DISPERSION RELATIONS FOR NONLINEAR WAVES AND SCHOTTKY PROBLEM

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Abstract.

An approach to the Schottky problem of specification of periods of holomorphic differentials on Riemann surfaces (or, equivalently, specification of Jacobians among all principaly polarized Abelian varieties) based on the theory of Kadomtsev - Petviashvili equation, is discussed.

Introduction. Dispersion relations for linear and nonlinear waves.

One of the first exercises in a course of PDE is in finding particular solutions. For linear PDE the simplest solutions can be found immediately using wellknown properties of the exponential. For example, for linear wave (or Helmholtz) equation

$$u_{tt} - u_{xx} + m^2 u = 0 \tag{0.1}$$

one can try to find a solution of the form

$$u(x,t) = Ae^{i(kx+\omega t)}. \tag{0.2}$$

Here A, k, ω are unknown parameters. After substitution in the equation one obtains a constraint for the parameters ω, k

$$\omega^2 - k^2 = m^2 \tag{0.3}$$

and no constraints for the amplitude A because of linearity of the equation. The solution (0.2) is called plane wave, or one-phase solution of (0.1). The parameters A, k, ω are the amplitude, the wave number† and the frequency of the plane wave. The equation (0.3) thus is the *dispersion relation* for the plane waves. The solution is $\frac{2\pi}{k}$-periodic in x and $\frac{2\pi}{\omega}$-periodic in t for real ω, k. The solution (0.2) is a complex one; to obtain a real solution one can take the real part of (0.2).

Multiphase quasi-periodic solutions of (0.1) are linear superpositions of plane waves

$$u(x,t) = \sum_s A_s e^{i(k_s x + \omega_s t)}, \tag{0.4a}$$

† In more standard terminology the wave number is $-k$.

$$A_s \text{ arbitrary}, \ \omega_s^2 - k_s^2 = m^2. \tag{0.4b}$$

Considering infinite sums (or integrals over s) for real k_s, ω_s one obtains a general solution of the Cauchy problem for the equation (0.1) for appropriate functional classes of initial data.

Nonlinear analogues of simple waves can be constructed for a wide class of PDE. A feature of them is that also the amplitude being involved in the dispersion relation for the nonlinear waves. To see it let us consider a nonlinear wave equation

$$u_{tt} - u_{xx} + V'(u) = 0. \tag{0.5}$$

Let us assume the potential $V(u)$ to satisfy the following condition: the equation

$$V(u) = E \tag{0.6a}$$

has two solutions

$$u_- = u_-(E) < u_+ = u_+(E) \tag{0.6b}$$

for some interval $E_{\min} < E < E_{\max}$. Then nonlinear simple waves have the form

$$u(x,t) = U(kx + \omega t + \phi_0; E) \tag{0.7a}$$

for arbitrary phase shift ϕ_0, where the function $U = U(\phi; E)$ has the form

$$\int_{u_-(E)}^{U} \frac{du}{\sqrt{2(E - V(u))}} = I(E)\phi, \tag{0.7b}$$

$$I(E) = \frac{1}{\pi} \int_{u_-(E)}^{u_+(E)} \frac{du}{\sqrt{2(E - V(u))}} \tag{0.7c}$$

and the parameters ω, k, E satisfy

$$\omega^2 - k^2 = I^{-2}(E). \tag{0.7d}$$

The function $U(\phi; E)$ plays the role of the exponential ($U = \frac{\sqrt{2E}}{m} \cos\phi$ for linear case). It is 2π-periodic in ϕ. So, again, the parameters ω, k have the sense of the frequency and wave number of the nonlinear waves (0.7). Shape of the wave is determined by the amplitude parameter E. The constraint (0.7d) for ω, k, E is nothing but nonlinear dispersion relation for the frequency, wave number and amplitude of the nonlinear waves.

