinates, $ds^2 = \eta_{\alpha\beta}dt^\alpha dt^\beta$, $\eta_{\alpha\beta} =$ const. The covariantly constant vector field e in the flat coordinates has constant components; using a linear change of the coordinates one can obtain $e^\alpha = \delta^\alpha_1$. The tensor $c_{\alpha\beta\gamma}(t)$ in these coordinates satisfies the condition

$$\partial_\delta c_{\alpha\beta\gamma} = \partial_\gamma c_{\alpha\beta\delta}. \tag{1.7a}$$

This means that $c_{\alpha\beta\gamma}(t)$ can be represented in the form

$$c_{\alpha\beta\gamma}(t) = \partial_\alpha\partial_\beta\partial_\gamma F(t) \tag{1.7b}$$

for some function $F(t)$ satisfying the WDVV equations.

The first step in solving WDVV is to obtain a "Lax pair" for these equations. The most convenient way is to represent them as the compatibility conditions of an overdetermined linear system depending on a spectral parameter λ.

Proposition 1.1. *The condition of symmetry of (1.5c) in the definition of Frobenius manifold holds iff the pencil of connections*

$$\tilde{\nabla}_u(\lambda)v = \nabla_u v + \lambda u \cdot v, \quad u,\ v \in Vect(M) \tag{1.8}$$

is flat identicaly in λ.

Indeed, WDVV is equivalent to compatibility of the following linear system

$$\tilde{\nabla}_\alpha(\lambda)\xi = 0, \quad \alpha = 1, ..., n, \tag{1.9a}$$

(here ξ is a covector field), or, equivalently, in the flat coordinates t^α

$$\partial_\alpha \xi_\beta = \lambda c^\gamma_{\alpha\beta}(t)\xi_\gamma. \tag{1.9b}$$

Compatibility of the system (1.9) (identicaly in the spectral parameter λ) together with the symmetry of the tensor $c_{\alpha\beta\gamma} = \eta_{\alpha\epsilon}c^\epsilon_{\beta\gamma}$ is equivalent to WDVV.

A suitable version of inverse spectral transform for the integrable system can be developed for the important class of *massive* Frobenius manifolds.

Definition 1.2. A Frobenius manifold is called massive if the algebra on T_xM is semisimple for any $x \in M$.

In physical language massive Frobenius manifolds are coupling spaces of massive TFT models.

Local classification of massive Frobenius manifolds in terms of inverse spectral transform was obtained in [21]. The crucial point in this classification is in constructing *canonical coordinates* on a massive Frobenius manifold.

Definition 1.3. Local coordinates $u^1(t)$, ..., $u^n(t)$ on a Frobenius manifold are called canonical if the structure tensor c in these coordinates has the constant form

$$c^k_{ij} = \delta_{ij}\delta^k_j. \tag{1.10}$$

It was proved in [21] that local canonical coordinates exist on any massive Frobenius manifold.

Here we will consider in more details the TCFT case. In this case there is a vector field v on the Frobenius manifold

$$v = \sum(1 - q_\alpha)t^\alpha\partial_\alpha \tag{1.11}$$

generating conformal transformations of the tensors c, η, e

$$\mathcal{L}_v c = c \tag{1.12a}$$

$$\mathcal{L}_v e = -e \tag{1.12b}$$

$$\mathcal{L}_v \eta = (2 - d)\eta. \tag{1.12c}$$

Here $\mathcal{L}_v$ means the Lie derivative along the vector field v. I will assume $q_\alpha \neq 1$ for all $\alpha = 1, ..., n$.

I will call M in this case *conformal invariant* Frobenius manifold. On a massive conformal invariant Frobenius manifold the canonical coordinates can be found explicitly (i.e. without quadratures).

Proposition 1.2. *The canonical coordinates $u^1, \dots, u^n$ on a massive conformal invariant Frobenius manifold coincide with eigenvalues of the matrix*

$$\tilde{U} = (\tilde{U}^\gamma_\beta(t)) = ((1 + q_\beta - q_\gamma) F^\gamma_\beta(t)) \tag{1.13a}$$

$$F^\gamma_\beta(t) = \eta^{\gamma\epsilon} \partial_\beta \partial_\epsilon F(t). \tag{1.13b}$$

This is the matrix of multiplication by the vector field v.

To complete local classification of massive conformal invariant Frobenius manifolds let us consider the following linear ordinary differential operator with rational coefficients

$$\Lambda = \partial_\lambda - U + \frac{V}{\lambda} \tag{1.14a}$$

$$U = \mathrm{diag}(u^1, \dots, u^n). \tag{1.14b}$$

$$V = -V^{\mathrm{T}} = (v_{ij}). \tag{1.14c}$$

The matrices U and V do not depend on λ. Solutions of the differential equation

$$\Lambda \psi(\lambda) = 0 \tag{1.15}$$

are multivalued functions in the complex domain. The equation has regular singularity at $\lambda = 0$, so

$$\psi \simeq \lambda^{-V}.$$

The infinite point of the λ-plane is an irregular singularity. There exist solutions ψ_1, ψ_2, ψ_3 of (1.15) with the asymptotics

$$\psi_k \simeq \exp \lambda U, \;\; k = 1, 2, 3$$

defined in certain sectors of the λ-plane near the infinity. These matrix solutions of (1.15) differ by constant matrix factors

$$\psi_{k+1} = \psi_k S_k$$

called *Stokes matrices.* There are $n(n-1)/2$ independent parameters in the Stokes matrices. They determine also the monodromy in the origin (see, e.g., [21]).

Theorem 1.1. *Let $\Lambda(u)$ be a family of operators of the form (1.14), $u = (u^1, ..., u^n)$, $V = V(u) = (v_{ij}(u))$ with the same monodromy (independent on u). Then the following procedure gives a massive conformal invariant Frobenius manifold: let $\psi_{i\alpha}(u)$, $\alpha = 1, ..., n$ be a basis of solutions of the linear system*

$$\partial_j \psi_{i\alpha} = \frac{v_{ij}}{u^i - u^j} \psi_{j\alpha}, \quad i \neq j, \tag{1.16a}$$

$$\sum_k \partial_k \psi_{i\alpha} = 0, \tag{1.16b}$$

$\alpha = 1, ..., n$. They can be chosen in such a way that

$$V \psi_\alpha = \mu_\alpha \psi_\alpha$$

where μ_1, ..., μ_n are the eigenvalues of V,

$$\mu_\alpha + \mu_{n-\alpha+1} = 0.$$

Putting

$$\eta_{\alpha\beta} = \sum_{i=1}^{n} \psi_{i\alpha}(u)\psi_{i\beta}(u), \tag{1.17b}$$

$$\frac{\partial t_\alpha}{\partial u^i} = \psi_{i1}(u)\psi_{i\alpha}(u), \tag{1.17c}$$

$$c_{\alpha\beta\gamma}(t(u)) = \sum_{i=1}^{n} \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}} \tag{1.17d}$$

we obtain a massive conformal invariant Frobenius manifold with the charges

$$q_\alpha = \mu_\alpha - \mu_1$$

and the dimension

$$d = -2\mu_1.$$

Any massive conformal invariant Frobenius manifold with $q_\alpha \neq 1$ *localy can be obtained by this procedure.*

The equations of isomonodromy deformations of the operator Λ have the form

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}, \ \ k \neq i, j \tag{1.18a}$$

$$\sum_{k=1}^{n} \partial_k \gamma_{ij} = 0 \tag{1.18b}$$

$$\sum_{k=1}^{n} u^k \partial_k \gamma_{ij} = -\gamma_{ij} \tag{1.18c}$$

where

$$\gamma_{ij} = \frac{v_{ij}(u)}{u^i - u^j} \tag{1.18d}$$

This system is equivalent in the massive case to the WDVV equations + quasihomogenuity. For $n = 2$ this gives $v_{12} = -v_{21} = id/2$. For the first nontrivial case $n = 3$ the system 1(18.) reads

$$\Gamma_1' = \Gamma_2\Gamma_3 \tag{1.19a}$$

$$(z\Gamma_2)' = -\Gamma_1\Gamma_3 \tag{1.19b}$$

$$((z-1)\Gamma_3)' = \Gamma_1\Gamma_2 \tag{1.19c}$$

where

$$v_{ij}(u) = \Gamma_k(z), \ i, \ j, \ k \text{ are distinct} \tag{1.19d}$$

$$z = \frac{u^1 - u^3}{u^2 - u^3}. \tag{1.19e}$$

Using the first integral

$$\Gamma_1^2 + (z\Gamma_2)^2 + ((z-1)\Gamma_3)^2 = \text{const} \tag{1.20}$$

one can reduce [14] the system (1.19) to a particular case of the Painlevé-VI equation. For $n > 3$ the system (1.18) of isomonodromy deformations can be considered as a high-order analogue of the Painlevé-VI. If the Stokes matrix of (1.14) is close to the identity then (1.18) can be reduced to linear integral equations [21].

Remark. The case where some of the charges $q_\alpha = 1$ can be included in the general picture if the equations (1.12) for some vector field v are postulated. The vector field v in the flat coordinates then should have the form

$$v = \sum_{\alpha=1}^{k}(1-q_\alpha)t^\alpha\partial_\alpha + \sum_{\alpha=k+1}^{n} r_\alpha\partial_\alpha, \tag{1.21}$$

where $q_\alpha \neq 1$, r_α are constants. Particularly, for the case $n = 2$, $d = q_2 = 1$ this will give only one solution: the primary free energy (3.5) of the CP^1-model. I do not know if there are physical motivations for the conformal invariance (1.12) w.r.t. (1.21). Interesting solutions of WDVV for $n = 8, 9$ were constructed in [49, 50].

In all the examples (below) of massive TFT the coupling space M (a massive Frobenius manifold) can be extended by adding certain locus M_{sing} (at least of real codimension 2). The structure of Frobenius manifold can be extended on $\hat{M} = M \cup M_{sing}$ but the algebra structure on the tangent spaces $T_x\hat{M}$ for $x \in M_{sing}$ has nilpotents. The flat metric $\eta_{\alpha\beta}$ is extended on $\hat{M}$ without degeneration. So $\hat{M}$ is still a localy Euclidean manifold.

Remark: Local classification of "massless" Frobenius manifolds where the multiplication on the tangent planes is nilpotent everywhere still is an open problem. These manifolds could depend on many functional parameters since the associativity equations (1.3b) are too "weak" in the nilpotent case. Local classification of massless Frobenius manifolds with an assumption of existence of a big group of algebraic symmetries was obtained in [44].

2. Systems of hydrodynamic type: their bi-hamiltonian formalism, solutions, and τ-functions. Coupling of a TCFT to topological gravity

Let us fix a Frobenius manifold (i.e. a solution of the WDVV equations). Considering this as the primary free energy of the matter sector of a 2D TFT model, let us try to calculate the tree-level (i.e., the zero-genus) approximation of the complete model obtained by coupling of the matter sector to topological gravity. The idea to use hierarchies of Hamiltonian systems of hydrodynamic type for such a calculation was proposed by E.Witten [28] for the case of topological sigma-models. An advantage of my approach is in effective construction of these hierarchies for any solution of WDVV. The tree-level free energy of the model will be identified with τ-function of a particular solution of the hierarchy. For a TCFT-model (i.e. for a conformal invariant Frobenius manifold) the hierarchy carries a bihamiltonian structure under a non-resonance assumption for charges and dimension of the model .

So let $c_{\alpha\beta}^{\gamma}(t)$, $\eta_{\alpha\beta}$ be a solution of WDVV, $t = (t^1, \ldots, t^n)$. I will construct a hierarchy of the first order PDE systems linear in derivatives (*systems of hydrodynamic type*) for functions $t^\alpha(T)$, T is an infinite vector

$$T = (T^{\alpha,p}), \quad \alpha = 1, \ldots, n, \quad p = 0, 1, \ldots; \; T^{1,0} = X,$$

$$\partial_{T^{\alpha,p}} t^\beta = c_{(\alpha,p)}{}_\gamma^\beta(t)\partial_X t^\gamma \tag{2.1a}$$

for some matrices of coefficients $c_{(\alpha,p)}{}_\gamma^\beta(t)$. The marked variable $X = T^{1,0}$ usualy is called *cosmological constant.*

I will consider the equations (2.1) as dynamical systems (for any (α, p)) on the space of functions $t = t(X)$ with values in the Frobenius manifold M.