One can try to look for multiphase oscillating solutions of a nonlinear PDE of the form (in the spatialy one-dimensional case)

$$u(x,t) = U(k_1 x + \omega_1 t + \phi_{10}, \ldots, k_m x + \omega_m t + \phi_{m0}; \mathbf{A}) \tag{0.8}$$

where the function $U(\phi_1, \ldots, \phi_m; \mathbf{A})$ is 2π-periodic in ϕ_1, ..., ϕ_m. The vector of parameters $\mathbf{A}$ plays the role of "amplitudes". It turns out that existence of such multiphase solutions for sufficiently big m (probably, for $m > 2$; see [16] for examples of 2-phase solutions of a nonintegrable equation) is a feature of integrable evolutionary equations (though this statement is still to be proved). For the KdV equation these are the famous finite-gap (or algebraic-geometrical) solutions that were constructed in the papers of 1974 - 1976 by S.Novikov and B.Dubrovin [1, 3-5], P.Lax [2], A.Its and V.Matveev [6], H.McKean and P.van Moerbeke [8]. On this basis a program of constructing and investigating multiphase solutions of nonlinear integrable systems was developed (see surveys [7, 9-12]). An extremely important step in development of this program was done by I.Krichever [13-14, 9]. He found a general approach to construct algebraic-geometrical solutions of spatialy 2-dimensional integrable systems such as the Kadomtsev - Petviashvili (KP) equation

$$\frac{3}{4}u_{yy} = \partial_x(u_t - \frac{1}{4}(6uu_x - u_{xxx})). \tag{0.9}$$

Multiphase solutions of this equation have the form

$$u(x,y,t) = U(k_1x + l_1y + \omega_1t + \phi_{10}, \ldots, k_mx + l_my + \omega_mt + \phi_{m0}; \mathbf{A}) \tag{0.10}$$

for arbitrary phase shifts ϕ_{10}, ..., ϕ_{m0}, where, as above, the function $U(\phi_1, \ldots, \phi_m; \mathbf{A})$ is 2π-periodic in each ϕ_1, ..., ϕ_m, and $\mathbf{A}$ is a vector of amplitude parameters. This function can be expressed via multidimensional theta-functions. Theta-functions are defined by a multiple Fourier series

$$\theta(\phi|\tau) = \sum_{-\infty<n_1,\ldots,n_m<\infty} \exp(\pi i \sum_{p,q=1}^{m} \tau_{pq}n_pn_q + \sum_{p=1}^{m} in_p\phi_p), \tag{0.11}$$

$$\phi = (\phi_1, \ldots, \phi_m), \ \tau = (\tau_{pq})_{1\le p,q\le m}.$$

Parameters of the theta-function form a *period matrix*, i.e. a symmetric $m \times m$ complex matrix $\tau = (\tau_{pq})$ with positive definite imaginary part. This is 2π-periodic in ϕ_1, ..., ϕ_m. It also posesses the quasi-periodicity property

$$\theta(\phi + N\tau|\tau) = \exp(-\pi i < N\tau, N > -i < N, \phi >)\theta(\phi|\tau) \tag{0.12}$$

for any integer vector $N = (N_1, \ldots, N_m)$, where the brackets $< \ , \ >$ mean the Euclidean inner product

$$< N, \phi >= \sum_1^m N_s\phi_s.$$

The Krichever's solutions of KP have the form (0.10) where

$$U = U(\phi; \tau) = -2\partial_k^2 \log\theta(\phi|\tau) + c, \tag{0.13}$$

$$\partial_k = \sum_{p=1}^m k_p \frac{\partial}{\partial\phi_p},$$

c is an arbitrary constant. (This can be killed by the transformation

$$\omega_p \mapsto \omega_p - \frac{3}{2} c k_p,\ p = 1, \ldots, m$$

$$u \mapsto u - c.)$$

So the period matrix τ can be considered as the amplitude of the multiphase solutions of KP. The main object of our investigation will be dispersion relations for these multiphase solutions

$$F_{\rm KP}(k, l, \omega, \tau) = 0$$

where $F_{\rm KP}(k, l, \omega, \tau)$ is a vector-valued analytic function. Explicit form of this system of equations will be given below. It turns out that these dispersion relations for $m \geq 4$ constrain also the period matrix τ. These constraints will give a solution of the classical Schottky problem (see Section 2 below) exactly specifying period matrices of holomorphic differentials on Riemann surfaces.