A. Construction of the systems. I define a Poisson bracket on the space of functions

$t = t(X)$ (i.e. on the loop space $\mathcal{L}(M)$) by the formula

$$\{t^\alpha(X), t^\beta(Y)\} = \eta^{\alpha\beta}\delta'(X - Y). \tag{2.2}$$

All the systems (2.1a) have hamiltonian form

$$\partial_{T^{\alpha,p}} t^\beta = \{t^\beta(X), H_{\alpha,p}\} \tag{2.1b}$$

with the Hamiltonians of the form

$$H_{\alpha,p} = \int h_{\alpha,p+1}(t(X))dX. \tag{2.3}$$

The generating functions of densities of the Hamiltonians

$$h_\alpha(t,\lambda) = \sum_{p=0}^{\infty} h_{\alpha,p}(t)\lambda^p, \ \alpha = 1,\ldots,n \tag{2.4}$$

coincide with the flat coordinates of the perturbed connection $\tilde{\nabla}(\lambda)$ (see (1.8)). That means that they are determined by the system (cf. (1.9))

$$\partial_\beta\partial_\gamma h_\alpha(t,\lambda) = \lambda c^\epsilon_{\beta\gamma}(t)\partial_\epsilon h_\alpha(t,\lambda). \tag{2.5}$$

This gives simple recurrence relations for the densities $h_{\alpha,p}$. Solutions of (2.5) can be normalized in such a way that

$$h_\alpha(t,0) = t_\alpha = \eta_{\alpha\beta}t^\beta, \tag{2.6a}$$

$$< \nabla h_\alpha(t,\lambda), \nabla h_\beta(t,-\lambda) >= \eta_{\alpha\beta}. \tag{2.6b}$$

Here ∇ is the gradient (in t). It can be shown that the Hamiltonians (2.3) are in involution. So all the systems of the hierarchy (2.1) commute pairwise.

B. Specification of a solution $t = t(T)$. The hierarchy (2.1) admits an obvious scaling group

$$T^{\alpha,p} \mapsto cT^{\alpha,p}, \quad t \mapsto t. \tag{2.7}$$

Let us take the nonconstant invariant solution for the symmetry

$$(\partial_{T^{1,1}} - \sum T^{\alpha,p}\partial_{T^{\alpha,p}})t(T) = 0 \tag{2.8}$$

(I identify $T^{1,0}$ and X. So the variable X is supressed in the formulae.) This solution can be found without quadratures from a fixed point equation for the gradient map

$$t = \nabla\Phi_T(t), \tag{2.9}$$

$$\Phi_T(t) = \sum_{\alpha,p} T^{\alpha,p}h_{\alpha,p}(t). \tag{2.10}$$

It can be proved the existence and uniqueness of such a fixed point for sufficiently small $T^{\alpha,p}$ for $p > 0$ (more precisely, in the domain: $T^{\alpha,0}$ are arbitrary, $T^{1,1} = o(1)$, $T^{\alpha,p} = o(T^{1,1})$ for $p > 0$).

C. τ-function. Let us define coefficients $V_{(\alpha,p),(\beta,q)}(t)$ from the expansion

$$(\lambda+\mu)^{-1}(<\nabla h_\alpha(t,\lambda),\nabla h_\beta(t,\mu)> -\eta_{\alpha\beta}) = \sum_{p,q=0}^{\infty} V_{(\alpha,p),(\beta,q)}(t)\lambda^p\mu^q \equiv V_{\alpha\beta}(t,\lambda,\mu). \tag{2.11}$$

The infinite matrix of coefficients $V_{(\alpha,p),(\beta,q)}(t)$ has a simple meaning: it is the energy-momentum tensor of the commutative Hamiltonian hierarchy (2.1). That means that a matrix entry $V_{(\alpha,p),(\beta,q)}(t)$ is the density of flux of the Hamiltonian $H_{\alpha,p}$ along the flow $T^{\beta,q}$:

$$\partial_{T^{\beta,q}} h_{\alpha,p+1}(t) = \partial_X V_{(\alpha,p),(\beta,q)}(t). \tag{2.12}$$

Then

$$\tau(T) = \frac{1}{2}\sum V_{(\alpha,p),(\beta,q)}(t(T))T^{\alpha,p}T^{\beta,q} + \sum V_{(\alpha,p),(1,1)}(t(T))T^{\alpha,p} + \frac{1}{2}V_{(1,1),(1,1)}(t(T)) \tag{2.13}$$

Remark. More general family of solutions of (2.1) has the form

$$\nabla[\Phi_T(t) - \Phi_{T_0}(t)] = 0 \tag{2.14}$$

for arbitrary constant vector $T_0 = T_0^{\alpha,p}$. For massive Frobenius manifolds these form a dense subset in the space of all solutions of (2.1) (see [21] and references therein). Formally they can be obtained from the solution (2.9) by a shift of the arguments $T^{\alpha,p}$. τ-function of the solution (2.14) can be formaly obtained from (2.13) by the same shift. For the example of topological gravity [2, 3, 28] such a shift is just the operation that relates the tree-level free energies of the topological phase of 2D gravity and of the matrix model. It should be taken in account that the operation of such a time shift in systems of hydrodynamic type is a subtle one: it can pass through a point of gradient catastrophe where derivatives become infinite. The correspondent solution of the KdV hierarchy has no gradient catastrophes but oscillating zones arise (see [16] for details).

Theorem 2.1. *Let*

$$\mathcal{F}(T) = \log\tau(T), \tag{2.15a}$$

$$<\phi_{\alpha,p}\phi_{\beta,q}\ldots>_0 = \partial_{T^{\alpha,p}}\partial_{T^{\beta,q}}\ldots\mathcal{F}(T). \tag{2.15b}$$

Then the following relations hold

$$\mathcal{F}(T)|_{T^{\alpha,p}=0 \text{ for } p>0,\ T^{\alpha,0}=t^\alpha} = F(t) \tag{2.16a}$$

$$\partial_X\mathcal{F}(T) = \sum T^{\alpha,p}\partial_{T^{\alpha,p-1}}\mathcal{F}(T) + \frac{1}{2}\eta_{\alpha\beta}T^{\alpha,0}T^{\beta,0} \tag{2.16b}$$

$$<\phi_{\alpha,p}\phi_{\beta,q}\phi_{\gamma,r}>_0 = <\phi_{\alpha,p-1}\phi_{\lambda,0}>_0 \eta^{\lambda\mu} <\phi_{\mu,0}\phi_{\beta,q}\phi_{\gamma,r}>_0 . \tag{2.16c}$$

Let me establish now a 1-1 correspondence between the statements of the theorem and the standard terminology of QFT. In a complete model of 2D TFT (i.e. a matter sector coupled to topological gravity) there are infinite number of operators. They usualy are denoted by $\phi_{\alpha,p}$ or $\sigma_p(\phi_\alpha)$. The operators $\phi_{\alpha,0}$ can be identified with the primary operators ϕ_α; the operators $\phi_{\alpha,p}$ for $p>0$ are called *gravitational descendants* of ϕ_α. Respectively one has infinite number of coupling constants $T^{\alpha,p}$. The formula (2.15a) expresses the tree-level (i.e. genus zero) partition function of the model of 2D TFT via logarythm of the τ-function (2.13). Equation (2.15b) is the standard relation between the correlators in the model and the free energy. Equation (2.16a) manifests that before coupling to gravity the partition function (2.15a) coincides with the primary partition function of the given matter sector. Equation (2.16b) is the string equation for the free energy [2, 3, 20, 28]. And equations (2.16c) coincide with the genus zero recursion relations for correlators of a TFT [3, 20, 28].

Particularly, from (2.15) one obtains

$$< \phi_{\alpha,p}\phi_{\beta,q} >_0 = V_{(\alpha,p),(\beta,q)}(t(T)), \tag{2.17a}$$

$$< \phi_{\alpha,p}\phi_{1,0} >_0 = h_{\alpha,p}(t(T)), \tag{2.17b}$$

$$< \phi_{\alpha,p}\phi_{\beta,q}\phi_{\gamma,r} >_0 = < \nabla h_{\alpha,p} \cdot \nabla h_{\beta,q} \cdot \nabla h_{\gamma,r}, [e - \sum T^{\alpha,p}\nabla h_{\alpha,p-1}]^{-1} > . \tag{2.17c}$$

The second factor of the inner product in the r.h.s. of (2.17c) is an invertible element (in the Frobenius algebra of vector fields on M) for sufficiently small $T^{\alpha,p}$, $p > 0$.

Up to now I even did not use the conformal invariance. It turns out that this gives rise to a bihamiltonian structure of the hierarchy (2.1).

Let us consider a conformal invariant Frobenius manifold, i.e. a TCFT model with charges q_α and dimension d. We say that a pair α, p is *resonant* if

$$\frac{d+1}{2} - q_\alpha + p = 0. \tag{2.18}$$

Here p is a nonnegative integer. The TCFT model is *nonresonant* if all pairs α, p are nonresonant. For example, models satisfying the inequalities

$$0 = q_1 \le q_2 \le \ldots \le q_n = d < 1 \tag{2.19}$$

all are nonresonant.

Theorem 2.2. *1) For a conformal invariant Frobenius manifold with charges q_α and dimension d the formula*

$$\{t^\alpha(X), t^\beta(Y)\}_1 = [(\frac{d+1}{2} - q_\alpha)F^{\alpha\beta}(t(X)) + (\frac{d+1}{2} - q_\beta)F^{\alpha\beta}(t(Y))]\delta'(X-Y) \tag{2.20}$$

$$F^{\alpha\beta}(t) = \eta^{\alpha\alpha'}\eta^{\beta\beta'}\partial_{\alpha'}\partial_{\beta'}F(t)$$

determines a Poisson bracket compatible with the Poisson bracket (2.2). 2) For a nonresonant TCFT model all the equations of the hierarchy (2.1) are Hamiltonian equations also with respect to the Poisson bracket (2.20).

The nonresonancy condition is essential: equations (2.1) with resonant numbers (α, p) do not admit another Poisson structure.

Remark: According to the theory [12, 13] of Poisson brackets of hydrodynamic type any such a bracket is determined by a flat Riemannian (or pseudo-Riemannian) metric $g_{\alpha\beta}(t)$ on the target space M (more precisely, one needs a metric $g^{\alpha\beta}(t)$ on the cotangent bundle to M). In our case the target space is the Frobenius manifold M. The first Poisson structure (2.2) is determined by the metric being specified by the double-point correlators $\eta_{\alpha\beta}$. The second flat metric for the Poisson bracket (2.20) on a conformal invariant Frobenius manifold M has the following geometrical interpretation. Let ω_1 and ω_2 be two 2-forms on M. We can multiply them $\omega_1, \omega_2 \mapsto \omega_1 \cdot \omega_2$ using the multiplication of tangent vectors and the isomorphism η between tangent and cotangent spaces. Then the new inner product $< \, , \, >_1$ is defined by the formula

$$< \omega_1, \omega_2 >_1 = i_v(\omega_1 \cdot \omega_2). \tag{2.21a}$$

Here i_v is the operator of contraction with the vector field v (the generator of conformal symmetries (1.12)). In the flat coordinates t^α the metric has (contravariant) components

$$g^{\alpha\beta}(t) = (d + 1 - q_\alpha - q_\beta)F^{\alpha\beta}(t). \tag{2.21b}$$

The metric (2.21) can be degenerate. The theorem states that, nevertheless, the Jacobi identity for the Poisson bracket (2.20) holds.