1. Dispersion relations for multiphase solutions of KP. Novikov's conjecture.

To obtain disperion relations for multiphase solutions (0.10), (0.13) let us substitute the theta-functional formula (0.11) to the KP equation (the constant c is assumed to equal zero). After substitution one obtains

$$\partial_x^2 [(\theta_{xxxx}\theta - 4\theta_{xxx}\theta_x - 3\theta_{xx}^2 - 4\theta_x\theta_t + 4\theta_{xt}\theta - 3\theta_{yy}\theta + 3\theta_y^2)/\theta^2] = 0 \qquad (1.1)$$

where

$$\theta = \theta(kx + ly + \omega t + \phi_0 | \tau)$$

ϕ_0 is an arbitrary complex vector. If the theta function is indecomposable (see below) then the expression in the square brackets equals a constant. Let us denote this integration constant by $-8d$. To reduce the obtained equality

$$\theta_{xxxx}\theta - 4\theta_{xxx}\theta_x - 3\theta_{xx}^2 - 4\theta_x\theta_t + 4\theta_{xt}\theta - 3\theta_{yy}\theta + 3\theta_y^2 + 8d\theta^2 = 0$$

to a finite number of dispersion relations for k, l, ω, and τ, let us introduce the theta-functions of the second order

$$\hat\theta[p](\phi|\tau) = \sum_{-\infty < n_1, \ldots, n_m < \infty} \exp(2\pi i \sum_{q,r=1}^m \tau_{qr}(n_q + \frac{p_q}{2})(n_r + \frac{p_r}{2}) + \sum_{q=1}^m i(n_q + \frac{p_q}{2})\phi_q). \qquad (1.2)$$

Here $p \in \mathbf{Z}_2^m$, i.e. it is an arbitrary m-vector with the coordinates being equal to 0 or 1. We have 2^m such theta-functions of the second order. Values of these theta-functions and of their derivatives in the origin $\phi = 0$ are called *theta-constants*. They are functions only on τ. For brevity let us omit arguments of the theta-constants:

$$\hat\theta[p] \equiv \hat\theta[p](0|\tau),$$

$$\hat{\theta}_{ij}[p] \equiv \frac{\partial^2}{\partial\phi_i\partial\phi_j}\hat{\theta}[p](0|\tau),$$

$$\hat{\theta}_{ijqr}[p] \equiv \frac{\partial^4}{\partial\phi_i\partial\phi_j\partial\phi_q\partial\phi_r}\hat{\theta}[p](0|\tau).$$

(The theta-functions (1.2) are even functions of ϕ, so only derivatives of even order in the origin could be nonzero.)

Theorem 1.[15] *Dispersion relations for the multiphase solutions (0.10), (0.13) have the form*

$$\partial_k^4\hat{\theta}[p] + \partial_k\partial_\omega\hat{\theta}[p] + \frac{3}{4}\partial_l^2\hat{\theta}[p] + d\hat{\theta}[p] = 0 \tag{1.3}$$

for arbitrary $p \in \mathbf{Z}_2^m$.

Here

$$\partial_k^4\hat{\theta}[p] = \sum_{i,j,q,r} k_i k_j k_q k_r \hat{\theta}_{ijqr}[p]$$

$$\partial_k\partial_\omega\hat{\theta}[p] = \sum_{i,j} k_i\omega_j\hat{\theta}_{ij}[p]$$

$$\partial_l^2\hat{\theta}[p] = \sum_{i,j} l_i l_j \hat{\theta}_{ij}[p].$$

The dispersion relations (1.3) are written in the form of a system of algebraic equations for the coordinates of the vectors k, l, and ω and for an auxillary unknown variable d with the coefficients depending on the period matrix τ. For $m = 1,\ 2,\ 3$ for generic matrix τ one can solve the dispersion relations in the form

$$k = k(\tau),\ l = l(\tau),\ \omega = \omega(\tau)$$

(in fact, one obtains a one-parameter family of solutions, see [15, 10, 17] and Section 3 below). This parametrization of 2-phase solutions of KP was used in [18] for constructing physicaly realistic models of nonlinear waves on shallow water.