Let us consider examples of the second hamiltonian structure (2.20). I start with the most elementary case $n = 1$ (the pure gravity). Let me redenote the coupling constant

$$u = t^1.$$

The Poisson bracket (2.20) for this case reads

$$\{u(X), u(Y)\}_1 = \frac{1}{2}(u(X) + u(Y))\delta'(X - Y). \tag{2.22}$$

This is nothing but the Lie – Poisson bracket on the dual space to the Lie algebra of one-dimensional vector fields.

For arbitrary graded Frobenius algebra A the Poisson bracket (2.20) also is linear in the coordinates t^α

$$\{t^\alpha(X), t^\beta(Y)\}_1 = [(\frac{d+1}{2} - q_\alpha)c_\gamma^{\alpha\beta}t^\gamma(X) + (\frac{d+1}{2} - q_\beta)c_\gamma^{\alpha\beta}t^\gamma(Y)]\delta'(X - Y). \tag{2.23}$$

It determines therefore a structure of an infinite dimensional Lie algebra on the loop space $\mathcal{L}(A^*)$ where A^* is the dual space to the graded Frobenius algebra A. Theory of linear Poisson brackets of hydrodynamic type and of corresponding infinite dimensional Lie algebras was constructed in [17] (see also [12]). But the class of examples (2.23) is a new one.

Let us come back to the general (i.e. nonlinear) case of a TCFT model. I will assume that the charges and the dimension are ordered in such a way that

$$0 = q_1 < q_2 \le \ldots \le q_{n-1} < q_n = d. \tag{2.24}$$

Since

$$\{t^\alpha(X), t^n(Y)\}_1 = [(\frac{d+1}{2} - q_\alpha)t^\alpha(X) + \frac{1-d}{2}t^\alpha(Y)]\delta'(X - Y), \tag{2.25}$$

the functional

$$P = \frac{2}{1-d}\int t^n(X)dX \tag{2.26}$$

generates spatial translations. We see that for $d \neq 1$ the Poisson bracket (2.20) can be considered as a nonlinear extension of the Lie algebra of one-dimensional vector fields.

3. Examples

I start with the most elementary examples of solutions of WDVV for $n = 2$. Only massive solutions are of interest here (a 2-dimensional nilpotent Frobenius algebra has no nontrivial deformations). The equations (1.18) in this case are linear. I consider only TCFT case (the similarity reduction of WDVV). Let us redenote the coupling constants

$$t^1 = u,\ t^2 = \rho. \tag{3.1}$$

For $d \neq 1$ the primary free energy F has the form

$$F = \frac{1}{2}\rho u^2 + \frac{g}{a(a+2)}\rho^{a+2}, \tag{3.2}$$

$$a = \frac{1+d}{1-d} \tag{3.3}$$

g is an arbitrary constant. The second term in the formula for the free energy should be understood as

$$\frac{g}{a(a+2)}\rho^{a+2} = \int\int\int g(a+1)\rho^{a-1}.$$

The linear system (2.5) can be solved via Bessel functions [21]. Let me give an example of equations of the hierarchy (2.1) (the $T = T^{1,1}$-flow)

$$u_T + uu_X + g\rho^a\rho_X = 0 \tag{3.4a}$$

$$\rho_T + (\rho u)_X = 0. \tag{3.4b}$$

These are the equations of isentropic motion of one-dimensional fluid with the dependence of the pressure on the density of the form $p = \frac{g}{a+2}\rho^{a+2}$. The Poisson structure (2.2) for these equations was proposed in [19]. For $a = 0$ (equivalently $d = -1$) the system coincides with the equations of waves on shallow water (the dispersionless limit [37] of the nonlinear Schrödinger equation (NLS)).

For $d = 1$ the primary free energy has the form

$$F = \frac{1}{2}\rho u^2 + ge^\rho. \tag{3.5}$$

This coincides with the free energy of the topological sigma-model with CP^1 as the target space. Note that this can be obtained from the same solution of the system (1.18) as the semiclassical limit of the NLS (the case $d = -1$ above) for different choices of the eigenfunction ψ_{1i} (in the notations of (1.17)). The corresponding $T = T^{2,0}$-system of the hierarchy (2.1) reads

$$u_T = g(e^\rho)_X$$

$$\rho_T = u_X.$$

Eliminating u one obtains the long wave limit

$$\rho_{TT} = g(e^\rho)_{XX} \tag{3.6}$$

of the Toda system

$$\rho_{n\,tt} = e^{\rho_{n+1}} - 2e^{\rho_n} + e^{\rho_{n-1}}. \tag{3.7}$$

(The 2-dimensional version of (3.6) was obtained in the formalism of Whitham-type equations in [26].) It would be interesting to prove that the nonperturbative free energy of the CP^1-model coincides with the τ-function of the Toda hierarchy.

Example 2. Topological minimal models. I consider here the A_n-series models only. The Frobenius manifold M here is the set of all polinomials (*Landau – Ginsburg superpotentials*) of the form

$$M = \{w(p) = p^{n+1} + a_1p^{n-1} + \ldots + a_n \mid a_1, \ldots, a_n \in \mathbf{C}\}. \tag{3.8}$$

For any $w \in M$ the Frobenius algebra $A = A_w$ is the algebra of truncated polynomials

$$A_w = \mathbf{C}[p]/(w'(p) = 0) \tag{3.9}$$

(the prime means derivative with respect to p) with the invariant inner product

$$<f,g>= \mathrm{res}_{p=\infty}\frac{f(p)g(p)}{w'(p)}. \tag{3.10}$$

The algebra A_w is semisimple if the polynomial $w'(p)$ has simple roots. The canonical coordinates (1.15) $u^1,\dots,u^n$ are the critical values of the polynomial $w(p)$

$$u^i = w(p_i),\ \text{where } w'(p_i)=0,\ i=1,\dots,n. \tag{3.11}$$

The metric on M is diagonal in the canonical coordinates

$$\sum_{i=1}^n \eta_{ii}(u)(du^i)^2, \quad \eta_{ii}(u)=[w''(p_i)]^{-1}. \tag{3.12}$$

The correspondent flat coordinates on M have the form

$$t^\alpha = -\frac{n+1}{n-\alpha+1}\mathrm{res}_{p=\infty}w^{\frac{n-\alpha+1}{n+1}}(p)dp,\ \alpha=1,\dots,n. \tag{3.13}$$

The metric (3.12) in these coordinates has the constant form

$$\sum_{i=1}^n \eta_{ii}(u)(du^i)^2 = \eta_{\alpha\beta}dt^\alpha dt^\beta, \quad \eta_{\alpha\beta}=\delta_{n+1,\alpha+\beta}. \tag{3.14}$$

The ortonormal basis in A_w with respect to this metric consists of the polynomials $\phi_1(p)$, ... , $\phi_n(p)$ of degrees 0, 1, ... , $n-1$ resp. where

$$\phi_\alpha(p) = \frac{d}{dp}[w^{\frac{\alpha}{n+1}}]_+, \quad \alpha=1,\dots,n, \tag{3.15}$$

Here $[\ ]_+$ means the polynomial part of the power series in p. This is a TCFT model with the charges and dimension

$$q_\alpha = \frac{\alpha-1}{n+1},\ d=q_n=\frac{n-1}{n+1}. \tag{3.16}$$

In fact one obtains a n-parameter family of TFT models with the same canonical coordinates u^i of the form (3.11) where

$$\eta_{ii}(u)\mapsto \eta_{ii}(u,c)=[w''(p_i)]^{-1}[\sum c_\alpha\phi_\alpha(p_i)]^2, \tag{3.17a}$$

$$t^\alpha \mapsto t^\alpha(c) = -\frac{n+1}{n-\alpha+1}\mathrm{res}_{p=\infty}w^{\frac{n-\alpha+1}{n+1}}(p)[\sum c_\gamma\phi_\gamma(p)]dp \tag{3.17b}$$

depending on arbitrary parameters c_1, ..., c_n. This reflects the ambiguity in the choice of the solution ψ_{i1} in the formulae (1.17). These models are conformal invariant if only one of the coefficients c_γ is nonzero.

The corresponding hierarchy of the systems of hydrodynamic type (2.1) coincides with the dispersionless limit of the Gelfand – Dickey hierarchy for the scalar Lax operator of order $n+1$. This essentialy follows from [4, 6]. I recall that the Gelfand – Dickey hierarchy for an operator

$$L = \partial^{n+1} + a_1(x)\partial^{n-1} + \ldots + a_n(x)$$

$$\partial = d/dx$$

has the form

$$\partial_{t^{\alpha,p}} L = c_{\alpha,p}[L, [L^{\frac{\alpha}{n+1}+p}]_+], \ \alpha = 1, \ldots, n, \ p = 0, 1, \ldots \tag{3.18}$$

for some constants $c_{\alpha,p}$. Here $[\ ]_+$ denotes differential part of the pseudodifferential operator. The dispersionless limit of the hierarchy is defined as follows: one should substitute

$$x \mapsto \epsilon x = X, \ t^{\alpha,p} \mapsto \epsilon t^{\alpha,p} = T^{\alpha,p} \tag{3.19}$$

and tend ϵ to zero. The dispersionless limit of τ-function of the hierarchy is defined [6, 34, 35] as

$$\log \tau_{\text{dispersionless}}(T) = \lim_{\epsilon \to 0} \epsilon^{-2} \log \tau(\epsilon t). \tag{3.20}$$

Modified minimal model (3.17) is related to the same Gelfand – Dickey hierarchy with the following modification of the L-operator

$$L \mapsto \tilde{L} = \sum c_\gamma [L^{\frac{\gamma}{n+1}}]_+. \tag{3.21}$$

The linear equation (2.5) for the minimal model can be solved in the form [21]

$$h_\alpha(t; \lambda) = -\frac{n+1}{\alpha} \text{res}_{p=\infty} w^{\frac{\alpha}{n+1}} {}_1F_1(1; 1 + \frac{\alpha}{n+1}; \lambda w(p)) dp. \tag{3.22}$$

Here ${}_1F_1(a; c; z)$ is the Kummer (or confluent hypergeometric) function [18]

$${}_1F_1(a; c; z) = \sum_{m=o}^{\infty} \frac{(a)_m}{(c)_m} \frac{z^m}{m!}, \tag{3.23a}$$

$$(a)_m = a(a+1)\ldots(a+m-1). \tag{3.23b}$$

The generating function (2.11) has the form

$$V_{\alpha\beta}(t; \lambda, \mu) = (\lambda + \mu)^{-1} [\eta^{\mu\nu} (\text{res}_{p=\infty} w^{\frac{\alpha}{n+1}-1} {}_1F_1(1; \frac{\alpha}{n+1}; \lambda w(p)) \phi_\mu(p) dp) \times$$

$$(\text{res}_{p=\infty} w^{\frac{\beta}{n+1}-1} {}_1F_1(1; \frac{\beta}{n+1}; \mu w(p)) \phi_\nu(p) dp) - \eta_{\alpha\beta}]. \tag{3.24}$$

From this one obtains formulae for the τ-function.