For $m > 3$ the dispersion relations are an overdeterminned system of algebraic equations for k, l, ω, d. Compatibility conditions of these overdetermined equations constrain the "amplitude" τ. It was conjectured by S.Novikov in 1980 that these constraints exactly specify periods of Riemann surfaces providing a solution of the classical Schottky problem. We are coming now to the formulation of this problem.

2. Periods of Riemann surfaces. Schottky problem.

A challenge of the theory of functions of the XIXth century is the problem of moduli of Riemann surfaces (still far from having been solved). Intuitively, the problem is to obtain a complete "list" of pairwise inequivalent Riemann surfaces of a given genus g. For $g = 0$ the "list" consists only of one point: the Riemann sphere ($= \mathbf{CP}^1$) since any Riemann surface of genus 0 is equivalent (biholomorphic) to the Riemann sphere. For $g = 1$ one obtains a 1-parameter family of Riemann surfaces (elliptic curves). If an elliptic curve is represented in the Weierstrass canonical form

$$y^2 = 4x^3 - g_2 x - g_3 \tag{2.1}$$

(where g_2 and g_3 are complex numbers) then the combination

$$J = \frac{g_2^3}{g_2^3 - 27g_3^2} \tag{2.2}$$

depends only on the equivalence class of the curve. So it can serve as the parameter of equivalence classes of elliptic curves (2.1) (in fact, any elliptic curve can be represented in the Weierstrass form (2.1)). Another choice of the parameter is the *period* of the elliptic curve

$$\tau = \oint_b \frac{dx}{y} : \oint_a \frac{dx}{y} \tag{2.3}$$

where a and b are basic cycles on the elliptic curve (i.e., on the torus) oriented in such a way that the intersection number $a \circ b = 1$. This is a complex number with positive imaginary part $\mathrm{Im}\tau > 0$. Ambiguity in the choice of the basis a, b provides the following transformation of the period τ

$$\tau \mapsto \frac{A\tau + B}{C\tau + D} \tag{2.4}$$

$$A,\ B,\ C,\ D \text{ are integers, } \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1.$$

Therefore the family of all elliptic curves can be represented as a quotient of the upper half plane $\mathbf{H}$ over the action (2.4) of the modular group $\mathbf{M}_1 = SL(2, \mathbf{Z})/(\pm 1)$.

For higher genera $g > 1$ the moduli space of Riemann surfaces has the complex dimension $3g - 3$ (see, e.g. [19]). Periods of a Riemann surface R of any genus $g > 0$ are natural parameters uniquely specifying the class of biholomorphic equivalence of the surface. These periods are defined as follows.

Let us fix a symplectic basis a_1, ..., a_g, b_1, ..., $b_g \in H_1(R, \mathbf{Z})$ of cycles on the surface R. That means that the intersection numbers of these cycles have the following canonical form

$$a_i \circ a_j = b_i \circ b_j = 0,\ a_i \circ b_j = \delta_{ij}.$$

Here δ_{ij} is the Kronecker delta. The basis of cycles a_i, b_j uniquely specifies a basis of holomorphic differentials (Abelian differentials of the first kind) on the surface R Ω_1, ..., Ω_g such that

$$\oint_{a_i} \Omega_j = \delta_{ij}. \tag{2.5}$$

The period matrix $\tau = (\tau_{ij})$ of the surface R (with respect to the symplectic basis a_i, b_j) has the form

$$\tau_{ij} = \oint_{b_i} \Omega_j, \quad i,\ j = 1, \ldots, g. \tag{2.6}$$

It is a symmetric matrix with positive definite imaginary part. For $g = 1$ it coincides with (2.3).