Example 3. $M_{g;n_0,\ldots,n_m}$-models [8, 9, 21, 44]. Let $M = M_{g;n_0,\ldots,n_m}$ be a moduli space of dimension

$$n = 2g + n_0 + \ldots + n_m + 2m \tag{3.25}$$

of sets

$$(C; \infty_0, \ldots, \infty_m; w; k_0, \ldots, k_m; a_1, \ldots, a_g, b_1, \ldots, b_g) \in M_{g;n_0,\ldots,n_m} \tag{3.26}$$

where C is a Riemann surface with marked points ∞_0, ..., ∞_m, and a marked meromorphic function

$$w : C \to CP^1, \ w^{-1}(\infty) = \infty_0 \cup \ldots, \cup \infty_m \tag{3.27}$$

having a degree $n_i + 1$ near the point ∞_i, and a marked symplectic basis $a_1, \ldots, a_g$, $b_1, \ldots, b_g \in H_1(C, \mathbf{Z})$, and marked branches of roots of w near ∞_0, ..., ∞_m of the orders

$n_0 + 1, \ldots, n_m + 1$ resp.,

$$k_i^{n_i+1}(P) = w(P), \ P \text{ near } \infty_i. \tag{3.28}$$

We need the critical values of w

$$u^j = w(P_j), \ dw|_{P_j} = 0, \ j = 1, \ldots, n \tag{3.29}$$

(i.e. the ramification points of the Riemann surface (3.27)) to be local coordinates in open domains in M where

$$u^i \neq u^j \text{ for } i \neq j \tag{3.30}$$

(due to the Riemann existence theorem). Another assumption is that the one-dimensional affine group acts on M as

$$(C; \infty_0, \ldots, \infty_m; w; \ldots) \mapsto (C; \infty_0, \ldots, \infty_m; aw + b; \ldots) \tag{3.31a}$$

$$u^i \mapsto au^i + b, \ i = 1, \ldots, n. \tag{3.31b}$$

The flat metric $\eta_{\alpha\beta}$ and the flat coordinates for these models are calculated in [8, 9, 21, 44]. For $g = 0$ and $m = 0$ one obtains the A_n minimal models (see above).

Remark. The above models with $m = 0$, $g > 0$ can be obtained [21] in a semiclassical description of correlators of multimatrix models (at the tree-level approximation for small couplings they correspond to various self-similar solutions of the hierarchy (2.1)) as functions of the couplings after passing through a point of gradient catastrophe. The idea of such a description is originated in the theory of a dispersive analogue of shock waves [16]; see also [12].

More general algebraic-geometrical examples of solutions of WDVV were constructed in [26]. In these examples M is a moduli space of Riemann surfaces of genus g with a marked normalized Abelian differential of the second kind dw with poles at marked points and with fixed b-periods

$$\oint_{b_i} = B_i, \ i = 1, \ldots, g.$$

For $B_i = 0$ one obtains the above Frobenius structures on $M_{g;n_0,\ldots,n_m}$. Unfortunately, for $B \neq 0$ the Frobenius structures of [26] does not admit a conformal invariance.

4. Calculation of the ground state metric and pluriharmonic maps

An additional structure that should be defined on a Frobenius manifold comes from the ground state metric of the family of TCFT models (the coupling space) as a Hermitean metric on the parametr space of the family. Equations of such a metric generalizing the equations of special geometry [22] were obtained in [23]. Their integrability was proved in [27]. Here I give a brief description of the underlined geometrical structure of these equations (see [27] for the details).

Let us consider the space $Q = Gl(n)/O(n)$ of real positive definite quadratic forms. This is a symmetric space. (In the tables of symmetric spaces usualy the correspondent irreducible symmetric space $\tilde{Q} = Sl(n)/O(n)$ of unimodular quadratic forms occurs.) A map

$$G : M \to Q$$

is called *pluriharmonic* if the restriction of it onto an arbitrary complex analytic curve is a harmonic map of this curve to Q. The class of pluriharmonic maps depends only on the complex structure on M.

The ground state metric on the Frobenius manifold is determined by a certain pluriharmonic map $G : M \to Q$ (I assume here M to be a simply connected manifold; otherwise one could meet with twisted pluriharmonic maps). The constraint imposed on the pluriharmonic map by the Frobenius structure on M can be described in terms of *Higgs bundles* [45]. I recall that a Higgs bundle on a complex manifold M is a pair (E, A) where $E \to M$ is a complex holomorphic bundle and A is a holomorphic section of the bundle $T_*^{1,0}M \otimes \mathrm{End}E$ such that: 1) $dA = 0$; 2) $A \wedge A = 0$.

Any pluriharmonic map $G : M \to Q$ determines a Higgs bundle over M according to the following construction. We put

$$E = M \times \mathbf{C}^n$$

where the holomorphic structure on E is determined by the d''-operator

$$d''_G = d\bar{z}^k \nabla_k,$$

where $(\nabla_k, \nabla_{\bar{k}}) =$ pull-back of the Levi-Civitá connection on Q;

$$A = G^{-1} d' G.$$

The equations $dA = 0$ and $A \wedge A = 0$ were proved in [27] (for pluriharmonic maps to a compact Lie group these were proved in [46]). This implies integrability of the equations of pluriharmopnic maps to Q.

In fact the above Higgs bundle carries an additional structure. We say that (E, A, G) is a *symmetric Higgs bundle* if (E, A) is a Higgs bundle and G is a holomorphic symmetric nondegenerate inner product on E and the operators A are symmetric w.r.t. G. In our example the symmetric inner product is the pull-back of the invariant metric on Q. There is a real subbundle $\mathrm{Re}E \subset E$ where G is a real positive definite quadratic form.

There is another symmetric Higgs bundle (E', A', G') on a Frobenius manifold:

$$E' = TM, \; (A')^i_j = c^i_{kj} dz^k, \; G'_{ij} = \eta_{ij}.$$

Now we can formulate the geometrical interpretation of the constraints for the pluriharmonic map $G : M \to Q$ equivalent to the tt^*-equations of [23]. We are looking for such a pluriharmonic map that the above symmetric Higgs bundles (E, A, G) and (E', A', G') are isomorphic. Then the quadratic form G can be extended from $\mathrm{Re}E$ onto $E \simeq TM$ also as a Hermitean positive definite form. This Hermitean metric satisfies the tt^*-equations of [23].

Local classification of pluriharmonic maps of a massive Frobenius manifold to Q can be reduced to the isomonodromy deformation machinery [27]. In the first nontrivial case $n = 2$ this gives the Painlevé-III equation; for $n > 2$ one obtains a high-order analogue of the Painlevé-III. Interesting results in global classification of solutions were recently obtained by Cecotti and Vafa [48].

5. Selection of solutions of WDVV

What the solutions of WDVV could be of special interest? The possible test for selection of solutions was proposed by C.Vafa [41]: to find solutions of WDVV for which the free energy $\mathcal{F}(T)$ can be expanded in a power series in T with rational coefficients? This test could be motivated by the interpretation of $\mathcal{F}(T)$ as the generated function of intersection numbers of cycles on certain moduli spaces of algebraic curves and their holomorphic maps [28]. From the constructions of Sect.2 above it follows

Proposition 5.1. *A TCFT corresponding to a solution $F(t)$ of WDVV passes through the Vafa's test in the tree-level approximation iff $F(t)$ is analytic in $t = 0$ and its Taylor expansion in the origin has rational coefficients.*

It turns out that analyticity of a solution of WDVV in the origin imposes very strong restriction for $F(t)$. Particularly, for $0 < d < 1$ for $n = 2,\ 3$ one obtains the following list of solutions analytic in the origin:

$n = 2$:

$$F = \frac{t_1^2 t_2}{2} + t_2^{k+2} \tag{5.1}$$

for integer $k \geq 2$;

$n = 3$: here one has only three solutions

$$F = \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^2 t_3^2}{4} + \frac{t_3^5}{60} \tag{5.2}$$

$$F = \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^3 t_3}{6} + \frac{t_2^2 t_3^3}{6} + \frac{t_3^7}{210} \tag{5.3}$$

$$F = \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^3 t_3^2}{6} + \frac{t_2^2 t_3^5}{20} + \frac{t_3^{11}}{3960}. \tag{5.4}$$

The formula (5.1) for $k = 2$ is the primary free energy of the A_2 topological minimal model, (5.2) is the primary free energy of the A_3 minimal model. Other solutions seems to be new.

V.I.Arnol'd recently brought my attention to a relation of degrees of the polynomials (5.2) - (5.4) to the Coxeter numbers of the three Coxeter groups in the 3-dimensional Euclidean space. Trying to explain this observation I found a general construction of polynomial solutions of WDVV for arbitrary Coxeter group G (finite group of linear transformations of a n-dimensional Euclidean space generated by reflections). In this construction the Frobenius manifold coincides with the space of orbits of the Coxeter group. The Euclidean coordinates x_1, ..., x_n will be the flat coordinates of the second flat metric (2.21). The first flat metric $\eta_{\alpha\beta}$ on the space of orbits can be defined using the affine structure on the space of orbits introduced by K.Saito [42]. The correspondent affine coordinates $t^1(x)$, ..., $t^n(x)$ were constructed in [43]. They are certain homogeneous polynomials in x_1, ..., x_n invariant w.r.t. G of the degrees

$$d_n = 2 < d_{n-1} \leq ... \leq d_2 < d_1 = h,\ d_\alpha = m_{n-\alpha+1} + 1, \tag{5.5}$$

h is the Coxeter number of G, m_α are the exponents of G (I reverse the standard order of the invariant polynomials!). It is important that the vector field

$$\partial_1 = \frac{\partial}{\partial t^1}$$

is well-defined within a factor due to the strict inequality $d_2 < d_1$. Let $g^{\alpha\beta}(t)$ is the (contravariant) Euclidean metric in the coordinates t^α

$$g^{\alpha\beta}(t) = (dt^\alpha(x), dt^\beta(x)) = \sum_{i=1}^n \frac{\partial t^\alpha}{\partial x_i}\frac{\partial t^\beta}{\partial x_i}. \tag{5.6}$$

(In the literature on reflection groups $(g^{\alpha\beta}(t))$ is called discriminant matrix. It degenerates on the discriminant of G, i.e. on the set of singular orbits where x_1, ..., x_n fail to be local coordinates on the space of orbits. In the canonical coordinates u^i of Prop. 1.2 the discriminant has the form $u^1...u^n = 0$.) The affine structure of Saito is uniquely defined by the following condition: the matrix $\partial_1 g^{\alpha\beta}(t)$ is a constant one. In these coordinates we put

$$F^{\alpha\beta}(t) = \frac{g^{\alpha\beta}(t)}{d_\alpha + d_\beta - 2}, \tag{5.7a}$$

$$\eta^{\alpha\beta} = \partial_1 F^{\alpha\beta}, \ (\eta_{\alpha\beta}) = (\eta^{\alpha\beta})^{-1} \tag{5.7b}$$

$$\frac{\partial^2 F}{\partial t^\alpha \partial t^\beta} = \eta_{\alpha\alpha'}\eta_{\beta\beta'}F^{\alpha'\beta'}. \tag{5.7c}$$

This gives a massive TCFT with a polynomial free energy F and with the dimensions and charges

$$d = 1 - \frac{2}{h}, \ q_\alpha = 1 - \frac{m_{n-\alpha+1}+1}{h}. \tag{5.8}$$

Example. For the group $I_2(k+1)$ of symmetries of a regular $(k+1)$-gone on the Euclidean plane the basic invariant polynomials are

$$t^1 = \frac{1}{k+1}\mathrm{Re}(x+iy)^{k+1}, \ t^2 = x^2+y^2,$$

$$d_1 = k+1, \ d_2 = 2.$$

The matrix (5.6) has the form

$$(g^{\alpha\beta}) = \begin{pmatrix} (t^2)^k & 2(k+1)t^1 \\ 2(k+1)t^1 & 4t^2 \end{pmatrix}.$$

The formulae (5.7) immediately give the solution (5.1) (up to rescaling of the couplings). Note that for any integer $k \geq 2$ (5.1) is the primary free energy of the topological Landau - Ginsburg model with the superpotential

$$w(p) = t^1 + (t^2)^{\frac{k+1}{2}} T_{k+1}((t^2)^{-1/2}p),$$

$T_{k+1}(\cos x) = \cos(k+1)x$ is the Tchebyscheff polynomial.

For the ADE Coxeter groups one obtains the corresponding topological minimal models [4, 20, 47]. The solutions (5.2) and (5.3) correspond to the groups B_3 and H_3 resp. Details of this construction will be given in a separate publication. Probably, my construction gives all polynomial solutions of WDVV (at least this is true for $n = 2, 3$.

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