A change of the symplectic basis a_i, b_j implies the following transformation of the period matrix

$$\tau \mapsto \tilde{\tau} = (A\tau + B)(C\tau + D)^{-1} \tag{2.7a}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbf{Z}) = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid \begin{pmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \}. \tag{2.7b}$$

According to the classical Torelli's theorem (see [19]) the class of (biholomorphic) equivalence of a Riemann surface R is uniquely determined by the class of equivalence (2.7) of the period matrix of the surface.

Remark. Using periods of a Riemann surface R as the parameters of the theta-function (0.11) (here $m = g$) one obtains the *theta-function of the Riemann surface R.* This is a very important special function associated with a Riemann surface both in algebro-geometrical calculations (see, e.g. [20]) and in application to nonlinear equations (see the next section). The above Novikov's conjecture means that *only* theta-functions of Riemann surfaces occur in the multiphase solutions of nonlinear equations (particularly, of KP). The advantage of KP (and, more generaly, of any 2+1 integrable system) is that the theta-function of *arbitrary* Riemann surface gives a multiphase solution of the equation. Multiphase solutions of a given 1+1 integrable system (like KdV) are expressed via more particular classes of theta-functions of Riemann surfaces represented as coverings of the Riemann sphere with fixed number of sheets. So dispersion relations of none of the 1+1 integrable systems can be used for specification of period matrices of *all* the family of Riemann surfaces.

Let us consider the family of *all* $g \times g$ symmetric matrices τ with positive definite imaginary part. They form *Siegel upper half plane* $\mathbf{H}_g$. The Siegel modular group $\mathbf{M}_g = Sp(g, \mathbf{Z})/(\pm 1)$ acts on the Siegel upper half plane by the transformations (2.7) (see [21]). Theta-functions $\theta(\phi|\tau)$ and $\theta(\tilde{\phi}|\tilde{\tau})$, where $\tilde{\phi} = \phi(C\tau + D)^{-1}$, for equivalent matrices τ and $\tilde{\tau}$ coincide up to a shift of the argument and multiplication by exponential of a quadratic form of ϕ (see [21] for the explicit formula of the transformation law).

A matrix $\tau \in \mathbf{H}_g$ is called *decomposable* if it is equivalent to a block-diagonal matrix

$$\tilde{\tau} = \begin{pmatrix} \tau' & 0 \\ 0 & \tau'' \end{pmatrix}.$$

The correspondent theta-function is factorized to a product of theta-functions with less than g arguments. The period matrices of a Riemann surface always is indecomposable [19].

We are ready now to formulate the Schottky problem [31]. The periods of Riemann surfaces of a given genus g determine the *period map*

$$\text{Moduli space of Riemann surfaces of genus } g \to \mathbf{H}_g/\mathbf{M}_g.$$

The Torelli theorem provides this map to be injective (in fact, being an analytic embedding of the complex varieties: see [19]). For $g = 1, 2, 3$ the image of the period map is an open dense subset in $\mathbf{H}_g/\mathbf{M}_g$ (the completion is empty for $g = 1$ and coincides with the sublocus of decomposable matrices τ for $m = 2,\ 3$). For $g > 3$ the dimension $3g - 3$ of the moduli space of Riemann surfaces is less than the dimension $g(g + 1)/2$ of the Siegel upper half plane. The Schottky problem reads: to specify the image of the period map for $g > 3$, i.e. to find a system of $g(g + 1)/2 - (3g - 3)$ equations for unknowns $\tau_{ij} \in \mathbf{H}_g$ specifying periods of Riemann surfaces.

The above Novikov's conjecture can be reformulated as follows: *the dispersion relations (1.3) for multiphase solutions of KP as equations for τ for $g = m > 3$ exactly specify periods of Riemann surfaces.*

The main motivation for this conjecture was the Krichever's construction of algebro-geometric solutions of the KP equation given in the next section.

3. Krichever's multiphase solutions of KP.

Let us come back in more details to the multiphase solutions of KP. In fact in [13] it was constructed a family of the multiphase solutions (0.10), (0.13) where the period matrix τ is just the period matrix of a Riemann surface and components of the wave numbers and frequency vectors are certain Abelian integrals on the Riemann surface. More precisely, let R be a Riemann surface of genus g with a marked point $\infty \in R$, and with a local parameter z on R near this point such that $z(\infty) = 0$, and with a marked symplectic basis of cycles $a_i,\ b_j$. Then for the Krichever's multiphase solutions (0.10), (0.13) $m = g$, τ is the period matrix of the surface R,

$$k_j = \oint_{b_j} \eta^{(1)} \tag{3.1a}$$

$$l_j = \oint_{b_j} \eta^{(2)} \tag{3.1b}$$

$$\omega_j = \oint_{b_j} \eta^{(3)} \tag{3.1c}$$

where $\eta^{(q)}$ are the normalized

$$\oint_{a_j} \eta^{(q)} = 0,\ j = 1, \dots, g,$$

Abelian differentials of the second kind with a pole only at ∞ with the principal parts

$$\eta^{(q)} = d(z^{-q}) + \text{regular terms},\ q = 1,\ 2,\ 3.$$

Theorem 2. [13] *For an arbitrary Riemann surface R of genus g with a marked point ∞ and with a marked local parameter z near this point and with a marked symplectic basis a_i, b_j the formulae (0.10), (0.13), (2.6), (3.1) with $m = g$ determine a multiphase solution of the KP equation.*

Remark. We consider here complex multiphase solutions of KP being meromorphic functions of complex variables ϕ_1, ..., ϕ_m. One should impose certain reality constraints for the data R, ∞, z and for the phase shift ϕ_0 to obtain real smooth multiphase solutions of the two real modifications of KP: the equations KP1 (coinciding with (0.9)) and KP2 (this can be obtained from (0.9) by the substitution $y \mapsto iy$). The reality constraints were obtained in [10, 22] (for periodic multiphase solutions also in [23]). In [23] it was proved that the multiphase double periodic (in x, y) solutions of the KP2 form a dense subset in the space of all double periodic solutions of this equation.

It turns out that the solution does not depend on a choice of the symplectic basis of cycles. A change of the local parameter

$$z \mapsto a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

reads as the following transformation of the solution

$$x \mapsto a_1 x + 2a_2 y + 3a_3 t$$

$$y \mapsto a_1^2 y + 3a_1 a_2 t$$

$$t \mapsto a_1^3 t$$

$$u \mapsto u a_1^{-2} + 2(a_2^2 - a_1 a_3) a_1^{-2}$$

(so u transforms like a projective connection on the Riemann surface).

4. KP equation and Schottky problem.

Let us come back to the Novikov's conjecture (see the end of Section 2 above). The system (1.3) has trivial solutions where the theta-function is a decomposable one. To get rid of these solutions let us impose the following *nondegeneracy condition* for the theta-function: the matrix of theta-constants

$$(\hat{\theta}_{11}[p], \hat{\theta}_{12}[p], \ldots, \hat{\theta}_{mm}[p], \hat{\theta}[p])$$

(the matrix has $\frac{m(m+1)}{2} + 1$ columns and 2^m lines enumerated by the vectors $p \in \mathbf{Z}_m$) has maximal rank $= \frac{m(m+1)}{2} + 1$. The nondegeneracy condition holds for period matrices of Riemann surfaces [17]. We will consider solutions of the system (1.3) only satisfying the nondegeneracy condition.

The system of dispersion relations is invariant with respect to the action of the group of changes of local parameter z

$$k \mapsto \lambda k \tag{4.1a}$$

$$l \mapsto \pm(\lambda^2 l + 2\alpha\lambda k) \tag{4.1b}$$

$$\omega \mapsto \lambda^3 \omega + 3\lambda^2 \alpha l + 3\lambda\alpha^2 k \tag{4.1c}$$

$$d \mapsto \lambda^4 d, \quad \tau \mapsto \tau \tag{4.1d}$$

and also with respect to the following action of the Siegel modular group (2.7)

$$\tau \mapsto (A\tau + B)(C\tau + D)^{-1} \tag{4.2a}$$

$$k \mapsto kM^{-1}, \text{ where } M = C\tau + D, \tag{4.2b}$$

$$l \mapsto lM^{-1} \tag{4.2c}$$

$$\omega \mapsto \omega M^{-1} + \frac{1}{3}\{k,k\}kM^{-1}, \text{ where } \{x,y\} = xM^{-1}Cy^{\mathrm{T}} \tag{4.2d}$$

$$d \mapsto d + \frac{3}{8}\{l,l\} - \frac{1}{2}\{k,\omega\} - \frac{3}{4}\{k,k\}^2. \tag{4.2e}$$

Hence eliminating the variables k, l, ω, d from the dispersion relations we obtain a system of equations (for $m > 3$) for the matrix τ being invariant with respect to the action of the modular group (i.e. if the system (1.3) is compatible for a period matrix τ then it will be compatible for any matrix $\tilde{\tau}$ being equivalent (2.7) to τ). So the compatibility conditions of the system (1.3) specify a sublocus

$$X_m \subset \mathbf{H}_m/\mathbf{M}_m.$$

Due to the Theorem 2 this sublocus contains period matrices of Riemann surfaces of genus $g = m$.

The first test of the Novikov's conjecture was done in [17]: it was shown that the irreducible component of X_m containing period matrices of Riemann surfaces consists only of period matrices of Riemann surfaces. Even this statement sounds surprising. Indeed, from the construction of the Section 3 it follows that for a given τ = period matrix of a Riemann surface the system (1.3) is more than compatible: it has a one-parameter family of solutions due to ambiguity in the choice of the marked point ∞ on the Riemann surface. So the crucial point to prove the Novikov's conjecture is to prove that from compatibility of (1.3) it follows that (1.3) has a one-parameter family of solutions $k = k(\tau, \epsilon)$, $l = l(\tau, \epsilon)$, $\omega = \omega(\tau, \epsilon)$. If we identify the parameter ϵ with the z-coordinate of the displaced marked point $\infty \mapsto \infty'$, $\epsilon = z(\infty')$, then the coefficients of the expansion

$$k(\tau, \epsilon) = k + \frac{\epsilon}{2}k^{(2)} + \frac{\epsilon^2}{2}k^{(3)} + \dots$$

are periods of the normalized Abelian differentials of the second kind with poles at ∞,

$$k_i^{(q)} = c_q \oint_{b_i} \eta^{(q)}, \ i = 1, \dots, g$$

(for certain constants c_q). One has $k^{(2)} = l$, $k^{(3)} = \omega$, other vectors $k^{(q)}$ are the frequencies of the multiphase solutions of the same form (0.10), (0.13) to the q-th equation of the KP hierarchy. Thus one needs to prove that any theta-function solution (0.10), (0.13) of KP can be extended to a solution of all the KP hierarchy.

The final step in proof of Novikov's conjecture was obtained by T.Shiota who proved that X_m has no extra irreducible components:

Theorem 3.[24] *An indecomposable theta-function gives a multiphase solution (0.10), (0.13) of the KP equation iff it is the theta-function of a Riemann surface.*

Because of limits of the paper we have no possibility to discuss here this remarkable theorem. Another proof of the Novikov's conjecture was obtained by E.Arbarello and C.De Concini [25].

Conclusion.

Investigating of dispertion relations of other integrable differential equations [26-28, 32-38] turned out to be fruitful for the algebraic geometry of Abelian varieties (e.g., Prym varieties [28]) as well as for the theory and applications of integrable systems. An approach to calculation parameters of *real* multiphase solutions of KP based on the classical theory of Schottky uniformization of Riemann surfaces and Burnside series was proposed in [29]. Recently it was found [30] that periods of Riemann surfaces with fixed number of sheets and fixed ramification at infinity as functions of moduli of these Riemann surfaces are themselves solutions of certain integrable systems arising in topological field theory. But this stuff, probably, should be a part of the next decade history of the soliton theory.

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